	Solution
Q1(i)	Let P_n be the statement, $8 \mid 4n(n+1)$ for $n \in \mathbf{Z}^+ \cup \{0\}$. Since $8 \mid 4(0)(1)$, P_0 is true. Suppose P_k is true for some $k \in \mathbf{Z}^+ \cup \{0\}$. Then, $\begin{aligned} 4(k+1)(k+2) &= 4(k)(k+1)+4(2)(k+1) \\ &= 4(k)(k+1)+8(k+1). \end{aligned}$ Hence $(\forall k \in \mathbf{Z}^+ \cup \{0\}.\, P_k \Rightarrow P_{k+1})$ is true. Since P_0 is true and $(\forall k \in \mathbf{Z}^+ \cup \{0\}.\, P_k \Rightarrow P_{k+1})$ is true, by the Principle of Mathematical Induction, P_n is true for $n \in \mathbf{Z}^+ \cup \{0\}$.
Q1(ii)	Let P_n be the statement, $2^{n+2} \mid \left(a^{2^n}-1\right)$ for positive odd integers *a* and positive integers *n*. Since *a* is a positive odd integer, let $a = 2m+1$ for some nonnegative integer *m*. By **(i)**, $2^3 \mid (2m+1)^2-1$. Hence P_1 is true. Suppose P_k is true for some $k \in \mathbf{Z}^+$. Then, $\begin{aligned} (2m+1)^{2^{k+1}}-1 &= \left[(2m+1)^{2^k}\right]^2-1 \\ &= \left[(2m+1)^{2^k}-1\right]\left[(2m+1)^{2^k}+1\right]. \end{aligned}$ We note that $2^{k+3} \mid \left[(2m+1)^{2^{k+1}}-1\right]$ since $2^{k+2} \mid \left[(2m+1)^{2^k}-1\right]$ by P_k and $2 \mid \left[(2m+1)^{2^k}+1\right]$. In particular, $(2m+1)^{2^k}$ is odd since powers of odd numbers are odd (because there does not exist a single copy of 2 in its prime factorization). Hence $(\forall k \in \mathbf{Z}^+.\, P_k \Rightarrow P_{k+1})$ is true. Since P_1 is true and $(\forall k \in \mathbf{Z}^+.\, P_k \Rightarrow P_{k+1})$ is true, by the Principle of Mathematical Induction, P_n is true for $n \in \mathbf{Z}^+$.
Q2(a)	By Euler's Theorem, we have $3^{\phi(10)} \equiv 1 \pmod{10} \Leftrightarrow 3^4 \equiv 1 \pmod{10}$. By the Division Algorithm, we have $\mathbb{Z}^+ = \{4k-3, 4k-2, 4k-1, 4k \mid k \in \mathbb{Z}^+\}$. It is clear by induction that $3^{4k-3} \equiv 3 \pmod{10}$ for all positive integers *k*. Then, **Case 1:** If $n = 4k-3$, then $3^n = 3^{4k-3} \equiv 3 \pmod{10}$.

