2017 JC2 Further Mathematics Prelim Paper 1

1	Let P_n be the statement that $\sum_{r=1}^{n}\frac{1}{r^2}<2-\frac{1}{n}$ for all integers $n\ge 2$. To show P_2 is true: LHS = $\sum_{r=1}^{2}\frac{1}{r^2}=\frac{5}{4}$. RHS = $2-\frac{1}{2}=\frac{3}{2}>\frac{5}{4}$. Hence P_2 is true. Suppose P_k is true for some integer $k\ge 2$, i.e. $\sum_{r=1}^{k}\frac{1}{r^2}<2-\frac{1}{k}$. Want to show P_{k+1} is true: $\sum_{r=1}^{k+1}\frac{1}{r^2}=\sum_{r=1}^{k}\frac{1}{r^2}+\frac{1}{(k+1)^2}$ $<2-\frac{1}{k}+\frac{1}{(k+1)^2}$ $=2-\frac{k^2+k+1}{k(k+1)^2}$ $<2-\frac{k^2+k}{k(k+1)^2}$ $=2-\frac{1}{k+1}$ Hence P_{k+1} is true. Since P_2 is true and P_k implies P_{k+1}, by the Principle of Mathematical Induction, P_n is true for all integers $n\ge 2$.	
2(i)	$(z^6+2z^3+4)(z^3-2)=z^9+2z^6+4z^3-2z^6-4z^3-8=z^9-8$	
2(ii)	$z^9=8=8e^{i0}$ Let $z=re^{i\theta}$ $r^9e^{i9\theta}=8e^{i0}=8e^{i2k\pi}$ $r^9=8$ $\quad 9\theta=2k\pi$ $r=\sqrt[9]{8}$ $\quad \theta=\frac{2k\pi}{9},k=0,\pm1,\pm2,\pm3\pm4$ $=\sqrt[3]{2}$ Hence the solutions are $z=\sqrt[3]{2}e^{i0}\left(=\sqrt[3]{2}\right),\sqrt[3]{2}e^{\pm i\frac{2\pi}{9}},\sqrt[3]{2}e^{\pm i\frac{4\pi}{9}},\sqrt[3]{2}e^{\pm i\frac{2\pi}{3}},\sqrt[3]{2}e^{\pm i\frac{8\pi}{9}}$.	
2(iii)	Solutions to $z^3-2=0$ are $z=\sqrt[3]{2}e^{i0},\sqrt[3]{2}e^{\pm i\frac{2\pi}{3}}$. Hence solutions to $z^6+2z^3+4=0$ are $z=\sqrt[3]{2}e^{\pm i\frac{2\pi}{9}},\sqrt[3]{2}e^{\pm i\frac{4\pi}{9}},\sqrt[3]{2}e^{\pm i\frac{8\pi}{9}}$. For a particular pair of conjugate solutions $z=\sqrt[3]{2}e^{\pm i\theta}$, we have	

	$\left(z-\sqrt[3]{2}\mathrm{e}^{\mathrm{i}\theta}\right)\left(z-\sqrt[3]{2}\mathrm{e}^{-\mathrm{i}\theta}\right)=z^2-\sqrt[3]{2}\left(\mathrm{e}^{\mathrm{i}\theta}+\mathrm{e}^{-\mathrm{i}\theta}\right)z+\left(\sqrt[3]{2}\mathrm{e}^{\mathrm{i}\theta}\right)\left(\sqrt[3]{2}\mathrm{e}^{-\mathrm{i}\theta}\right)$ $=z^2-\sqrt[3]{2}\left(2\cos\theta\right)z+\sqrt[3]{4}$ Hence the three quadratic factors are $z^2-\sqrt[3]{2}\left(2\cos\frac{2\pi}{9}\right)z+\sqrt[3]{4}$, $z^2-\sqrt[3]{2}\left(2\cos\frac{4\pi}{9}\right)z+\sqrt[3]{4}$, and $z^2-\sqrt[3]{2}\left(2\cos\frac{8\pi}{9}\right)z+\sqrt[3]{4}$.	
