NATIONAL JUNIOR COLLEGE
SENIOR HIGH 2 PRELIMINARY EXAMINATION
Higher 2

MATHEMATICS 9758/01

Paper 1 **23 August 2017**

3 hours

Additional Materials: Answer Paper
List of Formulae (MF26)
Cover Sheet

READ THESE INSTRUCTIONS FIRST

Write your name, registration number, subject tutorial group, on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for diagrams or graphs.
Do not use paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in the brackets [ ] at the end of each question or part question.

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1 Given that $\mathbf{p} = 2\mathbf{i} + \alpha\mathbf{j} + \mathbf{k}$ and $\mathbf{q} = \alpha\mathbf{i} + \mathbf{j} + 6\mathbf{k}$, where α is a real constant and $\mathbf{w}$ is the position vector of a variable point W relative to the origin such that $\mathbf{w} \times \mathbf{p} = \mathbf{q}$.

(i) Find the value of α. [2]

(ii) Find the set of vectors $\mathbf{w}$ in the form $\{\mathbf{w} : \mathbf{w} = \mathbf{a} + \lambda\mathbf{b}, \lambda \in \mathbb{R}\}$. [3]

2 **(a)** The sum, S_n, of the first n terms of a sequence $u_1, u_2, u_3, \ldots$ is given by $S_n = 3 + 7^{-2n}(n^2)$.

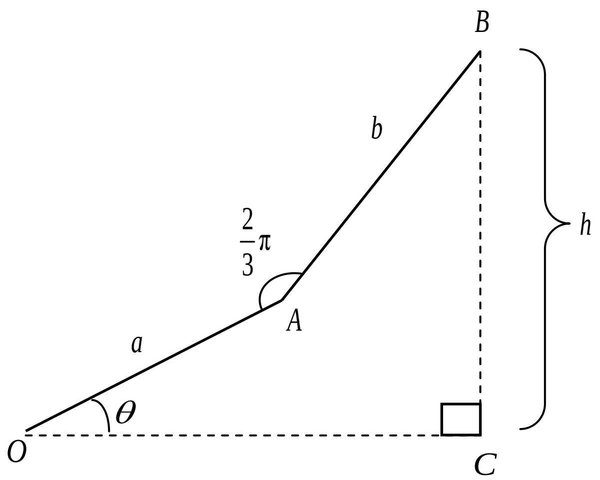
(i) Write down the value of $\sum_{r=1}^{\infty} u_r$. [1]

(ii) Find a formula for u_n for $n \geq 2$ and leave it in the form $7^{-2n}\,\mathrm{g}(n)$, where $\mathrm{g}(n)$ is an expression in terms of n. [2]

(b) Show that $\sum_{r=1}^{n}\left(\int_0^r \mathrm{e}^x - \mathrm{e}^{x-1}\,\mathrm{d}x\right) = \mathrm{e}^n + n\mathrm{e}^{-1} - (n+1)$.

Deduce the exact value of $\sum_{r=10}^{20}\left(\int_0^r \mathrm{e}^{x+2} - \mathrm{e}^{x+1}\,\mathrm{d}x\right)$. [5]

3 The diagram below shows two adjoining lines OA and AB where $OA = a$ m, $AB = b$ m and obtuse angle OAB is $\frac{2}{3}\pi$. C is a point such that OC and CB are perpendicular to each other, $BC = h$ m, and angle AOC is θ where $0 < \theta < \frac{\pi}{6}$.



(i) Show that $h = \sqrt{a^2 + ab + b^2}\sin(\theta + \alpha)$, where α is a constant to be determined in terms of a and b. [4]

It is given that $a = 1$ and $b = 2$.

(ii) Find the rate of change of θ when $\theta = \frac{\pi}{12}$ and h is decreasing at a rate of 0.5 m per minute. [3]

(iii) When θ is a sufficiently small angle, show that $h \approx p\theta^2 + q\theta + \sqrt{3}$, where constants p and q are to be determined exactly. [3]

[Turn over

4 A researcher is investigating the elasticity of a new material. In the experiment, he stretched an extensible string of length 30 cm using a machine.

Each stretch is followed by a contraction. The initial stretch leads to an elongation of 10 cm and is followed by a contraction of 0.1 cm. The elongation resulting from each subsequent stretch is $\frac{10}{11}$ of the elongation caused by the previous stretch. Each subsequent contraction is 0.001 cm less than the previous contraction.

