RAFFLES INSTITUTION
2017 YEAR 6 PRELIMINARY EXAMINATION
Higher 3

MATHEMATICS 9820

21 September 2017

3 hours

Additional materials: Answer Paper
Graph paper
List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name and CT group on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

RAFFLES INSTITUTION
Mathematics Department

1 **(a)** The function f is defined for all $x \in \mathbb{R}$ by

$$f(x) = \begin{cases} k & \text{for } |x| \le l, \\ 0 & \text{for } |x| > l, \end{cases}$$

where k and l are positive constants.

(i) Sketch the graph of $y = f\left(x + \frac{1}{2}l\right)$. [1]

The function g_n, where $n \ge 1$, is defined by

$$g_n(x) = \frac{1}{2}\left\{f\left(x + \frac{1}{n}l\right) + f\left(x - \frac{1}{n}l\right)\right\}.$$

(ii) Sketch the graph of $y = g_2(x)$. [2]

(iii) Show that $\int_{-\infty}^{\infty} g_n(x)\,dx$ is a constant independent of n. [2]

(b) Let $h(x) = ax^2 + bx + c$, where a, b and c are real numbers and $a \ne 0$.

(i) It is given that $h(x) = 0$ has no real roots, and the graph of $y = h(x)$ is reflected about the horizontal line passing through the stationary point to obtain the graph of $y = k(x)$.
Determine $k(x)$ in terms of a, b and c. [3]

(ii) Let $\alpha \pm \beta i$ be the roots of $h(x) = 0$, where α, β are real numbers. Find the roots of $k(x) = 0$ in terms of α and β. [2]

(iii) It is further given that $h(m + x) = h(m - x)$ and $h(0) = 2h(m)$, where m is a non-zero real number. Find the roots of $h(x) = 0$ in terms of m. [2]

2 An n-sided polygon is said to be *good* under the following conditions:

- it is equilateral, i.e. all its sides are of equal length,
- its interior angles are either 120° or 240°.

(i) Let l be the number of interior angles of 240° in a *good* n-sided polygon. By considering the sum of the angles in the polygon, show that $n = 2l + 6$. [1]

(ii) Show that there exists a 6-sided *good* polygon. [1]

(iii) Show that if there exists an n-sided *good* polygon, then there exists an $(n+4)$-sided *good* polygon. [2]

(iv) Hence determine all possible values of n such that there exists an n-sided *good* polygon. [3]

3 Let s_0 and t_0 be real numbers such that $0 < t_0 \le s_0$. Positive real-numbered sequences $s_0, s_1, s_2, \ldots$ and $t_0, t_1, t_2, \ldots$ are defined by

$$s_n = \frac{s_{n-1} + t_{n-1}}{2} \quad \text{and} \quad t_n = \frac{2s_{n-1}t_{n-1}}{s_{n-1} + t_{n-1}}, \quad \text{for } n \ge 1.$$

(i) Show that $s_n \ge t_n$ for $n \ge 0$. [2]

(ii) By considering $s_n - s_{n-1}$ and $s_n t_n$, deduce that $\{s_n\}$ is monotone decreasing and $\{t_n\}$ is monotone increasing. [2]

The *monotone convergence theorem* states that if a sequence is monotone increasing and bounded above, it will converge. Similarly, if a sequence is monotone decreasing and bounded below, it will converge.

(iii) By use of the monotone convergence theorem, show that both $\lim_{n\to\infty} s_n$ and $\lim_{n\to\infty} t_n$ exist. [2]

(iv) By showing $s_n - t_n \le \dfrac{s_0 - t_0}{2^n}$ for $n \ge 0$, prove that $\lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n$. [3]

(v) Find $\lim_{n\to\infty} s_n t_n$ and deduce the value of $\lim_{n\to\infty} s_n$ in terms of s_0 and t_0. [3]

[Turn over

4 Let $a_1, a_2, ..., a_n$ and $b_1, b_2, ..., b_n$ be real numbers such that $b_1 \le b_2 \le ... \le b_n$. Let $r < s$ and

$$S = \sum_{i=1}^{n} a_i b_i,$$

$$S' = \left(\sum_{\substack{i=1 \\ i \ne r,s}}^{n} a_i b_i \right) + a_s b_r + a_r b_s.$$