3(i)	$\mathbf{A}=\begin{pmatrix}1&2&0\\-2&0&1\\1&1&0\end{pmatrix}\begin{pmatrix}-1&0&0\\0&3&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&2&0\\-2&0&1\\1&1&0\end{pmatrix}^{-1}=\begin{pmatrix}7&0&-8\\-4&1&8\\4&0&-5\end{pmatrix}$	
3(ii)	$\mathbf{B}\begin{pmatrix}1\\-2\\1\end{pmatrix}=\begin{pmatrix}-4&0&6\\-2&3&4\\-3&0&5\end{pmatrix}\begin{pmatrix}1\\-2\\1\end{pmatrix}=\begin{pmatrix}2\\-4\\2\end{pmatrix}=2\begin{pmatrix}1\\-2\\1\end{pmatrix}$ $\mathbf{B}\begin{pmatrix}2\\0\\1\end{pmatrix}=\begin{pmatrix}-4&0&6\\-2&3&4\\-3&0&5\end{pmatrix}\begin{pmatrix}2\\0\\1\end{pmatrix}=\begin{pmatrix}-2\\0\\-1\end{pmatrix}=-\begin{pmatrix}2\\0\\1\end{pmatrix}$ $\mathbf{B}\begin{pmatrix}0\\1\\0\end{pmatrix}=\begin{pmatrix}-4&0&6\\-2&3&4\\-3&0&5\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix}=\begin{pmatrix}0\\3\\0\end{pmatrix}=3\begin{pmatrix}0\\1\\0\end{pmatrix}$ Hence, the corresponding eigenvectors are 2, -1 and 3 respectively.	
3(iii)	$\mathbf{AB}$ has the same eigenvectors as $\mathbf{A}$ and $\mathbf{B}$ with the corresponding eigenvalues -2, -3 and 3 respectively. Let $\mathbf{U}=\begin{pmatrix}1&2&0\\-2&0&1\\1&1&0\end{pmatrix}$. Then $\mathbf{U}^{-1}\left(\mathbf{AB}\right)\mathbf{U}=\begin{pmatrix}-2&0&0\\0&-3&0\\0&0&3\end{pmatrix}$.	

	$\left(\mathbf{U}^{-1}(\mathbf{AB})\mathbf{U}\right)^n = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 3 \end{pmatrix}^n = \begin{pmatrix} (-2)^n & 0 & 0 \\ 0 & (-3)^n & 0 \\ 0 & 0 & 3^n \end{pmatrix}$ $\mathbf{U}^{-1}(\mathbf{AB})^n\mathbf{U} = \begin{pmatrix} (-2)^n & 0 & 0 \\ 0 & (-3)^n & 0 \\ 0 & 0 & 3^n \end{pmatrix}$ $(\mathbf{AB})^n = \begin{pmatrix} 1 & 2 & 0 \\ -2 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} (-2)^n & 0 & 0 \\ 0 & (-3)^n & 0 \\ 0 & 0 & 3^n \end{pmatrix}\begin{pmatrix} 1 & 2 & 0 \\ -2 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}^{-1}$ $= \begin{pmatrix} 2(-3)^n-(-2)^n & 0 & 2(-2)^n-2(-3)^n \\ -(-2)^{n+1}-2(3^n) & 3^n & 2(-2)^{n+1}+4(3^n) \\ (-3)^n-(-2)^n & 0 & 2(-2)^n-(-3)^n \end{pmatrix}$	
4(i)	Equation of tangent at E is $\dfrac{x(a\sec\theta)}{a^2}-\dfrac{y(b\tan\theta)}{b^2}=1$ $\Rightarrow y=\dfrac{bx}{a\sin\theta}-\dfrac{b}{\tan\theta}$ Equation of line CD is $\dfrac{y-d}{x-a}=\dfrac{d-c}{2a}\Rightarrow y=\dfrac{d-c}{2a}x+\dfrac{d+c}{2}$ Comparing the two equations, $\dfrac{d-c}{2a}=\dfrac{b}{a\sin\theta}\Rightarrow\dfrac{d}{2}-\dfrac{c}{2}=\dfrac{b}{\sin\theta}$ $\dfrac{d}{2}+\dfrac{c}{2}=-\dfrac{b}{\tan\theta}$ Therefore $c=-\dfrac{b}{\sin\theta}-\dfrac{b}{\tan\theta}=-b\left(\dfrac{1+\cos\theta}{\sin\theta}\right)$ $d=\dfrac{b}{\sin\theta}-\dfrac{b}{\tan\theta}=b\left(\dfrac{1-\cos\theta}{\sin\theta}\right)$	
4(ii)	The foci lie on the x-axis and are equidistant from the origin, hence $OF=OF'$. M lies on the y-axis. Hence $\angle MOF=\angle MOF'=90^\circ$. Hence $\Delta MOF\equiv\Delta MOF'$ (SAS) Hence $MF=MF'$. Coordinates of foci are $(\pm ae,0)=\left(\pm\sqrt{a^2+b^2},0\right)$	

	Coordinates of M are $\left(\frac{a+(-a)}{2},\frac{c+d}{2}\right)=(0,-b\cot\theta)$ Distance $=\sqrt{a^2+b^2+b^2\cot^2\theta}=\sqrt{a^2+b^2\operatorname{cosec}^2\theta}$.	
