2017 SRJC H2 FM Prelim Paper 1 Question
Answer all questions [100 marks].

1	At the beginning of January 2017, a fish farm owner possesses 250 eels of a special breed. He rears and sells 40% of the eels to eel dealers and restaurants at the end of each month. To expand his farm, he adds 150 eels to his farm at the start of each month after January 2017. It is assumed that none of the eels die. Let U_n be the number of eels in the farm at the start of the n^{th} month after January 2017. **(i)** Write down a recurrence relation for U_n, stating clearly the initial condition(s). [2] **(ii)** Express U_n in the form $a(0.6)^n + b$, where a and b are constants to be determined. [3]
2	The equation $\mathrm{f}(x)=0$ where $\mathrm{f}(x)=\frac{1}{x}-2+2\ln x,\ x>0$ has exactly two real roots α and β, where $\alpha<2$. **(i)** Verify that α lies between 0 and 1. [1] The Newton-Raphson method is to be used to find an approximation to α, taking $x_0=0.4$ as a first approximation. **(ii)** Show, with the aid of a diagram, how Newton-Raphson method can be applied once to obtain a second approximation, x_1, to α, giving your answer correct to 2 decimal places. Explain why in this case, x_0 does not work well as an initial approximation. [4] **(iii)** Suggest a suitable initial approximation for the Newton-Raphson approximation to converge to α. [1]
3	The equation of a curve, in polar coordinates, is $$r=\mathrm{e}^{-2\theta},\ \text{for } 0\le\theta\le\pi.$$ **(i)** Sketch the curve, indicating clearly the polar coordinates of any axial intercepts. [3] **(ii)** The pole is O and points P and Q, with polar coordinates $\left(\frac{1}{2},\theta_1\right)$ and $\left(a,\theta_2\right)$ respectively, lie on the curve. Given that $0\le\theta_1<\theta_2\le\pi$ and a is a positive constant, show that the area of the region enclosed by the curve and the lines OP and OQ can be expressed as $k(1-4a^2)$ where k is a constant to be determined. [4]
4	The matrix $\mathbf{A}=\begin{pmatrix} x & y \\ z & w \end{pmatrix}$ is such that $$\mathbf{A}^2=\mathbf{A}$$ where x, y, z, w are constants, $x\ne 0$, $w\ne 0$.

(i) Prove that det **A** must be 1 or 0. [2]

(ii) **(a)** Prove that if det **A** = 1, then **A** = **I**. [2]

(b) Prove that if det **A** = 0, then $x + w = 1$. [3]

5 Suppose that $z = e^{i\theta}$.

(i) By considering $(z - \frac{1}{z})^3$, show that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$ [3]

(ii) Suppose that $a_r, b_r \in \mathbb{R}$ are constants for all r. Show the following identity:

$$(X^2-2)\sum_{r=0}^{k} a_r X^{2r} + \sum_{r=0}^{k-1} b_r X^{2r} = A + BX^{2k} + CX^{2k+2} + \sum_{r=1}^{k-1}(-2a_r + a_{r-1} + b_r)X^{2r}$$

where A, B and C are constants to be found. [2]

The function $U_n(x)$, known as Chebyshev Polynomial of Second Kind, is defined such that when $x = \cos\theta$, the following holds

$$\sin(n+1)\theta = \sin\theta\, U_n(x). \qquad \text{---------- (*)}$$

When $n = 0$, the equation gives $U_0(x) = 1$. When $n = 1$, the equation gives $U_1(x) = 2x$.

(iii) Verify that $U_2(x) = 4x^2 - 1$. [1]

It is known that the Chebyshev Polynomials follows the recurrence relation

$$U_{2n+2}(x) = \left(4x^2-2\right)U_{2n}(x) - U_{2n-2}(x), \quad \text{---------- (**)}$$

where $x = \cos\theta$.

(iv) By using the result in part (ii) with appropriate choice of X, a_r and b_r, prove using mathematical induction that

$$U_{2n}(x) = \sum_{r=0}^{n}(-1)^{n-r}\binom{n+r}{n-r}(2x)^{2r}$$

for $n \in \mathbb{Z}$, $n \geq 0$. [5]

[You may use the fact that $2\binom{k+r}{k-r} + \binom{k+r-1}{k-r+1} - \binom{k+r-1}{k-r-1} = \binom{k+r+1}{k-r+1}$.]