	Case 2: If $n=4k-2$, then $3^n=3^{4k-2}=\left(3^{4k-3}\right)(3)\equiv 9 \text{ (mod 10)}$. **Case 3:** If $n=4k-1$, then $3^n=3^{4k-1}=\left(3^{4k-3}\right)\left(3^2\right)\equiv 27\equiv 7 \text{ (mod 10)}$. **Case 4:** If $n=4k$, then $3^n=3^{4k}=\left(3^{4k-3}\right)\left(3^3\right)\equiv 81\equiv 1 \text{ (mod 10)}$. Hence the unit digits of $3^n, n\in\mathbf{Z}^+$ are 1,3,7 and 9.
Q2(b)	By the Division Algorithm, we have $\square^{+}=\left\{2k-1,2k \mid k\in\square^{+}\right\}$. It is clear by induction that $6^{2k-1}\equiv 6 \text{ (mod 10)}$ for any positive integer *k*. Since $6^{2k-1}\equiv 6 \text{ (mod 10)}$ and $6\equiv 6 \text{ (mod 10)}$, we have $6^{2k}\equiv 36\equiv 6 \text{ (mod 10)}$. Hence the unit digit of $6^n, n\in\square^{+}$ is 6.
Q3(a)	Let $n\in\mathbf{Z}$. By the Division Algorithm, *n* is of the form $3q, 3q+1, 3q+2$. Hence n^2 is of the form $9q^2, 9q^2+6q+1, 9q^2+12q+4$ $\Leftrightarrow 3\left(3q^2\right), 3\left(3q^2+2q\right)+1, 3\left(3q^2+4q+1\right)+1.$ This means that $n^2\equiv 0 \text{ (mod 3)}$ or $n^2\equiv 1 \text{ (mod 3)}$. Since $3a^2-1\equiv 2 \text{ (mod 3)}$, it shows that $3a^2-1$ is not a perfect square.
Q3(b)(i)	By applying the Division Algorithm repeatedly, we have $186=42(4)+18$ $42=18(2)+6$ $18=6(3)+0.$ The Euclidean Algorithm states that $\gcd(186,42)=\gcd(42,18)=\gcd(18,6)=\gcd(6,0)=6$ Working backwards, we have $6=42-18(2)$ $=42-(186-42(4))(2)$ $36=42(54)+186(-12).$
Q3(b)(ii)	It is known that a Diophantine equation of the form $ax+by=c$ has a solution if and only if $\gcd(a,b)\mid c$. Thus, $nx+42y=36$ has a solution if and only if $\gcd(n,42)\mid 36$. We note that $42=(2)(3)(7)$ and $36=\left(2^2\right)\left(3^2\right)$. Let $k\in\mathbf{Z}$. Since $\gcd(7k,42)$ picks the sevens from their respective prime factorizations, this means that $\gcd(7k,42)$ does not divide 36. Hence 182 and 189

	are among the values of n. Let m be a number from 176 to 195 excluding 182 and 189. Since m does not contain seven in its prime factorization, this means that $\gcd(m, 42)$ picks only two or three or both from their respective prime factorizations. Hence $\gcd(m, 42)$ divides 36.

Q4(i)	Case 1: Start with X or Y follow by (n-1) letters with same conditions. No. of ways $= 2P_{n-1}$ Case 2: Start with Z follow by even number of Zs. No. of ways $= 3^{n-1} - P_{n-1}$ Hence, $P_n = 2P_{n-1} + 3^{n-1} - P_{n-1}$ $P_n = 3^{n-1} + P_{n-1}, \quad n \geq 1, \quad P_1 = 1$
Q4(ii)	$P_n = 3^{n-1} + P_{n-1}$ $\therefore P_n - P_{n-1} = 3^{n-1}$ Hence, $P_2 - P_1 = 3$ $+ \ P_3 - P_2 = 3^2$ $+ \ P_4 - P_3 = 3^3$ $\vdots$ $+ \ P_n - P_{n-1} = 3^{n-1}$

