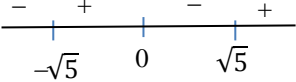


PRELIM Solutions

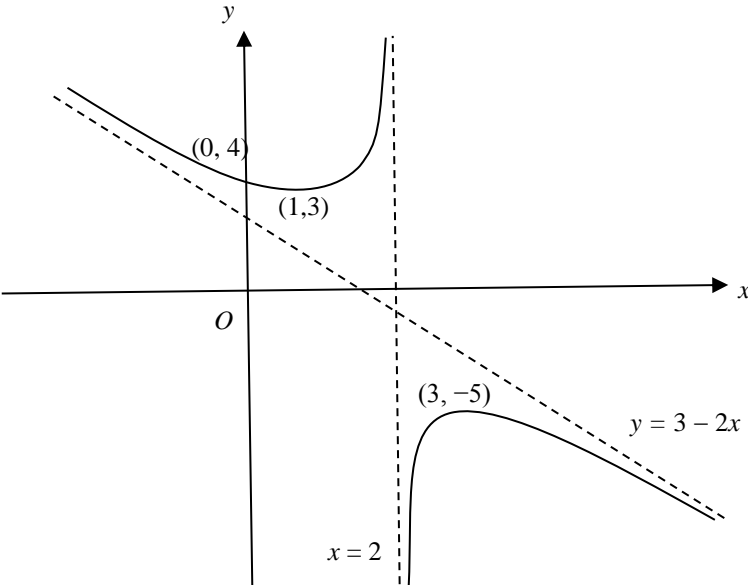
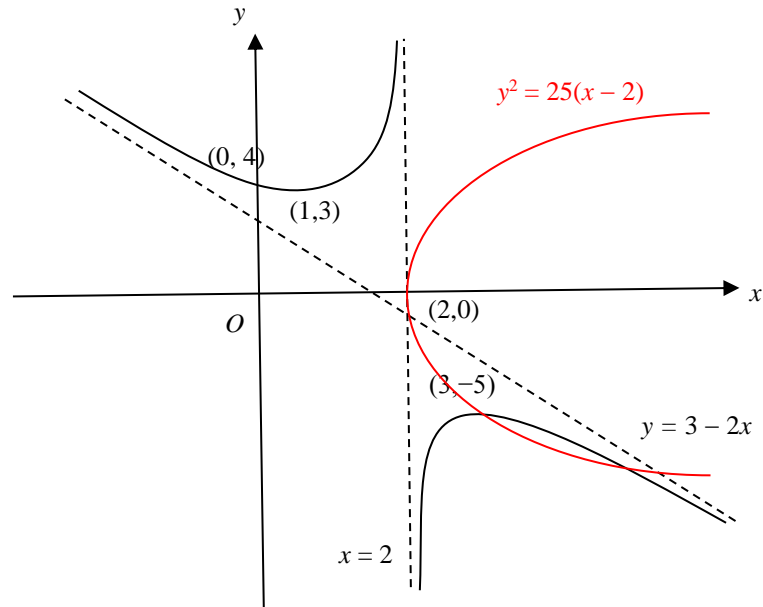
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Qn	Solution
<p>2(i)</p> <p>(ii)</p>	$\frac{x}{x^2-5} \leq 0$ $\frac{x}{(x-\sqrt{5})(x+\sqrt{5})} \leq 0$  <p>$\therefore x < -\sqrt{5}$ or $0 \leq x < \sqrt{5}$</p> $\frac{\sqrt{x}}{x-5} \leq 0$ $\frac{\sqrt{x}}{(\sqrt{x})^2-5} \leq 0$ <p>Replace x by \sqrt{x} in the result from (i),</p> $\sqrt{x} < -\sqrt{5} \quad \text{or} \quad 0 \leq \sqrt{x} < \sqrt{5}$ <p>(Reject $\because \sqrt{x} \geq 0$) or $0 \leq x < 5$</p> <p>Required set = $\{x \in \mathbf{R}: 0 \leq x < 5\}$</p>
3	$\overrightarrow{OP} = \frac{2}{5}\mathbf{a} \quad \overrightarrow{OQ} = \frac{1}{3}\mathbf{b}$ $\overrightarrow{OM} = \frac{1}{2}\left(\frac{2}{5}\mathbf{a} + \frac{1}{3}\mathbf{b}\right)$ <p>Area of triangle OMP</p> $= \frac{1}{2} \left \left(\frac{1}{2} \left(\frac{2}{5}\mathbf{a} + \frac{1}{3}\mathbf{b} \right) \right) \times \frac{2}{5}\mathbf{a} \right $ $= \frac{1}{2} \left \left(\left(\frac{1}{5}\mathbf{a} + \frac{1}{6}\mathbf{b} \right) \right) \times \frac{2}{5}\mathbf{a} \right $ $= \frac{1}{2} \left \frac{2}{25}\mathbf{a} \times \mathbf{a} + \frac{1}{15}\mathbf{b} \times \mathbf{a} \right $ $= \frac{1}{2} \left \frac{1}{15}\mathbf{b} \times \mathbf{a} \right $ $= \frac{1}{30} -\mathbf{a} \times \mathbf{b} $ $= \frac{1}{30} \mathbf{a} \times \mathbf{b} $

