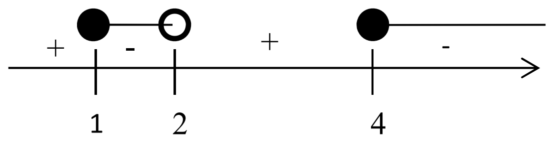
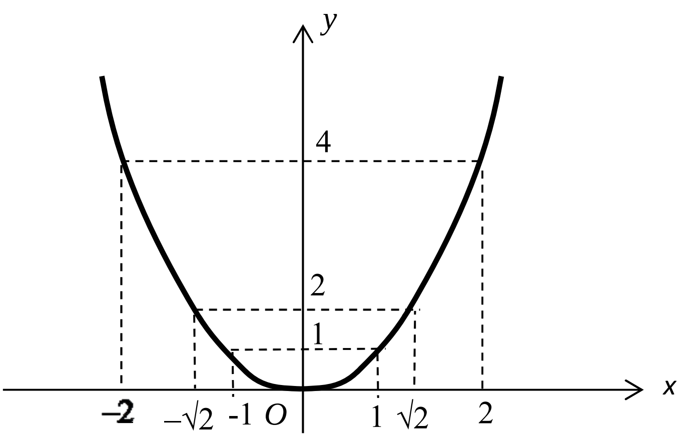
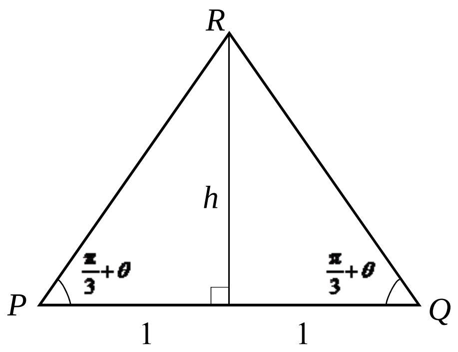
2017 Year 6 H2 Math Prelim Exam Paper 1 Mark Scheme

Qn	Solution
1	Consider replace k by $(k-1)$: $\sum_{k=1}^{n-1}(k+1)!(k^2+2k+2)=\sum_{k-1=1}^{k-1=n-1}(k-1+1)!((k-1)^2+2(k-1)+2)$ $=\sum_{k=2}^{n}k!(k^2+1)$ $=\sum_{k=1}^{n}k!(k^2+1)-1!(1^2+1)$ $=(n+1)!n-2$

Qn	Solution

2(i)	To prove AP, consider $U_{r+1} - U_r$ $= (\ln T_{r+1} - 3) - (\ln T_r - 3)$ $= \ln\left(\frac{T_{r+1}}{T_r}\right)$ $= \ln \mathrm{e}$ $=1$ Since difference is a **constant,** the sequence is arithmetic. (Proven)
(ii)	$\sum_{r=1}^{n-1}(W_{r+1} - W_r) = \sum_{r=1}^{n-1} U_r$ LHS $= \sum_{r=1}^{n-1}(W_{r+1} - W_r)$ $= W_2 - W_1$ $+ W_3 - W_2$ $+ W_4 - W_3$ $\vdots$ $+ W_n - W_{n-1}$ $= W_n - W_1$ $= W_n - \frac{1}{2}$ RHS $= \sum_{r=1}^{n-1} U_r$ $= U_1 + U_2 + \ldots + U_{n-1}$ $= \frac{n-1}{2}(2(1) + (n-2)1)$ $= \frac{n(n-1)}{2}$ Thus, $W_n - \frac{1}{2} = \frac{n(n-1)}{2}$ $\therefore W_n = \frac{1}{2}\left(n^2 - n + 1\right)$ (shown)

Qn	Suggested Solution
3(i)	$2-x \le \dfrac{x}{2-x}$ $2-x-\dfrac{x}{2-x} \le 0$ $\dfrac{(2-x)^2-x}{2-x} \le 0$ $\dfrac{x^2-5x+4}{2-x} \le 0$ $\dfrac{(x-4)(x-1)}{2-x} \le 0$  Set of values of *x*: $\{1 \le x < 2$ **or** $x \ge 4\}$
(ii)	Let $y = x^2$. $2-x^2 \le \dfrac{x^2}{2-x^2} \Rightarrow 2-y \le \dfrac{y}{2-y}$ $1 \le y < 2$ or $y \ge 4$  The range of values of *x* is $x \le -2$ or $-\sqrt{2} < x \le -1$ or $1 \le x < \sqrt{2}$ or $x \ge 2$