	$\Rightarrow P_n - P_1 = \dfrac{3\left(1-3^{n-1}\right)}{1-3}$ $\Rightarrow P_n = \dfrac{3}{2}\left(3^{n-1}-1\right)+1 \quad (\because P_1 = 1)$ $\Rightarrow P_n = \dfrac{1}{2}\left(3^n-1\right)$
Q5(a)(i)	By Pigeonhole Principle, 3(3) +1 = 10
Q5(a)(ii)	By Pigeonhole Principle, $6+\left[7\left(2\right)+1\right]=21$
Q5(b)	$A_1 = \{1\}$ $\quad \dfrac{1}{1} \in \left[\dfrac{2}{3},\dfrac{3}{2}\right]$ $A_2 = \{2,3\}$ $\quad \dfrac{2}{3}$ and $\dfrac{3}{2} \in \left[\dfrac{2}{3},\dfrac{3}{2}\right]$ $A_3 = \{4,5,6\}$ $\quad \dfrac{4}{6}$ and $\dfrac{6}{4} \in \left[\dfrac{2}{3},\dfrac{3}{2}\right]$ $A_4 = \{7,8,9,10\}$ $\quad \dfrac{7}{10}$ and $\dfrac{10}{7} \in \left[\dfrac{2}{3},\dfrac{3}{2}\right]$ $A_5 = \{11,12,\ldots,16\}$ $\quad \dfrac{11}{16}$ and $\dfrac{16}{11} \in \left[\dfrac{2}{3},\dfrac{3}{2}\right]$ $A_6 = \{17,18,\ldots,25\}$ $\quad \dfrac{17}{25}$ and $\dfrac{25}{17} \in \left[\dfrac{2}{3},\dfrac{3}{2}\right]$ $A_7 = \{26,27,\ldots,39\}$ $\quad \dfrac{26}{39}$ and $\dfrac{39}{26} \in \left[\dfrac{2}{3},\dfrac{3}{2}\right]$ $A_8 = \{40,41,\ldots,60\}$ $\quad \dfrac{40}{60}$ and $\dfrac{60}{40} \in \left[\dfrac{2}{3},\dfrac{3}{2}\right]$ $A_9 = \{61,62,\ldots,91\}$ $\dfrac{61}{91}$ and $\dfrac{91}{61} \in \left[\dfrac{2}{3},\dfrac{3}{2}\right]$ 9 pigeonholes and 10 numbers. There will be 2 numbers from the same pigeonhole whose ratio of these 2 numbers lies in the interval $\left[\dfrac{2}{3},\dfrac{3}{2}\right]$.
Q6(a)(i)	Case 1: aaaaaa $\dbinom{6}{1}=6$

	Case 2: aaaaab $\binom{6}{1}\binom{5}{1}$ or $\binom{6}{2}\binom{2}{1} = 30$ Case 3: aaaabb $\binom{6}{1}\binom{5}{1}$ or $\binom{6}{2}\binom{2}{1} = 30$ Case 4: aaaabc $\binom{6}{1}\binom{5}{2}$ or $\binom{6}{3}\binom{3}{1} = 60$ Case 5: aaabbb $\binom{6}{2} = 15$ Case 6: aaabbc $\binom{6}{1}\binom{5}{2}\binom{2}{1}$ or $\binom{6}{3}\binom{3}{1}\binom{2}{1} = 120$ Case 7: aaabcd $\binom{6}{1}\binom{5}{3}$ or $\binom{6}{4}\binom{4}{1} = 60$ Total = 321
Q6(a)(ii)	No.of ways to distribute 6 identical balls into 6 distinct boxes $= \binom{11}{6} = 462$ No digit occurs more than two times = 462 - 321 =141.
Q6(b)	$S_i =$ 'i' is in position i. No. of ways $= \lvert S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 \cup S_6 \rvert$ $= 6(5!) - \binom{6}{2}(4!) + \binom{6}{3}(3!) - \binom{6}{4}(2!) + \binom{6}{5}(1!) - 1$ $= 455$
Q6(b)	$D_6 = 6! - 455$ $= 265$
Q7(a)	(1) Write $2\sqrt{m+1} - 2\sqrt{m} = \dfrac{2\left(\sqrt{m+1}-\sqrt{m}\right)\left(\sqrt{m+1}+\sqrt{m}\right)}{\sqrt{m+1}+\sqrt{m}}$ $= \dfrac{2}{\sqrt{m+1}+\sqrt{m}} < \dfrac{2}{2\sqrt{m}} = \dfrac{1}{\sqrt{m}}$ (2) Similarly, $2\sqrt{m} - 2\sqrt{m-1} = \dfrac{2\left(\sqrt{m}-\sqrt{m-1}\right)\left(\sqrt{m}+\sqrt{m-1}\right)}{\sqrt{m}+\sqrt{m-1}}$ $= \dfrac{2}{\sqrt{m}+\sqrt{m-1}} > \dfrac{2}{2\sqrt{m}} = \dfrac{1}{\sqrt{m}}$

