# Solutions to 2017 Y6 H2 Maths Preliminary Exam II (Paper 2)

Question 1 [4 Marks]

At point $P$, $x=\cos(p)$, $y=\sin^3(p)$

$$\frac{dy}{dp}=3\cos(p)\sin^2(p)$$

$$\frac{dx}{dp}=-\sin(p)$$

$$\frac{dy}{dx}=-3\sin(p)\cos(p)=\frac{-3}{2}\sin(2p)$$

Let $z=\dfrac{dy}{dx}$

$$\begin{aligned}\frac{dz}{dt}&=\frac{dz}{dp}\cdot\frac{dp}{dt}\\&=-3\cos(2p)\cdot(0.5)\\&=\frac{-3}{2}\cos(2p)\end{aligned}$$

$$\left.\frac{dz}{dt}\right|_{p=\frac{\pi}{3}}=\frac{-3}{2}\cos\left(\frac{2\pi}{3}\right)=0.75$$

Therefore, $\dfrac{dy}{dx}$ is increasing at 0.75 units per second when $p=\dfrac{\pi}{3}$.

Question 2 [6 Marks]

Let the first term be $a$ and the common difference be $d$.

$$\sum_{k=5}^{14}u_k=u_{14}-u_5$$

$$S_{14}-S_4=\left|(a+13d)-(a+4d)\right|$$

$$\frac{14}{2}(2a+13d)-\frac{4}{2}(2a+3d)=|9d|$$

$$14a+91d-4a-6d=|9d|$$

$$10a+85d=|9d|$$

$$10a+85d=9d \quad \text{or} \quad 10a+85d=-9d$$

$$5a+38d=0\text{---- (1)} \quad \text{or} \quad 5a+47d=0\text{---- (1)}$$

$$u_3+u_5+u_{14}=19$$

$$(a+2d)+(a+4d)+(a+13d)=19$$

$$3a+19d=19 \quad \text{----- (2)}$$

Solving simultaneously, from GC,

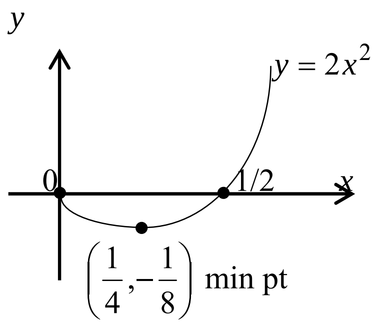
| | | |
|---|---|---|
| | $a=38,\ d=-5$ or $a=\frac{893}{46},\ d=-\frac{95}{46}$<br><br>Hence the common difference is $-5$ or $-\frac{95}{46}$ | |
| | $S_n>0$ or $S_n>0$<br>$\frac{n}{2}\left(2a+(n-1)d\right)>0$ or $\frac{n}{2}\left(2a+(n-1)d\right)>0$<br>$n(81-5n)>0$    $n\left(\frac{1881}{92}-\frac{95}{92}n\right)>0$<br>$0<n<\frac{81}{5}=16.2$ or $0<n<\frac{99}{5}=19.8$<br>Hence, the largest value of $n$ is 16 or 19. | |