Qn	Suggested Solution
4	 $h=\tan\left(\frac{\pi}{3}+\theta\right)$ $A=\frac{1}{2}(2)\tan\left(\frac{\pi}{3}+\theta\right)=\tan\left(\frac{\pi}{3}+\theta\right)$ $=\dfrac{\tan\left(\frac{\pi}{3}\right)+\tan\theta}{1-\tan\left(\frac{\pi}{3}\right)\tan\theta}=\dfrac{\sqrt{3}+\tan\theta}{1-\sqrt{3}\tan\theta}$ (shown) $\approx\dfrac{\sqrt{3}+\theta}{1-\theta\sqrt{3}}$ $=\left(\sqrt{3}+\theta\right)\left(1-\theta\sqrt{3}\right)^{-1}$ $\approx\left(\sqrt{3}+\theta\right)\left(1+\theta\sqrt{3}+3\theta^2\right)$ $=\sqrt{3}+4\theta+\left(4\sqrt{3}\right)\theta^2$

Qn	Solution
5(a)	$\operatorname{cosec} y = x$ Diff wrt x: $-\operatorname{cosec} y\ \cot y\frac{\mathrm{d}y}{\mathrm{d}x}=1$ $\therefore \frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{1}{\operatorname{cosec} y\ \cot y}$ Using $\cot^2 y+\operatorname{cosec}^2 y\equiv 1$, $\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{1}{\operatorname{cosec} y\sqrt{(\operatorname{cosec} y)^2-1}}$ [since $0<y<\frac{\pi}{2}\Rightarrow \tan y>0\Rightarrow \cot y>0\Rightarrow \cot y=\sqrt{(\operatorname{cosec} y)^2-1}$] $=-\frac{1}{x\sqrt{x^2-1}}$ (shown)

Since $y = \operatorname{cosec}^{-1} x$,

$$\frac{dy}{dx} = \frac{d}{dx}(\operatorname{cosec}^{-1} x) = -\frac{1}{x\sqrt{x^2-1}}$$

<u>Alternative</u>

$\operatorname{cosec} y = x$

Diff wrt x:

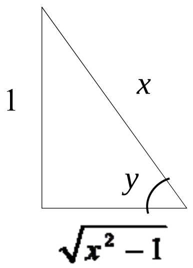
$$-\operatorname{cosec} y \cot y \frac{dy}{dx} = 1$$