4(iii)	$MC=MD=\sqrt{a^2+\left(b\cot\theta-b\left(\frac{\cos\theta-1}{\sin\theta}\right)\right)^2}$ $=\sqrt{a^2+b^2\left(\cot\theta+\frac{\cos\theta-1}{\sin\theta}\right)^2}$ $=\sqrt{a^2+b^2\left(\frac{1}{\sin\theta}\right)^2}$ $=\sqrt{a^2+b^2\operatorname{cosec}^2\theta}=MF=MF'$ Since CD is a straight line and M is its midpoint, a circle drawn with centre at M and radius MC would have CD as a diameter and F and F' on its circumference. Hence $\angle CFD$ and $\angle CF'D$ are both right angles.	
5(i)	$u_{n+2}=8u_{n+1}-15u_n$ $u_{n+2}-3u_{n+1}=5u_{n+1}-15u_n$ $=5(u_{n+1}-3u_n)$	
5(ii)	$v_{n+1}=5v_n=5^{n+1}v_0=5^{n+1}(u_1-3u_0)$ $v_n=5^n(u_1-3u_0)$ $=5^n(1-3)$ $=(-2)5^n$	
5(iii)	$u_{n+2}=8u_{n+1}-15u_n$ $u_{n+2}-5u_{n+1}=3u_{n+1}-15u_n$ $=3(u_{n+1}-5u_n)$ Let $w_n=u_{n+1}-5u_n$. $w_{n+1}=3w_n=3^{n+1}w_0=3^{n+1}(u_1-5u_0)$ $w_n=3^n(u_1-5u_0)$ $=3^n(1-5)$ $=(-4)3^n$	
5(iv)	$u_{n+1}-3u_n=(-2)5^n$ $u_{n+1}-5u_n=(-4)3^n$ By eliminating u_{n+1} : $u_n=\frac{1}{2}\left[(-2)5^n-(-4)3^n\right]$ $=2(3^n)-5^n$	

6(i)	$\begin{pmatrix} 2 & 1 & 4 & 3 \\ 1 & 1 & 3 & 2 \\ 1 & -1 & -1 & 0 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ Solution set is $\left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} : x = -z - w, y = -2z - w \right\}$. $\begin{pmatrix} -z-w \\ -2z-w \\ z \\ w \end{pmatrix} = z\begin{pmatrix} -1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + w\begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \end{pmatrix}$. Hence a basis for the null space is $\left\{ \begin{pmatrix} -1 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\}$.	
6(ii)	Gauss-Jordan Elimination of $\mathbf{A}^T$: $\begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & -1 \\ 4 & 3 & -1 \\ 3 & 2 & 0 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ A basis for R is $\left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} \right\}$.	
