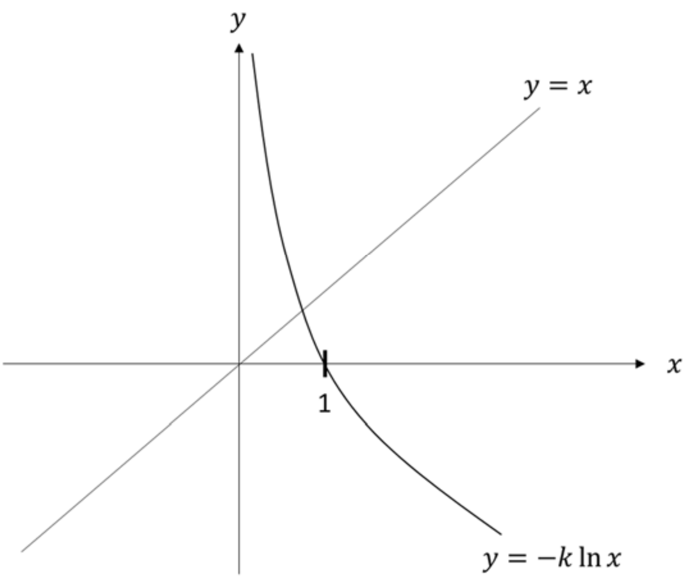
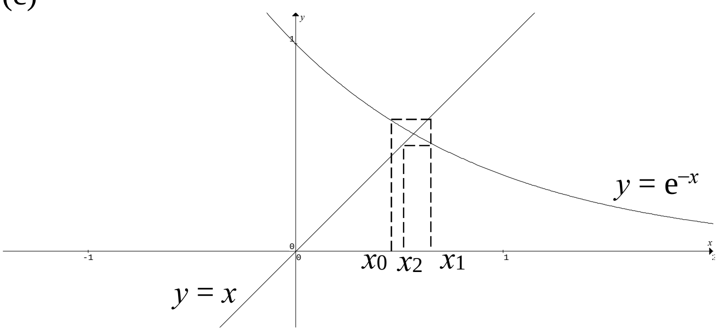
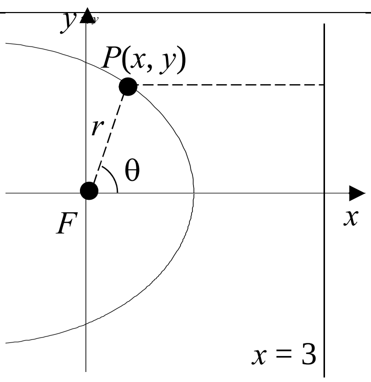
Further Maths Prelim Paper 2 Solution

1.	Let P_n be the proposition that $(x+1)^n - nx - 1$ is divisible by x^2, for $n \in □$, $n \geq 2$. P_2: $(x+1)^2 - 2x - 1 = x^2 + 2x + 1 - 2x - 1 = x^2$ $\therefore (x+1)^2 - 2x - 1$ is divisible by x^2, i.e. P_2 is true Assume P_k is true for some $k \in □$, $k \geq 2$, i.e. $(x+1)^k - kx - 1 = mx^2$ where *m* is some polynomial $m = a_1x^{k-2} + a_2x^{k-3} + \cdots + a_{k-2}x^1 + a_{k-1}$, with $a_1, a_2, \cdots, a_{k-1}$ as constants P_{k+1}: $(x+1)^{k+1} - (k+1)x - 1$ $= (x+1)^k(x+1) - kx - x - 1$ $= (x+1)\left[(x+1)^k - 1\right] - kx$ $= (x+1)\left[\left(mx^2 + kx + 1\right) - 1\right] - kx$ $= (x+1)mx^2 + (x+1)kx - kx$ $= (x+1)mx^2 + kx^2$ $= \left[(x+1)m + k\right]x^2$ $\therefore (x+1)^{k+1} - (k+1)x - 1$ is divisible by x^2, i.e. P_{k+1} is true if P_k is true. Since P_2 is true and P_{k+1} is true if P_k is true, by mathematical induction, P_n is true, i.e. $(x+1)^n - nx - 1$ is divisible by x^2 for all $n \in □$, $n \geq 2$.
2(i)	$x = -k \ln x$  From the graph, the graphs $y = x$ and $y = -k\ln x$ intersect only once. Hence there is exactly one real root.