$$\therefore \frac{dy}{dx} = -\frac{1}{\operatorname{cosec} y \cot y}$$

Since $\operatorname{cosec} y = x$,

$$\therefore \frac{1}{\sin y} = x$$

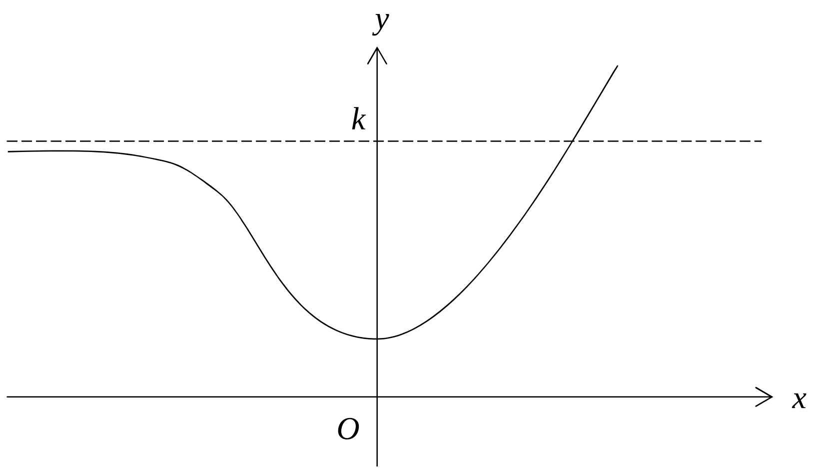
$$\therefore \sin y = \frac{1}{x}$$

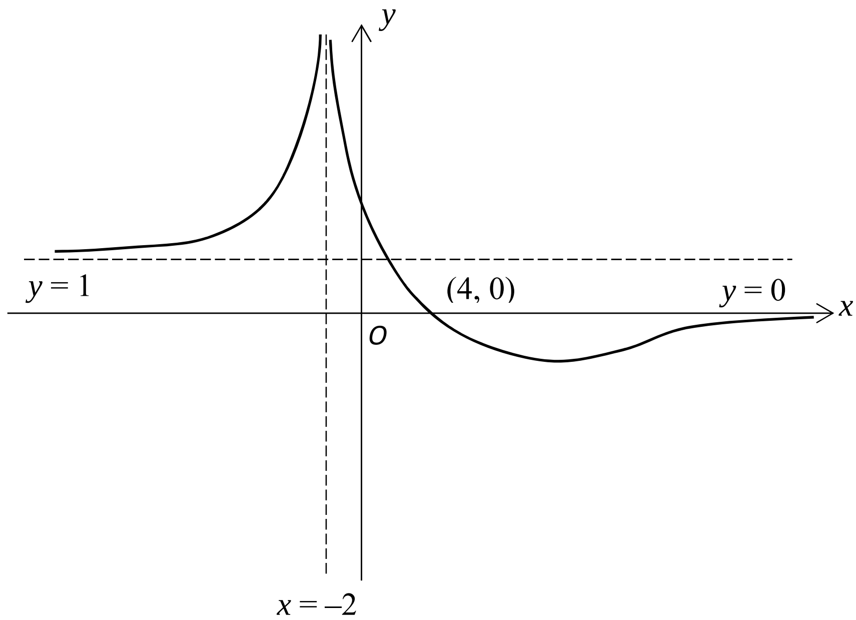


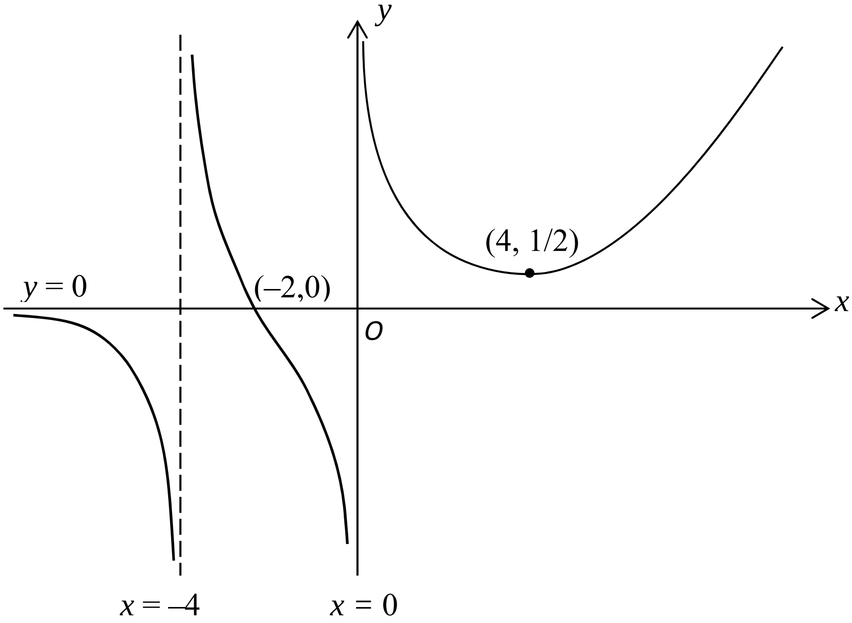
By constructing the right angle triangle, $\tan y = \frac{1}{\sqrt{x^2-1}}$

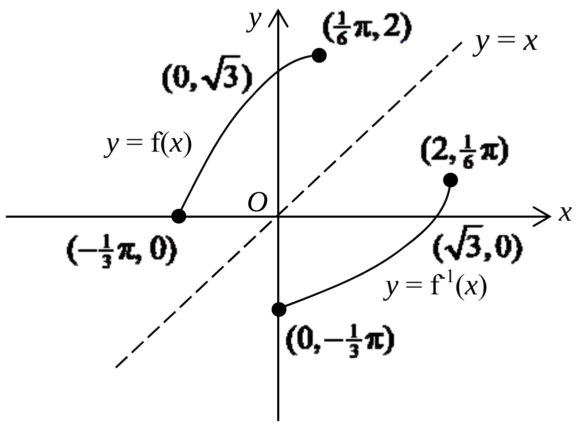
$$\frac{dy}{dx} = -\frac{1}{\operatorname{cosec} y \cot y} = -\frac{\tan y}{\operatorname{cosec} y} = -\frac{1}{x\sqrt{x^2-1}} \text{ (shown)}$$

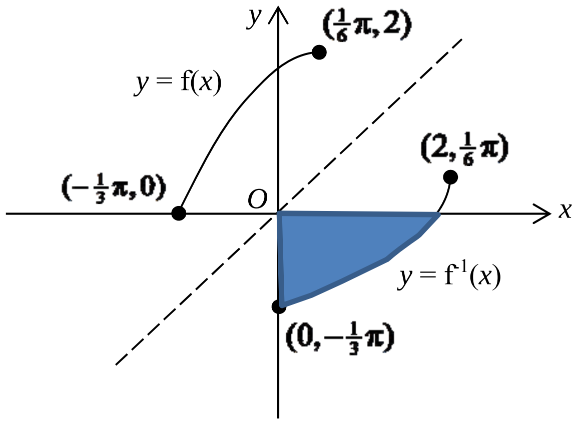
(b)



