No.	Suggested Solution
1	$\binom{n+p}{p} = \frac{(n+p)!}{n!\,p!}$ $= \frac{(n+p)(n+p-1)...(p+1)}{n(n-1)...(2)(1)}$ Observe that $n+p \equiv n \pmod p$ $n+p-1 \equiv n-1 \pmod p$ $n+p-2 \equiv n-2 \pmod p$ $\vdots$ $p+1 \equiv 1 \pmod p$. Therefore, we have $(n+p)(n+p-1)...(p+1) \equiv n! \pmod p$ $\frac{(n+p)(n+p-1)...(p+1)}{n!} n! \equiv n! \pmod p$ $\binom{n+p}{p} n! \equiv n! \pmod p$ $\binom{n+p}{p} \equiv 1 \left(\bmod \frac{p}{\gcd(n!, p)}\right)$ $\binom{n+p}{p} \equiv 1 \pmod p \quad \left(\begin{array}{l}\gcd(n!, p) = 1 \\ \because\ p \text{ is a prime and } p > n\end{array}\right)$

No.	Suggested Solution
2	Let $d_1 = \gcd(a,b,c)$ and $d_2 = \gcd[\gcd(a,b),c]$. By definition, $d_1 \mid a$ and $d_1 \mid b$. Therefore, $d_1 \mid \gcd(a,b)$. Also, $d_1 \mid c$. Since $d_1 \mid \gcd(a,b)$ and $d_1 \mid c$, we have $d_1 \mid \gcd[\gcd(a,b),c]$, i.e., $d_1 \mid d_2$. By definition, $d_2 \mid \gcd(a,b)$ and $d_2 \mid c$. Since $d_2 \mid \gcd(a,b)$ and $\gcd(a,b) \mid a$ and $\gcd(a,b) \mid b$, we have $d_2 \mid a$ and $d_2 \mid b$. Now, as $d_2 \mid a$, $d_2 \mid b$ and $d_2 \mid c$, it means that d_2 is a common divisor of *a*, *b* and *c*. Therefore, $d_2 \mid \gcd(a,b,c)$, i.e., $d_2 \mid d_1$. $d_1 \mid d_2$ and $d_2 \mid d_1$ implies that $d_1 = \pm d_2$. However, since d_1 and d_2 are greatest common divisors, d_1 and d_2 are positive. Hence, $d_1 = d_2$, i.e., $\gcd(a,b,c) = \gcd[\gcd(a,b),c]$.

No.	Suggested Solution
3	Suppose there are finitely many primes of the form $6k+5$. Denote these primes by $p_0, p_1, p_2, \ldots, p_n$ where $5 = p_0 < p_1 < p_2 < \ldots < p_n$. Consider the integer $M = 6p_1p_2 \ldots p_n + 5$. Since $M > p_n$ and *M* is of the form $6k+5$, *M* must be composite. $2 \mid 6p_1p_2\ldots p_n$ and $2 \nmid 5$ imply that $2 \nmid M$. Similarly, $3 \mid 6p_1p_2\ldots p_n$ and $3 \nmid 5$ imply that $3 \nmid M$. Also, $5 \nmid 6p_1p_2\ldots p_n$ and $5 \mid 5$ imply that $5 \nmid M$. Note that all primes greater than 3 are of the form $6k+1$ or $6k+5$. Claim: M has a prime divisor of the form $6k+5$. Proof: If *M* only has prime factors of the form $6k+1$, then, *M* will be of the form $6k+1$ as well since the product of two integers of this form, $\begin{aligned}(6a+1)(6b+1) &= 36ab+6a+6b+1 \\ &= 6(6ab+a+b)+1\end{aligned}$ is of the form $6k+1$. Therefore, *M* has at least one prime divisor of the form $6k+5$, call it *p*. Thus, $p \in \{p_0, p_1, p_2, \ldots, p_n\}$. However, since we have established that $5 \nmid M$, we can say that $p \neq 5$ and thus $p \in \{p_1, p_2, \ldots, p_n\}$. $\Rightarrow p \mid 6p_1p_2\ldots p_n$ and $p \nmid 5$ $\Rightarrow p \nmid 6p_1p_2\ldots p_n + 5$ $\Rightarrow p \nmid M$. This contradicts the statement that *M* has at least one prime divisor of the form $6k+5$ in the set $\{p_1, p_2, \ldots, p_n\}$. Therefore, *M* is prime (of the form $6k+5$), contradicting the statement that *M* is composite. Therefore, there are infinitely many primes of the form $6k+5$.

