2017 SRJC H2 FM Prelim Paper 2 Question

Section A: Pure Mathematics [50 marks]

1

(i) Show that the equation

$$\cos(x) = \ln(x+1)$$

has exactly one root in the interval [0, 1]. [3]

(ii) By applying linear interpolation once, find an approximation, x_1, to the root α, of the equation in part (i), correct to 3 decimal places.

Hence, determine whether the approximation obtained is an overestimation or underestimation of α. [3]

2

A recurrence relation is given by

$$F_1 = \sqrt{5},\ F_2 = \sqrt{5} \text{ and } F_{n+2} = F_{n+1} + F_n,\ \forall n \in \mathbb{Z}^+.$$

Denote the roots of the equation $m^2 - m - 1 = 0$ by ρ and ρ^*, where $\rho^* < \rho$.

(i) Write down the exact values of ρ and ρ^*. [1]

(ii) Prove, by mathematical induction

$$F_n = \rho^n - (\rho^*)^n \ \forall\ n \in \mathbb{Z}^+.$$ [5]

3

The complex number z is given by $z = e^{i\theta}$, where $0 < \theta < \frac{\pi}{2}$. The complex numbers s and t are given by $s = \left(-\sqrt{3} - i\right)z$ and $t = \left(-\sqrt{3} + i\right)z^*$ respectively.

(i) Draw an Argand diagram to show the loci of s and t as θ varies. You should identify the arguments and moduli of the end points of each locus. Hence, write down the value of θ when points representing the origin O, s and t are collinear on the Argand diagram. [6]

(ii) On the same Argand diagram, sketch the region representing all complex numbers w satisfying the following conditions.

$$\text{Re}\,(w) = 2 \text{ and } 1 < |w - 2e^{-i\frac{\pi}{3}}| < 2$$ [2]

4

Polynomials of degree up to 3, $P = \{ax^3 + bx^2 + cx + d \mid a, b, c, d \in \mathbb{R}\}$ forms a vector space with dimension 4 with standard basis $\{1, x, x^2, x^3\}$. The subspace Q of P is spanned by the set $B = \{x, x^2 - 1, x^3 - 3x\}$.

(i) Prove that B forms a linearly independent set of vectors in Q. Hence, deduce that Q is a subspace of dimension 3. [3]

A linear transformation on **T**: $P \to P$ is defined by

$$\mathbf{T}(ax^3 + bx^2 + cx + d) = (a + d)(x^3 - 3x) + (b + d)(x^2 - 1) + (c + d)x$$

(ii) Using the standard basis for P, show that the matrix **A** representing the linear transformation **T** is given by

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ -3 & 0 & 1 & -2 \\ 0 & -1 & 0 & -1 \end{pmatrix}$$ [2]

(iii) Find the rank of **A** and a basis for the null space of **A** in column vectors. [3]

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(iv) Explain why the range of **T** is the subspace Q. [1]

5 Suppose y is a function of x.

(i) Show that if $t = e^x$, then

$$\frac{dy}{dx} = t\frac{dy}{dt} \text{ and } \frac{d^2y}{dx^2} = t\left(\frac{dy}{dt} + t\frac{d^2y}{dt^2}\right)$$ [3]

(ii) Reduce the second order differential equation

$$\frac{d^2y}{dx^2} + \left(2e^x - 1\right)\frac{dy}{dx} + 2e^{2x}y = e^{2x}\sin 3e^x \qquad (*)$$

to the form

$$\frac{d^2y}{dt^2} + a\frac{dy}{dt} + by = \sin 3t,$$

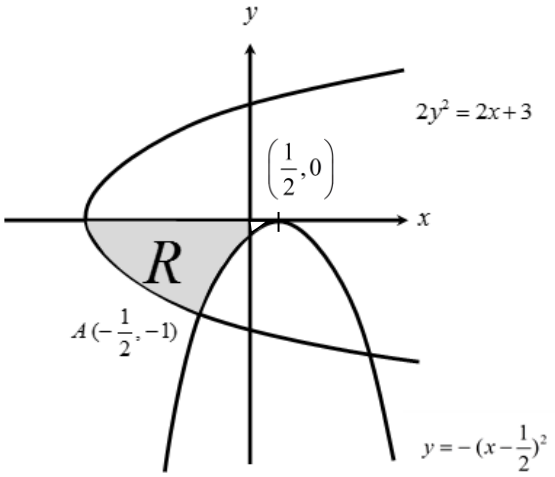
where a and b are integers to be determined. Hence find the general solution of the differential equation (*). [8]

