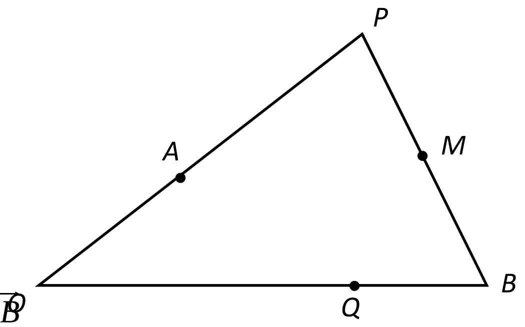
H2 Mathematics 2017 Prelim Exam Paper 2 Solution

1



$\overrightarrow{OP} = (\lambda+1)\mathbf{a}$

$\overrightarrow{OM} = \dfrac{\overrightarrow{OP}+\overrightarrow{OB}}{2}$

$= \dfrac{(\lambda+1)\mathbf{a}+\mathbf{b}}{2}$

area of triangle OPM $= \dfrac{1}{2}\left|\overrightarrow{OP}\times\overrightarrow{OM}\right|$

$= \dfrac{1}{2}\left|(\lambda+1)\mathbf{a}\times\dfrac{(\lambda+1)\mathbf{a}+\mathbf{b}}{2}\right|$

$= \dfrac{(\lambda+1)}{4}|\mathbf{a}\times\mathbf{b}|$

area of triangle OQM $= \dfrac{1}{2}\left|\overrightarrow{OQ}\times\overrightarrow{OM}\right|$

$= \dfrac{1}{2}\left|\dfrac{3}{4}\mathbf{b}\times\dfrac{(\lambda+1)\mathbf{a}+\mathbf{b}}{2}\right|$

$= \dfrac{3(\lambda+1)}{16}|\mathbf{a}\times\mathbf{b}|$

Ratio of the area of triangle OPM to the area of triangle OQM is

$\dfrac{(\lambda+1)}{4} : \dfrac{3(\lambda+1)}{16} = 4:3$ (Shown)

2

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\cos x\frac{\mathrm{d}y}{\mathrm{d}x}\right] = \cos x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - \sin x\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\cos x\frac{\mathrm{d}y}{\mathrm{d}x}\right] = \sec^2 x + \cos 2x$$

$$\cos x\frac{\mathrm{d}y}{\mathrm{d}x} = \int \sec^2 x + \cos 2x\,\mathrm{d}x$$

$$\cos x\frac{\mathrm{d}y}{\mathrm{d}x} = \tan x + \frac{1}{2}\sin 2x + C$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \sec x\tan x + \sin x + C\sec x$$

$$y = \int \sec x\tan x + \sin x + C\sec x\,\mathrm{d}x$$

$$y = \sec x - \cos x + C\ln|\sec x + \tan x| + D$$

3

(a)

Method 1

Step 1 : Translate by 1 unit in the direction of the *x*-axis.
Step 2 : reflection about the *x*-axis.
Step 3 : Translate by ln3 units in the direction of the *y*-axis.

$$y=\ln(2x+1)\to y=\ln\left[2(x-1)+1\right]\to y=-\ln\left(2x-1\right)\to y=\ln 3-\ln\left(2x-1\right)=\ln\left(\frac{3}{2x-1}\right)$$

Method 2

Step 1 : reflection about the *x*-axis.
Step 2 : Translate by 1 unit in the direction of the *x*-axis.
Step 3 : Translate by ln3 units in the direction of the *y*-axis.

$$y=\ln\left(2x+1\right)\to y=-\ln\left(2x+1\right)\to y=-\ln\left[2\left(x-1\right)+1\right]\to y=\ln 3-\ln\left(2x-1\right)=\ln\left(\frac{3}{2x-1}\right)$$

Method 3

Step 1 : reflection about the *x*-axis.
Step 2 : Translate by ln3 units in the direction of the *y*-axis.
Step 3 : Translate by 1 unit in the direction of the *x*-axis.

$$y=\ln\left(2x+1\right)\to y=-\ln\left(2x+1\right)\to y=\ln 3-\ln(2x+1)\to y=\ln 3-\ln\left[2\left(x-1\right)+1\right]=\ln\left(\frac{3}{2x-\right.$$