	Let $\mathrm{f}(x)=x+k\ln x$ $\mathrm{f}\left(\frac{1}{2}\right)=\frac{1}{2}+k\ln\frac{1}{2}<0 \quad (\because k>\frac{1}{2\ln 2})$ $\mathrm{f}(1)=1+k\ln 1>0$ Since f (x) is continuous on $\left[\frac{1}{2},1\right]$ and $\mathrm{f}\left(\frac{1}{2}\right)\mathrm{f}(1)<0$, by Intermediate Value Theorem, there exists a real root α in the interval $\left(\frac{1}{2},1\right)$.
(ii)	(a) $x+\ln x=0$ $x=-\ln x$ $x_1=0.6931,\ x_2=0.3665$ (4 d.p.) If the iteration continues, $x_3=1.0037,\ x_4=-0.037<0$ and the iteration cannot continue since $\ln x$ is not defined for $x<0$.
	(b) $-x=\ln x$ $x=\mathrm{e}^{-x}\Rightarrow \mathrm{F}(x)=\mathrm{e}^{-x}$ By G.C., $\alpha=0.567$ (3 d.p.)
	(c) 
3(i)	 **Method 1:** $r=\frac{0.5\times 3}{1+0.5\cos\theta}$ $=\frac{1.5}{1+0.5\cos\theta}$

Method 2:

$$\frac{r}{3-r\cos\theta}=\frac{1}{2}$$

$$2r=3-r\cos\theta$$

$$r(2+\cos\theta)=3$$

$$r=\frac{3}{2+\cos\theta}$$

(ii)

Method 1:

Since $0 < e < 1$, the conic is an ellipse.

When $\theta=0, r=1=a-c$.

When $\theta=\pi, r=3=a+c$.

Solving gives $a=\frac{1+3}{2}=2, c=\frac{3-1}{2}=1$.

$$a^2=b^2+c^2$$

$$2^2=b^2+1^2$$

$$b^2=4-1=3$$

Cartesian equation is $\frac{(x+1)^2}{4}+\frac{y^2}{3}=1$

Method 2:

Since $0 < e < 1$, the conic is an ellipse.

When $\theta=0, r=1$. When $\theta=\pi, r=3$.

Centre of ellipse = $\left(\frac{1-3}{2},0\right)=(-1,0)$.