No.	Suggested Solution
4(i)	Let $\mathrm{P}(n)$ be the statement $1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{2^n} > \frac{n+1}{2}$ for all positive integers *n*. When $n=1$, $1+\frac{1}{2}=\frac{3}{2}>\frac{2}{2}=\frac{1+1}{2}$. Therefore, $\mathrm{P}(1)$ is true. Suppose $\mathrm{P}(k)$ is true for some $k \in \mathbb{Z}^+$. i.e., $1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{2^k} > \frac{k+1}{2}$. We wish to prove $\mathrm{P}(k+1)$ is true, i.e., $1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{2^k}+\frac{1}{2^k+1}+\frac{1}{2^k+2}+\ldots+\frac{1}{2^{k+1}} > \frac{k+2}{2}$.

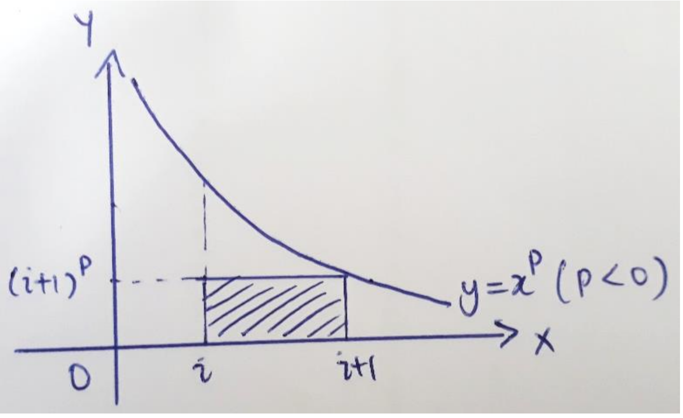
	Therefore, $\mathrm{P}(k)$ is true $\Rightarrow$ $1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{2^k}+\frac{1}{2^k+1}+\frac{1}{2^k+2}+\ldots+\frac{1}{2^{k+1}}$ $>\frac{k+1}{2}+\frac{1}{2^k+1}+\frac{1}{2^k+2}+\ldots+\frac{1}{2^{k+1}}$ (by the induction hypothesis) $>\frac{k+1}{2}+\underbrace{\frac{1}{2^{k+1}}+\frac{1}{2^{k+1}}+\ldots+\frac{1}{2^{k+1}}}_{2^k \text{ terms}}$ $=\frac{k+1}{2}+\frac{2^k}{2^{k+1}}$ $=\frac{k+1}{2}+\frac{1}{2}$ $=\frac{k+2}{2}$ $\mathrm{P}(k+1)$ is true . Since $\mathrm{P}(1)$ is true and $\mathrm{P}(k)$ is true $\Rightarrow \mathrm{P}(k+1)$ is true, by mathematical induction, $\mathrm{P}(n)$ is true for all $n \in \mathbb{Z}^+$.
(ii)	For any $p \in \mathbb{Z}^+$, there exists a greatest $q \in \mathbb{Z}^+$ such that $2^q \le p$. Therefore, $\sum_{n=1}^{p}\frac{1}{n}=1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{p}$ $\ge 1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{2^q}$ $>\frac{q+1}{2}$ (by part (i)) As $q \to \infty$, we have $p \to \infty$ and $\sum_{n=1}^{p}\frac{1}{n}>\frac{q+1}{2} \to \infty$. Therefore, $\sum_{n=1}^{\infty}\frac{1}{n}$ is a divergent series.