Method 4

Step 1 : Translate by 1 unit in the direction of the *x*-axis.
Step 2 : Translate by $-\ln 3$ units in the direction of the *y*-axis.
Step 3 : reflection about the *x*-axis.

$$y=\ln(2x+1)\to y=\ln\left[2(x-1)+1\right]\to y=-\ln 3+\ln\left(2x-1\right)\to y=\ln 3-\ln\left(2x-1\right)=\ln\left(\frac{3}{2x-1}\right.$$

Method 5

Step 1 : Translate by $-\ln 3$ units in the direction of the *y*-axis.
Step 2 : Translate by 1 unit in the direction of the *x*-axis.
Step 3 : reflection about the *x*-axis.

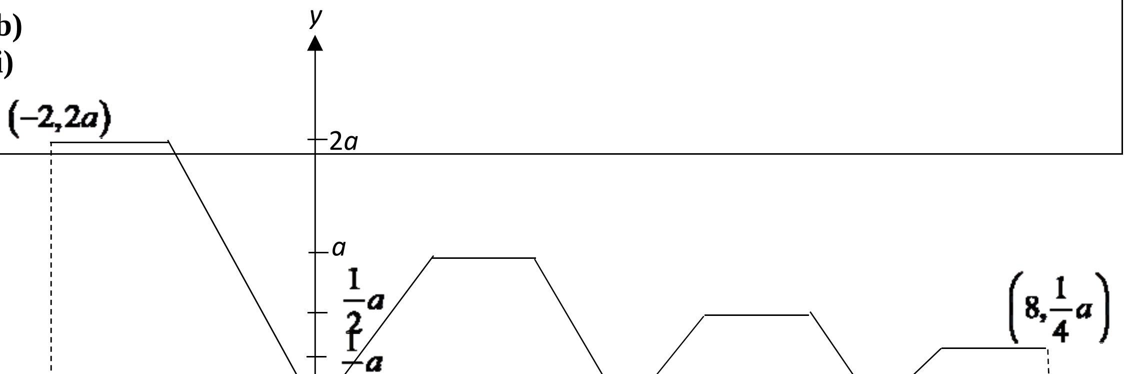
$$y=\ln\left(2x+1\right)\to y=-\ln 3+\ln(2x+1)\to y=-\ln 3+\ln\left[2\left(x-1\right)+1\right]\to y=\ln 3-\ln\left(2x-1\right)=\ln$$

Method 6

Step 1 : Translate by $-\ln 3$ units in the direction of the *y*-axis.
Step 2 : reflection about the *x*-axis.
Step 3 : Translate by 1 unit in the direction of the *x*-axis.

$$y=\ln\left(2x+1\right)\to y=-\ln 3+\ln(2x+1)\to y=\ln 3-\ln(2x+1)\to y=\ln 3-\ln\left[2\left(x-1\right)+1\right]=\ln\left(\right.$$

(b)
(i)



(ii)

$$\int_0^2 \mathrm{f}\left(-x\right)\,\mathrm{d}x = \frac{1}{2}\left(2+1\right)\left(2a\right) = 3a$$

(iii)

$$\int_0^\infty \mathrm{f}\left(x\right)\,\mathrm{d}x = 16$$

$$2\left(a + \frac{1}{2}a + \frac{1}{4}a + ...\right) = 16$$

$$2a\left(\frac{1}{1-\frac{1}{2}}\right) = 16$$

$$4a = 16$$

$$a = 4$$

4

(i)

$$\frac{z^2}{z^*} = \frac{\left(-1+\mathrm{i}c\right)^2}{\left(-1-\mathrm{i}c\right)}$$

$$= \frac{1-\mathrm{i}2c-c^2}{\left(-1-\mathrm{i}c\right)}$$

$$= \frac{1-\mathrm{i}2c-c^2}{\left(-1-\mathrm{i}c\right)} \times \frac{\left(-1+\mathrm{i}c\right)}{\left(-1+\mathrm{i}c\right)}$$

$$= \frac{-1+\mathrm{i}c+\mathrm{i}2c+2c^2+c^2-\mathrm{i}c^3}{1+c^2}$$

Since $\frac{z^2}{z^*}$ is purely real,

$$\frac{3c-c^3}{1+c^2} = 0$$

$$c\left(3-c^2\right) = 0$$

$c = 0$ (rej since c is non-zero) $c = \pm\sqrt{3}$

(ii)