Qn	Solution
4(a)	$\int \cos(\ln x) \, dx = x \cos(\ln x) - \int -x \sin(\ln x) \cdot \frac{1}{x} \, dx$ $= x \cos(\ln x) + \int \sin(\ln x) \, dx$ $= x \cos(\ln x) + x \sin(\ln x) - \int \cos(\ln x) \, dx$ $2 \int \cos(\ln x) \, dx = x \cos(\ln x) + x \sin(\ln x) + \text{constant}$ $\int \cos(\ln x) \, dx = \frac{1}{2} x [\cos(\ln x) + \sin(\ln x)] + C$
(b)	$\int \frac{1-2x}{2x^2+1} \, dx$ $= \frac{1}{2} \int \frac{1}{x^2 + \frac{1}{2}} \, dx - \frac{1}{2} \int \frac{4x}{2x^2+1} \, dx$ $= \frac{\sqrt{2}}{2} \tan^{-1} \sqrt{2}x - \frac{1}{2} \ln(2x^2+1) + c$
5(i)	$ z = \sqrt{3} + i = \sqrt{3+1} = 2,$ $ w = -1 + i = \sqrt{1+1} = \sqrt{2}$ $\arg(z) = \arg(\sqrt{3} + i)$ $= \tan^{-1} \frac{1}{\sqrt{3}} = \frac{\pi}{6}$ $\arg(w) = \arg(-1 + i)$ $= \pi - \tan^{-1} 1 = \frac{3\pi}{4}$ $\frac{z^2}{w^*} = \frac{\left(2e^{i\frac{\pi}{6}}\right)^2}{\sqrt{2}e^{i\left(-\frac{3\pi}{4}\right)}}$ $= 2^{\frac{3}{2}} e^{i\frac{13\pi}{12}}$ $= 2^{\frac{3}{2}} e^{-i\frac{11\pi}{12}}$
(ii)	$\arg\left(1 - \frac{q}{z}\right) = \arg\left(\frac{z-q}{z}\right)$ $= \arg(z-q) - \arg(z) = \frac{\pi}{12}$ $\arg(z-q) = \frac{\pi}{12} + \frac{\pi}{6} = \frac{\pi}{4}$ $\arg((\sqrt{3}-q) + i) = \frac{\pi}{4}$ $\sqrt{3}-q = 1 \Rightarrow q = \sqrt{3}-1$