Qn	Suggested Solution
6(a)	$x^2+\frac{1}{3}(y-2)^2=1$ $\downarrow$ Replace x by $x-2$ $(x-2)^2+\frac{1}{3}(y-2)^2=1$ $\downarrow$ Replace y by $y+2$ $(x-2)^2+\frac{1}{3}(y)^2=1$ $\downarrow$ Replace y by $\sqrt{3}y$ $(x-2)^2+y^2=1$ 1. Translate 2 units in the positive x-direction 2. Translate 2 units in the negative y-direction 3. Scale by a factor of $\dfrac{1}{\sqrt{3}}$ parallel to the y-direction
b(i)	

(ii)	

Qn	Solution
7(i)	Using R formula, $\sin x + \sqrt{3}\cos x = 2\sin(x + \frac{1}{3}\pi)$
(ii) **(iii)**	Graphs of $y = \mathrm{f}(x)$ and $y = \mathrm{f}^{-1}(x)$:  To find f^{-1}: Let $y = 2\sin(x + \frac{1}{3}\pi)$ $\therefore x = -\frac{1}{3}\pi + \sin^{-1}(\frac{1}{2}y)$ $\mathrm{f}^{-1}(x) = -\frac{1}{3}\pi + \sin^{-1}(\frac{1}{2}x)$ $\mathrm{D}_{\mathrm{f}^{-1}} = \mathrm{R}_{\mathrm{f}} = [0, 2]$

(iv)	For the area bounded by the graph of f^{-1} and the axes:  **By symmetry,** Area $= \int_{-\frac{\pi}{3}}^{0} \mathrm{f}(x)\, \mathrm{d}x$ $= \int_{-\frac{\pi}{3}}^{0} (\sin x + \sqrt{3}\cos x)\, \mathrm{d}x$ $= \left[-\cos x + \sqrt{3}\sin x\right]_{-\frac{\pi}{3}}^{0}$ $= (-1+0) - \left(-\frac{1}{2} - \frac{3}{2}\right)$ $= 1$ **<u>Alternative</u>** Area $= \int_{-\frac{\pi}{3}}^{0} x\, \mathrm{d}y$ $= \int_{-\frac{\pi}{3}}^{0} \mathrm{f}(y)\, \mathrm{d}y \quad [\text{since } y = \mathrm{f}^{-1}(x) \Rightarrow x = \mathrm{f}(y)]$ $= \int_{-\frac{\pi}{3}}^{0} (\sin y + \sqrt{3}\cos y)\, \mathrm{d}y$ $= \left[-\cos y + \sqrt{3}\sin y\right]_{-\frac{\pi}{3}}^{0}$ $= (-1+0) - \left(-\frac{1}{2} - \frac{3}{2}\right)$ $= 1$
(v)	gf^{-1} exists if $\mathrm{R}_{\mathrm{f}^{-1}} \subseteq \mathrm{D}_{\mathrm{g}}$. Since $\mathrm{R}_{\mathrm{f}^{-1}} = [-\frac{1}{3}\pi, \frac{1}{6}\pi]$ $\mathrm{D}_{\mathrm{g}} = (-2, \infty)$,

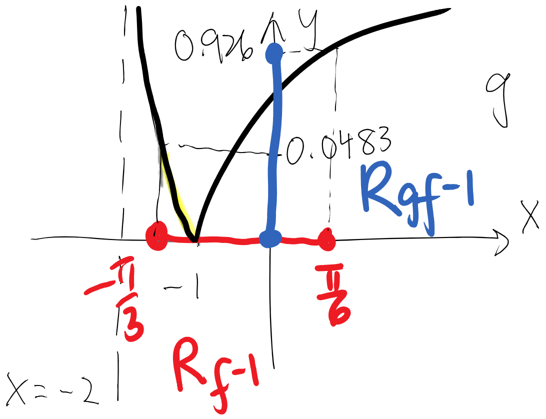
Ie. $R_{f^{-1}} \subseteq D_g \Rightarrow gf^{-1}$ exists

To find the range of gf^{-1}:

Method 1 (two stage mapping method)

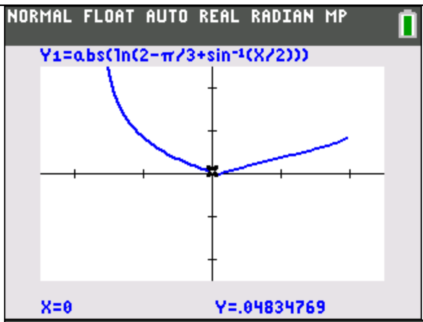
$$D_{f^{-1}} \xrightarrow{f^{-1}} R_{f^{-1}} \xrightarrow{g} R_{gf^{-1}}$$

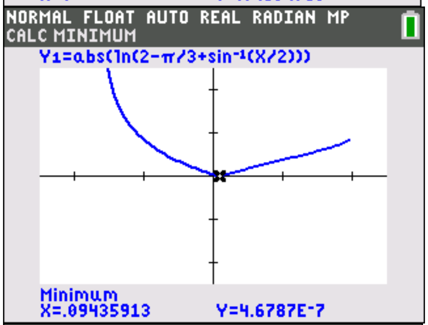
$[0,2]$ $\quad [-\frac{1}{3}\pi, \frac{1}{6}\pi]$ $\quad$?