$$z = -1 + \mathrm{i}\sqrt{3}$$

$|z| = 2$, $\qquad \arg\left(z\right) = \frac{2\pi}{3}$

	$\dfrac{z^n}{z^*}=\dfrac{\left(2\mathrm{e}^{\mathrm{i}\frac{2\pi}{3}}\right)^n}{2\mathrm{e}^{\mathrm{i}\left(-\frac{2\pi}{3}\right)}}=2^{n-1}\mathrm{e}^{\mathrm{i}\left(\frac{2n\pi}{3}+\frac{2\pi}{3}\right)}$ Since $\dfrac{z^n}{z^*}$ is purely real, $\arg\left(\dfrac{z^n}{z^*}\right)=0,\pm\pi,\pm2\pi,\pm3\pi,\pm4\pi,\pm5\pi,\pm6\pi...$ or $\sin\left[(n+1)\dfrac{2\pi}{3}\right]=0$ $(n+1)\dfrac{2\pi}{3}=0,\pm\pi,\pm2\pi,\pm3\pi,\pm4\pi,\pm5\pi,\pm6\pi...$ $n+1=0,\pm\dfrac{3}{2},\pm3,\pm\dfrac{9}{2},\pm6,...$ Considering positive integer values only, $n+1=3,6,9...$ Three smallest positive integer values of *n* are $2,5,8$
5	**(i)** $y=\sec 2x$ $\dfrac{\mathrm{d}y}{\mathrm{d}x}=2\sec 2x\tan 2x=2y\tan 2x$ $\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2=4y^2\tan^2 2x$ $=4y^2(\sec^2 2x-1)$ $=4y^2(y^2-1)$ **(ii)** $2\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)\left(\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)=16y^3\dfrac{\mathrm{d}y}{\mathrm{d}x}-8y\dfrac{\mathrm{d}y}{\mathrm{d}x}$ $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=8y^3-4y$ $\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}=24y^2\dfrac{\mathrm{d}y}{\mathrm{d}x}-4\dfrac{\mathrm{d}y}{\mathrm{d}x}$ $\dfrac{\mathrm{d}^4y}{\mathrm{d}x^4}=24y^2\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+48y\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2-4\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ When $x=0$, $y=1$, $\dfrac{\mathrm{d}y}{\mathrm{d}x}=0$, $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=4$, $\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}=0$, $\dfrac{\mathrm{d}^4y}{\mathrm{d}x^4}=80$ $y=1+\dfrac{x^2}{2!}(4)+\dfrac{x^4}{4!}(80)+...$

$y=1+2x^2+\dfrac{10}{3}x^4+\ldots$

(iii)

$\sec 2x=\dfrac{1}{\cos 2x}$

$\approx \dfrac{1}{1-\dfrac{(2x)^2}{2!}+\dfrac{(2x)^4}{4!}}$

$=(1-2x^2+\dfrac{2}{3}x^4)^{-1}$

$=1+(-1)(-2x^2+\dfrac{2}{3}x^4)+\dfrac{(-1)(-2)}{2!}\left(-2x^2+\dfrac{2}{3}x^4\right)^2+\ldots$

$=1+2x^2-\dfrac{2}{3}x^4+4x^4+\ldots$

$\approx 1+2x^2+\dfrac{10}{3}x^4$

6 **(i)**

		Score on the die			
	S	**1** ($\frac{1}{5}$)	**2** (p)	**3** ($\frac{1}{5}$)	**4** (q)
Dots on the disc	**1**($\frac{1}{2}$)	3 ($\frac{1}{10}$)	4 ($\frac{1}{2}p$)	5 ($\frac{1}{10}$)	6 ($\frac{1}{2}q$)
	2 ($\frac{1}{2}$)	5 ($\frac{1}{10}$)	6 ($\frac{1}{2}p$)	7 ($\frac{1}{10}$)	8 ($\frac{1}{2}q$)

Since total probability = 1,

$\dfrac{1}{5}+p+\dfrac{1}{5}+q=1 \Rightarrow p+q=\dfrac{3}{5}$

OR: $\left(\dfrac{1}{10}\right)+\left(\dfrac{1}{2}p\right)+\left(\dfrac{1}{5}\right)+\left(\dfrac{1}{2}p+\dfrac{1}{2}q\right)+\left(\dfrac{1}{10}\right)+\left(\dfrac{1}{2}q\right)=1 \Rightarrow p+q=\dfrac{3}{5}$

Hence $P(S=6)=\dfrac{1}{2}p+\dfrac{1}{2}q=\dfrac{1}{2}\times\dfrac{3}{5}=\dfrac{3}{10}$.