Qn	Solution
6(i)	$y = \ln(3 + e^x)$ $\frac{dy}{dx} = \frac{e^x}{3 + e^x}$ $(3 + e^x) \frac{dy}{dx} = e^x$ $\frac{d^2y}{dx^2} (3 + e^x) + e^x \frac{dy}{dx} = e^x$ $\frac{d^2y}{dx^2} + \frac{e^x}{(3 + e^x)} \frac{dy}{dx} = \frac{e^x}{(3 + e^x)}$ $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx} \right)^2 = \frac{dy}{dx} \quad (\text{proved})$
(ii)	$\frac{d^3y}{dx^3} + 2 \left(\frac{dy}{dx} \right) \frac{d^2y}{dx^2} = \frac{d^2y}{dx^2}$ <p>When $x = 0$, $y = \ln 4$, $\frac{dy}{dx} = \frac{1}{4}$, $\frac{d^2y}{dx^2} = \frac{3}{16}$, $\frac{d^3y}{dx^3} = \frac{3}{32}$</p> $y = \ln 4 + x \left(\frac{1}{4} \right) + \frac{x^2}{2!} \left(\frac{3}{16} \right) + \frac{x^3}{3!} \left(\frac{3}{32} \right) + \dots$ $= \ln 4 + \frac{1}{4}x + \frac{3}{32}x^2 + \frac{1}{64}x^3 + \dots$
(iii)	$\frac{e^x}{3 + e^x} = \frac{1}{4} + \frac{3}{16}x + \frac{3}{64}x^2 + \dots$ $\frac{e^{-2x}}{3 + e^{-2x}} = \frac{1}{4} + \frac{3}{16}(-2x) + \frac{3}{64}(-2x)^2 + \dots$ $= \frac{1}{4} - \frac{3}{8}x + \frac{3}{16}x^2 + \dots$
7(i)	$a = -2$ <p>By long division, $y = (b - 4) - 2x + \frac{2b - 16}{x - 2}$.</p> $b - 4 = 3 \Rightarrow b = 7 \quad (\text{shown})$

Qn	Solution
(ii)	<p>The equation is $y = \frac{-2x^2 + 7x - 8}{x - 2}$</p> 
(iii)	<p> $(-2x^2 + 7x - 8)^2 - 25(x - 2)^3 = 0$ $(-2x^2 + 7x - 8)^2 = 25(x - 2)^3$ $\left(\frac{-2x^2 + 7x - 8}{x - 2}\right)^2 = 25(x - 2)$ </p> <p>Add graph of $y^2 = 25(x - 2)$</p>  <p>From the graphs, the number of real roots is 2.</p>

Qn	Solution
8(i)	<p>This is an AP with $a = 1, d = 1$.</p> <p>For $S_n \leq 1016$</p> $\frac{n}{2}(1+n) \leq 1016$ $n^2 + n - 2032 \leq 0$ <p>From GC, $-45.58 \leq n \leq 44.58$</p> <p>She can complete a maximum of 44 rows.</p> $S_{44} = \frac{44}{2}(1+44) = 990$ <p>Number of bricks left = $1016 - 990 = 26$</p>
(ii)	<p>The sequence is a GP with common ratio 2</p> $S_{2k-1} = 1016$ $\frac{m[2^{2k-1} - 1]}{2 - 1} = 1016$ $m[2^{2k-1} - 1] = 1016 \quad \text{--- (1)}$ $T_k = 64 \Rightarrow m2^{k-1} = 64 \quad \text{---- (2)}$ <p>(1) \div (2):</p> $\frac{2^{2k-1} - 1}{2^{k-1}} = \frac{1016}{64}$ <p>From GC, $k = 4$</p> <p>Sub. into (2): $m2^{4-1} = 64$</p> $\Rightarrow m = 8$ <p>No. of bags = $2(4) - 1 = 7$</p>
9(a)	$\frac{dx}{d\theta} = 3 \sec \theta \tan \theta$ $\int_{3\sqrt{2}}^6 \frac{3x+1}{\sqrt{x^2-9}} dx$ $= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{9 \sec \theta + 1}{\sqrt{9 \sec^2 \theta - 9}} (3 \sec \theta \tan \theta) d\theta$ $= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{9 \sec \theta + 1}{3 \tan \theta} (3 \sec \theta \tan \theta) d\theta$