$$2a=1+3\Rightarrow a=2$$

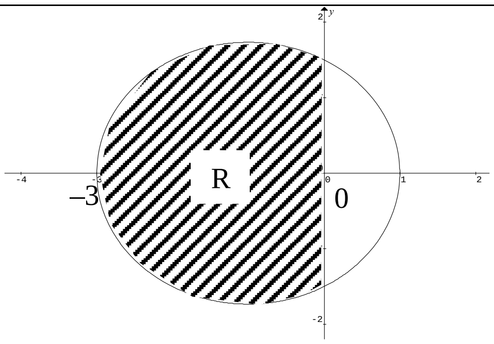
$$a^2=b^2+c^2$$

$$2^2=b^2+1^2$$

$$b^2=4-1=3$$

Cartesian equation is $\frac{(x+1)^2}{4}+\frac{y^2}{3}=1$

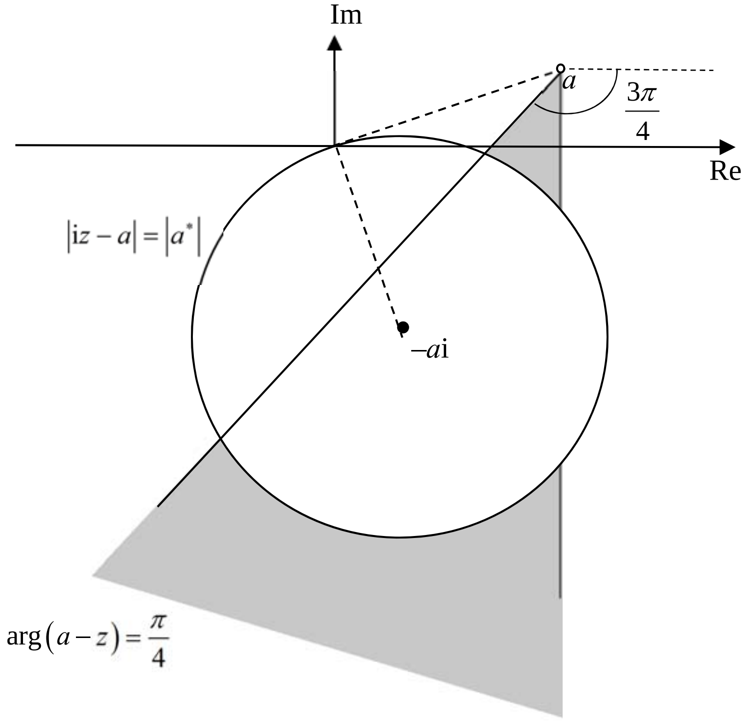
(iii)



$$\frac{(x+1)^2}{4}+\frac{y^2}{3}=1$$

$$y^2=3-\frac{3(x+1)^2}{4}$$

	Volume $= 2\left\|\int_{-3}^{0} 2\pi xy \ \mathrm{d}x\right\|$ $= 4\pi\int_{-3}^{0} -x\sqrt{3-\frac{3(x+1)^2}{4}} \ \mathrm{d}x$ $= 2\pi\sqrt{3}\int_{0}^{-3} x\sqrt{4-(x+1)^2} \ \mathrm{d}x$ $\therefore k = 2\pi\sqrt{3}$. Let $x+1 = 2\sin t \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t} = 2\cos t$ When $x=-3, t=-\frac{\pi}{2}$ When $x=0, t=\frac{\pi}{6}$ Volume $= 2\pi\sqrt{3}\int_{\pi/6}^{-\pi/2}\left(2\sin t-1\right)\sqrt{4-4\sin^2 t} \ 2\cos t \ \mathrm{d}t$ $= 8\pi\sqrt{3}\int_{-\pi/2}^{\pi/6}\left(1-2\sin t\right)\cos^2 t \ \mathrm{d}t$ $= 8\pi\sqrt{3}\int_{-\pi/2}^{\pi/6}\cos^2 t \ \mathrm{d}t - 16\pi\sqrt{3}\int_{-\pi/2}^{\pi/6}\sin t\cos^2 t \ \mathrm{d}t$ $= 4\pi\sqrt{3}\int_{-\pi/2}^{\pi/6} 1+\cos 2t \ \mathrm{d}t + 16\pi\sqrt{3}\left[\frac{\cos^3 t}{3}\right]_{-\pi/2}^{\pi/6}$ $= 4\pi\sqrt{3}\left[t+\frac{\sin 2t}{2}\right]_{-\pi/2}^{\pi/6} + \frac{16\pi\sqrt{3}}{3}\left[\frac{3\sqrt{3}}{8}-0\right]$ $= 4\pi\sqrt{3}\left[\frac{\pi}{6}+\frac{\sqrt{3}}{4}+\frac{\pi}{2}\right]+6\pi$ $= 4\pi\sqrt{3}\left[\frac{2\pi}{3}+\frac{\sqrt{3}}{4}\right]+6\pi$ $= \frac{8\pi^2}{\sqrt{3}}+9\pi$
4a)(i)	$\left\|\mathrm{i}z-a\right\| = \left\|a^*\right\| \quad \Rightarrow \left\|\mathrm{i}\right\|\left\|z+a\mathrm{i}\right\| = \left\|a\right\| \ \Rightarrow \left\|z-\left(-a\mathrm{i}\right)\right\| = \left\|a\right\|$
a)(ii)	$\arg\left(a-z\right) = \frac{\pi}{4} \Rightarrow \arg\left(-1\right)+\arg\left(z-a\right) = \frac{\pi}{4}$ $\Rightarrow \arg\left(z-a\right) = \frac{\pi}{4}-\pi = -\frac{3\pi}{4}$



