HWA CHONG INSTITUTION
JC2 Preliminary Examination
Higher 3

MATHEMATICS **9820/01**

Paper 1 **21 September 2017**

3 hours

Additional Materials: Answer Paper
List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name and CT group on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphic calculator.
Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **5** printed pages.

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MATHEMATICS

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Name: | **CT:** | | | | |

C OVER PAGE

1. Write your name, CT group and calculator model(s) in the spaces provided.
2. Arrange your answers in numerical order.
3. Place this cover page on top of your answer sheets and fasten them securely together with the string provided.

For Examiner's Use			
Question No.	**Marks Obtained**	**Total Marks**	**Remarks**
1		10	
2		9	
3		11	
4		10	
5		9	
6		11	
7		12	
8		14	
9		14	
TOTAL		**100**	

Graphing Calculator Model:

Scientific Calculator Model:

1. **(i)** Prove that 8 divides $4n(n+1)$ for all nonnegative integers *n*. [5]

(ii) Hence or otherwise, prove that 2^{n+2} divides $a^{2^n}-1$ for all positive odd integers *a* and for all positive integers *n*. [5]

2. **(a)** **Definition:** For $n \in \mathbf{Z}^+$, let $\phi(n)$ denote the number of positive integers not exceeding *n* that are coprime to *n*. For example, $\phi(1)=1=|\{1\}|$ since $\gcd(1,1)=1$ and $\phi(3)=2=|\{1,2\}|$ since $\gcd(1,3)=1$ and $\gcd(2,3)=1$.

Result: (*Euler's Theorem*). For $a \in \mathbf{Z}$ and $n \in \mathbf{Z}^+$, if $\gcd(a,n)=1$, then

$$a^{\phi(n)} \equiv 1 \pmod{n}.$$

Use Euler's Theorem to find the unit digits of 3^n for all positive integers *n*.

[5]

(b) Prove that the unit digit of 6^n is 6 for all positive integers *n*. [4]

3. **(a)** Prove that $3a^2-1$ is never a perfect square for all integers *a*. [4]

(b) A *Diophantine equation* is an equation in which one or more unknowns are to be solved in the integers.

(i) Find a solution for the Diophantine equation, $186x+42y=36$. [4]

(ii) For $n \in \mathbf{Z}^+$ with $176 \le n \le 195$, find the values of *n* for which the Diophantine equation, $nx+42y=36$ does not have a solution. [3]

4. Let P_n be the number of *n*-letter words that use letters from the set $\{X, Y, Z\}$ and contain an odd number of Zs.

(i) Find a recurrence relation satisfied by P_n. [5]

(ii) Find a non-recursive formula for P_n, for all $n \ge 1$. [5]

5. **(a)** There are 27 sweets in a jar, of which 6 are strawberry-flavoured, 9 are orange-flavoured and 12 are grape-flavoured. A child takes out some sweets from the jar without looking into the jar. Find the number of sweets the child must take out from the jar to get

(i) 4 sweets of the same flavour; [2]

(ii) 8 sweets of the same flavour. [2]

(b) Ten numbers are chosen from the first 91 positive integers. Prove that there exists two of these numbers whose ratios lie in the interval $\left[\frac{2}{3},\frac{3}{2}\right]$.

For example, if the two numbers are 8 and 10, then their ratios $\frac{8}{10}$ and $\frac{10}{8}$ lie in the interval $\left[\frac{2}{3},\frac{3}{2}\right]$. However, if the two numbers are 5 and 10, then both their ratios $\frac{5}{10}$ and $\frac{10}{5}$ do not lie in the interval $\left[\frac{2}{3},\frac{3}{2}\right]$. [5]

6. **(a)** Find the number of six-digit combinations from the set $\{3,4,5,6,7,8\}$ such that

(i) some digits occur at least three times. [3]

(ii) No digit occurs more than two times. [2]

(b) Find the number of arrangements of the string 345678 in which at least one of the digits is in its original position. [4]

Hence find the derangement number D_6. [2]

(A permutation $a_1a_2\cdots a_n$ of the numbers 1, 2, 3, …, n is called a derangement if $a_i \neq i$ for each $i=1$, 2, …, n. D_n is the number of derangements of 1, 2, 3, …, n).

7. **(a)** Prove that $2\sqrt{m+1}-2\sqrt{m}<\frac{1}{\sqrt{m}}<2\sqrt{m}-2\sqrt{m-1}$ for $m\in\mathbb{Z}^+$. Hence show that

$$2\sqrt{n}-3<\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{n}}<2\sqrt{n}-2,\text{ where } n\in\mathbb{Z}^+.$$

Deduce the greatest value of a and least value of b such that

$$a<1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{10}<b.$$

[7]

(b) Let $u_1=1$, $u_2=3$ and $u_n=u_{n-2}+u_{n-1}$ for all $n\geq 3$, $n\in\mathbb{Z}^+$. Given α,β are roots of the equation $x^2-x-1=0$. By considering sum and product of roots of a quadratic equation, prove that $u_n=\alpha^n+\beta^n$ for $n\geq 1$ by mathematical induction.

[5]

8. Let $\mathrm{f}:\mathbb{R}\to\mathbb{R}$ be a non-constant function which satisfies the following conditions:

(1) $\mathrm{f}(x+y)=\mathrm{f}(x)+\mathrm{f}(y)+\mathrm{f}(x)\mathrm{f}(y)$ for all $x,y\in\mathbb{R}$.

(2) $\lim_{h\to 0}\frac{\mathrm{f}(h)}{h}=a$, where a is a real constant.

(i) Prove that $\mathrm{f}(0)=0$, and $\mathrm{f}(x)\neq -1$ for all $x\in\mathbb{R}$. [4]

(ii) Show that $\mathrm{f}(x)>-1$ for all $x\in\mathbb{R}$. [3]

(iii) Show that $\mathrm{f}'(x)=a\left[1+\mathrm{f}(x)\right]$ and deduce that $a\neq 0$. [4]

(iv) By considering the derivative of $\ln\left[1+\mathrm{f}(x)\right]$, show that $\mathrm{f}(x)=\mathrm{e}^{ax}-1$. [3]

9. **(a)** Let a,b,c,d and e be real numbers such that $a+b+c+d+e=8$ and $a^2+b^2+c^2+d^2+e^2=16$. By considering the Cauchy-Schwarz inequality, find the respective values of a,b,c and d so that the greatest value of e is attained.

[4]

(b) Let $\lfloor x\rfloor$ be the greatest integer less than or equal to x where $x\in\mathbb{R}$. It is given that $\mathrm{f}:\mathbb{R}\to\mathbb{R}$ is a function whose f' exists for all $x\in\mathbb{R}$.

(i) For any $k\in\mathbb{Z}$, prove that $\int_k^{k+1}\lfloor t\rfloor\,\mathrm{f}'(t)\,\mathrm{d}t=k\left(f(k+1)-f(k)\right)$. Hence, show that for all real numbers $x\geq 2$, $\int_1^x\lfloor t\rfloor\,\mathrm{f}'(t)\,\mathrm{d}t=\lfloor x\rfloor\,\mathrm{f}(x)-\sum_{k=1}^{\lfloor x\rfloor}\mathrm{f}(k)$.

[7]

(ii) By using the substitution $u=\mathrm{e}^t$ and the earlier result, or otherwise, find the exact value of $\int_0^1\lfloor \mathrm{e}^t\rfloor\,\mathrm{d}t$. [3]

END