No.	**Suggested Solution**
5(i) **(a)**	Suppose there exist $b \in S$ such that $ab \in T$ is a multiple of *p*. By Euclid's Lemma, $p \mid a$ or $p \mid b$. However, this gives rise to a contradiction as it is given that $p \nmid a$ and $0<b<p \Rightarrow p \nmid b$. Therefore, the set *T* does not contain any multiple of *p*.
(i) **(b)**	$S=\{1,2,3,\ldots,p-1\}$ and $T=\{as \mid s \in S\}$. Suppose that there exist $i, j \in S$ with $i \ne j$ (and thus $ai, aj \in T$) such that $ai \equiv aj \pmod p$. Then, we have $i \equiv j \left(\bmod \frac{p}{\gcd(a,p)}\right)$. Since $p \nmid a$, $\gcd(a,p)=1$. Therefore, $i \equiv j \pmod p$. Along with the fact that $1 \le i, j \le p-1$, we have $i=j$. This contradicts $i \ne j$. Therefore, the elements of the set *T* are all distinct modulo *p*.

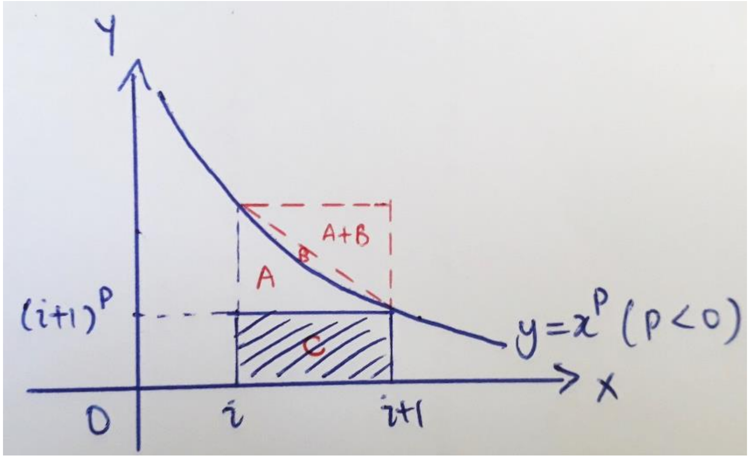
(ii)	Let a be an integer such that $p \nmid a$. By part **(i)**, we have $1a \times 2a \times 3a \times ... \times (p-1)a \equiv 1 \times 2 \times 3 \times ... \times (p-1) \pmod{p}$ $a^{p-1} \equiv 1 \left(\bmod \dfrac{p}{\gcd((p-1)!, p)} \right)$ Since $\gcd(m, p) = 1$ for $m \in \{1, 2, 3, ..., p-1\}$, we have $\gcd((p-1)!, p) = 1$. Therefore, $a^{p-1} \equiv 1 \pmod{p}$.
(iii)	Since 17 is a prime and $17 \nmid 3$, by Fermat's little theorem, we have $3^{16} \equiv 1 \pmod{17}$, so, $(3^{16})^{347} \equiv 1^{347} \pmod{17}$ $3^{5552} \equiv 1 \pmod{17}$ $(3^{5552})(3^3) \equiv 3^3 \pmod{17}$ $3^{5555} \equiv 27 \pmod{17}$ $\equiv 10 \pmod{17}$ Therefore, the remainder when 3^{5555} is divided by 17 is 10.

Qn	**Suggested Solution**
6(i)	Let $x = \sqrt{u}$. Then $\dfrac{\mathrm{d}x}{\mathrm{d}u} = \dfrac{1}{2\sqrt{u}} = \dfrac{1}{2x}$. When $u = 4,\ x = 2$ and $u = 9,\ x = 3$. Therefore $\displaystyle\int_4^9 \frac{u}{\sqrt{u}-1}\ \mathrm{d}u = \int_2^3 \frac{x^2}{x-1}(2x)\ \mathrm{d}x$ $\displaystyle= \int_2^3 \frac{2x^3}{x-1}\ \mathrm{d}x$ $\displaystyle= \int_2^3 2x^2 + 2x + 2 + \frac{2}{x-1}\ \mathrm{d}x$ $\displaystyle= \left[\frac{2}{3}x^3 + x^2 + 2x + 2\ln\lvert x-1\rvert\right]_2^3$ $\displaystyle= \left[\frac{2}{3}(27) + 9 + 6 + 2\ln 2\right] - \left[\frac{2}{3}(8) + 4 + 4 + 2\ln 1\right]$ $\displaystyle= \frac{59}{3} + 2\ln 2$
6(ii)	Let $y = xu$. Then $\dfrac{\mathrm{d}y}{\mathrm{d}x} = x\dfrac{\mathrm{d}u}{\mathrm{d}x} + u$. Hence $\dfrac{1}{x}\dfrac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}\left(\dfrac{y}{x}\right) + \dfrac{y}{x^2}$ becomes