b	$C=\sum_{k=0}^{11}(-1)^k\cos(2k+1)\theta$ $S=\sum_{k=0}^{11}(-1)^k\sin(2k+1)\theta$ $C+\mathrm{i}S=\sum_{k=0}^{11}(-1)^k\,\mathrm{e}^{(2k+1)\theta\mathrm{i}}$ $=\frac{\mathrm{e}^{\theta\mathrm{i}}\left(1-\left(-\mathrm{e}^{2\theta\mathrm{i}}\right)^{12}\right)}{1-\left(-\mathrm{e}^{2\theta\mathrm{i}}\right)}$ $=\frac{\mathrm{e}^{\theta\mathrm{i}}\left(1-\mathrm{e}^{24\theta\mathrm{i}}\right)}{1+\mathrm{e}^{2\theta\mathrm{i}}}$ $=\frac{\mathrm{e}^{\theta\mathrm{i}}\mathrm{e}^{12\theta\mathrm{i}}\left(\mathrm{e}^{-12\theta\mathrm{i}}-\mathrm{e}^{12\theta\mathrm{i}}\right)}{\mathrm{e}^{\theta\mathrm{i}}\left(\mathrm{e}^{-\theta\mathrm{i}}+\mathrm{e}^{\theta\mathrm{i}}\right)}$ $=\frac{\mathrm{e}^{12\theta\mathrm{i}}\left(-2\mathrm{i}\sin12\theta\right)}{2\cos\theta}=-\frac{\mathrm{e}^{12\theta\mathrm{i}}\left(\sin12\theta\right)\mathrm{i}}{\cos\theta}$ Taking the imaginary part, $S=-\frac{\cos12\theta\sin12\theta}{\cos\theta}$ $=-\frac{1}{2}\sin24\theta\sec\theta$ $a=-\frac{1}{2}\quad b=1\quad c=24$

5(a)	(i) First of all, $y=0$ satisfies the differential equation, hence S contains the zero-vector. Let $y_1, y_2 \in S$, i.e. $\frac{\mathrm{d}^2 y_1}{\mathrm{d}x^2}+5\frac{\mathrm{d}y_1}{\mathrm{d}x}+6y_1=0$ & $\frac{\mathrm{d}^2 y_2}{\mathrm{d}x^2}+5\frac{\mathrm{d}y_2}{\mathrm{d}x}+6y_2=0$, $\frac{\mathrm{d}^2}{\mathrm{d}x^2}(y_1+y_2)+5\frac{\mathrm{d}}{\mathrm{d}x}(y_1+y_2)+6(y_1+y_2)$ $=\left(\frac{\mathrm{d}^2 y_1}{\mathrm{d}x^2}+5\frac{\mathrm{d}y_1}{\mathrm{d}x}+6y_1\right)+\left(\frac{\mathrm{d}^2 y_2}{\mathrm{d}x^2}+5\frac{\mathrm{d}y_2}{\mathrm{d}x}+6y_2\right)$ $=0$ So $y_1+y_2\in S$ Let $y\in S$, i.e. $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}+5\frac{\mathrm{d}y}{\mathrm{d}x}+6y=0$, $\frac{\mathrm{d}^2}{\mathrm{d}x^2}(ky)+5\frac{\mathrm{d}}{\mathrm{d}x}(ky)+6(ky)$ $=k\left(\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}+5\frac{\mathrm{d}y}{\mathrm{d}x}+6y\right)$ $=0$ So $ky\in S$ Hence, S is a subspace of C.
	(ii) $\lambda^2+5\lambda+6=0$ $(\lambda+2)(\lambda+3)=0$ $\lambda=-2$ or $\lambda=-3$ Hence, $y=A\mathrm{e}^{-2x}+B\mathrm{e}^{-3x}$. The basis is $\left\{\mathrm{e}^{-2x},\mathrm{e}^{-3x}\right\}$.
	(iii) $y=0$ is not a solution of the differential equation. Since zero-vector is not in the solution set, the solution set is not a subspace of C.
(b)	(i) $T\left(k\begin{pmatrix}a\\b\\c\\d\end{pmatrix}\right)=T\left(\begin{pmatrix}ka\\kb\\kc\\kd\end{pmatrix}\right)$ $=ka+kb+kc+kd$ $=k(a+b+c+d)$ $=kT\left(\begin{pmatrix}a\\b\\c\\d\end{pmatrix}\right)$ ---(1)