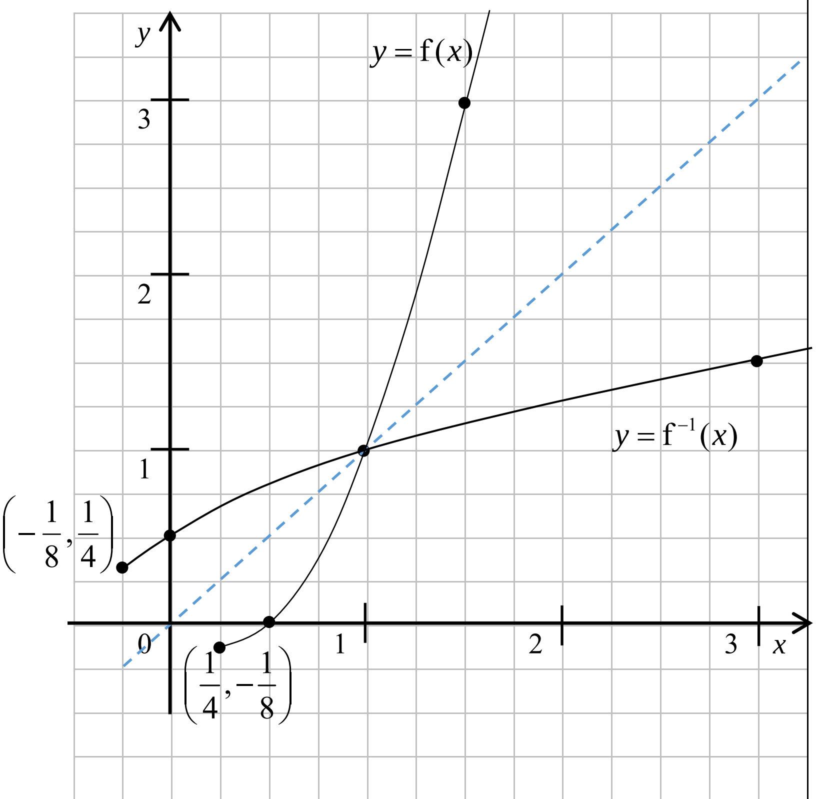
| Question 3 [7 Marks] | | |
|---|---|---|
| (i) | Let $\frac{4r+6}{(r+1)(r+2)(r+3)}=\frac{A}{r+1}+\frac{B}{r+2}+\frac{C}{r+3}$<br>Then by cover up rule, $A=1, B=2, C=-3$<br>Hence, $\frac{4r+6}{(r+1)(r+2)(r+3)}=\frac{1}{r+1}+\frac{2}{r+2}-\frac{3}{r+3}$ | |
| (ii) | $\sum_{r=1}^{n}\frac{4r+6}{(r+1)(r+2)(r+3)}=\sum_{r=1}^{n}\frac{1}{r+1}+\frac{2}{r+2}-\frac{3}{r+3}$<br>$=\frac{1}{2}+\frac{2}{3}-\frac{3}{4}$<br>$+\frac{1}{3}+\frac{2}{4}-\frac{3}{5}$<br>$+\frac{1}{4}+\frac{2}{5}-\frac{3}{6}$<br>$+\frac{1}{5}+\ldots\ldots\quad\ldots\ldots$<br>$+\ldots\ldots\quad\ldots\ldots-\frac{3}{n}$<br>$+\frac{1}{n-1}+\frac{2}{n}-\frac{3}{n+2}$<br>$+\frac{1}{n}+\frac{2}{n+1}-\frac{3}{n+2}$<br>$+\frac{1}{n+1}+\frac{2}{n+2}-\frac{3}{n+3}$<br>$=\frac{1}{2}+\frac{2}{3}+\frac{1}{3}-\frac{3}{n+2}+\frac{2}{n+2}-\frac{3}{n+3}$<br>$=\frac{3}{2}-\left(\frac{1}{n+2}+\frac{3}{n+3}\right)$ | |

| | | |
|---|---|---|
| (iii) | $\frac{3}{1\times2\times3}+\frac{5}{2\times3\times4}+\frac{7}{3\times4\times5}+\frac{9}{4\times5\times6}+\ldots$<br>$=\frac{3}{1\times2\times3}+\frac{1}{2}\left(\frac{10}{2\times3\times4}+\frac{14}{3\times4\times5}+\frac{18}{4\times5\times6}+\ldots\right)$<br>$=\frac{1}{2}+\frac{1}{2}\sum_{r=1}^{\infty}\frac{4r+6}{(r+1)(r+2)(r+3)}$<br>$=\frac{1}{2}+\frac{1}{2}\lim_{n\to\infty}\left(\frac{3}{2}-\left(\frac{1}{n+2}+\frac{3}{n+3}\right)\right)$<br>$=\frac{1}{2}+\frac{1}{2}\times\frac{3}{2}$<br>$=\frac{5}{4}$ | |