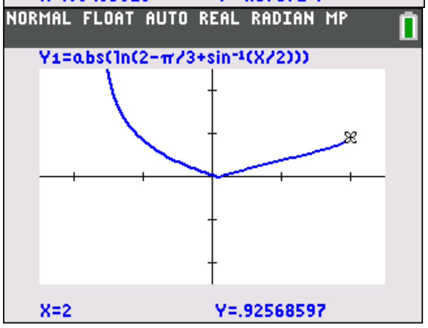


$R_{gf^{-1}} = [0,\ 0.926]$

Method 2 (find gf^{-1}) (need to use GC to see shape)







$\mathrm{gf}^{-1}(x) = \left|\ln(2 - \tfrac{1}{3}\pi + \sin^{-1}(\tfrac{1}{2}x))\right|$

$D_{\mathrm{gf}^{-1}} = D_{\mathrm{f}^{-1}} = [0, 2]$

$R_{\mathrm{gf}^{-1}} = [0, 0.926]$

Qn	Suggested Solution
8(a)	Since the curve shows only one x-intercept, it means that <u>**there is only one real root**</u> in the equation $\mathrm{f}(x) = 0$. Since the equation has all real coefficients, then **the two other roots must be <u>non-real and they are conjugate pair.</u>**
(b)	Since $z = 3 + 4\mathrm{i}$ is a root of $2z^2 - (7 + 6\mathrm{i})z + 11 + \mathrm{i}c = 0$,

$2(3+4\mathrm{i})^2-(7+6\mathrm{i})(3+4\mathrm{i})+11+\mathrm{i}c=0$

$2(9+24\mathrm{i}-16)-(21+28\mathrm{i}+18\mathrm{i}-24)+11+\mathrm{i}c=0$

Comparing the Im - part,

$2+c=0$

$\therefore c=-2$ (shown)

Since $z=3+4\mathrm{i}$ is a root of $2z^2-(7+6\mathrm{i})z+11-2\mathrm{i}=0$,

$2z^2-(7+6\mathrm{i})z+11-2\mathrm{i}=\left(z-(3+4\mathrm{i})\right)\left(2z-a\right)$, where $a \in \mathbb{C}$

Comparing the coefficient of constant term,

$11-2\mathrm{i}=a(3+4\mathrm{i})$

$$a=\frac{11-2\mathrm{i}}{3+4\mathrm{i}}$$

$$=\frac{(11-2\mathrm{i})(3-4\mathrm{i})}{25}$$

$$=\frac{25-50\mathrm{i}}{25}$$

$$=1-2\mathrm{i}$$

$2z-(1-2\mathrm{i})=0 \Rightarrow z=\frac{1}{2}-\mathrm{i}$

Therefore, the other root is $\frac{1}{2}-\mathrm{i}$.

Replace z by $\mathrm{i}w$

$2(\mathrm{i}w)^2-(7+6\mathrm{i})(\mathrm{i}w)+11-2\mathrm{i}=0$

$-2w^2-(-6+7\mathrm{i})w+11-2\mathrm{i}=0$

$2w^2+(-6+7\mathrm{i})w-11+2\mathrm{i}=0$

$\mathrm{i}w=3+4\mathrm{i} \Rightarrow w=4-3\mathrm{i}$ or $\mathrm{i}w=\frac{1}{2}-\mathrm{i} \Rightarrow w=-1-\frac{1}{2}\mathrm{i}$

$\therefore$ the roots of the equation are $4-3\mathrm{i}$ and $-1-\frac{1}{2}\mathrm{i}$.