	$T\left(\begin{pmatrix}a_1\\b_1\\c_1\\d_1\end{pmatrix}+\begin{pmatrix}a_2\\b_2\\c_2\\d_2\end{pmatrix}\right)=T\left(\begin{pmatrix}a_1+a_2\\b_1+b_2\\c_1+c_2\\d_1+d_2\end{pmatrix}\right)$ $=a_1+a_2+b_1+b_2+c_1+c_2+d_1+d_2$ $=(a_1+b_1+c_1+d_1)+(a_2+b_2+c_2+d_2)$ $=T\left(\begin{pmatrix}a_1\\b_1\\c_1\\d_1\end{pmatrix}\right)+T\left(\begin{pmatrix}a_2\\b_2\\c_2\\d_2\end{pmatrix}\right)---(2)$ Hence, T is a linear transformation.
	(ii) T is represented by $\begin{pmatrix}1 & 1 & 1 & 1\end{pmatrix}$. $\begin{pmatrix}1 & 1 & 1 & 1\end{pmatrix}\begin{pmatrix}a\\b\\c\\d\end{pmatrix}=0$ $a+b+c+d=0$ $\begin{pmatrix}a\\b\\c\\d\end{pmatrix}=\begin{pmatrix}-b-c-d\\b\\c\\d\end{pmatrix}=b\begin{pmatrix}-1\\1\\0\\0\end{pmatrix}+c\begin{pmatrix}-1\\0\\1\\0\end{pmatrix}+d\begin{pmatrix}-1\\0\\0\\1\end{pmatrix}$ Hence, the null space has the basis $\left\{\begin{pmatrix}-1\\1\\0\\0\end{pmatrix},\begin{pmatrix}-1\\0\\1\\0\end{pmatrix},\begin{pmatrix}-1\\0\\0\\1\end{pmatrix}\right\}$.
6(i)	Let μ be the population mean mass of the mini chocolate bars (in grams). $H_0:\ \mu=8.0$ $H_1:\ \mu>8.0$ at 1% level of significance Test statistic $T=\dfrac{\overline{X}-8.0}{S/\sqrt{10}}\sim t(9)$ From GC, $t=3.308>2.539$ p-value $=0.00455<0.01$ $\overline{x}=8.15,\ s=0.14337$ $\therefore$ reject H_0 at 1% level of significance and conclude that there is sufficient evidence to say that the mean mass of the mini chocolate bars exceeds 8 grams.
(ii)	From GC, 98% confidence interval for μ