| Question 4 [11 Marks] | | |
|---|---|---|
| | $y=\mathrm{e}^{\sqrt{(1-x)^3}}$<br>$\frac{\mathrm{d}y}{\mathrm{d}x}=\mathrm{e}^{\sqrt{(1-x)^3}}\left(\frac{3}{2}\right)(1-x)^{\frac{1}{2}}(-1)=\frac{-3}{2}y\sqrt{1-x}$<br>$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{-3}{2}\frac{\mathrm{d}y}{\mathrm{d}x}\sqrt{1-x}+\frac{-3}{2}y\frac{-1}{2\sqrt{1-x}}=\frac{3y}{4\sqrt{1-x}}-\frac{3\sqrt{1-x}}{2}\frac{\mathrm{d}y}{\mathrm{d}x}$<br>$4\sqrt{1-x}\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=3y-6(1-x)\frac{\mathrm{d}y}{\mathrm{d}x}$<br>Thus,<br>$4\sqrt{1-x}\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+6(1-x)\frac{\mathrm{d}y}{\mathrm{d}x}-3y=0$ (shown) | |
| | When $x=0$,<br>$y=\mathrm{e}$<br>$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-3\mathrm{e}}{2}$<br>$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=3\mathrm{e}$<br>$\mathrm{f}(x)\approx\mathrm{f}(0)+\mathrm{f}'(0)x+\frac{\mathrm{f}''(0)}{2!}x^2$<br>$\approx\mathrm{e}-\frac{3\mathrm{e}}{2}x+\frac{3\mathrm{e}}{2}x^2$ | |

| | | |
|---|---|---|
| | $(1-x)^{\frac{3}{2}} \approx 1-\frac{3}{2}x+\frac{\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)}{2!}(-x)^2$<br>$=1-\frac{3}{2}x+\frac{3}{8}x^2$<br>$e^{\sqrt{(1-x)^3}} \approx e^{1-\frac{3}{2}x+\frac{3}{8}x^2}$<br>$=e\left(e^{-\frac{3}{2}x+\frac{3}{8}x^2}\right)$<br>$\approx e\left(1+\left(-\frac{3}{2}x+\frac{3}{8}x^2\right)+\frac{\left(-\frac{3}{2}x+\frac{3}{8}x^2\right)^2}{2!}\right)$<br>$\approx e\left(1-\frac{3}{2}x+\frac{3}{8}x^2+\frac{9}{8}x^2\right)$<br>$=e\left(1-\frac{3}{2}x+\frac{3}{2}x^2\right)$<br>$=e-\frac{3e}{2}x+\frac{3e}{2}x^2$<br>which is the same as the above series expansion of $f(x)$ | |

| Question 5 [12 Marks] | | |
|---|---|---|
| (i) | As shown in the following sketch:<br><br>Any horizontal line of the form $y=k$ where $-\frac{1}{4}<k\le 0$ will intersect the curve at 2 points.<br>Thus, f is not one-one and hence $f^{-1}$ does not exist. | |
| (ii) | From the sketch of the curve, we deduce that the least value of $k=\frac{1}{4}$ for $f^{-1}$ to exist.<br>Next let $y=2x^2-x$. Then we have<br>$y=2\left(x^2-\frac{1}{2}x\right)$<br>$=2\left(x^2-\frac{1}{2}x+\left(\frac{1}{4}\right)^2-\left(\frac{1}{4}\right)^2\right)$<br>$=2\left(x-\frac{1}{4}\right)^2-\frac{1}{8}$ | |

