**ANGLO-CHINESE JUNIOR COLLEGE**
**JC2 PRELIMINARY EXAMINATION**

Higher 2

# FURTHER MATHEMATICS 9649/01

Paper 1 **23 August 2017**

**3 hours**

Additional Materials: Cover Sheet
Answer Paper
List of Formulae (MF26)

**READ THESE INSTRUCTIONS FIRST**

Write your index number, class and name on the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

**ANGLO-CHINESE JUNIOR COLLEGE**
**MATHEMATICS DEPARTMENT**
**JC2 Preliminary Examination 2017**

**FURTHER MATHEMATICS 9649**
**Higher 2**
**Paper 1**

/ 100

Index No: | | | | |

Form Class: ____________

Name: ____________________________

Calculator model: ____________________

Arrange your answers in the same numerical order.

Place this cover sheet on top of them and tie them together with the string provided.

| Question No. | Marks |
|---|---|
| 1 | /6 |
| 2 | /8 |
| 3 | /9 |
| 4 | /9 |
| 5 | /9 |
| 6 | /11 |
| 7 | /14 |
| 8 | /12 |
| 9 | /10 |
| 10 | /12 |

| Summary of Areas for Improvement | | | |
|---|---|---|---|
| Knowledge (K) | Careless Mistakes (C) | Read/Interpret Qn wrongly (R) | Presentation (P) |
| | | | |

**1** Show by induction that $\sum_{r=1}^{n} \frac{1}{r^2} < 2 - \frac{1}{n}$ for all integers $n \geq 2$. [6]

**2** **(i)** Show that $\frac{z^9 - 8}{z^3 - 2} = z^6 + 2z^3 + 4$. [1]

**(ii)** Find all complex roots of the equation $z^9 - 8 = 0$. [4]

**(iii)** Hence, express $z^6 + 2z^3 + 4$ as a product of quadratic factors of the form $z^2 + pz + q$, where $p$ and $q$ are real numbers to be found exactly. [3]

**3** The matrix **A** has eigenvectors $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$, and $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ with corresponding eigenvalues $-1$, 3 and 1 respectively.

**(i)** Find **A**. [3]

**(ii)** Given that the matrix

$$\mathbf{B} = \begin{pmatrix} -4 & 0 & 6 \\ -2 & 3 & 4 \\ -3 & 0 & 5 \end{pmatrix}$$

has the same eigenvectors as **A**, find the eigenvalues of **B**. [2]

**(iii)** Write down $3 \times 3$ matrices **D** and **U**, where **D** is a diagonal matrix, for which

$$\mathbf{AB} = \mathbf{UDU}^{-1},$$

and show how this form can be used to determine $(\mathbf{AB})^n$ for any positive integer $n \geq 2$. [4]

**4** The hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ has foci $F$ and $F'$. The points $C$ and $D$ have coordinates $(-a, c)$ and $(a, d)$ respectively, and the line $CD$ is a tangent to the hyperbola at point $E(a\sec\theta, b\tan\theta)$.

**(i)** Show that $c = -b\left(\frac{1+\cos\theta}{\sin\theta}\right)$ and $d = b\left(\frac{1-\cos\theta}{\sin\theta}\right)$. [4]

**(ii)** Let $M$ be the midpoint of $CD$. Show that $M$ is the same distance from both foci, and find this distance. [3]

**(iii)** Show that $MC = MF$, and hence, that $\angle CFD$ and $\angle CF'D$ are both right angles. [2]

**5** A sequence $u_0, u_1, u_2, \ldots$ is given by

$$u_{n+2} = 8u_{n+1} - 15u_n,\ u_0 = u_1 = 1.$$

**(i)** Show that $u_{n+2} - 3u_{n+1} = 5\left(u_{n+1} - 3u_n\right)$. [1]

**(ii)** Let $v_n = u_{n+1} - 3u_n$. Write down a first-order recurrence relation for $v_n$, and hence express $v_n$ in terms of $n$. [3]

**(iii)** Let $w_n = u_{n+1} - 5u_n$. Write down a first-order recurrence relation for $w_n$, and hence show that $w_n = (-4)3^n$. [3]

**(iv)** By finding and solving a pair of simultaneous linear equations in $u_{n+1}$ and $u_n$, or otherwise, express $u_n$ in terms of $n$ for $n \geq 0$. [2]

**6** A linear transformation $\mathrm{T}: \mathbb{R}^4 \to \mathbb{R}^3$ is given by the matrix

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 4 & 3 \\ 1 & 1 & 3 & 2 \\ 1 & -1 & -1 & 0 \end{pmatrix}.$$

**(i)** Find a basis for the null space of T. [2]

**(ii)** Find a basis for the range space $R$ of T. [2]

**(iii)** The vector $\begin{pmatrix} \lambda \\ \mu \\ 0 \end{pmatrix}$, where $\lambda$ is a constant, belongs to $R$. Find $\mu$ in terms of $\lambda$ and find the set of vectors $\mathbf{x}$ such that

$$\mathrm{T}(\mathbf{x}) = \begin{pmatrix} \lambda \\ \mu \\ 0 \end{pmatrix}.$$ [4]

**(iv)** Determine, with reasons, whether the set

$$\left\{ \mathbf{x} \in \mathbb{R}^4 : \mathrm{T}(\mathbf{x}) = \begin{pmatrix} \lambda \\ \mu \\ 0 \end{pmatrix}, \lambda \in \mathbb{R} \right\}$$

is a subspace of $\mathbb{R}^4$. [3]

**7** Using *Newton's Law of Gravitation*, it can be shown that the polar equation of a planet's orbit around a sun located at the pole of the coordinate system satisfies the equation

$$r = \frac{1}{A\cos\theta + B\sin\theta + k},$$

where *A*, *B*, and *k* are constants.