6(iii)	$\begin{pmatrix} \lambda \\ \mu \\ 0 \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix}$ $a = \lambda, b = \mu \Rightarrow 2\lambda - 3\mu = 0 \Rightarrow \mu = \frac{2}{3}\lambda$,. $\left(\begin{array}{cccc\|c} 2 & 1 & 4 & 3 & \lambda \\ 1 & 1 & 3 & 2 & \frac{2}{3}\lambda \\ 1 & -1 & -1 & 0 & 0 \end{array}\right) \xrightarrow{RREF} \left(\begin{array}{cccc\|c} 1 & 0 & 1 & 1 & \frac{1}{3}\lambda \\ 0 & 1 & 2 & 1 & \frac{1}{3}\lambda \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$ Solution set is $\left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} : x = \frac{1}{3}\lambda - z - w, y = \frac{1}{3}\lambda - 2z - w \right\}$. May also be written as	

	$\left\{\lambda\begin{pmatrix}\frac{1}{3}\\\frac{1}{3}\\0\\0\end{pmatrix}+z\begin{pmatrix}-1\\-2\\1\\0\end{pmatrix}+w\begin{pmatrix}-1\\-1\\0\\1\end{pmatrix}:z,w\in\mathbb{R}\right\}.$	
6(iv)	Let $V=\left\{\mathbf{x}\in\mathbb{R}^4:\mathrm{T}(\mathbf{x})=\begin{pmatrix}\lambda\\\mu\\0\end{pmatrix},\lambda\in\mathbb{R}\right\}$ If $\mathbf{x}_1,\mathbf{x}_2\in V$, then $\mathbf{x}_1=\lambda_1\begin{pmatrix}1\\1\\0\\0\end{pmatrix}+z_1\begin{pmatrix}-1\\-2\\1\\0\end{pmatrix}+w_1\begin{pmatrix}-1\\-1\\0\\1\end{pmatrix}$ and $\mathbf{x}_2=\lambda_2\begin{pmatrix}1\\1\\0\\0\end{pmatrix}+z_2\begin{pmatrix}-1\\-2\\1\\0\end{pmatrix}+w_2\begin{pmatrix}-1\\-1\\0\\1\end{pmatrix}$ for $\lambda_1,z_1,w_1,\lambda_2,z_2,w_2\in\mathbb{R}$. $\mathbf{x}_1+\mathbf{x}_2=(\lambda_1+\lambda_2)\begin{pmatrix}1\\1\\0\\0\end{pmatrix}+(z_1+z_2)\begin{pmatrix}-1\\-2\\1\\0\end{pmatrix}+(w_1+w_2)\begin{pmatrix}-1\\-1\\0\\1\end{pmatrix}\in V$ And $k\mathbf{x}_1+\mathbf{x}_2=k\lambda_1\begin{pmatrix}1\\1\\0\\0\end{pmatrix}+kz_1\begin{pmatrix}-1\\-2\\1\\0\end{pmatrix}+kw_1\begin{pmatrix}-1\\-1\\0\\1\end{pmatrix}\in V$ for all $k\in\mathbb{R}$. Hence V is a subspace of $\mathbb{R}^4$, as it is closed under addition and multiplication by a scalar.	
7(i)	When $\theta=0, r=r_0$ $r_0=\dfrac{1}{A\cos(0)+B\sin(0)+k}=\dfrac{1}{A+k}$ $A=\dfrac{1}{r_0}-k$ When $\theta=0,\dfrac{dr}{d\theta}=0$ $\dfrac{B\cos(0)-A\sin(0)}{\left(A\cos(0)+B\sin(0)+k\right)^2}=0\Rightarrow B=0$	

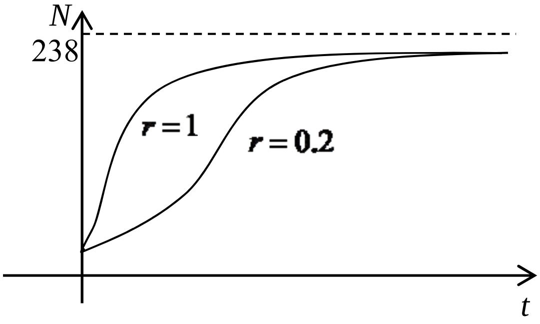