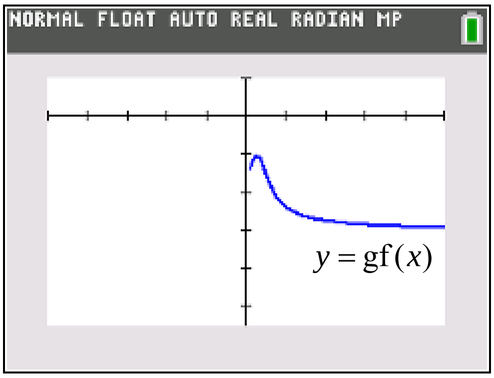
| | | |
|---|---|---|
| | $\left(x-\frac{1}{4}\right)^2=\frac{1}{2}\left(y+\frac{1}{8}\right)$<br>$\Rightarrow x=\frac{1}{4}\pm\sqrt{\frac{8y+1}{16}}=\frac{1}{4}+\frac{\sqrt{8y+1}}{4}$ since $x\geq\frac{1}{4}$<br>Hence, $\mathrm{f}^{-1}:x\mapsto\frac{1+\sqrt{8x+1}}{4},x\geq-\frac{1}{8}$.<br>$D_{\mathrm{f}^{-1}}=R_{\mathrm{f}}=\left[-\frac{1}{8},\infty\right)$ | |
| (iii) | Sketch of $y=\mathrm{f}(x)$ and $y=\mathrm{f}^{-1}(x)$:<br><br> | |
| (iv) | From the sketch in part (iii) we note that to solve the equation $\mathrm{f}(x)=\mathrm{f}^{-1}(x)$, we can also solve $\mathrm{f}(x)=x$<br>Thus, $2x^2-x=x\Rightarrow 2x(x-1)=0$<br>Therefore, in the restricted domain of $x\geq\frac{1}{4}$,<br>the solution is $x=1$. | |
| (v) | For $\mathrm{f}:x\mapsto 2x^2-x,\ x\in\mathbb{R},\ x\geq 0, R_{\mathrm{f}}=\left[-\frac{1}{8},\infty\right)$<br>Also, for $\mathrm{g}:x\mapsto -3+\frac{1}{\sqrt{2x+\frac{1}{2}}},\ x\in\mathbb{R},\ x>-\frac{1}{4}$, | |

$D_g = \left(-\frac{1}{4}, \infty\right)$

Since $R_f \subseteq D_g$, the composite function gf exists.

Then,

$$\text{gf}(x) = -3 + \frac{1}{\sqrt{2(2x^2 - x) + \frac{1}{2}}} = -3 + \frac{1}{\sqrt{4\left(x - \frac{1}{4}\right)^2 + \frac{1}{4}}}$$




Since $D_{gf} = [0, \infty)$ and $\text{gf}\left(\frac{1}{4}\right) = -3 + \frac{1}{\sqrt{\frac{1}{4}}} = -3 + 2 = -1$,

we have $R_{gf} = (-3, -1]$

<u>ALT</u>

$[0, +\infty) \to [-\frac{1}{8}, +\infty) \to (-3, -1]$

$R_{gf} = (-3, -1]$

<table>
<tr><td colspan="3">Question 6 [8 Marks]</td></tr>
<tr><td>(i)</td><td>$(8-1)! = 5040$</td><td></td></tr>
<tr><td>(ii)</td><td>No. of ways with \$0 and \$5 segments adjacent<br>$= (7-1)!2!$<br>$= 1440$<br>No. of ways without identical segments adjacent<br>= total no. of ways − no. of ways with identical segments adjacent<br>$= 5040 - 1440$<br>$= 3600$</td><td></td></tr>
<tr><td>(iii)</td><td><u>Case 1: no segment separating them</u><br>$(7-1)!2! = 1440$</td><td></td></tr>
</table>

| | | |
|---|---|---|
| | Case 2: exactly 1 segment separating them<br>$\binom{6}{1}2!(6-1)!=1440$<br><br>Total number of ways = $5040-1440-1440$<br>= 2160<br><br>ALT<br><br>Case 1: exactly 2 segments separating them<br>$\binom{6}{2}2!2!(5-1)!=1440$<br>Case 2: exactly 3 segments separating them<br>$\dfrac{\binom{6}{3}3!2!(4-1)!}{2}=720$<br><br>Therefore, total number of ways = 2160 | |
| (iv) | The segments are \$0, \$0, \$0, \$5, \$10, \$15, \$20, \$25<br>$\dfrac{(8-1)!}{3!}=840$ | |
| (v) | Arrange the other 5 objects in (5 – 1)! = 24 ways<br>Choose 3 spaces for the \$0 in ${}^5C_3$ = 10 ways<br>Total = 240 ways | |

| Question 7 [10 Marks] | | |
|---|---|---|
| (i) | Probability distribution for *A*: | |

| $a$ | 0 | 1 | 2 |
|---|---|---|---|
| $P(A = a)$ | 5/8 | 2/8 | 1/8 |

Probability distribution for *D*:

| $d$ | 0 | 1 | 2 |
|---|---|---|---|
| $P(D = d)$ | 2/8 | 4/8 | 2/8 |

$$E(A)=\left(\frac{5}{8}\right)(0)+\left(\frac{2}{8}\right)(1)+\left(\frac{1}{8}\right)(2)=\frac{1}{2}$$

$$E(D)=\left(\frac{2}{8}\right)(0)+\left(\frac{4}{8}\right)(1)+\left(\frac{2}{8}\right)(2)=1$$

$$E(A-D)=E(A)-E(D)=\frac{-1}{2}$$

$$E(A^2)=\left(\frac{5}{8}\right)(0)^2+\left(\frac{2}{8}\right)(1)^2+\left(\frac{1}{8}\right)(2)^2=\frac{3}{4}$$

$$E(D^2)=\left(\frac{2}{8}\right)(0)^2+\left(\frac{4}{8}\right)(1)^2+\left(\frac{2}{8}\right)(2)^2=\frac{3}{2}$$

| | | |
|---|---|---|
| | $\text{Var}(A)=\text{E}(A^2)-\text{E}(A)^2=\frac{3}{4}-\left(\frac{1}{2}\right)^2=\frac{1}{2}$<br>$\text{Var}(D)=\text{E}(D^2)-\text{E}(D)^2=\frac{3}{2}-1^2=\frac{1}{2}$<br>$\text{Var}(A-D)=\text{Var}(A)+\text{Var}(D)=\frac{1}{2}+\frac{1}{2}=1$ | |
| (ii) | Probability distribution for *X*: | |
| (iii) | If the score on the defence die is more than the score on the attack die, the damage dealt will be zero. So even though sometimes $A - D$ will be less than zero, that is never considered when dealing damage. Hence, the expected damage must be greater than zero. | |

Probability distribution for *X* (part (ii)):

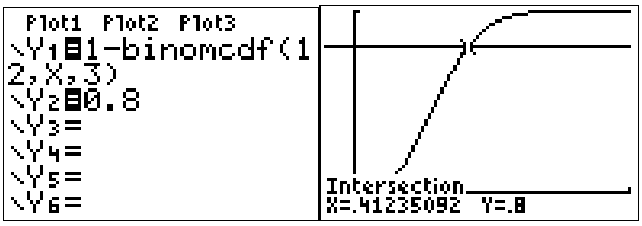
| $x$ | 0 | 1 | 2 |
|---|---|---|---|
| $\text{P}(X=x)$ | $\text{P}(A=0)$ $+\text{P}(A=1)\text{P}(D\geq 1)$ $+\text{P}(A=2)\text{P}(D=2)$ $=\frac{27}{32}$ | $\text{P}(A=1)\text{P}(D=0)$ $+\text{P}(A=2)\text{P}(D=1)$ $=\frac{1}{8}$ | $\text{P}(A=2)\text{P}(D=0)$ $=\frac{1}{32}$ |

$$\text{E}(X)=\left(\frac{27}{32}\right)(0)+\left(\frac{1}{8}\right)(1)+\left(\frac{1}{32}\right)(2)=\frac{3}{16}$$

$$\text{E}(X^2)=\left(\frac{27}{32}\right)(0)^2+\left(\frac{1}{8}\right)(1)^2+\left(\frac{1}{32}\right)(2)^2=\frac{1}{4}$$

$$\text{Var}(X)=\frac{1}{4}-\left(\frac{3}{16}\right)^2=\frac{55}{256}$$

| Question 8 [9 marks] | | |
|---|---|---|
| (i) | The 2 assumptions needed for *X* to be well modelled by a binomial distribution are as follow:<br>1. The occupancy of any particular parking lot in the car park is *independent* of that of another lot.<br>2. The probability of a parking lot being occupied in a day is *constant* for all the car park lots in the car park. | |
| (ii) | Since for 80% of the days in the survey period, there are at least 4 occupied lots for each day, we can infer that $P(X\geq 4)=1-P(X\leq 3)=0.8$ for $X\sim \text{B}(12, p)$.<br><br>We then use GC to plot the graph involving binomial cdf and determine the *x* coordinate of the intersection of the curve and the line $y = 0.8$ as shown below: | |




Hence, the value of $p$ is 0.412 (3 s.f.)