If the planet is nearest to the sun when its polar coordinates are $(r_0, 0)$, find *A* in terms of *k* and $r_0$, and show that $B = 0$. [3]

Use these values of *A* and *B* for the rest of the question.

**(i)** Given that *k* is positive, find the range of values of *k* such that the planet's orbit is elliptical. [4]

**(ii)** Suppose $k = \dfrac{2}{3r_0}$. Show that the distance moved by the planet around the sun in one complete revolution is

$$6r_0 \int_0^{\pi} \frac{\sqrt{4\cos\theta + 5}}{(\cos\theta + 2)^2} \, \mathrm{d}\theta. \qquad [3]$$

**(iii)** Using Simpson's rule with strips of width $\frac{1}{4}\pi$, evaluate the integral in **(iii)**, giving your answer in the form $cr_0$, where $c$ is to be determined to 3 significant figures. [4]

**8** It is given that $y = \mathrm{f}(x)$, where $\mathrm{f}(x) = \sin(\mathrm{e}^x) - kx$ for a constant *k*.

**(i)** For $0 < k < 2$, show algebraically that $y = \mathrm{f}(x)$ has exactly 1 real root in the interval $\left(\ln\dfrac{\pi}{2}, \ln\pi\right)$. [3]

For the rest of this question, let $k = 1$.

**(ii)** Show that $\cos x - \sin x = \sqrt{2}\sin\left(\dfrac{\pi}{4} - x\right)$. [1]

**(iii)** By carrying out linear interpolation once, approximate the root of $y = \mathrm{f}(x)$ between $x = \ln\dfrac{\pi}{2}$ and $x = \ln\pi$, giving your answer to 3 decimal places.

Without using your graphing calculator, explain why your approximated root of $y = \mathrm{f}(x)$ is an underestimation. [5]

**(iv)** By using two iterations of the Newton-Raphson method with $x_1 = \ln\pi$, find an approximation for $\alpha$ such that $\mathrm{f}(\alpha) = 0$, giving your answer to 3 decimal places. [3]

**9** A hyperbola has equation $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ where $a$ and $b$ are both positive constants with $a>b$.

**(i)** Prove that an ellipse with the equation $\frac{x^2}{2a^2}+\frac{y^2}{a^2-b^2}=1$ and the above hyperbola have the same foci. [2]

The point $P$ lies on both the above ellipse and hyperbola.

**(ii)** Show that $(PF)(PF')=a^2$, where $F$ and $F'$ are the foci. [2]

**(iii)** By finding $FF'$, or otherwise, find the cosine of $\angle F'PF$ in terms of $a$ and $b$. [3]

**(iv)** Hence, or otherwise, find the area of triangle $F'PF$ in terms of $a$ and $b$. [3]

**10** In November 2002, there was an epidemic outbreak of Severe Acute Respiratory Syndrome (SARS) in China which subsequently spread to the rest of the world in 2003. The first infected case in Singapore was in 1st March 2003. Measures such as school closures and home quarantines were enacted by the Singaporean government. By the time the World Health Organisation removed Singapore from the list of 'Infected Areas' on May 30th 2003 (after a period of more than 10 days with no new infections), a total of 238 people has been infected and of which, 33 died (the others recovered).

To study the spread of this epidemic in Singapore, a simple model is proposed:

$$\text{(A): } \frac{\mathrm{d}N}{\mathrm{d}t} = rN \text{ for some constant } r,$$

where $N$ denotes the total number of infected cases (recovered or otherwise) at a time $t$ days after the first infected case was discovered.

**(i)** State a limitation of model (A) and explain why it does not fit this context. [2]

A second model for the spread of SARS in Singapore is proposed instead:

$$\text{(B): } \frac{\mathrm{d}N}{\mathrm{d}t} = \frac{r}{k}N\left(k-N\right) \text{ for some constants } r \text{ and } k.$$

**(ii)** Assume that when $t = 0, N = 1$. Show that the particular solution of differential equation (B) is $N = \dfrac{k}{\left(k-1\right)\mathrm{e}^{-rt}+1}$. [4]

**(iii)** The constant $k$ is also known as the *carrying capacity* of the logistic growth model. Given the context, state the value of $k$. Hence sketch the graphs of $N$ against $t$ for $r = 0.2$ and $r = 1$ on the same diagram.

On 22nd March 2003, the number of infected cases in Singapore was 44.

Explain which value of $r$ used will model the data better. [3]

Another model for the spread of diseases in a given population is given by the following differential equation:

$$\text{(C): } \frac{\mathrm{d}N}{\mathrm{d}t} = \left(0.1+0.1t\right)N\left(1+\frac{N}{40}\right),$$

where $N$ denotes the total number of infected cases (in thousands) and $t$ denotes the time elapsed (in weeks) from the day the first infection was diagnosed.

**(iv)** Assuming that there were 1000 infected cases initially (when $t = 0$), use two iterations of the *Euler Method* to estimate the total number of infected cases after 1 week, giving your answer to the nearest whole number. [3]