	$=\left(8.15\pm 2.8214\dfrac{0.14337}{\sqrt{10}}\right)$ $=(8.022, 8.278)$ $=(8.02, 8.28)$
(iii)	Since 8.0 is less than the lower bound, this implies that the alternative hypothesis $\mu > 8.0$ will be accepted at 1% level of significance.
7i)	P(a winner is found after each player tossed once) $=\mathrm{P}\left(\text{one Head and others all Tail}\right)$ $+\mathrm{P}\left(\text{one Tail and others all Head}\right)$ $=\left(\dfrac{1}{2}\left(\dfrac{1}{2}\right)^{n-1}+\dfrac{1}{2}\left(\dfrac{1}{2}\right)^{n-1}\right)n=n\left(\dfrac{1}{2}\right)^{n-1}$ $\mathrm{P}\left(X=r\right)=n\left(\dfrac{1}{2}\right)^{n-1}\left(1-n\left(\dfrac{1}{2}\right)^{n-1}\right)^{r-1}$
ii)a)	Letting $p=\dfrac{1}{2}\left(\dfrac{1}{2}\right)^{n-1}$ and using (i) and (ii), $\mathrm{E}\left(X\right)=\dfrac{1}{n\left(\frac{1}{2}\right)^{n-1}}=\dfrac{2^{n-1}}{n}=\dfrac{1}{8}\left(2^{7}\right)=16$
ii)b)	$\mathrm{P}\left(X=10 \mid X>6\right)$ $=\mathrm{P}\left(X=4\right)$ $=q^{3}p$ $=\left(\dfrac{15}{16}\right)^{3}\dfrac{1}{16}=\dfrac{3375}{65536}$
ii)c)	<table><tr><td>m</td><td>P($X \le m$)</td></tr><tr><td>14</td><td>0.5949 < 0.6</td></tr><tr><td>15</td><td>0.6202 > 0.6</td></tr></table>Largest integer m = 14 **Method 2:** $\mathrm{P}\left(X\le m\right)<0.6$ $p+qp+q^{2}p+\ldots+q^{m-1}p<0.6 \quad (*)$ $\dfrac{p\left(1-q^{m}\right)}{1-q}<0.6$ $1-q^{m}<0.6$ $q^{m}>0.4$ $m<\dfrac{\ln 0.4}{\ln q}=\dfrac{\ln 0.4}{\ln\frac{15}{16}}=14.2$

<table>
<tr><td>8(a)</td><td>$1 = k\int_0^2 5 - x^2 \ dx = k\left[5x - \frac{1}{3}x^3\right]_0^2$
$k = \dfrac{3}{22}$</td></tr>
<tr><td>(b)</td><td>$\int_0^m \frac{3}{22}\left(5 - x^2\right) \ dx = \frac{1}{2}$
$\dfrac{3}{22}\left[5m - \dfrac{m^3}{3}\right] = \dfrac{1}{2}$
$m^3 - 15m + 11 = 0$
Using GC, choices of positive m values are
m = 0.763 or m = 3.43 (reject as $3.34 \notin [0, 2]$)
Median of X is 76.3 hours.</td></tr>
<tr><td>(c)</td><td>P(X > 1.8) = $\int_{1.8}^2 \frac{3}{22}\left(5 - x^2\right) \ dx = 0.03781818$
Let N be the number of observations of X such that X > 1.8 out of 100 observations of X.
N~B(100 , p) where p = P(X > 1.8)
P(5 < N < 12) = P($N \le 11$) − P($N \le 5$) = 0.178</td></tr>
<tr><td>(d)</td><td>The model does not allow for P(X > 2.5) > 0 since P(X > 2) = 0. Thus the model is not suitable.</td></tr>
<tr><td>9(a)</td><td>$\bar{x} = \dfrac{\sum fx}{\sum f} = \dfrac{405}{450} = 0.9$
<table>
<tr><td>No of hits</td><td>0</td><td>1</td><td>2</td><td>3</td><td>4 or more</td></tr>
<tr><td>O_i</td><td>182</td><td>171</td><td>67</td><td>22</td><td>8</td></tr>
<tr><td>E_i</td><td>182.956</td><td>164.661</td><td>74.097</td><td>22.229</td><td>6.056</td></tr>
</table>
Let X denote the number of hits
H_0: X follows Poisson Distribution
H_1: X does not follows Poisson Distribution
Degree of freedom = 5 – 2 = 3
By GC, the p-value = 0.670 > 0.10.
Thus we do not reject H_0 at 10% level of significance, and Poisson distribution is adequate model for the data.</td></tr>
<tr><td>(b)</td><td>The table of expected frequencies is given below.
<table>
<tr><td></td><td>A</td><td>B</td><td>C</td><td>Total</td></tr>
<tr><td>North</td><td>57.96</td><td>58.32</td><td>45.72</td><td>162</td></tr>
<tr><td>East</td><td>53.667</td><td>54</td><td>42.333</td><td>150</td></tr>
<tr><td>South & West</td><td>49.373</td><td>49.68</td><td>38.947</td><td>138</td></tr>
<tr><td>Total</td><td>161</td><td>162</td><td>127</td><td>450</td></tr>
</table>
H_0: Zone is independent of bomb type
H_1: Zone is not independent of bomb type
$\chi^2_{calc} = 7.84$
Degree of freedom = 4</td></tr>
</table>