	From (1) and (2), it implies $2\sqrt{m+1}-2\sqrt{m}<\dfrac{1}{\sqrt{m}}<2\sqrt{m}-2\sqrt{m-1}$. (*) Substituting $m=2,3,\ldots,n$ into (*), $2\sqrt{3}-2\sqrt{2}<\dfrac{1}{\sqrt{2}}<2\sqrt{2}-2\sqrt{1}$ $2\sqrt{4}-2\sqrt{3}<\dfrac{1}{\sqrt{3}}<2\sqrt{3}-2\sqrt{2}$ $\vdots$ $2\sqrt{n+1}-2\sqrt{n}<\dfrac{1}{\sqrt{n}}<2\sqrt{n}-2\sqrt{n-1}$ Adding all the inequalities above, $2\sqrt{n+1}-2\sqrt{2}<\dfrac{1}{\sqrt{2}}+\dfrac{1}{\sqrt{3}}+\cdots+\dfrac{1}{\sqrt{n}}<2\sqrt{n}-2\sqrt{1}$ Since $2\sqrt{n}<2\sqrt{n+1}$ and $2\sqrt{2}<3$, we have $2\sqrt{n}-3<2\sqrt{n+1}-2\sqrt{2}$ Therefore, $2\sqrt{n}-3<\dfrac{1}{\sqrt{2}}+\dfrac{1}{\sqrt{3}}+\cdots+\dfrac{1}{\sqrt{n}}<2\sqrt{n}-2$. Now, add 1 throughout we have $2\sqrt{n}-2<\dfrac{1}{\sqrt{2}}+\dfrac{1}{\sqrt{3}}+\cdots+\dfrac{1}{\sqrt{n}}<2\sqrt{n}-1$ Set $n=100$, $2\sqrt{100}-2<1+\dfrac{1}{\sqrt{2}}+\dfrac{1}{\sqrt{3}}+\cdots+\dfrac{1}{\sqrt{100}}<2\sqrt{100}-1$ $18<1+\dfrac{1}{\sqrt{2}}+\dfrac{1}{\sqrt{3}}+\cdots+\dfrac{1}{10}<19$ $a=18, b=19$
Q7(b)	Let $\mathrm{P}(n)$ be "$u_n=\alpha^n+\beta^n$ for $n\geq 1, n\in\mathbb{Z}^{+}$". For $n=1$, $\mathrm{LHS}=u_1=1$ $\mathrm{RHS}=\alpha+\beta=-(-1)=1$ For $n=2$, $\mathrm{LHS}=u_2=3$ $\mathrm{RHS}=\alpha^2+\beta^2$ $=(\alpha+\beta)^2-2\alpha\beta$ $=3$ $\Rightarrow \mathrm{P}(1), \mathrm{P}(2)$ are true. Assume $u_{k-1}=\alpha^{k-1}+\beta^{k-1}$ and $u_k=\alpha^k+\beta^k$ for some $k\geq 2$. (*) For $n=k+1$,

