Name	( )	Class	

RIVER VALLEY HIGH SCHOOL
2017 Year 6 Preliminary Examination II
Higher 2

FURTHER MATHEMATICS 9649/01

Paper 1

14 Sep 2017

3 hours

Additional Materials: Answer Paper
List of Formulae (MF26)
Cover Page

READ THESE INSTRUCTIONS FIRST

Do not open this booklet until you are told to do so.
Write your name, class and index number in the space at the top of this page.

Write your name and class on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphic calculator.
Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, place the cover page on top of your answer paper and fasten all your work securely together.

The number of marks is given in brackets [ ] at the end of each question or part question.

1. Prove by mathematical induction that $8^{2n} - 3(7^n) + 2$ is divisible by 9 for all positive integer n. [5]

2. A curve C is defined parametrically by

$$x = \sin 2t, \quad y = \cos 2t - \ln(\cot t), \quad \text{where } 0 < t < \frac{\pi}{2}.$$

(i) Show that $\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 = 4\cot^2 2t$. [3]

(ii) The arc of C joining the point where $t = \frac{\pi}{12}$ to the point where $t = \frac{\pi}{6}$ is rotated completely about the y-axis. Find the exact value of the surface area generated in terms of π. [3]

3. Show that the general solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 2xy - 2x\left(x^2 + 1\right) = 0$$

can be expressed in the form $y = x^2 + Ce^{-x^2}$, where C is an arbitrary constant. [4]

Deduce, with reasons, the number of stationary points of the solution curves of the equation when

(i) $C \le 1$;

(ii) $C > 1$. [3]

4. A discrete random variable X is said to have a Poisson distribution Po(λ), where $\lambda > 0$, if it has probability distribution function $\mathrm{P}(X = x) = \dfrac{\mathrm{e}^{-\lambda}\lambda^{x}}{x!}$, for $x = 0, 1, 2, \ldots$

Given that $\mathrm{P}(X \le 1) = 0.35093$, correct to 5 decimal places, write down an equation in terms of λ. [1]

Show that the equation has a root, α, in the interval (2, 2.5). [1]

A recurrence relation

$$\lambda_0 = 2 \text{ and } \lambda_{n+1} = \mathrm{F}\left(\lambda_n\right) \text{ for } n \ge 0$$

is used to generate a sequence of approximations $\left\{\lambda_n\right\}$ to find α to 3 decimal places.

(i) Explain, with the aid of a diagram, why $\mathrm{F}\left(\lambda\right) = 0.35093\mathrm{e}^{\lambda} - 1$ would not be appropriate in finding α. [2]

(ii) Show that $\mathrm{F}\left(\lambda\right)$ can be expressed as $\mathrm{F}\left(\lambda\right) = \ln\left(\dfrac{c+\lambda}{d}\right)$, where c and d are constants to be determined. Using these values of c and d, find the value of α to 3 decimal places. Demonstrate how the correctness of this value can be verified. [3]

5. **Do not use a calculator to answer this question.**

(a) Show that if $z = \mathrm{e}^{\mathrm{i}\theta}$ then for all integers n,

$$\tan\left(n\theta\right) = \frac{1 - z^{2n}}{1 + z^{2n}}\mathrm{i}\,.$$

Use this result to prove the identity

$$\tan\left(2\theta\right) \equiv \frac{2\tan\left(\theta\right)}{1 - \tan^{2}\left(\theta\right)}\,. \quad [4]$$

(b) (i) Find the roots of $w^3 = -1$, leaving your answer in the form $r\mathrm{e}^{\mathrm{i}\theta}$ where $r > 0$ and $-\pi < \theta \le \pi$. [2]

(ii) Hence, or otherwise, find the exact value of

$$w + w^2 + w^3 + \ldots + w^{99}$$

where $\arg(w) < 0$. [3]

6. The complex numbers z and w are such that

$$|z+3\mathrm{i}|=2 \text{ and } \arg(w-a)=\theta$$

where a is a complex number and $-\pi<\theta\le\pi$.

(i) Given $\theta=\frac{\pi}{2}$, find the set of values of a for which the loci of z and w have exactly one point of intersection. [3]

(ii) It is instead given $\theta=-\frac{\pi}{6}$ and a = 1. By sketching the loci for z and w, find the range of values for $\arg(w-z)$. [5]

7. (i) A point P lies on a hyperbola H. The difference of the distances of P from the foci $(0,c)$ and $(0,-c)$ is $2a$. Find the equation of the hyperbola H, in terms of a and c. [2]

(ii) Given that the line $y = mx + d$ is a tangent to H, show that

$$\left(c^2-a^2\right)m^2+d^2-a^2=0. \quad [3]$$

(iii) The locus of points at which the perpendicular tangents to a hyperbola meet is called its *orthoptic*. Find the equation of the orthoptic of H. Deduce that the orthoptic of H exists only if $-\sqrt{2}a<c<\sqrt{2}a$. [4]

8. The curves C_1 and C_2 have equations

$$C_1: \ r = a(1+\cos\theta),$$
$$C_2: \ r = \frac{9a}{4(1+\cos\theta)},$$

where $a > 0$ is a constant and $-\pi < \theta \le \pi$.