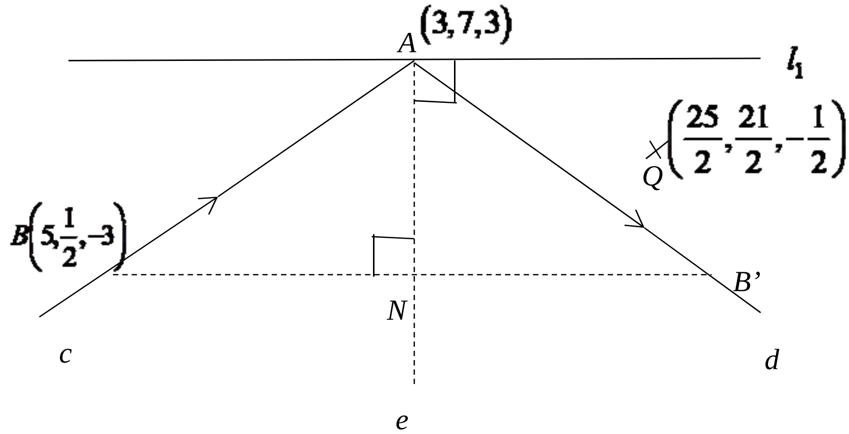
<u>Alternative Method:</u>

	$2z^2-(7+6\text{i})z+11-2\text{i}=0$ Let the other root be $a+b\text{i}$. Sum of the roots $=3+4\text{i}+a+b\text{i}=\dfrac{7+6\text{i}}{2}=\dfrac{7}{2}+3\text{i}$ Comparing real and imaginary parts: $a+3=\dfrac{7}{2}\Rightarrow a=\dfrac{1}{2}$ $4+b=3\Rightarrow b=-1$ The other root is $\dfrac{1}{2}-\text{i}$
(c)i)	$z=1+\text{e}^{\text{i}\alpha}$ $=\text{e}^{\text{i}\frac{\alpha}{2}}(\text{e}^{-\text{i}\frac{\alpha}{2}}+\text{e}^{\text{i}\frac{\alpha}{2}})$ $=\text{e}^{\text{i}\frac{\alpha}{2}}\left[2\text{Re}\left(\text{e}^{\text{i}\frac{\alpha}{2}}\right)\right]$ $=2\cos\dfrac{\alpha}{2}\text{e}^{\text{i}\frac{\alpha}{2}}$ (shown) **<u>Alternative Method:</u>** $z=1+\text{e}^{\text{i}\alpha}$ $=1+\cos\alpha+\text{i}\sin\alpha$ $=1+2\cos^2\dfrac{\alpha}{2}-1+\text{i}\left(2\sin\dfrac{\alpha}{2}\cos\dfrac{\alpha}{2}\right)$ $=2\cos\dfrac{\alpha}{2}\left(\cos\dfrac{\alpha}{2}+\text{i}\sin\dfrac{\alpha}{2}\right)$ $=2\cos\dfrac{\alpha}{2}\text{e}^{\text{i}\frac{\alpha}{2}}$ (shown)

(ii)	$\left\|\left(\frac{z}{w^3}\right)^*\right\|$ $=\left\|\left(\frac{z}{w^3}\right)\right\|$ $=\frac{\|z\|}{\|w\|^3}$ $=\frac{\left\|2\cos\frac{\pi}{6}\right\|}{\left(\sqrt{1+3}\right)^3}$ $=\frac{2\left(\frac{\sqrt{3}}{2}\right)}{(2)^3}$ $=\frac{\sqrt{3}}{8}$ $\arg\left(\frac{z}{w^3}\right)^*$ $=-\arg\left(\frac{z}{w^3}\right)$ $=-\left[\arg(z)-3\arg(w)\right]$ $=-\frac{\alpha}{2}+3\left(-\frac{2\pi}{3}\right)$ $=-\frac{\pi}{6}-2\pi$ $\therefore \arg\left(\frac{z}{w^3}\right)^*=-\frac{\pi}{6}$

	Suggested Solution
9(i)	A vector perpendicular to p_1 is $\begin{pmatrix}3\\4\\1\end{pmatrix}\times\begin{pmatrix}1\\-2\\2\end{pmatrix}=\begin{pmatrix}10\\-5\\-10\end{pmatrix}=5\begin{pmatrix}2\\-1\\-2\end{pmatrix}$ Eqn of p_1 $\mathbf{r}.\begin{pmatrix}2\\-1\\-2\end{pmatrix}=\begin{pmatrix}3\\7\\3\end{pmatrix}.\begin{pmatrix}2\\-1\\-2\end{pmatrix}=-7$ Cartesian eqn : $2x-y-2z=-7$
(ii)	Distance of l_2 to $p_1=\left\|\left[\begin{pmatrix}2\\3\\5\end{pmatrix}-\begin{pmatrix}3\\7\\3\end{pmatrix}\right].\begin{pmatrix}2\\-1\\-2\end{pmatrix}\right\|/3$ $=\left\|\begin{pmatrix}-1\\-4\\2\end{pmatrix}.\begin{pmatrix}2\\-1\\-2\end{pmatrix}\right\|/3$ $=\frac{2}{3}$
(iii)	A vector perpendicular to p_2 $=\begin{pmatrix}3\\4\\1\end{pmatrix}\times\begin{pmatrix}2\\-1\\-2\end{pmatrix}=\begin{pmatrix}-7\\8\\-11\end{pmatrix}$ Equation of p_2 $\mathbf{r}.\begin{pmatrix}-7\\8\\-11\end{pmatrix}=\begin{pmatrix}3\\7\\3\end{pmatrix}.\begin{pmatrix}-7\\8\\-11\end{pmatrix}=2$

(iv)



line *d* is a reflection of line *c* in the line *e* through *A* perpendicular to l_1 and lying in p_2.