6

(i) Express $\dfrac{5x^2-3}{x-x^3}$ in partial fraction. [2]

(ii) Using the substitution $v = \dfrac{y}{x}$, $x \neq 0$, show that the differential equation

$$y\left(x^2+y^2\right) + x\left(3x^2-5y^2\right)\frac{dy}{dx} = 0$$

can be reduced to

$$\left(5v^2-3\right)\frac{\mathrm{d}v}{\mathrm{d}x}=4\left(\frac{v-v^3}{x}\right).$$

Hence, using the results in part (i), show that the general solution is given by $y^3\left(x^2-y^2\right)=Cx$, where C is an arbitrary constant. [7]

7 A hyperbola, H, has its foci at the origin and (6, 0), and its directrix is parallel to the y-axis. It is given further that H is the locus of points whose difference in distances from the foci is 2 units.

(i) By finding two points on H lying on the x-axis, show that the polar equation of H is

$$r=\frac{8}{1+3\cos\theta}.$$ [3]

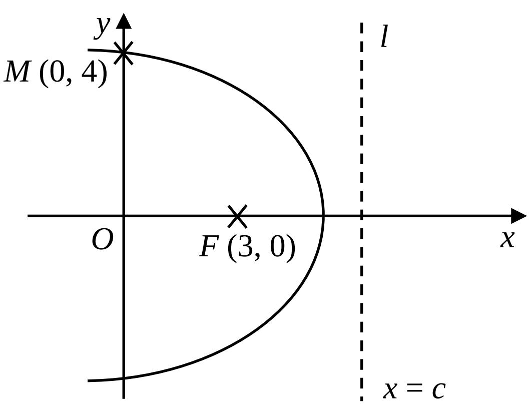
(ii) Show that the arc length, C, of the hyperbola for $-\frac{\pi}{6}\le\theta\le\frac{\pi}{3}$ is given by

$$C=8\sqrt{2}\int_{-\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{\sqrt{3\cos\theta+5}}{(1+3\cos\theta)^2}\,\mathrm{d}\theta.$$ [2]

(iii) Using trapezium rule approximation, with 6 trapezia, find an estimate for C correct to 4 decimal places. [2]

(iv) Using Simpson's rule approximation, with 3 parabolas, find an estimate for C correct to 4 decimal places. [2]

8 Suppose given a point F (3, 0) and a line l with equation $x = c$, where $c \geqslant 5$ is a constant. A conic section, S, is given by the locus of points P whose ratio of distance from F to the distance from l is the constant eccentricity e, as shown in the diagram below.



It is given further that the point M (0, 4) lies on S.

(i) By considering $PF = ePl$, where $P(x, y)$ is any point on S, show that a Cartesian equation for S is

$$(c^2-25)x^2+2(25-3c)cx+c^2y^2=16c^2.$$ [2]

Given that $c = 5$,

(ii) determine whether S is an ellipse, parabola or hyperbola. [1]

The point $E(8, 0)$ lies on the straight line l' with equation $y = \frac{\sqrt{1-a^2}}{2}(x-8)$ where $-1 < a < 1$ is a constant.

(iii) Show that if $a = 0$, the line l' intersect S at exactly one point. [1]

When $a \neq 0$, the line l' intersect S at two distinct points Q and R.

(iv) Find the equation of tangents to S at Q and R, l_Q and l_R, in the form

$$2y\sqrt{1-a^2} = Ax + B,$$

where A and B are constants in terms of a to be determined in each case, simplifying your answers. [5]

(v) Prove that the locus of points of intersection between l_Q and l_R as a varies is the y-axis. [1]

9 A group of Arts students is tasked to design an art ornament using two curves C_1 and C_2 defined as follows

$$C_1: \ x = t^2 - 2\ln t \ , \ y = 4(t-1) \text{ where } t \geq 1,$$

$$C_2: \ y = \frac{1}{x}, \ x \geq \frac{1}{4}.$$

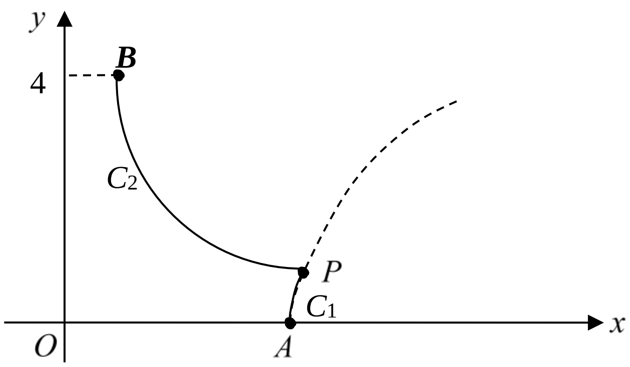
In a sketch of the 2 graphs on the same diagram, the group discovers that there is a common point P where $t = \alpha$ which could be useful to their design.

