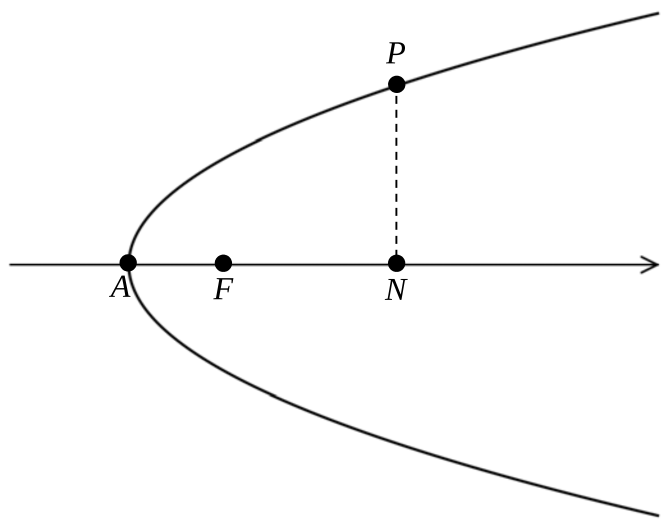
Raffles Institution

H2 Further Mathematics 2017 Preliminary Exam Paper 1 Question

1	The equations of three planes p, q and r are $$2x+y+3z=4,$$ $$8x+6y+5z=\mu,$$ $$-4x+8y+\lambda z=7,$$ respectively, where λ and μ are constants. Determine the conditions on λ and μ such that the three planes **(i)** intersect at exactly one point, **(ii)** intersect at a line, **(iii)** have no point in common. [6]
2	The complex number z is given by $\cos\frac{\pi}{4}+\mathrm{i}\sin\frac{\pi}{4}$. **(i)** Show that $$z+z^3+z^5+\cdots+z^{2n-1}=k\sin\frac{n\pi}{4}\left(\cos\frac{n\pi}{4}+\mathrm{i}\sin\frac{n\pi}{4}\right),$$ where k is a positive constant to be determined. [3] **(ii)** Hence find, in terms of n and θ, $\sum_{r=1}^{n}\cos\left(\theta+\frac{2r-1}{4}\pi\right)$. [3]
3	**Do not use a calculator in answering this question.** The complex number z satisfies the inequalities $$\lvert z\rvert\le 1 \quad \text{and} \quad \frac{\pi}{2}\le\arg(\mathrm{i}z)\le\frac{3\pi}{4}.$$ On an Argand diagram, sketch the region in which the point representing z can lie. [3] Given that $w=\frac{\sqrt{3}}{2}+\mathrm{i}\frac{\sqrt{3}}{2}$, find **(i)** the greatest value of $\lvert z+w\rvert$, [2] **(ii)** the value of $\tan\theta$, where θ is the least value of $\arg(z+w)$. [3]
4	**(a)** Let N be the foot of perpendicular from a point P on a parabola, with vertex A and focus F, to the axis of the parabola. Show that $$PN^2=4AF\times AN.$$ [4] 

(b) The Richard D. Swensen sundial is a vertical sundial attached to the wall of the Kleinpell Fine Arts Building in the University of Wisconsin River Falls in recognition of the services of Richard D. Swensen (see Fig. 1). Aluminum rods form the hyperbolas traced by the shadow cast by the sun on the summer solstice, June 21.