(i) Show that the length of the string after two stretches is 48.892 cm correct to 3 decimal places. [2]

(ii) Find the length of the string after it has been stretched n times, in terms of n. [3]

(iii) The string loses its elasticity completely when contraction exceeds elongation in a stretch. Find the minimum number of stretches for the string to lose its elasticity. [2]

(iv) The researcher coats a new string of the same initial length with another material. Now the string does not contract after every stretch while its elongation properties remain unchanged. Justify why it is impossible for the string to be elongated beyond 140 cm. [1]

5 **Do not use a calculator in answering this question.**

(a) Showing your working clearly, find the complex numbers z and w which satisfy the simultaneous equations

$$\mathrm{i}z + w = 2 \quad \text{and}$$

$$zw^* = 2 + 4\mathrm{i},$$

where w^* is the complex conjugate of w. [5]

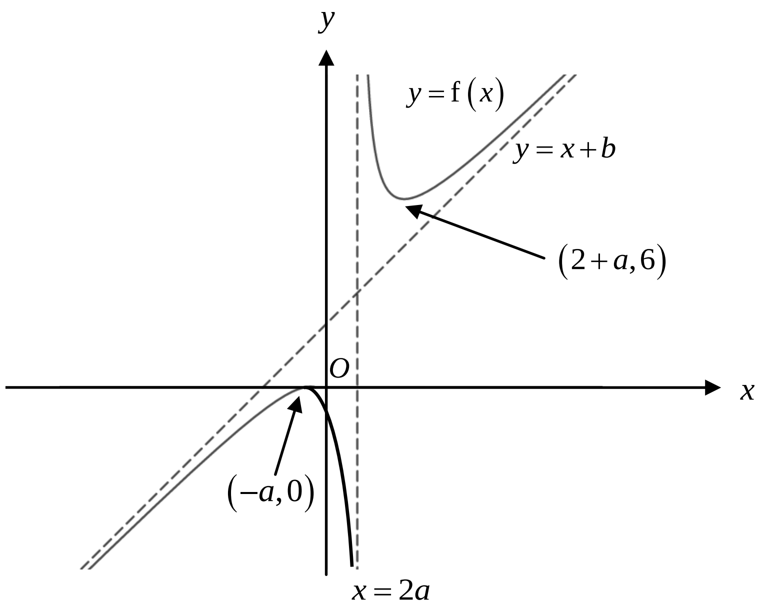
(b) The complex number p is given by $a + \mathrm{i}b$, where $a > 0$, $b < 0$, $a^2 + b^2 > 1$ and $\tan^{-1}\left(\frac{b}{a}\right) = -\frac{2\pi}{9}$.

(i) Express the complex number $\frac{1}{p^2}$ in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where r is in terms of a and b, and $-\pi < \theta \le \pi$. [2]

(ii) On a single Argand diagram, illustrate the points P and Q representing the complex numbers p and $\frac{1}{p^2}$ respectively, labelling clearly their modulus and argument. [2]

(iii) It is given that $\angle OPQ = \alpha$. Using sine rule, show that $|p|^3 \approx \frac{\sqrt{3}}{2\alpha} - \frac{1}{2} - \frac{\alpha}{2\sqrt{3}}$ where α is small. [4]

6 The diagram shows the graph of the function $y = \mathrm{f}(x)$ where, $a,\ b \in \mathbb{R}$, $b \geq 2$ and $0 < a < 1$. The coordinates of the minimum point and maximum point on the curve are $(-a,\ 0)$ and $(2+a,\ 6)$ respectively. The equations of the asymptotes are $y = x + b$ and $x = 2a$.



On separate diagrams, sketch the graphs of the following functions, labelling the coordinates of any points of intersection with the x-axis, the coordinates of any turning points and the equations of any asymptotes.

(i) $y = \mathrm{f}(2x-1) + 1$, [3]

(ii) $y = \dfrac{1}{\mathrm{f}(x)}$. [3]

The two asymptotes of $y = \mathrm{f}(x)$ intersect at point P. Show that P lies on the line $y = mx + (b + 2a - 2am)$ for all real values of m. Hence, state the range of values of m for which the line $y = mx + (b + 2a - 2am)$ does not cut the curve $y = \mathrm{f}(x)$. [3]

7 **(a)** Find $\int \mathrm{e}^{x} \cos(2x)\, \mathrm{d}x$. [3]

(b) The curve C has parametric equations

$$x = t - \mathrm{e}^{t}, \quad y = 3\cos^2 t - 1, \quad \text{for } 0 < t < \pi.$$

(i) Use differentiation to find the exact x-coordinate of any turning point and determine the nature of the turning point. [3]