	$u_{k+1} = u_k + u_{k-1}$ $= \alpha^k + \beta^k + \alpha^{k-1} + \beta^{k-1}$ $= \alpha^k + \alpha^{k-1} + \beta^k + \beta^{k-1}$ $= \alpha^{k-1}(\alpha+1) + \beta^{k-1}(\beta+1)$ $= \alpha^{k-1}\alpha^2 + \beta^{k-1}\beta^2$ $= \alpha^{k+1} + \beta^{k+1}$ $\mathrm{P}(k+1)$ is true. By MI, $\mathrm{P}(n)$ is true for $n \geq 1, n \in \mathbb{Z}^+$. (*)
Q8(i)	Putting $y = 0$, $\mathrm{f}(x) = \mathrm{f}(x) + \mathrm{f}(0) + \mathrm{f}(x)\mathrm{f}(0)$ $\mathrm{f}(0)\left[1 + \mathrm{f}(x)\right] = 0$ $\Rightarrow \mathrm{f}(0) = 0$ or $\mathrm{f}(x) = -1$ (rejected since f is a non-constant function) And now suppose on the contrary, $\mathrm{f}(r) = -1$ for some $r \in \mathbb{R}$, $\mathrm{f}(x+r) = \mathrm{f}(x) + \mathrm{f}(r) + \mathrm{f}(x)\mathrm{f}(r)$ $\mathrm{f}(x+r) = \mathrm{f}(r) = -1$ Putting $x = -r$, we have $\mathrm{f}(0) = -1$, which is a contradiction. Hence $\mathrm{f}(x) \neq -1$ for all $x \in \mathbb{R}$.
Q8(ii)	Let $\mathrm{f}(x) = \mathrm{f}\left(\frac{x}{2} + \frac{x}{2}\right)$. $\mathrm{f}(x) = \mathrm{f}\left(\frac{x}{2}\right) + \mathrm{f}\left(\frac{x}{2}\right) + \mathrm{f}\left(\frac{x}{2}\right)\mathrm{f}\left(\frac{x}{2}\right)$ $\mathrm{f}(x) = \mathrm{f}\left(\frac{x}{2}\right) + \mathrm{f}\left(\frac{x}{2}\right) + \mathrm{f}\left(\frac{x}{2}\right)\mathrm{f}\left(\frac{x}{2}\right) + 1 - 1$ $= \left(\mathrm{f}\left(\frac{x}{2}\right) + 1\right)^2 - 1$ > -1 since $\mathrm{f}(x) \neq -1$.
Q8(iii)	$\mathrm{f}'(x) = \lim_{h \to 0} \frac{\mathrm{f}(x+h) - \mathrm{f}(x)}{h}$

	$=\lim_{h\to 0}\frac{\mathrm{f}(h)+\mathrm{f}(x)\mathrm{f}(h)}{h}$ $=\lim_{h\to 0}\frac{\mathrm{f}(h)\left[1+\mathrm{f}(x)\right]}{h}$ $=a\left[1+\mathrm{f}(x)\right]$ Suppose $a=0$, $\Rightarrow \mathrm{f}'(x)=0$ for all $x\in\mathbb{R}$. But f is non-constant, therefore $a\neq 0$.
8(iv)	$\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln\left[1+\mathrm{f}(x)\right]\right)=\frac{\mathrm{f}'(x)}{1+\mathrm{f}(x)}=a$ $\Rightarrow \ln\left[1+\mathrm{f}(x)\right]=ax+c$ Putting $\mathrm{f}(0)=0$, we have $c=0$. $\ln\left[1+\mathrm{f}(x)\right]=ax$ $\Rightarrow \mathrm{f}(x)=\mathrm{e}^{ax}-1$
Q9(a)	By Cauchy- Schwarz Inequality, $(a\cdot 1+b\cdot 1+c\cdot 1+d\cdot 1)^2\leq (a^2+b^2+c^2+d^2)(1^2+1^2+1^2+1^2)$ $(8-e)^2\leq 4(16-e^2)$ $5e^2-16\leq 0$ $e(5e-16)\leq 0$ $0\leq e\leq\frac{16}{5}$ Therefore the greatest value of e is $\frac{16}{5}$. Equality holds if $\frac{a}{1}=\frac{b}{1}=\frac{c}{1}=\frac{d}{1}$ and $e=\frac{16}{5}$. Since $a+b+c+d+e=8$, $a=b=c=d=\frac{6}{5}$.
Q9(b)(i)	For $k\leq t<k+1$, we have $\lfloor t\rfloor=k$. Thus, $\int_k^{k+1}\lfloor t\rfloor \mathrm{f}'(t)\,\mathrm{d}t=\int_k^{k+1}k\,\mathrm{f}'(t)\,\mathrm{d}t$ $=k\left[\mathrm{f}(t)\right]_k^{k+1}$ $=k\left[\mathrm{f}(k+1)-\mathrm{f}(k)\right]$ For $k\geq 2$, $\int_1^x\lfloor t\rfloor \mathrm{f}'(t)\,\mathrm{d}t=\int_1^{\lfloor x\rfloor}\lfloor t\rfloor \mathrm{f}'(t)\,\mathrm{d}t+\int_{\lfloor x\rfloor}^x\lfloor t\rfloor \mathrm{f}'(t)\,\mathrm{d}t$