Qn	Solution
	$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} 9 \sec^2 \theta + \sec \theta \, d\theta$ $= \left[9 \tan \theta + \ln \sec \theta + \tan \theta \right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$ $= 9 \tan \frac{\pi}{3} + \ln \left \sec \frac{\pi}{3} + \tan \frac{\pi}{3} \right - \left(9 \tan \frac{\pi}{4} + \ln \left \sec \frac{\pi}{4} + \tan \frac{\pi}{4} \right \right)$ $= 9\sqrt{3} + \ln 2 + \sqrt{3} - (9 + \ln \sqrt{2} + 1)$ $= 9\sqrt{3} - 9 + \ln \frac{2 + \sqrt{3}}{\sqrt{2} + 1}$
(b)(i)	<p>Consider $y = 2 \pm 2\sqrt{1 - \frac{x^2}{16}}$</p> <p>Required area $= \int_{-3}^3 2 + 2\sqrt{1 - \frac{x^2}{16}} \, dx - 2(6)$</p> $= 10.753 \text{ (3 dp)}$ <p><u>Alternative</u></p> <p>Consider $\frac{x^2}{16} + \frac{y^2}{4} = 1 \Rightarrow y = \pm 2\sqrt{1 - \frac{x^2}{16}}$</p> <p>Required area $= \int_{-3}^3 2\sqrt{1 - \frac{x^2}{16}} \, dx$ or $4 \int_0^3 \sqrt{1 - \frac{x^2}{16}} \, dx$</p> $= 10.753 \text{ (3 dp)}$
(ii)	<p>When $x = 3$, $y = 2 + 2\sqrt{1 - \frac{9}{16}} = 2 + \frac{1}{2}\sqrt{7}$</p> <p>When $x = 0$, $y = 4$</p> <p>Required Volume $= \frac{\sqrt{7}}{2} \pi (3^2) + \pi \int_{2+\frac{1}{2}\sqrt{7}}^4 16 \left(1 - \frac{(y-2)^2}{4} \right) dy$</p> $= \frac{9\sqrt{7}}{2} \pi + 16\pi \int_{2+\frac{1}{2}\sqrt{7}}^4 1 - \frac{(y-2)^2}{4} \, dy$ $= \frac{9\sqrt{7}}{2} \pi + 16\pi \left[y - \frac{(y-2)^3}{12} \right]_{2+\frac{1}{2}\sqrt{7}}^4$ $= \frac{9\sqrt{7}}{2} \pi + 16\pi \left[\left(4 - \frac{2}{3} \right) - \left(2 + \frac{\sqrt{7}}{2} - \frac{7\sqrt{7}}{96} \right) \right]$ $= \frac{1}{3} (64 - 7\sqrt{7}) \pi$

