NATIONAL JUNIOR COLLEGE
SENIOR HIGH 2 Preliminary Examination
Higher 2

FURTHER MATHEMATICS 9649 / 02

11 September 2017

3 hours

Additional Materials: Answer Paper
List of Formulae (MF26)
Cover Sheet

READ THESE INSTRUCTIONS FIRST

Write your name, registration number, subject tutorial group, on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use an HB pencil for diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in the brackets [ ] at the end of each question or part question.

Section A: Pure Mathematics [50 marks]

1 **(i)** On the same Argand diagram, sketch the loci of points given by each of the following equations.

$$L_1 : |z-2+\mathrm{i}| = 2,$$
$$L_2 : \left|\frac{z+1-2\mathrm{i}}{z+3+2\mathrm{i}}\right| = 1. \qquad [3]$$

(ii) It is given that two complex numbers w_1 and w_2 satisfy both

$$|w_i - 2 + \mathrm{i}| \le 2 \text{ and } \left|\frac{w_i+1-2\mathrm{i}}{w_i+3+2\mathrm{i}}\right| \le 1 \text{ for } i = 1, 2.$$

Find the greatest possible value of $\arg\left(\frac{w_1}{w_2}\right)$, leaving your answer in the form $\cos^{-1} a$, where a is a negative real number to be determined exactly. [4]

2 It is given that $\frac{\mathrm{d}^m}{\mathrm{d}x^m}\left(x^m\right) = m!$, where m is positive integer.

Prove by induction that $\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(x^n \ln x\right) = n!\left(\ln x + \sum_{r=1}^{n}\frac{1}{r}\right)$ for every positive integer n. [5]

Hence, find $\int\int\int\int \ln x \,\mathrm{d}x \,\mathrm{d}x \,\mathrm{d}x \,\mathrm{d}x$. [3]

3 **Do not use a calculator in answering this question.**

The three planes p_1, p_2 and p_3 have equations

$$x + 2y - z = 1,$$
$$2x + 5y + az = 3,$$
$$3x + ay + bz = a - 3,$$

respectively, where a and b are real constants.

It is given that the three planes do not have a unique point of intersection. By considering the augmented matrix of the system, or otherwise, find b in terms of a. Describe the nature of the intersection between the three planes in this case. [4]

It is further given that a point with coordinates $(c, b, 5)$ is common to all three planes. Find the possible values of a, b and c, and determine the set of points that are common to all three planes in each case. [7]

4 **(a)** Find the general solution of the differential equation

$$(1+x)y - x\frac{\mathrm{d}y}{\mathrm{d}x} = x^3 - x^2,$$

leaving your answer in the form $y = \mathrm{f}(x)$. [5]

Sketch and label clearly the equations of 2 **distinct** members of the family of solution curves where their nature of stationary points differ from each other. [2]

(b) It is given that $\frac{\mathrm{d}N}{\mathrm{d}t} = 3t - N^2$ and $N(0) = 1$. Use two iterations of the improved Euler's method to find an approximate value of $N(0.4)$, correct to 4 decimal places. [4]

5 A curve C is defined parametrically by

$$x = r(\theta - \sin\theta),\ y = r(1 - \cos\theta),$$

where r is a positive constant and $0 \le \theta \le 2\pi$.

(i) The curve C is rotated through one revolution about the x-axis. Given that the area of the surface generated is 432π units^2, find the exact value of r. [7]

Suppose instead that $r = 2$.

(ii) Let L be the line that passes through the origin and the point $(3\pi + 2, 2)$ on C. The region R is enclosed by C and L. Calculate the volume of the solid generated when R is rotated through 2π radians about the y-axis. [6]

Section B: Statistics [50 marks]

6 An amateur statistician wanted to investigate whether a particular tetrahedral die numbered '1', '2', '3' and '4' is fair. He repeated an experiment 100 times, in which the die was tossed until an even number appeared. Let X denote the number of tosses required in each experiment. Its values and the corresponding frequencies are recorded in the table below.