Eqn of this line *e* : $\mathbf{r}=\begin{pmatrix}3\\7\\3\end{pmatrix}+\mu\begin{pmatrix}2\\-1\\-2\end{pmatrix}$

Foot of perpendicular, *N*, from $\left(5,\frac{1}{2},-3\right)$ to line *e*

$$\left[\begin{pmatrix}3\\7\\3\end{pmatrix}+\mu\begin{pmatrix}2\\-1\\-2\end{pmatrix}-\begin{pmatrix}5\\0.5\\-3\end{pmatrix}\right].\begin{pmatrix}2\\-1\\-2\end{pmatrix}=0$$

$$\left[\begin{pmatrix}-2\\6.5\\6\end{pmatrix}+\mu\begin{pmatrix}2\\-1\\-2\end{pmatrix}\right].\begin{pmatrix}2\\-1\\-2\end{pmatrix}=0$$

$$-4-6.5-12+\mu(4+1+4)=0$$

$$\mu=\frac{45}{18}=\frac{5}{2}$$

$$\overrightarrow{ON}=\begin{pmatrix}3\\7\\3\end{pmatrix}+\frac{5}{2}\begin{pmatrix}2\\-1\\-2\end{pmatrix}=\begin{pmatrix}8\\4.5\\-2\end{pmatrix}$$

$$\overrightarrow{ON}=\frac{1}{2}\left(\overrightarrow{OB}+\overrightarrow{OB'}\right)$$

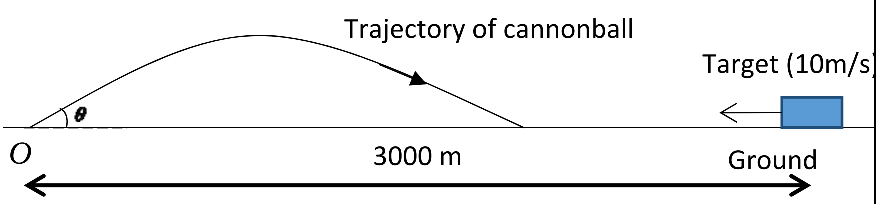
$$\overrightarrow{OB'}=2\begin{pmatrix}8\\4.5\\-2\end{pmatrix}-\begin{pmatrix}5\\0.5\\-3\end{pmatrix}=\begin{pmatrix}11\\8.5\\-1\end{pmatrix}$$

$$\overrightarrow{AB'}=\begin{pmatrix}11\\8.5\\-1\end{pmatrix}-\begin{pmatrix}3\\7\\3\end{pmatrix}=\begin{pmatrix}8\\1.5\\-4\end{pmatrix}//\begin{pmatrix}16\\3\\-8\end{pmatrix}$$

Direction cosines of line *d* are $\frac{16}{\sqrt{329}},\frac{3}{\sqrt{329}}$ and $-\frac{8}{\sqrt{329}}$

	[Alternative to find $\overrightarrow{ON}$] Eqn of this line e : $\mathbf{r}=\begin{pmatrix}3\\7\\3\end{pmatrix}+\mu\begin{pmatrix}2\\-1\\-2\end{pmatrix}$ Eqn of line BN: $\mathbf{r}=\begin{pmatrix}5\\0.5\\-3\end{pmatrix}+\alpha\begin{pmatrix}3\\4\\1\end{pmatrix}$ At N, $\begin{pmatrix}3\\7\\3\end{pmatrix}+\mu\begin{pmatrix}2\\-1\\-2\end{pmatrix}=\begin{pmatrix}5\\0.5\\-3\end{pmatrix}+\alpha\begin{pmatrix}3\\4\\1\end{pmatrix}$ $2\mu-3\alpha=2$ $\mu+4\alpha=6.5$ $2\mu+\alpha=6$ Solving, $\mu=2.5,\alpha=1$ $\overrightarrow{ON}=\begin{pmatrix}3\\7\\3\end{pmatrix}+\frac{5}{2}\begin{pmatrix}2\\-1\\-2\end{pmatrix}=\begin{pmatrix}8\\4.5\\-2\end{pmatrix}$
(v)	Shortest distance from Q to line d $=\left\lvert\dfrac{\overrightarrow{AQ}\times\begin{pmatrix}16\\3\\-8\end{pmatrix}}{\sqrt{329}}\right\rvert$ $=\dfrac{1}{\sqrt{329}}\left\lvert\left[\begin{pmatrix}\frac{25}{2}\\\frac{21}{2}\\-\frac{1}{2}\end{pmatrix}-\begin{pmatrix}3\\7\\3\end{pmatrix}\right]\times\begin{pmatrix}16\\3\\-8\end{pmatrix}\right\rvert$ $=\dfrac{1}{\sqrt{329}}\left\lvert\begin{pmatrix}\frac{19}{2}\\\frac{7}{2}\\-\frac{7}{2}\end{pmatrix}\times\begin{pmatrix}16\\3\\-8\end{pmatrix}\right\rvert=\dfrac{1}{2\sqrt{329}}\left\lvert\begin{pmatrix}19\\7\\-7\end{pmatrix}\times\begin{pmatrix}16\\3\\-8\end{pmatrix}\right\rvert$

