NATIONAL JUNIOR COLLEGE
SENIOR HIGH 2 PRELIMINARY EXAMINATION
Higher 3

MATHEMATICS 9820/01

15 September 2017

3 hours

Additional Materials: Answer Paper
List of Formulae (MF26)
Cover Sheet

READ THESE INSTRUCTIONS FIRST

Write your name, registration number, subject tutorial group, on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for diagrams or graphs.
Do not use paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in the brackets [ ] at the end of each question or part question.

This document consists of **5** printed pages.

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Answer **all** the questions.

1 Let p be a prime and n be an integer such that $0 \le n < p$. Prove that

$$\binom{n+p}{p} \equiv 1 \pmod{p}.$$ [5]

2 Let $\gcd(a,b,c)$ denote the greatest common divisor of a, b and c, where a, b and c are integers.

Show that $\gcd(a,b,c) = \gcd\left[\gcd(a,b), c\right]$. [5]

3 Prove that there are infinitely many primes of the form $6k+5$, where $k \in \mathbb{Z}^+ \cup \{0\}$. [6]

4 **(i)** Prove by mathematical induction that

$$1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{2^n} > \frac{n+1}{2}$$

for all positive integers n. [5]

(ii) Hence, explain why $\sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent series. [2]

5 Fermat's little theorem is given as follows:

"If a is an integer and p is a prime such that $p \nmid a$, then $a^{p-1} \equiv 1 \pmod{p}$."

(i) Let the sets S and T be defined as $S = \{1, 2, 3, \ldots, p-1\}$ and $T = \{as \mid s \in S\}$, where where a is an integer and p is a prime such that $p \nmid a$.

Prove that

(a) the set T does not contain any multiple of p, [2]

(b) the elements of the set T are all distinct modulo p. [4]

(ii) Hence, prove Fermat's little theorem. [3]

(iii) Use Fermat's little theorem to find the remainder when 3^{5555} is divided by 17. [2]

6 **(i)** Find the exact value of $\int_4^9 \frac{u}{\sqrt{u}-1}\, du$. [5]

(ii) Show that the differential equation

$$\frac{1}{x}\frac{dy}{dx} = f\left(\frac{y}{x}\right) + \frac{y}{x^2}$$

can be transformed into the equation $\frac{du}{dx} = f(u)$ by the substitution $y = xu$. [3]

(iii) A solution curve of the differential equation

$$\frac{dy}{dx} = x\sqrt{\frac{x}{y}} + \frac{y}{x} - \frac{x^2}{y}$$

passes through the point $\left(\frac{1}{3}, \frac{4}{3}\right)$. Find the exact value of the x-coordinate of the point where this curve intersects the line $y = 9x$. [4]

7 Let p be a negative number.

(i) Use a sketch graph of $y = x^p$ to explain why, for any positive number i

(a) $(i+1)^p < \int_i^{i+1} x^p\, dx$, [2]

(b) $2\int_i^{i+1} x^p\, dx < i^p + (i+1)^p$. [2]

(ii) For $p < -1$, prove that the sum of the infinite series $1^p + 2^p + \ldots$ is less than $\frac{p}{1+p}$.
[4]

(iii) For $p > -1$, prove that

$$\frac{n^p+1}{2} + \frac{n^{p+1}-1}{p+1} < 1^p + 2^p + \ldots + n^p < 1 + \frac{n^{p+1}-1}{p+1}.$$

Hence, find the value of $\frac{1^p + 2^p + \ldots + n^p}{n^{p+1}}$ as n tends to infinity. [5]

8 Prizes are to be awarded to a group of 15 people. Find the number of ways in which the prizes can be awarded if

(i) there are one 1st prize, two 2nd prizes and one 3rd prize, which are to be awarded to different people, [2]

(ii) there are 6 identical prizes which each person can get any number of prizes. [2]

The group of 15 people, including Alice and Barry, are to be randomly allocated into five teams of three people. Find the probability that Alice and Barry are in the same team. [3]

Alice's team wins a prize, and is allowed to pick *n* envelopes from *m* envelopes each containing a different sponsored item arranged in a row. Let $A(m,n)$ be the number of ways in which no two adjacent envelopes are picked.

(a) State the value of $A(m,1)$ and of $A(2n-1,n)$. [2]

(b) Show that $A(m,n)=A(m-1,n)+A(m-2,n-1)$ for $2\le n\le \frac{m}{2}$. [2]

Hence or otherwise, find $A(8,3)$. [2]

9 There are *n* six-sided fair dice ($n\ge 6$) distinguishable by size.

(i) State the number of ways to place all the dice in $(n-1)$ identical boxes such that no box is empty. [1]

(ii) Suppose instead all the dice are **randomly** placed in $(n-2)$ identical boxes such that no box is empty. Find the probability that the biggest and the smallest dice are in the same box. [6]

All the *n* dice are tossed and let $p(n)$ denote the probability that all the six numbers have appeared at least once.

Find $p(n)$ in the form $\sum_{r=0}^{5}\mathrm{f}(n,r)$, where $\mathrm{f}(n,r)$ is an expression to be determined. [6]

[Turn over

10 **(i)** Suppose that $a_i > 0$ for all $i > 0$. Show that

$$a_1 a_2 \leq \left(\frac{a_1 + a_2}{2} \right)^2 .$$ [2]

(ii) Prove by induction that for all positive integers m,

$$a_1 a_2 a_3 ... a_{2^m} \leq \left(\frac{a_1 + a_2 + a_3 + ... + a_{2^m}}{2^m} \right)^{2^m} .$$ [7]

(iii) If $n < 2^m$, and we let $b_1 = a_1$, $b_2 = a_2$, …, $b_n = a_n$ and $b_{n+1} = b_{n+2} = ... = b_{2^m} = A$, where

$$A = \frac{a_1 + a_2 + ... + a_n}{n},$$

show that $a_1 a_2 ... a_n A^{2^m - n} \leq A^{2^m}$. [4]

(iv) Deduce the arithmetic mean – geometric mean inequality

$$\left(a_1 a_2 ... a_n \right)^{\frac{1}{n}} \leq \frac{a_1 + a_2 + ... + a_n}{n} .$$ [2]

—— End of Paper ——