	$r=\dfrac{1}{\left(\dfrac{1}{r_0}-k\right)\cos\theta+k}$ $=\dfrac{\frac{1}{k}}{\left(\dfrac{1}{kr_0}-1\right)\cos\theta+1}$ Hence, eccentricity is $\dfrac{1}{kr_0}-1$. To have an elliptical orbit, $0\le\dfrac{1}{kr_0}-1<1$ $\dfrac{1}{2r_0}<k\le\dfrac{1}{r_0}$	
7(ii)	When $k=\dfrac{2}{3r_0}$, $r=\dfrac{\frac{3}{2}r_0}{\frac{1}{2}\cos\theta+1}$. Arc length of one revolution is given by $\int_0^{2\pi}\sqrt{r^2+\left(\dfrac{dr}{d\theta}\right)^2}\,d\theta$ $=\int_0^{2\pi}\sqrt{\left(\dfrac{\frac{3}{2}r_0}{\frac{1}{2}\cos\theta+1}\right)^2+\left(\dfrac{\frac{3}{4}r_0\sin\theta}{\left(\frac{1}{2}\cos\theta+1\right)^2}\right)^2}\,d\theta$ $=r_0\int_0^{2\pi}\dfrac{\frac{3}{2}}{\frac{1}{2}\cos\theta+1}\sqrt{1+\dfrac{\sin^2\theta}{4\left(\frac{1}{2}\cos\theta+1\right)^2}}\,d\theta$ $=r_0\int_0^{2\pi}\dfrac{3}{\cos\theta+2}\sqrt{1+\dfrac{\sin^2\theta}{(\cos\theta+2)^2}}\,d\theta$ $=r_0\int_0^{2\pi}\dfrac{3}{(\cos\theta+2)}\sqrt{\dfrac{\sin^2\theta+\cos^2\theta+4\cos\theta+4}{(\cos\theta+2)^2}}\,d\theta$ $=r_0\int_0^{2\pi}\dfrac{3\sqrt{4\cos\theta+5}}{(\cos\theta+2)^2}\,d\theta$ Because of symmetry of $\cos\theta$ around $\theta=\pi$, $r_0\int_0^{2\pi}\dfrac{3\sqrt{4\cos\theta+5}}{(\cos\theta+2)^2}\,d\theta=r_0\int_0^{\pi}\dfrac{6\sqrt{4\cos\theta+5}}{(\cos\theta+2)^2}\,d\theta$.	

7(iii)	$\int_0^{\pi} \mathrm{f}(\theta)\,d\theta = \frac{\pi}{12}\left[\mathrm{f}(0)+4\mathrm{f}\left(\frac{\pi}{4}\right)+2\mathrm{f}\left(\frac{\pi}{2}\right)+4\mathrm{f}\left(\frac{3\pi}{4}\right)+\mathrm{f}(\pi)\right]$ $= \frac{\pi r_0}{12}[45.029]$ $= 11.8 r_0$	
8(i)	$\mathrm{f}(x)=\sin\left(\mathrm{e}^{x}\right)-kx$ $\mathrm{f}\left(\ln\frac{\pi}{2}\right)=\sin\left(\frac{\pi}{2}\right)-k\ln\frac{\pi}{2}=1-k\ln\frac{\pi}{2}>0$ for $k<2$. $\mathrm{f}(\ln\pi)=\sin(\pi)-k\ln\pi=-k\ln\pi<0$ for $k>0$. Since f is continuous on $\left(\ln\frac{\pi}{2},\ln\pi\right)$ and $\mathrm{f}\left(\ln\frac{\pi}{2}\right)\mathrm{f}(\ln\pi)<0$, f has at least a real root in the interval $\left(\ln\frac{\pi}{2},\ln\pi\right)$. $\mathrm{f}'(x)=\mathrm{e}^{x}\cos\left(\mathrm{e}^{x}\right)-k$ For $x\in\left(\ln\frac{\pi}{2},\ln\pi\right)$, $\frac{\pi}{2}<\mathrm{e}^{x}<\pi$, $-1<\cos\mathrm{e}^{x}<0$, Hence $\mathrm{f}'(x)<0$ for $0<k<2$ (i.e. f is strictly decreasing on the interval). Therefore, f has exactly one real root in the interval $\left(\ln\frac{\pi}{2},\ln\pi\right)$.	