	p-value = 0.0976 Since p-value > 0.05, we do not reject H_0 at 5% level of significance. Thus the zone and type of bomb are independent of each other.
	Method 1: For H_0 to be rejected, p-value 0.0976 ≤ $\frac{\alpha}{100}$ Smallest integer $\alpha = 10$
	Method 2: Using the χ^2 table values at $\nu = 4$ from MF26, we obtain by linear interpolation the following: $\frac{y - 7.779}{9.488 - 7.779} = \frac{x - 0.9}{0.95 - 0.9}$ $y = 7.779 + \frac{1.709}{0.05}(x - 0.9)$ From the tables in GC, we have <table><tr><th>x</th><th>y</th></tr><tr><td>0.90</td><td>7.779</td></tr><tr><td>0.91</td><td>8.121</td></tr><tr><td>0.92</td><td>8.463</td></tr></table>Comparing $\chi^2 = 7.84$ and the table above, for H_0 to be rejected, minimum value of $\alpha = 10$.
10(i)	Participants may be better or more skillful at a certain task such that they generally complete one task faster than the other. The tasks should be assigned randomly such that around half the participants (7 or 8 people) do task A first while the other half do task B first.
(ii)	Let X and Y be the time taken for the task before and after the exercise respectively, and let $D = X - Y$. Let μ_d and m_d be the population mean and median of D respectively. The hypotheses for the paired sample t-test is on the mean, while those of the Wilcoxon signed rank test is on the median, i.e., For paired sample t-test, $H_0: \mu_d = 0,\ H_1: \mu_d > 0$ For Wilcoxon signed rank test, $H_0: m_d = 0,\ H_1: m_d > 0$ The paired sample t-test requires the difference in time taken to follow a normal distribution.
(a)	Test statistic $T = \frac{\bar{D} - 0}{S/\sqrt{15}} \sim t(14)$ From GC, $t = 1.781 > 1.761$ $p\text{-value} = 0.0483 < 0.05$ $\bar{d} = 1.5333,\ s = 3.33524$ $\therefore$ reject H_0 at 5% level of significance and conclude that there is sufficient evidence to say that the mean time taken for the second task is less than the first task, i.e. the exercise is effective at improving concentration.

(b)	Test statistic T = smaller of sum of ranks corresponding to positive differences or negative differences

x	25	27	24	25	32	19	21	27
y	22	20	25	25	25	17	24	27
d	3	7	-1	0	7	2	-3	0

x	25	23	24	29	26	26	23
y	23	22	24	23	29	22	25
d	2	1	0	6	-3	4	-2

Removing zeroes and arranging by magnitude,

d	-1	1	2	2	-2
	1	2	3	4	5
Signed rank	-1.5	1.5	4	4	-4

d	-3	-3	3	4	6	7	7
	6	7	8	9	10	11	12
Signed rank	-7	-7	7	9	10	11.5	11.5

$T = Q = 1.5 + 4 + 7 + 7 = 19.5 > 17$ (where n = 12)

$\therefore$ do not reject H_0 at 5% level of significance and conclude that there is insufficient evidence to say that the mean time taken for the second task is less than the first task, i.e. the exercise is NOT effective at improving concentration.