(i) An iterative sequence t_n to approximate α is proposed as follows

$$t_{n+1} = 1 + \frac{1}{\mathrm{h}(t_n)}.$$

Find a possible $\mathrm{h}(t_n)$ such that if the above sequence converge, it converges to α showing your working clearly. [2]

(ii) Using the iterative process defined in part (i) and $t_0 = 1$ as the initial approximation, find α correct to three decimal places. [2]

	The group of students decides to use the arcs AP and PB as the outline for their design as shown in the diagram below. Point A lies on the x-axis and point B has y-coordinates equal to 4.  **(iii)** State the coordinates of points A and B. Using the value of α found in (ii), find the coordinates of point P, giving your answer to three decimal places. [2] **(iv)** Find the perimeter of the outline of the design defined by the arc APB. [3] To obtain the shape of the ornament, the arc APB is rotated through one revolution about the y-axis. **(v)** Find the area of the curve surface generated. [3]
10	A general second order linear recurrence relation is given by $u_{n+1} = 2a\, u_n + b\, u_{n-1}$, where a, b are constants such that $b > -a^2$. The above recurrence relation can be expressed using the following equation $$\mathbf{A}\begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix} = \begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix}, \qquad \text{----------------} \ (1)$$ where $\mathbf{A} = \begin{pmatrix} x_1 & x_2 \\ 1 & 0 \end{pmatrix}$. **(i)** Find the matrix entries $x_1,\ x_2$ in terms of a and b such that equation (1) holds. [2] **(ii)** Find all the eigenvalues and corresponding eigenvectors of $\mathbf{A}$, in terms of a and b. [5] **(iii)** Write down a diagonal matrix $\mathbf{D}$ and a non-singular matrix $\mathbf{P}$, in terms of a and b such that $$\mathbf{A} = \mathbf{PDP}^{-1}.$$ [2] Taking $u_0 = a,\ u_1 = -a$ and $b = 3a^2$, **(iv)** find u_n in terms of a and n. [3]
11	A model was used to predict how an information gets passed among people in a community. The number of people who has the knowledge of a particular information is denoted by x. The model assumes that the information gets spread at a rate proportional to the number of people who has the information, x, after t days. That is, $$\frac{\mathrm{d}x}{\mathrm{d}t} = kx, \text{ where } k \text{ is a positive constant.} \quad \text{-----} \ (1)$$ Initially, two persons has the knowledge of the information. After 5 days, 64 people in the community has knowledge of the information.

(i) Write down the general solution of equation (1) and hence find x in term of t. [2]

The spread of the information is kept within the community of a fixed number of people. A logistic model is used to account for this fact using the following equation:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = (\ln 2)\left(1 - \frac{x}{2002}\right)x \quad \text{----- (2)}$$

For equation (2), only the initial condition of equation (1) holds.

(ii) Find x in terms of t, and hence, estimate, based on this model, the number of people who has the information after 10 days, correct to the nearest whole number. [4]

A more detailed investigation into the spread of information attempted to account for a small group of individuals in the community who has higher capacity to spread information outside the community. The modified model is proposed to be

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \mathrm{f}(t, x) = kx\left(1 - \frac{x}{2002}\right) + \frac{1}{2}t^2 \quad \text{----- (3)}$$

The Euler method is used to estimate the number of people who has the information after t days, with the same initial condition as in equation (1) and (2).

(iii) Using the Euler method, estimate x when $t = 2$ using step size $h = 0.5$, correct to the nearest whole number. [2]

(iv) Comment on the estimation using this method for large values of t. [1]

Denote $x_{n+1}^* = x_n + h\,\mathrm{f}(t_n, x_n)$.

(v) Copy the table given below onto your answer script and fill the table using the improved Euler Method using step size 0.5.

n	x_n	$\mathrm{f}(t_n, x_n)$	x_{n+1}^*	$\mathrm{f}(t_{n+1}, x_{n+1}^*)$	x_{n+1}
0	2	1.3849	2.6924		2.8434
1	2.8434	2.0931		3.1910	4.1644
2	4.1644		5.8547	5.1571	6.3024
3		5.4797	9.0423	8.2393	
4		8.7130	14.088	12.821	15.115
5	15.115	13.523	21.877	19.498	23.371

[3]

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