Qn	Solution
	<p><u>Alternative</u></p> <p>When $x = 3$, $y = 2\sqrt{1 - \frac{9}{16}} = \frac{1}{2}\sqrt{7}$</p> <p>When $x = 0$, $y = 2$</p> <p>Required Volume $= \frac{\sqrt{7}}{2}\pi(3^2) + \pi \int_{\frac{1}{2}\sqrt{7}}^2 16\left(1 - \frac{y^2}{4}\right) dy$</p> $= \frac{9\sqrt{7}}{2}\pi + 16\pi \int_{\frac{1}{2}\sqrt{7}}^2 1 - \frac{y^2}{4} dy$ $= \frac{9\sqrt{7}}{2}\pi + 16\pi \left[y - \frac{y^3}{12} \right]_{\frac{1}{2}\sqrt{7}}^2$ $= \frac{9\sqrt{7}}{2}\pi + \frac{1}{6}[128 - 41\sqrt{7}]\pi$ $= \frac{1}{3}(64 - 7\sqrt{7})\pi$
10(a)	<p>$z = x - y \Rightarrow \frac{dz}{dx} = 1 - \frac{dy}{dx}$</p> <p>$\Rightarrow \frac{dy}{dx} = 1 - \frac{dz}{dx}$</p> <p>$\frac{dy}{dx} = \frac{x - y - 1}{x - y + 1}$</p> <p>$\Rightarrow 1 - \frac{dz}{dx} = \frac{z - 1}{z + 1}$</p> <p>$\Rightarrow \frac{dz}{dx} = 1 - \frac{z - 1}{z + 1}$</p> <p>$\Rightarrow \frac{dz}{dx} = \frac{2}{z + 1}$</p> <p>$\int (z + 1) dz = \int 2 dx$</p> <p>$\frac{z^2}{2} + z = 2x + C$</p> <p>$\frac{(x - y)^2}{2} + x - y = 2x + C$ where C is a constant</p> <p>$\frac{(x - y)^2}{2} - x - y = C$</p> <p>When $x = 1$, $y = 1$,</p> <p>$\Rightarrow C = -2$</p> <p>Therefore $\frac{(x - y)^2}{2} - x - y = -2$</p>

Qn	Solution
(b)	<p> $\frac{dv}{dt} = 10 - kv^2$, where $k > 0$ When $v = 50$, $\frac{dv}{dt} = 7.5$ $7.5 = 10 - k(50)^2$ $\Rightarrow k = 0.001$ $\therefore \frac{dv}{dt} = 10 - 0.001v^2$ </p> <p> $\int \frac{1}{10 - 0.001v^2} dv = \int 1 dt$ $\frac{1}{0.001} \int \frac{1}{10000 - v^2} dv = \int 1 dt$ $1000 \int \frac{1}{100^2 - v^2} dv = \int 1 dt$ $\frac{1000}{200} \ln \left \frac{100+v}{100-v} \right = t + C$ $\ln \left \frac{100+v}{100-v} \right = \frac{1}{5}t + \frac{1}{5}C$ $\left \frac{100+v}{100-v} \right = e^{\frac{1}{5}t + \frac{1}{5}C}$ $\frac{100+v}{100-v} = \pm e^{\frac{1}{5}t + \frac{1}{5}C}$ $\frac{100+v}{100-v} = Ae^{0.2t}$ where $A = \pm e^{0.2C}$ </p> <p> When $t = 0$, $v = 0$ then $A = 1$ $\frac{100+v}{100-v} = e^{0.2t} \Rightarrow \frac{100-v}{100+v} = e^{-0.2t}$ $e^{-0.2t}(100+v) = 100-v$ $v(1+e^{-0.2t}) = 100(1-e^{-0.2t})$ $v = \frac{100(1-e^{-0.2t})}{1+e^{-0.2t}}$ </p> <p> As $t \rightarrow \infty$, $e^{-0.2t} \rightarrow 0$ and $v \rightarrow 100$ The sky diver's speed would increase to a limit of 100 m/s long after he has descended and before he deployed his parachute. </p>