	$\frac{1}{x}\left(x\frac{\mathrm{d}u}{\mathrm{d}x}+u\right)=\mathrm{f}(u)+\frac{u}{x}$ $\Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x}+\frac{u}{x}=\mathrm{f}(u)+\frac{u}{x}$ $\Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x}=\mathrm{f}(u)$
6(iii)	$\frac{\mathrm{d}y}{\mathrm{d}x}=x\sqrt{\frac{x}{y}+\frac{y}{x}-\frac{x^2}{y}}$ $\Rightarrow \frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}x}=\sqrt{\frac{x}{y}-\frac{x}{y}+\frac{y}{x^2}}$ $\Rightarrow \frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}x}=\mathrm{f}\left(\frac{y}{x}\right)+\frac{y}{x^2}$, where $\mathrm{f}(x)=\sqrt{\frac{1}{x}-\frac{1}{x}}$ From (ii), we know that this DE will reduce to $\frac{\mathrm{d}u}{\mathrm{d}x}=\sqrt{\frac{1}{u}-\frac{1}{u}}$, where $u=\frac{y}{x}$. $\frac{\mathrm{d}u}{\mathrm{d}x}=\sqrt{\frac{1}{u}-\frac{1}{u}}=\frac{\sqrt{u}-1}{u}$ $\Rightarrow \int_4^9 \frac{u}{\sqrt{u}-1}\,\mathrm{d}u=\int_{1/3}^{x} 1\,\mathrm{d}x$ $\Rightarrow \frac{59}{3}+2\ln 2=x-\frac{1}{3}$ $\Rightarrow x=20+2\ln 2$

No.	Suggested Solution
7(i) **(a)**	 Area of the shaded rectangle = $(i+1)^p$ Area of region under $y=x^p$ from $x=i$ to $x=i+1$ $=\int_i^{i+1} x^p\,\mathrm{d}x$ From the graph above, we can see that $(i+1)^p<\int_i^{i+1} x^p\,\mathrm{d}x$.

7(i) (b)



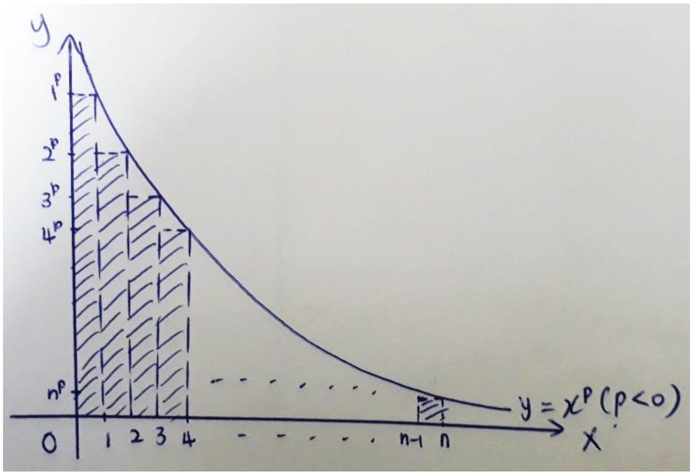
Let *a*, *b*, *c* denote the areas of region A, B and C respectively. Hence we know that $a,b,c>0$

$$\text{LHS}=2\int_i^{i+1} x^p \ \mathrm{d}x$$
$$=2\left(a+c\right)$$

$$\text{RHS}=i^p+\left(i+1\right)^p$$
$$=2\left(a+b\right)+c+c$$
$$=2\left(a+c\right)+2b$$

Hence it is quite obvious that LHS < RHS since b > 0.