(ii) Find the exact area of the region bounded by the curve C and the line $y = 2$, expressing your answer in the form $a\pi + b + c\mathrm{e}^{\pi}$, where a, b and c are rational numbers to be determined. [5]

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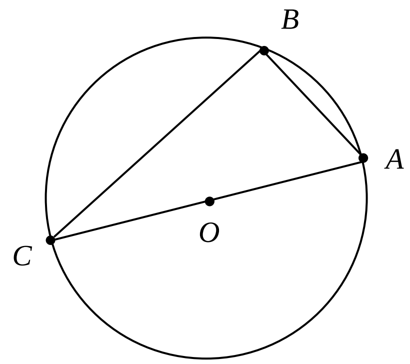


The diagram above shows the cross-section of a sphere containing the centre O of the sphere. The points A, B and C are on the circumference of the cross-section with the line segment AC passing through O. It is given that $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.

(i) Find $\overrightarrow{BC}$ in terms of $\mathbf{a}$ and $\mathbf{b}$. [1]

(ii) D is a point on BC such that ΔOCD is similar to ΔACB. Find $\overrightarrow{OD}$ in terms of $\mathbf{a}$ and $\mathbf{b}$. [2]

Point B lies on the x-z plane and has a positive z-component. It is also given that $\overrightarrow{OC} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$ and $\angle OCB = \dfrac{\pi}{6}$.

(iii) Show that $\overrightarrow{OB} = \begin{pmatrix} -1 \\ 0 \\ \sqrt{3} \end{pmatrix}$. [4]

(iv) Hence, determine whether the line passing through O and B and the line $\dfrac{x-2}{3} = \dfrac{y}{3} = z-1$ are skew. [3]

9 The parametric equations of the curve C are

$$x = 2\sec t \text{ and } y = 3\tan t\text{, where } -\pi < t \le \pi,\ t \ne \pm\frac{\pi}{2}.$$

(i) Write down the Cartesian equation of C. [1]

(ii) Sketch the curve C, stating the equations of the asymptotes and the coordinates of the points where C crosses the axes, if any. [2]

(iii) The line $y = \sqrt{3}x + k$, where $k < 0$, is a tangent to C. Show that $k = -\sqrt{3}$. [3]

The region bounded by this tangent, the curve C and the x-axis is rotated completely about the x-axis. Calculate the volume obtained. [4]

10 **(a)** By using the substitution $u = \dfrac{y}{x}$, show that the differential equation

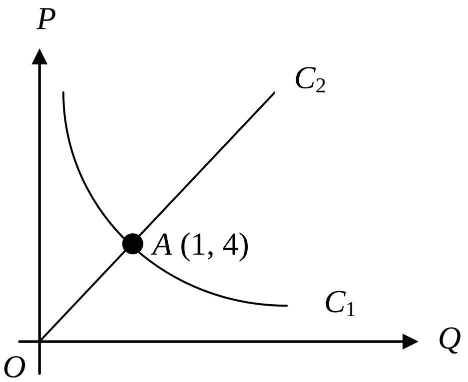
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2 + xy + x^2}{x^2}, \text{ where } x > 0,$$

can be reduced to $\dfrac{1}{u^2+1}\dfrac{\mathrm{d}u}{\mathrm{d}x} = \dfrac{1}{x}$. Hence, find y in terms of x. [5]

(b) In the diagram below, the curve C_1 and the line C_2 illustrate the relationship between price (P dollars per kg) and quantity (Q tonnes) for consumers and producers respectively.

The curve C_1 shows the quantity of rice that consumers will buy at each price level while the line C_2 shows the quantity of rice that producers will produce at each price level. C_1 and C_2 intersect at point A, which has the coordinates (1, 4).

The quantity of rice that consumers will buy is inversely proportional to the price of the rice. The quantity of rice that producers will produce is directly proportional to the price.



(i) Interpret the coordinates of A in the context of the question. [1]

(ii) Solve for the equations of C_1 and C_2, expressing Q in terms of P. [2]

Shortage occurs when the quantity of rice consumers will buy exceeds the quantity of rice producers will produce. It is known that the rate of increase of P after time t months is directly proportional to the quantity of rice in shortage.

(iii) Given that the initial price is \$3 and that after 1 month, the price is \$3.65, find P in terms of t and sketch this solution curve, showing the long-term behaviour of P. [7]

Suggest a reason why producers might use $P = aQ + b$, where $a, b \in \mathbb{R}^+$, instead of C_2 to model the relationship between price and quantity of rice produced. [1]

--- END OF PAPER ---