	$= \dfrac{1}{2\sqrt{329}} \left\| \begin{pmatrix} -35 \\ 40 \\ -55 \end{pmatrix} \right\| = 2.11 \text{ (3 s.f.)}$

Qn	Solution
10i)	To determine range of cannonball, we consider $y = 0$: $0 = (v\sin\theta)t - 5t^2$ $0 = t\left[v\sin\theta - 5t\right]$ $\therefore t = 0 \quad \text{or} \quad v\sin\theta - 5t = 0$ $\therefore 5t = v\sin\theta$ $\therefore t = \dfrac{v\sin\theta}{5}$ When $t = \dfrac{v\sin\theta}{5}$, $x = (v\cos\theta)t$ $= (v\cos\theta)\dfrac{v\sin\theta}{5}$ $= \dfrac{v^2\sin\theta\cos\theta}{5} \qquad \therefore d = \dfrac{v^2\sin\theta\cos\theta}{5}$
ii)	 Time taken for cannonball to hit the ground = time taken for the target to reach the point of impact of the cannonball. $\dfrac{v\sin\theta}{5} = \dfrac{3000-d}{10}$ $2v\sin\theta = 3000 - \dfrac{v^2\sin\theta\cos\theta}{5}$ $\dfrac{(200)^2\sin\theta\cos\theta}{5} + 400\sin\theta = 3000$ Possible angles are $22.7°$ (to 1 dp) or $69.5°$ (to 1 dp). (shown)
iii)	Since $t = \dfrac{v\sin\theta}{5}$ when cannon hits target and $\dfrac{v\sin 22.7°}{5} < \dfrac{v\sin 69.5°}{5}$ Therefore to hit target earlier, cannonball should be fired at $22.7°$

iv)	$x=(200\cos 22.7°)t \qquad y=(200\sin 22.7°)t-5t^2$ $\frac{dx}{dt}=184.51 \qquad \frac{dy}{dt}=77.181-10t$ $\therefore \frac{dy}{dx}=\frac{77.181-10t}{184.51}$ When $x=370, \quad 184.51t=370 \Rightarrow t=2.0053$ $\therefore \frac{dy}{dx}=\frac{77.181-10(2.0053)}{184.51}=0.30962$ Let the required angle be α . $\tan\alpha=0.30962 \Rightarrow \alpha=17.2°$ (to 1dp)

Qn	**Solution**
11ai	$a+b+c=25000$ ----------(1) $0.02a+0.06b-0.02c=0$ ----------(2) [or $1.02a+1.06b+0.98c=25000$] Solving SLE, $a=37500-2c$ $b=c-12500$
ii	Since a and b must both be positive, it implies that: c must lie between 12500 and 18750.
bi	Since Mr Lee invested in a period of five years, the average return per year will be 1.6%. Total amount of interest earned $=(1.016)^5(55000)-55000$ $=4543$ (to the nearest dollar)
ii	Amount of money in the normal savings account at the end of n years $=1000(1.0019+1.0019^2+1.0019^3+...+1.0019^n)$ $=1000(1.0019)\left(\frac{1.0019^n-1}{1.0019-1}\right)$ $=527315.79(1.0019^n-1)$ Amount of money in the special savings account at the end of n years $=10000(1.018)\left(\frac{1.018^n-1}{1.018-1}\right)$ $=565555.56\left(1.018^n-1\right)$
iii	Total interest earned from dual-savings account $=527315.79(1.0019^n-1)+565555.56\left(1.018^n-1\right)-11000n$

	$527315.79(1.0019^{n}-1)+565555.56\left(1.018^{n}-1\right)-11000n>4543$ From GC, $n \geq 7$ Least value of n is 7.