7(ii)



For $n>1$,

Area of the *n* rectangles

$$=1^p+2^p+\ldots+n^p$$
$$<1^p+\int_1^n x^p \ \mathrm{d}x$$
$$=1+\left[\frac{x^{p+1}}{p+1}\right]_1^n$$
$$=1+\left[\frac{n^{p+1}}{p+1}-\frac{1^{p+1}}{p+1}\right]=1-\frac{1}{p+1}+\frac{n^{p+1}}{p+1}$$

	Hence as $n \to \infty$ and $p < -1$, we have $= 1^p + 2^p + \ldots$ $< \lim_{n\to\infty}\left(1 - \frac{1}{p+1} + \frac{n^{p+1}}{p+1}\right)$ $= 1 - \frac{1}{p+1}$ (since $n^{p+1} \to 0$) $= \frac{p}{p+1}$
7(iii)	Using (ii), we have already shown $1^p + 2^p + \ldots + n^p < 1 - \frac{1}{p+1} + \frac{n^{p+1}}{p+1} = 1 + \frac{n^{p+1}-1}{p+1}$. Using (i), we have $2\int_1^n x^p \,\mathrm{d}x < 1^p + 2\left(2^p + 3^p + \ldots + (n-1)^p\right) + n^p$ $\Rightarrow 1^p + n^p + 2\int_1^n x^p \,\mathrm{d}x < 2\left(1^p + 2^p + \ldots + (n-1)^p + n^p\right)$ $\Rightarrow \frac{1^p + n^p}{2} + \int_1^n x^p \,\mathrm{d}x < 1^p + 2^p + \ldots + (n-1)^p + n^p$ $\Rightarrow \frac{1+n^p}{2} + \frac{n^{p+1}-1}{p+1} < 1^p + 2^p + \ldots + (n-1)^p + n^p$ $\lim_{n\to\infty}\left[\frac{1+n^p}{2n^{p+1}} + \frac{n^{p+1}-1}{(p+1)n^{p+1}}\right]$ $= \lim_{n\to\infty}\left[\frac{1}{2n^{p+1}} + \frac{1}{2n} + \frac{1}{p+1} - \frac{1}{(p+1)n^{p+1}}\right]$ $= \frac{1}{p+1}$ $\lim_{n\to\infty}\left[\frac{1}{n^{p+1}} + \frac{n^{p+1}-1}{(p+1)n^{p+1}}\right]$ $= \lim_{n\to\infty}\left[\frac{1}{n^{p+1}} + \frac{1}{p+1} - \frac{1}{(p+1)n^{p+1}}\right]$ $= \frac{1}{p+1}$ Since the limits on both sides of the inequality is $\frac{1}{p+1}$, therefore $\frac{1^p + 2^p + \ldots + n^p}{n^{p+1}}$ tends to $\frac{1}{p+1}$ as $n \to \infty$.

No.	Suggested Solution
8(i)	Number of ways = $\binom{15}{4}\times\binom{4}{2}\times 2!=16380$ OR Number of ways = $\binom{15}{1}\times\binom{14}{2}\times\binom{12}{1}=16380$
8(ii)	This is equivalent to distributing 6 identical balls into 15 distinct boxes with empty boxes allowed. $\binom{15+6-1}{6}=38760$
	Number of teaming without restriction $\dfrac{\binom{15}{3}\binom{12}{3}\binom{9}{3}\binom{6}{3}\binom{3}{3}}{5!}=1401400$ Number of teaming in which Alice and Barry are in the same team $\dfrac{\binom{13}{1}\binom{12}{3}\binom{9}{3}\binom{6}{3}\binom{3}{3}}{4!}=200200$ Required probability = $\dfrac{200200}{1401400}=\dfrac{1}{7}$
8(a)	$A(m,1)=m$, $A(2n-1,n)=1$
8(b)	Case 1: If the first envelope is chosen, then the second envelope cannot be chosen. Number of ways for Alice to choose from the remaining $m-2$ envelopes is $A(m-2,n-1)$. Case 2: If the first envelope is not chosen, Alice can choose from the remaining $m-1$ envelopes so that no two adjacent envelopes are selected and the number of ways to do so is given by $A(m-1,n)$. By AP, $A(m,n)=A(m-1,n)+A(m-2,n-1)$
	$A(8,3)=A(7,3)+A(6,2)$ $=A(6,3)+A(5,2)+A(6,2)$ $=A(5,3)+A(4,2)+A(5,2)+A(6,2)$ $=1+A(4,2)+A(5,2)+\left[A(5,2)+A(4,1)\right]$ $=1+A(4,2)+2A(5,2)+4$ $=5+A(4,2)+2\left[A(4,2)+A(3,1)\right]$ $=5+3A(4,2)+6$ $=11+3\left[A(3,2)+A(2,1)\right]$ $=11+3(1+2)$ $=20$