(ii)

$P(S=4)=\dfrac{1}{6} \Rightarrow \dfrac{1}{2}p=\dfrac{1}{6} \Rightarrow p=\dfrac{1}{3}$

Since $p+q=\dfrac{3}{5}$, then $q=\dfrac{3}{5}-\dfrac{1}{3}=\dfrac{4}{15}$.

(iii)

s	3	4	5	6	7	8
$P(S=s)$	$\frac{1}{10}$	$\frac{1}{6}$	$\frac{1}{5}$	$\frac{3}{10}$	$\frac{1}{10}$	$\frac{2}{15}$

7 **(i)**

	X ~ mass of a black sea bass fish. $X \sim \text{N}\left(1.1, 0.2^2\right)$ Y ~ mass of a red tilapia fish. $Y \sim \text{N}\left(0.55, 0.05^2\right)$ Let T be the total cost of 2 black sea bass and 3 red tilapia. Then $T = 12\left(X_1 + X_2\right) + 9\left(Y_1 + Y_2 + Y_3\right)$ $\text{E}\left(T\right) = (12)(2)\text{E}(X) + (9)(3)\text{E}(Y)$ $= 26.4 + 14.85$ $= 41.25$ $\text{Var}\left(T\right) = (12)^2(2)\text{Var}(X) + (9)^2(3)\text{Var}(Y)$ $= 11.52 + 0.6075$ $= 12.1275$ Thus $T \sim \text{N}\left(41.25\ ,\ 12.1275\right)$. P($T$ > 40) = 0.64018 ≈ 0.640 (3 s.f.) An assumption needed is the price / mass of all fish are independent of one another. **(ii)** Probability required = $\frac{4!}{2!2!}$ $[\text{P}(Y > 0.5)]^2\ [\text{P}(Y < 0.5)]^3 \approx 0.0170$
8	**(i)** X ~ amount of cholesterol in one standard fillet of raw red snapper. unbiased estimate of population variance σ^2 is $s^2 = \frac{n}{n-1}\sigma_x{}^2 = \frac{50}{49}\left(2\right)^2 = \frac{200}{49}$ Test $\text{H}_0: \mu = w$ vs $\text{H}_1: \mu \neq w$ Since $n = 50$ is large, by Central Limit Theorem, $\overline{X} \sim \text{N}\left(w, \frac{\frac{200}{49}}{50}\right)$ approx. $\overline{X} \sim N\left(w, \frac{4}{49}\right)$ approx. Level of significance: 5 % Critical region is $z < -1.9600$ or $z > 1.9600$ Standardised test statistic: $z = \frac{78.5 - w}{\frac{2}{7}}$ Since H_0 is not rejected, z lies outside the critical region. $-1.9600 < \frac{78.5 - w}{\frac{2}{7}} < 1.9600$ $-1.96\left(\frac{2}{7}\right) < 78.5 - w < 1.96\left(\frac{2}{7}\right)$ $-1.96\left(\frac{2}{7}\right) - 78.5 < -w < 1.96\left(\frac{2}{7}\right) - 78.5$