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$$= \sum_{k=1}^{\lfloor x \rfloor -1} \int_k^{k+1} \lfloor t \rfloor \mathrm{f}'(t)\,\mathrm{d}t + \int_{\lfloor x \rfloor}^{x} \lfloor x \rfloor \mathrm{f}'(t)\,\mathrm{d}t$$

$$= \sum_{k=1}^{\lfloor x \rfloor -1} k\left(\mathrm{f}(k+1) - \mathrm{f}(k)\right) + \left[\lfloor x \rfloor \mathrm{f}(t)\right]_{\lfloor x \rfloor}^{x}$$

$$= \sum_{k=1}^{\lfloor x \rfloor -1} k\,\mathrm{f}(k+1) - \sum_{k=1}^{\lfloor x \rfloor -1} k\,\mathrm{f}(k) + \lfloor x \rfloor \mathrm{f}(x) - \lfloor x \rfloor \mathrm{f}(\lfloor x \rfloor)$$

$$= \sum_{k=2}^{\lfloor x \rfloor} (k-1)\mathrm{f}(k) - \left(\sum_{k=1}^{\lfloor x \rfloor -1} k\,\mathrm{f}(k) + \lfloor x \rfloor \mathrm{f}(\lfloor x \rfloor)\right) + \lfloor x \rfloor \mathrm{f}(x)$$

$$= \sum_{k=2}^{\lfloor x \rfloor} (k-1)\mathrm{f}(k) - \sum_{k=1}^{\lfloor x \rfloor} k\,\mathrm{f}(k) + \lfloor x \rfloor \mathrm{f}(x)$$

$$= \lfloor x \rfloor \mathrm{f}(x) - \sum_{k=1}^{\lfloor x \rfloor} \mathrm{f}(k)$$

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Q9(b)(ii)

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Use substitution $u = \mathrm{e}^t$, $\int_0^1 \lfloor \mathrm{e}^t \rfloor \,\mathrm{d}t = \int_1^{\mathrm{e}} \lfloor u \rfloor \frac{1}{u}\,\mathrm{d}u$ --------(*)

Let $\mathrm{f}(u) = \ln u$ so that $\mathrm{f}'(u) = \frac{1}{u}$.

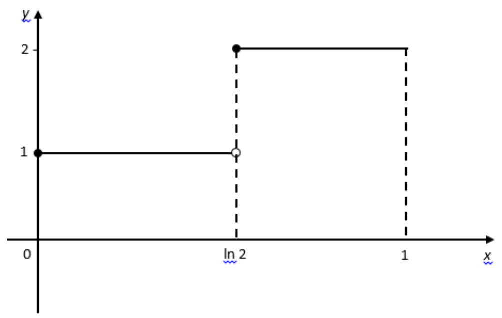
From (i), we have $\lfloor e \rfloor = 2$ (*) becomes

$$\int_0^1 \lfloor \mathrm{e}^t \rfloor \,\mathrm{d}t = \int_1^{\mathrm{e}} \lfloor u \rfloor \frac{1}{u}\,\mathrm{d}u = \int_1^{\mathrm{e}} \lfloor u \rfloor \mathrm{f}'(u)\,\mathrm{d}u$$

$$= 2\ln \mathrm{e} - \sum_{k=1}^{2} \ln k$$

$$= 2 - (\ln 1 + \ln 2) = 2 - \ln 2$$

Alternative solution (by graph):
sketch $y = \lfloor e^x \rfloor$ for $0 \le x \le 1$



$\int_0^1 \lfloor \mathrm{e}^t \rfloor \,\mathrm{d}t$ = Area under the graph

$= 2(1 - \ln 2) + \ln 2$

$= 2 - \ln 2$

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