	Or by inserting 3 selected books in between 5 unselected books (6 slots), we have $\binom{6}{3}=20$.

No.	Suggested Solution
9(i)	This is equivalent to choosing two dice to put in the same box, i.e. $\binom{n}{2}=\frac{n(n-1)}{2}$
9(ii)	To find the number of ways without restriction, we consider 2 cases: Case 1: Exactly one box containing 3 dice. No. of ways = $\binom{n}{3}=\frac{n(n-1)(n-2)}{6}$. Case 2: Exactly two boxes containing 2 dice each. No. of ways = $\frac{\binom{n}{2}\binom{n-2}{2}}{2!}=\frac{n(n-1)(n-2)(n-3)}{8}$ By AP, number of ways without restriction is $\frac{n(n-1)(n-2)}{6}+\frac{n(n-1)(n-2)(n-3)}{8}$ $=\frac{n(n-1)(n-2)(3n-5)}{24}$ To find the number of ways satisfying the restriction, we consider 2 cases: Case 1: Exactly one box containing 3 dice, with the smallest and biggest dice inside. No. of ways = $\binom{n-2}{1}=n-2$. Case 2: Exactly two boxes containing 2 dice each, one of the boxes containing and the biggest and the smallest dice. No. of ways = $\binom{n-2}{2}=\frac{(n-2)(n-3)}{2}$. By AP, number of ways satisfying the restriction is $n-2+\frac{(n-2)(n-3)}{2}=\frac{(n-1)(n-2)}{2}$. The required probability $=\frac{\frac{(n-1)(n-2)}{2}}{\frac{n(n-1)(n-2)(3n-5)}{24}}$ $=\frac{12}{n(3n-5)}$
	Let A_r be the event that the number r does not appear on any of the n dice, $r=1,2,3,4,5,6$.

	$p(n)=\dfrac{\left\lvert\bigcap_{r=1}^{6}\overline{A_r}\right\rvert}{6^n}$, where $\left\lvert\bigcap_{r=1}^{6}\overline{A_r}\right\rvert=\left\lvert\overline{\bigcup_{r=1}^{6}A_r}\right\rvert$ $=6^n-\left\lvert\bigcup_{r=1}^{6}A_r\right\rvert$ $=6^n-\binom{6}{1}\lvert A_1\rvert+\binom{6}{2}\lvert A_1\cap A_2\rvert-\binom{6}{3}\lvert A_1\cap A_2\cap A_3\rvert$ $+\binom{6}{4}\lvert A_1\cap A_2\cap A_3\cap A_4\rvert$ $-\binom{6}{5}\lvert A_1\cap A_2\cap A_3\cap A_4\cap A_5\rvert$ $+\binom{6}{6}\lvert A_1\cap A_2\cap A_3\cap A_4\cap A_5\cap A_6\rvert$ $=\binom{6}{0}6^n-\binom{6}{1}5^n+\binom{6}{2}4^n-\binom{6}{3}3^n+\binom{6}{4}2^n-\binom{6}{5}1^n$ Thus, $p(n)=\dfrac{\binom{6}{0}6^n-\binom{6}{1}5^n+\binom{6}{2}4^n-\binom{6}{3}3^n+\binom{6}{4}2^n-\binom{6}{5}1^n}{6^n}$ $=\sum_{r=0}^{5}(-1)^r\binom{6}{r}\left(\dfrac{6-r}{6}\right)^n$