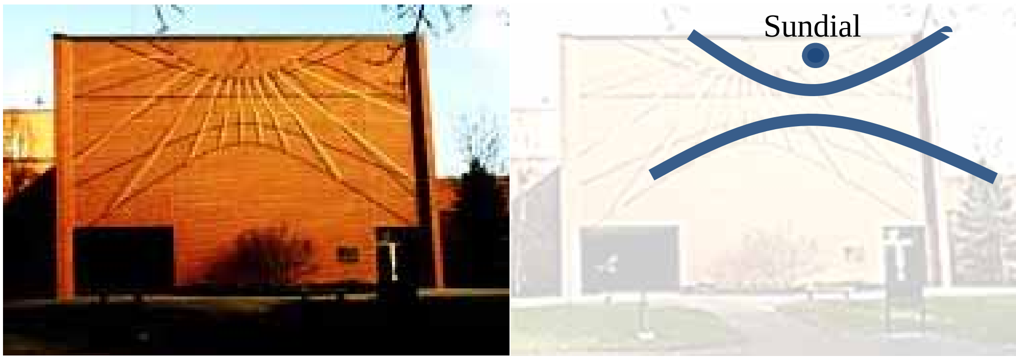
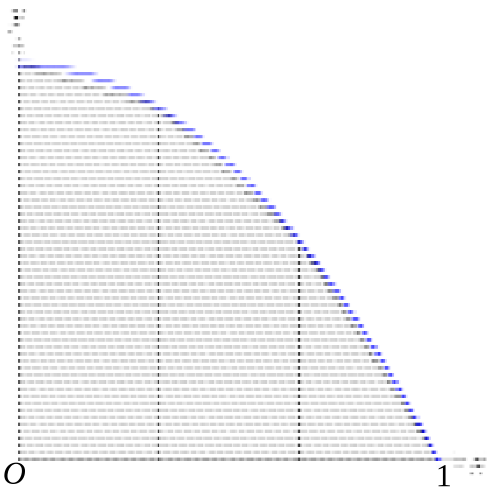


Fig. 1

One focus of the summer solstice hyperbola is 525 cm above the ground. The vertex of the aluminum branch is 675 cm above the ground. If the x-axis is 900 cm above the ground and the centre of the hyperbola is at the origin, find the cartesian equation for the summer solstice hyperbola. [4]

5 **(a)** The diagram below shows part of the graph of $y = 1 - x^2$, with 3 trapezia of equal widths approximating the area under the curve between $x = 0$ and $x = 1$.



If the x-axis between $x = 0$ and $x = 1$ is divided into n equal parts, the area under the curve may be approximated by the total area, A_n, of n trapezia each of width $\frac{1}{n}$.

(i) Given that $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$, show that

$$A_n = \frac{(2n-1)(2n+1)}{6n^2}.$$ [4]

(ii) Explain briefly how the exact value of $\int_0^1 (1 - x^2)\, dx$ may be deduced from A_n.

Deduce the exact value of $\int_0^1 (1 - x^2)\, dx$. [2]

	(b) It is given that $y = \mathrm{f}(x)$ is a particular solution of the differential equation $$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x - y^2$$ with the initial condition $\mathrm{f}(0) = 1$. **(i)** Use the Euler method with two steps of equal size, starting at $x = 0$, to approximate $\mathrm{f}(-0.2)$. [2] **(ii)** Find $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ in terms of x and y. [1] Determine whether the approximation found in **(b)(i)** is an under-estimate or an over-estimate. [2]
6	Find the eigenvalues and corresponding eigenvectors of the matrix **A** where $$\mathbf{A} = \begin{pmatrix} 6 & 1 & 1 \\ 7 & 0 & 11 \\ 3 & -1 & 8 \end{pmatrix}.$$ [6] Hence, write down a non-singular matrix **Q** and a diagonal matrix **D** such that $\mathbf{A} = \mathbf{QDQ}^{-1}$. [2] By expressing $\begin{pmatrix} -4 \\ 13 \\ 6 \end{pmatrix}$ as a linear combination of the eigenvectors of **A**, find $\mathbf{A}^4 \begin{pmatrix} -4 \\ 13 \\ 6 \end{pmatrix}$. [3]
7	**(a)** Two sequences of positive integers $x_1, x_2, x_3, \ldots$ and $y_1, y_2, y_3, \ldots$ are given by $$x_1 = 3, \quad y_1 = 2,$$ $$x_n = 3x_{n-1} + 4y_{n-1}, \quad \text{for } n \geq 2,$$ $$y_n = 2x_{n-1} + 3y_{n-1}, \quad \text{for } n \geq 2.$$ **(i)** Use the method of mathematical induction to prove that $$x_n^2 - 2y_n^2 = 1, \quad \text{for } n \geq 1.$$ [4] **(ii)** Explain how x_n and y_n can be used to obtain an approximation to $\sqrt{2}$ in the form of a rational number. [2] **(b)** Prove by the method of mathematical induction that $\dbinom{2n}{n} < 2^{2n-2}$, for $n \geq 5$. [5]
8	Ben, with k dollars initially, repeatedly wagers on a game of chance until he has no more money left, or achieves a total amount of N dollars. In each game he wins a dollar with probability p or loses a dollar with probability $1 - p$. It can be assumed that each game is