6 **(a)** **(i)** By using the substitution $u = \sqrt{x + \frac{3}{2}}$, show that

$$\int x\sqrt{x+\frac{3}{2}}\ dx = \frac{2}{5}\left(x+\frac{3}{2}\right)^{\frac{5}{2}} - \left(x+\frac{3}{2}\right)^{\frac{3}{2}} + C,$$

where C is an arbitrary constant. [2]

(ii) The diagram below shows the shaded region R that is bounded by the curves $2y^2 = 2x + 3$, $y = -(x - \frac{1}{2})^2$ and the axes. Both curves intersects at $A\left(-\frac{1}{2}, -1\right)$ as shown in the diagram.



Using the method of cylindrical shells, find the exact volume generated when the region R is rotated through 2π radians about the y-axis. [3]

(b) The curve C is defined parametrically by

$$x = -\sin t, \quad y = e^{-2t} \quad \text{where} \quad 0 \le t \le \frac{\pi}{2}.$$

(i) Sketch the curve C. [2]

(ii) The region S, bounded by the curve C, the line $y = \mathrm{e}^{-\pi}$ and the y-axis, is rotated through 2π radians about the x-axis. Using the method of cylindrical shells, find the volume generated correct to 3 significant figures. [3]

7 During a training session, Jeremy, a soccer player, practices penalty shooting repeatedly until he scores a goal. The probability that Jeremy scores a goal in each attempt is 0.4. Let X denotes the number of shots he makes until he scores a goal.

(i) State an assumption needed for X to be well modelled by a geometric distribution. [1]

(ii) Find the probability that his first goal will not occur until at least the fourth shot. [3]

(iii) Find the least number of shots Jeremy needs to make for him to have more than 95% chance of scoring a goal. [3]

8 A meat wholesaler sells remnants of meat in bags. The amount, in kg, of inedibles (that is, bone and gristle) in a bag, is a random variable, X, with probability density function given by

$$\mathrm{f}(x) = \begin{cases} a(x-2)(4-x), & 2 \le x \le 4 \\ 0 \quad , & \text{otherwise.} \end{cases}$$

(i) Show that $a = \frac{3}{4}$. [2]

(ii) State the mean of X. [1]

(iii) Find the probability that X is greater than 3.5 kg, giving your answer to 4 decimal places. [2]

A butcher buys the bags and he can use them in one of the following two processes:

Process (I): The complete contents (including inedibles) are ground up and made into standard sausages. If the bag contains more than 3.5 kg of inedibles, the resulting sausages are unsaleable and a loss of \$1.90 is made on the bag. Otherwise, a profit of 80 cents is made on each bag.

Process (II): The inedibles are extracted and the remaining contents are made into premium sausages. The profit on this process is $(190 - 50X)$ cents per bag.

By comparing the expected profit per bag, state which process the butcher should choose to use. [4]

9 Changes in university rankings (current numerical ranking subtract previous ranking) by 9 different agencies A to I for Harvey University is tabulated as shown below.

Agent	A	B	C	D	E	F	G	H	I
Change in Uni. ranking	+3	+4	+2	−4	+11	−2	+6	0	+5
Rank									

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(i) Copy the table and compute the rank for the differences in order for a Wilcoxon matched-pair signed rank test to be carried out. Indicating by "NA" if the data is to be omitted from the test. (You do not need to carry out the test.) [1]

A newly released non-zero change in university ranking of Harvey University by agency J was combined with the data above, and a sign test of whether Harvey University has fallen in university ranking produced a *p*-value greater than 0.1.