8(ii)	$\text{RHS}=\sqrt{2}\sin\left(\frac{\pi}{4}-x\right)$ $=\sqrt{2}\sin\frac{\pi}{4}\cos x-\sqrt{2}\sin x\cos\frac{\pi}{4}$ $=\cos x-\sin x=\text{LHS}$	
8(iii)	Carry out linear interpolation for the approximate root (let it be *c*) of $y=\mathrm{f}(x)$ in the interval $\left(\ln\frac{\pi}{2},\ln\pi\right)$ with the points $\left(\ln\frac{\pi}{2},1-\ln\frac{\pi}{2}\right)$ and $(\ln\pi,-\ln\pi)$. Then $c=\dfrac{-\ln\frac{\pi}{2}\ln\pi-\ln\pi\left(1-\ln\frac{\pi}{2}\right)}{-\ln\pi-\left(1-\ln\frac{\pi}{2}\right)}=\dfrac{-\ln\pi}{\ln\frac{1}{2}-1}\approx 0.676$. $\mathrm{f}(x)=\sin\left(\mathrm{e}^{x}\right)-x$	

	$\mathrm{f}'(x)=\mathrm{e}^x\cos(\mathrm{e}^x)-1$ $\mathrm{f}''(x)=\mathrm{e}^x\cos(\mathrm{e}^x)-\mathrm{e}^{2x}\sin(\mathrm{e}^x)$ $<\mathrm{e}^x\left[\cos(\mathrm{e}^x)-\sin(\mathrm{e}^x)\right]$ since $\mathrm{e}^{2x}>\mathrm{e}^x$, and $\sin(\mathrm{e}^x)>0$ $=\sqrt{2}\mathrm{e}^x\sin\left(\frac{\pi}{4}-\mathrm{e}^x\right)$ For $x\in\left(\ln\frac{\pi}{2},\ln\pi\right)$, $-\pi<-\mathrm{e}^x<-\frac{\pi}{2}$, $-\frac{3\pi}{4}<\frac{\pi}{4}-\mathrm{e}^x<-\frac{\pi}{4}$, $\sin\left(\frac{\pi}{4}-\mathrm{e}^x\right)<0$. Hence $\mathrm{f}''(x)<0$ for $x\in\left(\ln\frac{\pi}{2},\ln\pi\right)$, which implies that the graph of $y=\mathrm{f}(x)$ is concave downwards on the interval. From **3(i)**, $\mathrm{f}'(x)<0$ for $x\in\left(\ln\frac{\pi}{2},\ln\pi\right)$. Since the graph of $y=\mathrm{f}(x)$ is decreasing and concave downwards on the interval, the method of linear interpolation would then have led to an underestimated approximation of the root.	
8(iv)	$\mathrm{f}(x)=\sin(\mathrm{e}^x)-x$, $\mathrm{f}'(x)=\mathrm{e}^x\cos(\mathrm{e}^x)-1$ $\mathrm{f}(\ln\pi)=-\ln\pi$, $\mathrm{f}'(\ln\pi)=-\pi-1$, $x_2=x_1-\frac{\mathrm{f}(x_1)}{\mathrm{f}'(x_1)}$ $x_2=\ln\pi-\frac{-\ln\pi}{-\pi-1}=\ln\pi-\frac{\ln\pi}{\pi+1}=\ln\pi\left(\frac{\pi}{\pi+1}\right)$ $x_3=\ln\pi\left(\frac{\pi}{\pi+1}\right)-\frac{\mathrm{f}(x_2)}{\mathrm{f}'(x_2)}=0.802$.	
9(i)	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, $a>b$ Foci of hyperbola are $(\pm c,0)$, where $c^2=a^2+b^2$ $\frac{x^2}{2a^2}+\frac{y^2}{a^2-b^2}=1$ Foci of ellipse are $(\pm c,0)$, where $c^2=2a^2-(a^2-b^2)=a^2+b^2$ Hence the two conic sections have the same foci.	
9(ii)	$PF+PF'=2\sqrt{2}a$ (property of ellipse) ---(1) $\lvert PF-PF'\rvert=2a$ (property of hyperbola) ---(2) $(1)^2-(2)^2: 4(PF)(PF')=4a^2 \Rightarrow (PF)(PF')=a^2$.	