	played independently of another. The probability that Ben achieves N dollars given that he has k dollars initially is given by R_k. It is known that R_k satisfies the recurrence relation $$R_k=\left(\frac{1}{p}\right)R_{k-1}-\left(\frac{1-p}{p}\right)R_{k-2},$$ with boundary conditions $R_0=0$ and $R_N=1$. **(i)** Given that $p\neq\frac{1}{2}$, solve the above recurrence relation, giving your answer in terms of p. [6] **(ii)** If $p=\frac{9}{19}$, explain whether it is more likely for Ben to achieve 20 dollars with an initial 10 dollars, or to achieve 120 dollars with an initial 100 dollars. [3] **(iii)** If $p=\frac{1}{2}$, find the probability to achieve N dollars given that he starts with k dollars. [3]
9	**(a)** **(i)** Find $\int x(x+3)^{\frac{3}{2}}\,\mathrm{d}x$. [2] **(ii)** Hence find the exact volume of the solid generated when the finite region bounded by the curve $y^2=(1-x)^2(x+3)$ is rotated through one revolution about the line $x=-3$. [3] **(b)** A curve is defined parametrically by $x=(1+t)^{\frac{3}{2}}$, $y=(5-t)^{\frac{1}{2}}$. **(i)** Find the area of the surface generated when the portion of the curve from $t=-1$ to $t=5$ is rotated through one revolution about the x-axis. [2] **(ii)** Show that the mean value of y with respect to x over the interval $-1\le t\le 2$ is $$\frac{1}{p}\int_{-1}^{2}\sqrt{\left(9-(t-2)^2\right)}\,\mathrm{d}t,$$ where p is a constant to be determined. Hence, using the substitution $t-2=3\sin\theta$, find the exact mean value. [6] **[**The mean value of a function $y=\mathrm{f}(x)$ over the interval $a\le x\le b$ is given by $\frac{1}{b-a}\int_a^b \mathrm{f}(x)\,\mathrm{d}x$.**]**
10	The growth of a population, $P(t)=P$, where t is the time in years, of fish in a lake is governed by the logistic equation $$\frac{\mathrm{d}P}{\mathrm{d}t}=\alpha P-\beta P^2,$$ where α and β are positive constants. **(a)** **(i)** State, in terms of α and β, the carrying capacity of the lake and explain what

	the carrying capacity refers to in the context of the question. [2] **(ii)** Find P in terms of t, given that there are $\dfrac{4\alpha}{5\beta}$ fish in the lake initially. [5] **(iii)** Hence show that the population of the fish in the lake will approach the carrying capacity stated in **(a)(i)** eventually. [2] **(b)** In the case where $\alpha = \dfrac{1}{4}$, $\beta = \dfrac{1}{72000}$ and there are 5000 fish in the lake initially, fishing is allowed in the lake at a fixed rate of h fish per annum. Find the largest integral value of h such that **(i)** there will be a positive growth in the fish population in the lake, [3] **(ii)** there will still be fish in the lake for people to fish in the long run. [2]