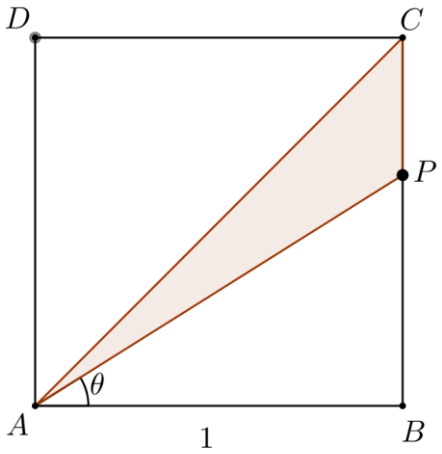
x	1	2	3	4	5	6
f_i	45	31	11	4	4	5

(i) What should the distribution of X be if the die is fair? State the value of any parameter of the distribution that you use. [1]

(ii) Show that, at the 10% level of significance, your answer to part **(i)** is a good fit for the distribution of X. [5]

Explain whether we can evidently conclude that the die is fair based on part **(ii)**. [1]

7 The diagram shows a square *ABCD* with length of side 1. *P* is a point on *BC* such that angle *PAB* is θ radians, which is a continuous random variable with a uniform distribution in the interval $\left[0, \frac{\pi}{4}\right]$.



The area of triangle ACP is denoted by *S*.

(i) Show that the probability density function of *S* is

$$\mathrm{f}(s) = \begin{cases} \dfrac{4}{\pi\left(1-2s+2s^2\right)}, & 0 < s < \dfrac{1}{2}, \\ 0. & \text{otherwise.} \end{cases}$$

[6]

(ii) Hence, find the exact value of $\mathrm{E}(S)$. [3]

8 The probability that an event occurs in any time interval of length δt is $\lambda \delta t$, where λ is a constant.

Let $p_x(t)$ denote the probability that there are *x* independent events in the interval $(0, t)$. Prove that

(a) $p_x(t+\delta t) \approx (1-\lambda\delta t)\, p_x(t) + \lambda\delta t p_{x-1}(t)$ for $x = 1, 2, \ldots,$ [2]

(b) $p_0(t+\delta t) \approx (1-\lambda\delta t)\, p_0(t)$. [1]

State a necessary assumption in your proofs. Verify that $p_x(t) = \dfrac{\mathrm{e}^{-\lambda t}(\lambda t)^x}{x!}$ satisfies part **(b)** when δt is small enough. [3]

You may now assume $p_x(t) = \dfrac{\mathrm{e}^{-\lambda t}(\lambda t)^x}{x!}$ is the solution to parts **(a)** and **(b)**.

It is given that the event is the occurrence of an earthquake in Country X, with $\lambda = 1.3$ and *t* being measured in years. Find the probability that in the next decade, there will be at least one year in which more than two earthquakes occur. [3]

9 A bar owner suspects that consumption of alcohol reduces a person's accuracy of throwing a dart. Eight people are randomly selected to each throw a single dart aiming at the centre of the dart-board. They then each have a pint of beer and then throw the dart once more. The distances (in cm) from the centre of the dart-board are recorded in the table below.

Volunteer	A	B	C	D	E	F	G	H
Before Beer	5.0	2.4	6.9	0.3	4.1	1.2	2.0	4.2
After Beer	3.7	7.6	7.1	5.3	4.7	1.1	2.4	7.9

(i) Conduct a paired-sample *t*-test at the 5% significance level to examine the bar owner's suspicion. Comment on the validity of this test. [6]

(ii) Test the bar owner's suspicion using the Wilcoxon matched-pairs signed rank test at the same level of significance, stating explicitly the null and alternative hypotheses. [4]

State an assumption that you have made in **(ii)**. [1]

10 **(a)** A telecom company claims that the choice of mobile phone (2G, 3G or 4G) depends on the age group in a large city. In order to investigate this claim, the age profiles of 300 randomly selected mobile phone users are compared in the table below.

Age group	4G mobile phone user	3G mobile phone user	2G mobile phone user
16 – 21	45	30	5
22 – 34	65	34	1
35 – 49	41	31	8
50 and above	19	15	6

Use a χ^2 test, at the 5% level of significance, to determine whether there is an association between the age profile and the choice of mobile phone. [7]

Interpret your result for the 22-34 age group. [2]

(b) A marketing company needs to estimate the proportion of residents who own a 4G mobile phone in another large city. It wishes to estimate this proportion to be within 0.05 from its point estimate with a confidence of 98%.

Given that the proportion is known to be close to 70% in 2017, estimate the minimum sample size required in order to meet the company's need. [3]

This proportion is expected to increase next year. Explain what will happen to the minimum sample size required to construct such an interval in 2018. [2]

--- END OF PAPER ---