9(iii)	$FF' = 2c = 2\sqrt{a^2+b^2}$ Since $(PF)(PF') = a^2$, $(PF+PF')^2 = 8a^2 \;\Rightarrow\; (PF)^2 + (PF')^2 = 6a^2$ Then $(FF')^2 = (PF)^2 + (PF')^2 - 2(PF)(PF')\cos\angle F'PF$ $4a^2 + 4b^2 = 6a^2 - 2a^2\cos\angle F'PF$ $\cos\angle F'PF = \dfrac{a^2-2b^2}{a^2}\left(=1-\dfrac{2b^2}{a^2}\right).$	
9(iv)	$\sin^2\angle F'PF = 1-\cos^2\angle F'PF$ $= 1-\left(\dfrac{a^4-4a^2b^2+4b^4}{a^4}\right)$ $= \dfrac{4a^2b^2-4b^4}{a^4}$ $\therefore \sin\angle F'PF = \dfrac{2b\sqrt{a^2-b^2}}{a^2}$ Area of triangle $F'PF$ $= \dfrac{1}{2}(PF)(PF')\sin\angle F'PF$ $= \dfrac{1}{2}(a^2)\left(\dfrac{2b\sqrt{a^2-b^2}}{a^2}\right)$ $= b\sqrt{a^2-b^2}.$ **OR (without using 'Hence')** $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1;\quad \dfrac{x^2}{2a^2}+\dfrac{y^2}{a^2-b^2}=1$ $\dfrac{1}{2}\left(\dfrac{x^2}{a^2}\right)+\dfrac{y^2}{a^2-b^2}=1$ $\dfrac{1}{2}\left(1+\dfrac{y^2}{b^2}\right)+\dfrac{y^2}{a^2-b^2}=1$ $\dfrac{y^2}{2b^2}+\dfrac{y^2}{a^2-b^2}=\dfrac{1}{2}$ $\dfrac{(a^2-b^2)y^2+2b^2y^2}{2b^2(a^2-b^2)}=\dfrac{1}{2}$ $y^2=\dfrac{b^2(a^2-b^2)}{a^2+b^2}$ Area of triangle $F'PF$	

	$=\frac{1}{2}(FF')\left(\frac{b\sqrt{\left(a^2-b^2\right)}}{\sqrt{a^2+b^2}}\right)$ $=\frac{1}{2}\left(2\sqrt{a^2+b^2}\right)\left(\frac{b\sqrt{\left(a^2-b^2\right)}}{\sqrt{a^2+b^2}}\right)$ $=b\sqrt{a^2-b^2}$.	
10(i)	The proposed model assumes that the population (infected people) can grow indefinitely. In this context, N did not grow exponentially indefinitely as Singapore was taken off the ‘infected areas’ list after a few months (when there were no more new infections). **OR** The proposed model assumes that r, the growth rate (per-capita growth rate), is constant. In this context, r is unlikely to stay constant as it changes over time due to measures taken by the Singaporean government.	
10(ii)	$\frac{\mathrm{d}N}{\mathrm{d}t}=\frac{r}{k}N\left(k-N\right)$ $\int\frac{1}{N\left(k-N\right)}\,\mathrm{d}N=\int\frac{r}{k}\,\mathrm{d}t$ $\frac{1}{k}\int\frac{1}{N}+\frac{1}{k-N}\mathrm{d}N=\int\frac{r}{k}\,\mathrm{d}t$ $\ln\left\|\frac{N}{k-N}\right\|=rt+c$ for some constant c $\frac{N}{k-N}=A\mathrm{e}^{rt}$ where $A=\pm\mathrm{e}^{c}$ $N=\left(k-N\right)A\mathrm{e}^{rt}$ $N=\frac{kA\mathrm{e}^{rt}}{1+A\mathrm{e}^{rt}}=\frac{k}{B\mathrm{e}^{-rt}+1}$ where $B=\frac{1}{A}$ $t=0, N=1: B=k-1$ $\therefore N=\frac{k}{\left(k-1\right)\mathrm{e}^{-rt}+1}$	
10(iii)	$k=238$ For $r=1$, the model predicts 44 infected cases after about 4 days. For $r=0.2$, the model predicts 44 infected cases after about 20 days. The given data showed that 44 infected cases were present after 21 days. Thus, the value of $r=0.2$ used models the data better.	

		
10(iv)	$\frac{dN}{dt}=(0.1+0.1t)N\left(1+\frac{N}{40}\right),\ N(0)=1$. Step size: 0.5. $N(0.5)=N(0)+0.5N'(0,1)$ $=1+0.5(0.1)(1)\left(1+\frac{1}{40}\right)=1.05125$ $N(1)=N(0.5)+0.5N'(0.5,1.05125)$ $=1.05125+0.5(0.1+0.05)(1.05125)\left(1+\frac{1.05125}{40}\right)$ ≈ 1.1322 Hence, the number of infected cases after 1 week will be 1132.	