(ii) Determine whether the newly released ranking of Harvey University by agency J has changed for the better or worse. [2]

(iii) Suggest an advantage of using non-parametric tests here over using a *t*-test. [1]

Agency A, being located in Asia, intended to test the changes in the ranking of 350 Asian universities included its ranking system. A Wilcoxon matched-pair signed rank test is to be used.

(iv) Suppose that agency A found that the sum of ranks of positive changes (current ranking subtract previous ranking is positive) is 27150. Using a suitable approximation, test at 5% level of significance whether there is sufficient evidence that the ranking of Asian universities has improved. [4]

10 Statistics on marriages and divorces in Singapore from 2010 to 2015 is given below.

KEY INDICATORS ON MARRIAGES AND DIVORCES, 2010-2015

Total Marriages & Divorces	2010	2011	2012	2013	2014	2015
Number of Marriages	24,363	27,258	27,936	26,254	28,407	28,322
Number of Divorces and Annulments	7,338	7,604	7,237	7,525	7,307	7,522
Mean Age at First Marriage (Years)						
Grooms	30.0	30.1	30.1	30.2	30.2	30.3
Brides	27.7	28.0	28.0	28.1	28.2	28.2

Source from http://www.singstat.gov.sg

The number of divorces and annulments per year is x with mean μ.

(i) Using the given set of 6-years data, find the unbiased estimate of mean and variance. Hence, construct a 95% confidence interval for μ, giving your answer to 2 decimal places, stating any assumptions made. [4]

(ii) A two-tailed test of the null hypothesis H_0: $\mu = 7420$ is to be carried out at the 5% level of significance. Using the confidence interval obtained in (i), state the conclusion of the test. [2]

(iii) Alex discovered that the actual mean number of divorces and annulments per year in Singapore is 7586. Explain why the actual figure does not lie in the confidence interval found in (i). [1]

(iv) The mean age at first marriage per year of brides and groom is assumed to follow a normal distribution. John claims that the mean age at first marriage for grooms in Singapore is at most 2 more than that of the corresponding values for the brides in the same year. Using the 6-year data, test at the 5% level of significance, whether John's claim is valid. State any assumption made. [5]

11 A terminal at the servicing port of country *S* consists of 8 servicing berths and it handles only container ships, oil tankers and coaster vessels. On average, 4 container ships, 3 oil tankers and 2 coaster vessels arrive at the terminal per hour.

(i) State an assumption for the arrivals of any of the 3 types of ships at the terminal to be well-modelled by a Poisson distribution. [1]

Due to the excellent port service quality, servicing one ship of any type takes exactly 45 minutes.

(ii) Find the most probable number of arrivals at the terminal within 45 minutes.

Hence, state the most probable number of ships to be found concurrently at the servicing berths. [3]

With a philosophy of constant improvement in service quality, the port management wishes to determine the maximum t minutes, so that at least 95% of the time, there are no more than 8 arrivals in t minutes.

(iii) Find the value of t, correct to 2 decimal places. [3]

Due to creation of new shipping routes, the number of ship arrivals changed drastically. All the terminals at the service port receive an overall mean of 2 container ships, 1 oil tanker and no coaster vessel every hour. The waiting time for the arrival of a container ship is denoted by C and the waiting time for the arrival of an oil tanker is denoted by D.

(iv) Given that there is no arrival of container ships for an hour, find the probability that there is at most 1.5 hour of waiting for arrival of one container ship. [2]

(v) The times of arrival of 200 ships are recorded.

Observed frequency	Time of Arrival		
	00 00 – 08 00	08 00 – 16 00	16 00 – 00 00
Oil Tanker	35	50	15
Container Ship	45	30	25

Hence, using a χ^2-test, determine at 5% level of significance, whether the number of arrival time and type of ship are independent. [5]

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