	$-79.06 < -w < -77.94$ $77.94 < w < 79.06$ **(ii)** No. It is not necessary to assume that amount of cholesterol in a standard fillet follows a normal distribution since sample size is large, by Central Limit Theorem, sample mean is normally distributed approximately.
9	**(i)** No. of ways = $4^{11} = 4194304$ **(ii)** Case 1: four 0s, four 2s, with one 1, two 3s **or** one 3, two 1s No. of ways = $\dfrac{11!}{4!4!2!} \times 2 = 69300$ Case 2: four 0s, four 2s, with three 1s **or** three 3s No. of ways = $\dfrac{11!}{4!4!3!} \times 2 = 23100$ Hence the total number of ways is $69300 + 23100 = 92400$ **Alternative** No. of ways = ${}^{11}C_4 \times {}^{7}C_4 \times 2^3 = 92400$ Choose 4 slots from 11 slots to place fou 0s. Choose 4 slots from remaining 7 slot place four 2s. 2 choices (digit 1 or 3) for each of the remaining 3 slots. **(iii)** No. of ways = no. of ways without restriction – no. of ways with no consecutive digits that are the same $= 4194304 - 4 \times 3^{10} = 3958108$
10	**(i)** Least value of $\text{P}(A \cap B) = \text{P}(A) + \text{P}(B) - 1 = \dfrac{11}{20} + \dfrac{1}{2} - 1 = \dfrac{1}{20}$ Greatest value of $\text{P}(A \cap B) = \text{P}(B) = \dfrac{1}{2}$ **(ii)** $\text{P}(B \cap A') = \text{P}(B \mid A')\text{P}(A') = \dfrac{7}{9} \times \dfrac{9}{20} = \dfrac{7}{20}$ $\text{P}(A \cup B) = \text{P}(B \cap A') + \text{P}(A) = \dfrac{7}{20} + \dfrac{11}{20} = \dfrac{9}{10}$ **(iii)** $\because \text{P}(B \mid A') = \dfrac{7}{9} \neq \dfrac{1}{2} = \text{P}(B)$, then B and A' are not independent events. Hence A and B are not independent events.

OR:

$$P(A\cup B)=P(A)+P(B)-P(A\cap B)$$

$$\frac{9}{10}=\frac{11}{20}+\frac{1}{2}-P(A\cap B)$$

$$P(A\cap B)=\frac{3}{20}$$

$$P(A)P(B)=\frac{11}{20}\times\frac{1}{2}=\frac{11}{40}$$

$$P(A\cap B)\neq P(A)P(B)$$

Hence, A and B are not independent events.

(iv)

$$P\left[C\cap\left(A\cup B\right)'\right]=P(A\cup B\cup C)-P(A\cup B)=\frac{19}{20}-\frac{9}{10}=\frac{1}{20}$$

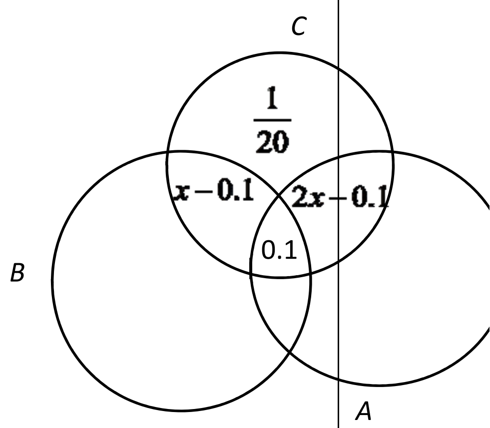
Let $P(B\cap C)=x$, then $P(A\cap C)=2x$

$$P\left(C\right)=\frac{1}{20}+\frac{1}{10}+\left(x-\frac{1}{10}\right)+\left(2x-\frac{1}{10}\right)$$

$$\frac{2}{5}=3x-\frac{1}{20}$$

$$x=0.15$$

$$P(A\cap C)=2x=0.3$$



Or

$$P(A\cup B\cup C)=P(A)+P(B)+P(C)-P(A\cap B)-P(A\cap C)-P(B\cap C)+P(A\cap B\cap C)$$

$$\frac{19}{20}=\frac{11}{20}+\frac{1}{2}+\frac{2}{5}-\frac{3}{20}-2P(B\cap C)-P(B\cap C)+\frac{1}{10}$$

$$3P(B\cap C)=\frac{9}{20}$$

$$P(B\cap C)=\frac{3}{20}$$

$$P(A\cap C)=\frac{6}{20}=\frac{3}{10}$$

11 2 assumptions:

- Occurrence of show / no show is independent among passengers.
- Probability that a passenger does not show up is constant.

(i)

X ~ number of passengers with reservation, who show up, out of 245.

$X \sim B(245, 0.93)$

$P(X>232)=1-P(X\leq 232)=0.118761\approx 0.119$ (3 s.f.)

(ii)

W ~ number of passengers with reservation, who show up, out of n.

