NANYANG JUNIOR COLLEGE

JC2 PRELIMINARY EXAMINATION

Higher 3

MATHEMATICS **9820/01**

Paper 1 **27th September 2017**

3 Hours

Additional Materials: Answer Paper

List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name and class on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphic calculator.
Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

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Internal Examinations

1 **(i)** Prove that for any integers x and y and for any prime p,

$$(x+y)^p \equiv x^p + y^p \pmod{p}. \qquad [2]$$

(ii) Using induction, show that for any prime p, $a^p \equiv a \pmod{p}$ for all positive integers a. [4]

(iii) Show that if n is not a multiple of 4, then $\sum_{i=1}^{4} i^n \equiv 0 \pmod{5}$. [4]

2 A differential equation of the form

$$\frac{\mathrm{d}z}{\mathrm{d}x} + zP(x) = Q(x)$$

where $P(x)$ and $Q(x)$ are some functions of x is said to be a *linear equation of the first order*.

(i) Write down an expression, in terms of $z, P(x)$ and $\frac{\mathrm{d}z}{\mathrm{d}x}$, for $\frac{\mathrm{d}}{\mathrm{d}x}\left\{z\mathrm{e}^{\int P(x)\mathrm{d}x}\right\}$. [1]

(ii) Hence show that

$$z = \frac{\int\left\{Q(x)\mathrm{e}^{\int P(x)\mathrm{d}x}\right\}\mathrm{d}x}{\mathrm{e}^{\int P(x)\mathrm{d}x}}. \qquad [2]$$

(iii) Use the substitution $z = \mathrm{f}(y)$ to transform the differential equation

$$\mathrm{f}'(y)\frac{\mathrm{d}y}{\mathrm{d}x} + \mathrm{f}(y)P(x) = Q(x)$$

to a linear equation of the first order in x and z. [1]

(iv) The *Bernoulli equation*, named after the Swiss mathematician Jacob Bernoulli who studied it in 1695, has the form

$$y^{-n}\frac{\mathrm{d}y}{\mathrm{d}x} + y^{-n+1}P(x) = Q(x)$$

where n is a real number such that $n \neq 0, 1$.

Show that a general solution of the above Bernoulli equation is given by

$$y = \left\{\frac{(1-n)\int\left\{Q(x)\mathrm{e}^{(1-n)\int P(x)\mathrm{d}x}\right\}\mathrm{d}x}{\mathrm{e}^{(1-n)\int P(x)\mathrm{d}x}}\right\}^{\frac{1}{1-n}}. \qquad [2]$$

(v) Solve the differential equation

$$\frac{1}{y^3}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{xy^2} = x^2\cos x \quad (x > 0)$$

given that $y = 1$ when $x = \pi$. [4]

3 The sequence of real numbers $a_0, a_1, a_2, \ldots$ satisfies the relationship

$$a_{m+n} + a_{m-n} = \frac{1}{2}\left(a_{2m} + a_{2n}\right)$$

for all non-negative integers *m* and *n* with n £ m. It is also given that $a_1 = 1$.

(i) Determine a_0, a_2, a_3 and a_4. [4]

(ii) Show that $a_{2m} = 4a_m$ for all non-negative integers *m*. [1]

(iii) Conjecture a formula for a_n in terms of *n* and prove your conjecture using strong induction. [5]

(iv) Prove that $a_{m+n} - a_{m-n} = 4mn$ for all non-negative integers *m* and *n* with n £ m.
Hence or otherwise, find the exact value of the sum

$$\sum_{0 \le n \le m \le 3} \left(a_{m+n} - a_{m-n}\right).$$ [3]

[For example, $(n, m) = (0, 1)$ and $(2, 2)$ are two possible pairs of values of *n* and *m*.]

4 Throughout this question, *n* denotes a non-negative integer.

Let $I_n = \displaystyle\int_0^{\pi} \frac{\cos nx}{5 - 4\cos x}\, \mathrm{d}x$.