No.	Suggested Solution
10(i)	$\left(\dfrac{a_1+a_2}{2}\right)^2-a_1a_2$ $=\dfrac{a_1^{\ 2}+a_2^{\ 2}+2a_1a_2}{4}-a_1a_2$ $=\dfrac{a_1^{\ 2}+a_2^{\ 2}-2a_1a_2}{4}$ $=\dfrac{(a_1-a_2)^2}{4}$ ≥ 0, where equality is achieved when $a_1=a_2$.

10 (ii)	Let P$_m$ denote $a_1a_2a_3...a_{2^m} \le \left(\dfrac{a_1+a_2+a_3+...+a_{2^m}}{2^m}\right)^{2^m}$ for $m \in \mathbf{Z}^+$. For $m=1$, LHS of P$_1$ = a_1a_2 and RHS of P$_1$ = $\left(\dfrac{a_1+a_2}{2}\right)^2$ From result in (i), P$_1$ is true. Suppose P$_k$ is true for some $k \in \mathbf{Z}^+$. $\left(a_1a_2a_3...a_{2^k}\right)\left(a_{2^k+1}a_{2^k+2}...a_{2^k+2^k}\right)$ $\le \left(\dfrac{a_1+a_2+a_3+...+a_{2^k}}{2^k}\right)^{2^k}\left(\dfrac{a_{2^k+1}+a_{2^k+2}+...+a_{2^k+2^k}}{2^k}\right)^{2^k}$ $= \left[\left(\dfrac{a_1+a_2+...+a_{2^k}}{2^k}\right)\left(\dfrac{a_{2^k+1}+a_{2^k+2}+...+a_{2^k+2^k}}{2^k}\right)\right]^{2^k}$ $= \left[\dfrac{1}{2^{2k}}\left(a_1+a_2+...+a_{2^k}\right)\left(a_{2^k+1}+a_{2^k+2}+...+a_{2^k+2^k}\right)\right]^{2^k}$ $\le \left[\dfrac{1}{2^{2k}}\left(\dfrac{a_1+a_2+...+a_{2^k}+a_{2^k+1}+a_{2^k+2}+...+a_{2^k+2^k}}{2}\right)^2\right]^{2^k}$ $= \left[\dfrac{\left(a_1+a_2+...+a_{2^k}+a_{2^k+1}+a_{2^k+2}+...+a_{2^k+2^k}\right)^2}{2^{2(k+1)}}\right]^{2^k}$ $= \left[\dfrac{a_1+a_2+...+a_{2^k}+a_{2^k+1}+a_{2^k+2}+...+a_{2^{k+1}}}{2^{k+1}}\right]^{2^{k+1}}$ Therefore P$_{k+1}$ is true when P$_k$ is true. Since P$_1$ is true, then by induction, P$_m$ is true for all $m \in \mathbf{Z}^+$.
10 (iii)	$b_1b_2...b_nA^{2^m-n} \le \left(\dfrac{b_1+b_2+...+b_n+\left(2^m-n\right)A}{2^m}\right)^{2^m}$ $\Rightarrow a_1a_2...a_nA^{2^m-n} \le \left(\dfrac{a_1+a_2+...+a_n+\left(2^m-n\right)A}{2^m}\right)^{2^m}$ $= \left(\dfrac{nA+\left(2^m-n\right)A}{2^m}\right)^{2^m} = A^{2^m}$

10 (iv)	$a_1 a_2 \ldots a_n A^{2^m - n} \le A^{2^m}$ $\Rightarrow a_1 a_2 \ldots a_n \le A^{2^m - \left(2^m - n\right)} = A^n = \left(\dfrac{a_1 + a_2 + \ldots + a_n}{n} \right)^n$ $\Rightarrow \left(a_1 a_2 \ldots a_n \right)^{\frac{1}{n}} \le \dfrac{a_1 + a_2 + \ldots + a_n}{n}$