$W \sim B(n\ ,0.93)$

$P(W > 232) < 0.01$

$1 - P(W \leq 232) < 0.01$

$P(W \leq 232) > 0.99$

Using GC,

When $n = 239$, $P(W \leq 232) = 0.998 > 0.99$

When $n = 240$, $P(W \leq 232) = 0.995 > 0.99$

When $n = 241$, $P(W \leq 232) = 0.989 < 0.99$

When $n = 242$, $P(W \leq 232) = 0.977 < 0.99$

Hence the maximum reservations that should be accepted is 240.

(iii)

Y ~ number of flights which is overbooked, out of 7.

$Y \sim B(7\ ,0.118761)$

$P(Y = 0) = 0.41272 \approx 0.413$ (3 s.f.)

(iv)

Since n = 52 is large, by CLT,

$\overline{Y} \sim N(E(Y), \frac{Var(Y)}{52})$ approximately

$\overline{Y} \sim N\left((7)(0.118761), \frac{(7)(0.118761)(0.881239)}{52}\right)$ approximately

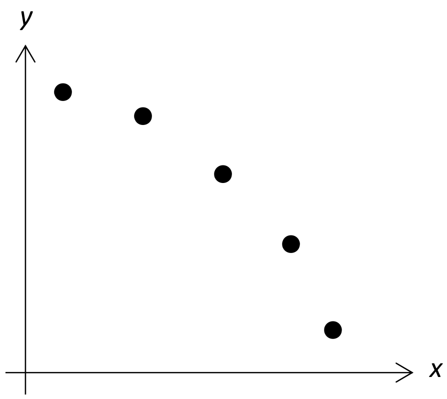
$\overline{Y} \sim N\left(0.831327, \frac{0.732598}{52}\right)$ approximately

$\overline{Y} \sim N(0.831327, 0.014088)$ approximately

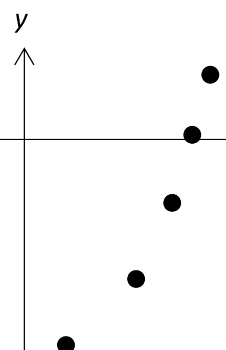
$P(\overline{Y} \leq 1) = 0.9223521 \approx 0.922$ (3 s.f.)

12 **(i)**

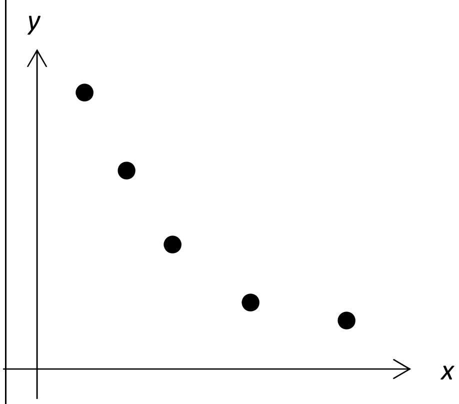
$y = p + qx^2$



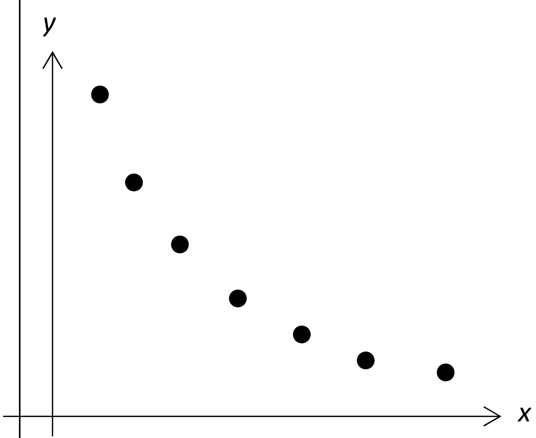
$y = r + se^x$



$y = t + \frac{u}{x}$



(ii)



(iii)

As x increases, y decreases at a decreasing rate. Hence, model (C) is the most appropriate.
Using GC, $r = 0.984$

(iv)

	Equation of regression line: $y = 69.425 + \frac{5.75555}{x} \approx 69.4 + \frac{5.76}{x}$ When $x = 10$, $y = 69.425 + \frac{5.75555}{10} = 70.0$ **(v)** Replace x with $\frac{x}{7}$, New equation: $y = 69.425 + \frac{5.75555}{\frac{x}{7}} \approx 69.4 + \frac{40.3}{x}$