(i) Find the polar coordinates of the points on C_1 that have tangents parallel to the initial line. [3]

(ii) Find the exact length of C_1 between $\theta = -\frac{\pi}{3}$ and $\theta = \frac{\pi}{3}$ that lies entirely in the first and fourth quadrants. [3]

(iii) The region R bounded by C_1, C_2 and the initial line lies entirely in the first quadrant. Using trapezium rule with 4 stripes, find an approximation to the area of R, expressing your answer in the form ka^2 where k is a constant to be determined. Give your answer for k to 4 decimal places. [4]

9. (a) A set of equations is given by

$$x - 2y + 2z = 2$$
$$4x - 7y + \lambda z = 5$$
$$3x + \lambda y - 7z = 3$$

Determine the value(s) of the constant λ so that the set of equations has

(i) no solution,
(ii) exactly one solution. [4]

[Question 9 continues on next page.]

(b) The linear transformation $\mathrm{T} : \mathbb{R}^4 \to \mathbb{R}^4$ is represented by the matrix $\mathbf{M}$ where

$$\mathbf{M} = \begin{pmatrix} 1 & -2 & 2 & 2 \\ 4 & -7 & -5 & 5 \\ 3 & -5 & -7 & 3 \end{pmatrix}.$$

The range space of T is denoted by R. The set $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for R.

(i) Find the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ which are of the form $\begin{pmatrix} 1 \\ 4 \\ p \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ q \end{pmatrix}$ where p and q are integers to be determined. [3]

(ii) W is the vector space spanned by the vector $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$.

Find a basis for the vector space $R \cap W$.
Giving clear reasons, what can you say about the relationship between the vector spaces W and R? [2]

(iii) Find the set of vectors $\mathbf{x}$ such that $\mathbf{Mx} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and state with justification, whether this set forms a vector space. [3]

10 Two scientists A and B are investigating the change in population of a certain species of fish. At the end of the year 2013, the scientists started tracking the population of the species of fish and found it to be b thousands. The term u_n denotes the population of the species of fish, in thousands at the end of the nth year, starting from 2014.

(a) Scientist A suggests that the fish population grows at a natural rate of 10% over the course of a year, every year. Commercial fishing activity on the species of fish, leads to the depletion of p thousands of the population of the species of fish annually. It is assumed that for every year, the depletion to the population of the species of fish occurs at the start of the year.

(i) Write down a recurrence relation for u_n and solve it, in terms of b and p. [4]

(ii) Explain with clear reasoning, whether the population of the species of fish will stabilize in the long run when $b = 6600$ and $p = 100$. [2]

(iii) State with clear reasoning, a relationship between b and p, such that the commercial fishing activity is sustainable in the long run. [1]

(b) Scientist B suggests that if all commercial fishing activities on the species of fish cease, the number of fish at the end of a particular year can be modelled by adding 110% of the number of fish at the end of the previous year with 12% of the number of fish that were present at the end of the year before that.
Scientist B estimates the number of fish at the end of the year, in 2013 and 2014 to be 6 600 000 and 8 700 000 respectively.

(i) Verify that the number of fish at the end of 2015 is 10 362 000. [1]

(ii) Based on Scientist B's findings, write down a recurrence relation for u_n and solve it. [5]

11. The population of cats, x, on an island at time t years is modelled by the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = ax(1-bx),$$

where a and b are positive constants.

(i) State the meanings of a and b in the context of the question. [2]

(ii) With the aid of a suitable diagram, explain the behaviour of the population of cats over time with different initial values. [4]

It is given that the values of a and b are 0.2 and 0.0005 respectively. At the start of a research on the population of cats on the island, it was observed that there were 380 of them.

(iii) Copy and complete the table, showing the use of the improved Euler method with step size 0.5 to estimate the population of cats on the island one year into the research. [4]

t	x	$\frac{\mathrm{d}x}{\mathrm{d}t}$	$\tilde{x}$	$\frac{\Delta x}{\Delta t}$
0	380	61.56	410.78	63.4210
0.5	411.7105	65.3915		
1				

One year into the research, the cats are hunted by predators newly introduced into their habitat. The predators prey on the cats at a rate equals to k% of the population of cats present.

(iv) Using the values of a and b given above, find the range of values of k such that the cats on the island will become extinct. [2]

(v) For $k = 30$, sketch the graph of x against t for $t \geq 0$. [2]

– End of Paper –