Prove that

$$I_n = \begin{cases} \displaystyle\int_0^{\pi} \frac{\cos nx}{5 + 4\cos x}\, \mathrm{d}x, & n \text{ is even,} \\ -\displaystyle\int_0^{\pi} \frac{\cos nx}{5 + 4\cos x}\, \mathrm{d}x, & n \text{ is odd.} \end{cases}$$ [3]

Hence or otherwise,

(i) find, in the case when *n* is even, the exact value of

$$\int_0^{\pi} \frac{\cos x \cos nx}{25 - 16\cos^2 x}\, \mathrm{d}x,$$

showing your working clearly, [3]

(ii) show that

$$\int_0^{\pi} \frac{\cos^2 x}{25 - 16\cos^2 x}\, \mathrm{d}x = \frac{1}{4}\int_0^{\pi} \frac{\cos x}{5 - 4\cos x}\, \mathrm{d}x$$

and use the substitution $t = \tan\dfrac{x}{2}$ to evaluate the integral

$$\int_0^{\pi} \frac{\cos^2 x}{25 - 16\cos^2 x}\, \mathrm{d}x$$

exactly, giving your answer in the form $k\pi$ where *k* is a rational number to be determined. [7]

5 A function defined on the set of positive integers is called an *arithmetic function*. Arithmetic functions play an important role in number theory.

Let f be an arithmetic function having its range in the set of positive integers. That is,

$$\mathrm{f}:\mathbb{Z}^+ \to \mathbb{Z}^+.$$

Given also that f satisfies the following relation:

$$\mathrm{f}(n)+2\mathrm{f}(\mathrm{f}(n))=3n+5 \text{ for all } n\in\mathbb{Z}^+. \qquad (*)$$

(i) Explain why $\mathrm{f}(1)$ is an even number between 1 and 6 (inclusive). [2]

(ii) Show that $\mathrm{f}(1)\neq 4,6$. [4]

(iii) Find $\mathrm{f}(2)$ and $\mathrm{f}(3)$. [3]

(iv) Find all functions $\mathrm{f}(n)$ that satisfy (*). [3]

6 Let S be the set of 3-digit numbers each of whose digits are different and non-zero. For example, $123\in S$, $312\in S$ and $473\in S$.

(i) How many numbers are there in S. [1]

(ii) Find the sum of all the numbers in S. [2]

(iii) How many numbers in S are

(a) divisible by 2, [2]

(b) divisible by 3, [4]

(c) **not** divisible by any of 2 or 3 or 5. [5]

7 Let $P(r,n)$ denote the number of ways of distributing r identical objects into n identical boxes so that no box is empty.

(i) **(a)** Explain why $P(r,2)$ is $\frac{1}{2}r$ in the case that r is even. [2]

(b) Write down a formula for $P(r,2)$ in the case that r is odd. [1]

(ii) Prove that $P(r,n)=P(r-1,n-1)+P(r-n,n)$, for $1<n\leq r$. [4]

(iii) Let k be a fixed positive integer and let m increase from 1 through integer values.

(a) Prove that $P(m+k,m)$ eventually reaches a constant value. [3]

(b) Write down the least value of m such that $P(m+k,m)=P(m-1+k,m-1)$. [1]

8 **(a)** Let $\{a_i\}_{i=1,2,\ldots,n}$ be a sequence of increasing positive real numbers. Define

$$G=\left(a_1a_2\cdots a_n\right)^{\frac{1}{n}}, \quad A=\frac{1}{n}\left(a_1+a_2+\ldots+a_n\right).$$

(i) Show that $a_1 \le G \le a_n$ and explain why there exists $k \in \{1,2,\ldots,n-1\}$ such that $a_k \le G \le a_{k+1}$. [3]

(ii) Hence show that for the k described in **(i)**,

$$\sum_{i=1}^{k}\left\{\int_{a_i}^{G}\left(\frac{1}{t}-\frac{1}{G}\right)\mathrm{d}t\right\}+\sum_{i=k+1}^{n}\left\{\int_{G}^{a_i}\left(\frac{1}{G}-\frac{1}{t}\right)\mathrm{d}t\right\}\ge 0\,.\;(*)$$ [3]

(iii) Prove that (*) is equivalent to

$$\sum_{i=1}^{n}\left\{\int_{G}^{a_i}\frac{1}{G}\,\mathrm{d}t\right\}\ge\sum_{i=1}^{n}\left\{\int_{G}^{a_i}\frac{1}{t}\,\mathrm{d}t\right\}.$$ [3]

(iv) Hence deduce the AM-GM inequality

$$A \geq G,$$

with equality being achieved when $a_i = G$ for all $i = 1,2,\ldots,n$. [4]

(b) Use the AM-GM inequality to prove that amongst all cuboids with a fixed volume V, the one that has the minimum surface area is the cube and state this minimum surface area in terms of V only. [4]

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