(i) Show that $S \ge S'$ if and only if $a_s \ge a_r$. [2]

(ii) Hence show that if $a_1 \le a_2 \le ... \le a_n$ and $b_1 \le b_2 \le ... \le b_n$ are real numbers arranged in ascending order, for any permutation $(a_1', a_2', ..., a_n')$ of $(a_1, a_2, ..., a_n)$, we have

$$a_1 b_1 + a_2 b_2 + ... + a_n b_n \ge a_1' b_1 + a_2' b_2 + ... + a_n' b_n \ge a_n b_1 + a_{n-1} b_2 + ... + a_1 b_n.$$ [2]

(iii) Deduce that if $x_1, x_2, ..., x_n$ are distinct positive integers,

$$\frac{x_1}{1^2} + \frac{x_2}{2^2} + ... + \frac{x_n}{n^2} \ge \frac{1}{1} + \frac{1}{2} + ... + \frac{1}{n}.$$ [3]

(iv) Let $\sigma : \mathbb{N} \to \mathbb{N}$ be a bijection. Define the sequence T_n by $T_n = \sum_{k=1}^{n} \frac{\sigma(k)}{k^2}$. State, with a reason, if the sequence $T_1, T_2, T_3, ...$ converges. [2]

5 Let p be a prime number.

(i) Show that if n is a number in the set P, where $P = \{1,\ 2,\ ...,\ p-1\}$, then there exists a unique integer $\theta(n)$ in P such that $n\theta(n) \equiv 1 \pmod{p}$. [4]

(ii) Deduce that $(p-1)! \equiv -1 \pmod{p}$. [2]

(iii) Hence show that if p is a prime such that $p \equiv 1 \pmod 4$, there exists an integer x such that $x^2 \equiv -1 \pmod{p}$. [2]

Fermat's Little Theorem states that if p is a prime and a is an integer coprime to p, then $a^{p-1} \equiv 1 \pmod{p}$.

(iv) Use Fermat's Little Theorem to show that if p is an odd prime and there exists an integer x such that $x^2 \equiv -1 \pmod{p}$, then $p \equiv 1 \pmod 4$. [3]

A graphing calculator must not be used for question 6.

6 **(i)** Given that f is a continuous function, explain, with the aid of a sketch, why the value of

$$\lim_{n\to\infty}\frac{1}{n}\left\{\mathrm{f}\left(\frac{1}{n}\right)+\mathrm{f}\left(\frac{2}{n}\right)+\ldots+\mathrm{f}\left(\frac{n}{n}\right)\right\}$$

is $\int_0^1 \mathrm{f}\left(x\right)\ \mathrm{d}x$. [2]

(ii) Hence evaluate $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\frac{1}{\sqrt{(3n^2+k^2)}}$, leaving your answer in exact form. [4]

(iii) **(a)** By considering the function $\mathrm{g}(x)=x-\sin x$, show that $\sin x\le x$ for $x\ge 0$.

Hence or otherwise, show that $x-\frac{x^3}{6}\le \sin x$ for $x\ge 0$. [4]

(b) Deduce that $\sum\limits_{k=1}^{n}\left(\frac{k}{n^2}-\frac{k^3}{6n^6}\right)\sin\frac{k}{n}\le\sum\limits_{k=1}^{n}\sin\frac{k}{n^2}\sin\frac{k}{n}\le\sum\limits_{k=1}^{n}\frac{k}{n^2}\sin\frac{k}{n}$. [1]

(c) Hence determine the exact value of $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\sin\frac{k}{n^2}\sin\frac{k}{n}$. [3]

7 In a race, there are r racers. Let $\mathrm{f}(r,p)$ be the number of ways the r racers can complete the race in exactly p placings, where it is possible for multiple racers to be in the same placing. For example, $\mathrm{f}(2,1)=1$ as both racers are tied in first placing, and $\mathrm{f}(2,2)=2$ as either racer can be first and the other is second.

(i) Explain combinatorically the recurrence

$$\mathrm{f}(r,p)=p\left[\mathrm{f}(r-1,p-1)+\mathrm{f}(r-1,p)\right].$$ [2]

(ii) Let $\mathrm{F}(r)$ be the total number of ways for r racers to complete the race with any number of ties.