Qn	Solution
11(i)	$V = 2\left[\pi(3r)^2 y\right] + (\pi r^2 \times 15)$ $k = 18\pi y r^2 + 15\pi r^2$ $y = \frac{1}{18\pi r^2}(k - 15\pi r^2)$ $A = 4\left[\pi(3r)^2\right] - 2\pi r^2 + 15(2\pi r) + 2y\left[2\pi(3r)\right]$ $= 34\pi r^2 + 30\pi r + 2\left[\frac{1}{18\pi r^2}(k - 15\pi r^2)\right]\left[2\pi(3r)\right]$ $= 34\pi r^2 + 30\pi r + \frac{2}{3r}(k - 15\pi r^2)$ $= 34\pi r^2 + 20\pi r + \frac{2k}{3r}$ $\frac{dA}{dr} = 68\pi r + 20\pi - \frac{2k}{3r^2}$ <p>At minimum area, $\frac{dA}{dr} = 0$</p> $68\pi r + 20\pi - \frac{2k}{3r^2} = 0$ $204\pi r^3 + 60\pi r^2 - 2k = 0$ $102\pi r^3 + 30\pi r^2 - k = 0 \text{ (shown)}$
(ii)	$102\pi r^3 + 30\pi r^2 - 450 = 0$ <p>From GC, $r = 1.03$ (3 s.f.)</p>
(iii)	<p>Volume of water pumped after 1 min = 15 (60) = 900 cm³</p> <p>Volume of a weight = $\pi(3 \times 2)^2 \times 7 = 791.68$ cm³</p> <p>Volume of the handle = $\pi(2)^2 \times 15 = 188.50$ cm³</p> <p>Since $900 < 791.68 + 188.50 = 980.18$, the water level is at the handle at 1 min.</p> <p>Let W = volume of water in the handle and h = depth of water from the base of the handle</p> $W = \pi(2)^2 h = 4\pi h$ $\frac{dW}{dh} = 4\pi$ $\frac{dh}{dt} = \frac{dW}{dt} \times \frac{dh}{dW}$ $= 15 \times \frac{1}{4\pi}$ <p>Thus the depth of the water is increasing at a rate of $\frac{15}{4\pi}$ cm s⁻¹.</p>

Qn	Solution
12(i)	<p>Let θ be the acute angle between the plane and the incident beam.</p> $\sin \theta = \frac{\left \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} \right }{\sqrt{1+4+1}\sqrt{1+4+9}}$ $= \frac{6}{\sqrt{84}}$ <p>Therefore $\theta = 40.9^\circ$</p>
(ii)	<p>Let F be the foot of the perpendicular from O to the plane.</p> $\overrightarrow{OF} = \lambda \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}, \text{ for some } \lambda \in \mathbb{R}$ <p>F is on plane $\Rightarrow \overrightarrow{OF} \cdot \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} = 12$</p> $\Rightarrow \lambda \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} = 12$ $14\lambda = 12$ $\lambda = \frac{6}{7}$ $\overrightarrow{OF} = \frac{6}{7} \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$ $\overrightarrow{OP} = \mu \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \text{ for some } \mu \in \mathbb{R}$ <p>P is on plane $\Rightarrow \overrightarrow{OP} \cdot \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} = 12$</p> $\Rightarrow \mu \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} = 12$ $6\mu = 12$ $\mu = 2$ $\overrightarrow{OP} = \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}$

Qn	Solution
	$\overrightarrow{OP} = \frac{2}{7} \begin{pmatrix} 13 \\ 2 \\ -11 \end{pmatrix}$ <p>Hence $l : \mathbf{r} = \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix} + \gamma \begin{pmatrix} 13 \\ 2 \\ -11 \end{pmatrix}, \gamma \in \mathbb{R}$</p>
(iii)	<p>Let $B \equiv (3, 1, 0)$.</p> <p>Shortest distance of B from incident beam</p> $= \frac{\left \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right }{\sqrt{1+4+1}} = \frac{\left \begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix} \right }{\sqrt{6}} = \sqrt{\frac{35}{6}} > 2$ $\overrightarrow{PB} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix}$ <p>Shortest distance of B from reflected beam</p> $= \frac{\left \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix} \times \begin{pmatrix} 13 \\ 2 \\ -11 \end{pmatrix} \right }{\sqrt{169+4+121}} = \frac{\left \begin{pmatrix} 37 \\ -15 \\ 41 \end{pmatrix} \right }{\sqrt{294}} = \sqrt{\frac{3275}{294}} > 2$ <p>Hence sensor will not work properly</p>