(a) Show that $\mathrm{F}(r)=\sum\limits_{p=1}^{r}p!\mathrm{S}(r,p)$ where $\mathrm{S}(r,n)$ denote the number of ways of distributing r distinct objects into n identical boxes so that no box is empty. [2]

(b) Find the total number of ways for 5 racers to complete the race with any number of ties. [3]

(iii) Let $r\ge 2$. Find an expression, in terms of the function F, for r racers to complete the race with any number of ties, if a particular racer will never finish first in the race. Explain clearly how this expression is derived. [3]

[Turn over

8 Euler's totient-function $\varphi(n)$ is defined as the number of integers in $\{1, 2, \ldots, n\}$ that are coprime to n. Let A_i be the set of positive integers that are multiples of i which are less than or equal to n.

(i) Show that $\left|A_j \cap A_k\right| = \left|A_{jk}\right|$ if j and k are coprime. [3]

(ii) Hence show that if $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$, where $p_1, p_2, \ldots, p_k$ are the distinct prime factors of n, $\varphi(n) = n\left(1-\dfrac{1}{p_1}\right)\left(1-\dfrac{1}{p_2}\right)\cdots\left(1-\dfrac{1}{p_k}\right)$. [4]

(iii) A function $\mathrm{f}: \square \to \square$ is said to be **multiplicative** if $\mathrm{f}(1) = 1$ and $\mathrm{f}(ab) = \mathrm{f}(a)\mathrm{f}(b)$ holds for all coprime positive integers a and b. The function f is said to be **completely multiplicative** if $\mathrm{f}(1) = 1$ and $\mathrm{f}(ab) = \mathrm{f}(a)\mathrm{f}(b)$ holds for all positive integers a and b.

State whether the following assertions are true or false. Prove those which you consider to be true and give counter-examples for those which you consider to be false.

(a) φ is multiplicative. [1]

(b) φ is completely multiplicative. [1]

(c) If f is multiplicative, $\mathrm{f}(a)\mathrm{f}(b) = \mathrm{f}(\gcd(a,b))\mathrm{f}(\operatorname{lcm}(a,b))$ holds for all positive integers a and b. [2]

[Question 9 is printed on the next page]

9 The Farey sequences $F_1, F_2, \ldots,$ is a list of sequences consisting of all irreducible fractions between 0 and 1. In particular, F_n contains all such fractions with denominators less than or equal to *n*, arranged in ascending order. The first four sequences are therefore

$$F_1 = \left\{\frac{0}{1}, \frac{1}{1}\right\},$$
$$F_2 = \left\{\frac{0}{1}, \frac{1}{2}, \frac{1}{1}\right\},$$
$$F_3 = \left\{\frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}\right\},$$
$$F_4 = \left\{\frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1}\right\}.$$

(i) If $\frac{a}{b}$ and $\frac{c}{d}$ are consecutive terms in F_n where $\frac{a}{b} < \frac{c}{d}$ and $bc - ad = 1$, show that $\frac{a+c}{b+d}$ is also an irreducible fraction and $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$. [2]

(ii) Show that for any irreducible fraction $\frac{m}{n}$ such that $\frac{a}{b} < \frac{m}{n} < \frac{c}{d}$ and $bc - ad = 1$, there exists positive integers *q* and *r* such that $m = qa + rc$ and $n = qb + rd$. [3]

(iii) Hence show that if $\frac{a}{b}$ and $\frac{c}{d}$ are consecutive terms in F_n where $\frac{a}{b} < \frac{c}{d}$ and $bc - ad = 1$, then either $\frac{a}{b}$ and $\frac{c}{d}$ are consecutive terms in F_{n+1} or $\frac{a}{b}, \frac{a+c}{b+d}, \frac{c}{d}$ are consecutive terms in F_{n+1}, where $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$. [1]

(iv) Hence prove by mathematical induction that if $\frac{a}{b}$ and $\frac{c}{d}$ are consecutive terms in F_n such that $\frac{a}{b} < \frac{c}{d}$, then $bc - ad = 1$, for $n \in \mathbb{Z}^+$. [4]

(v) Determine coprime positive integers *m* and *n* satisfying

(a) $n \le 10$, [2]

(b) $n \le 100$, [2]

so that $\left|\pi - \frac{m}{n}\right|$ is minimum.

[END OF PAPER]