(iii) Let $X \sim \mathrm{B}(12, p)$

The required conditional probability, f($p$)

$$= \mathrm{P}(2 \le X < 4 \mid X \ge 1)$$

$$= \frac{\mathrm{P}(X = 2 \text{ or } X = 3)}{\mathrm{P}(X \ge 1)}$$

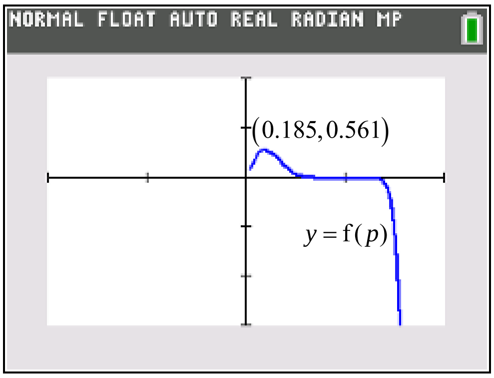
$$= \frac{\mathrm{P}(X = 2 \text{ or } X = 3)}{1 - \mathrm{P}(X = 0)}$$

$$= \frac{\binom{12}{2} p^2 (1-p)^{10} + \binom{12}{3} p^3 (1-p)^9}{1 - \left[\binom{12}{0} p^0 (1-p)^{12}\right]}$$

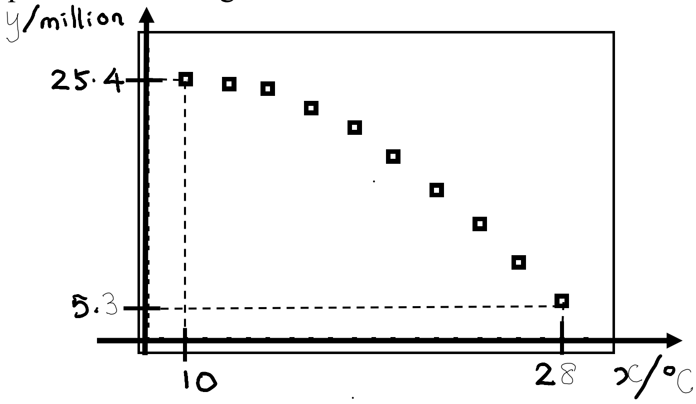
$$= \frac{66 p^2 (1-p)^{10} + 220 p^3 (1-p)^9}{1-(1-p)^{12}}$$

$$= \frac{22 p^2 (1-p)^9 \left[3(1-p) + 10p\right]}{1-(1-p)^{12}}$$

$$= \frac{22 p^2 (1-p)^9 (3+7p)}{1-(1-p)^{12}}. \quad \text{(Shown)}$$




$p \approx 0.185$ give the maximum probability.

| Question 9 [9 Marks] | | |
|---|---|---|
| (i) | The required scatter diagram is as shown below:<br><br> | |
| (ii) | From GC, the correlation coefficient $r = -0.973$.<br>Although the value of $r$ is close to $-1$ and suggests a strong negative linear relationship between $x$ and $y$, the scatter diagram shows a curvilinear relationship between $x$ and $y$.<br>Thus, the a linear relationship between $x$ and $y$ is not appropriate. | |
| (iii) | The scatter diagram shows that when $x$ increases, $y$ decreases at increasing rate. Thus, the model with $y = a - bx^2$ where $a$, $b$ are positive constants is more appropriate.<br>Using GC, we found that $a = 29.98560169 = 30.0$ (3 s.f.)<br>and $b = 0.0307756388 = 0.0308$ (3 s.f.)<br>(For $a$, $b > 0$, $y = a + \frac{b}{x}$ decreases at a decreasing rate when $x$ increases) | |
| (iv) | As $x$ is the independent variable and $y$ is the dependent variable, we will still use the regression line $y = 30.0 - 0.0308x^2$ to estimate the value of $x$.<br>Thus, when $y = 10$, $x = 25.5$ °C (3 s.f.)<br>The anwer is reliable for the following reasons:<br><br>i) correlation coefficient $r = -0.995$ has absolute value close to 1<br>ii) the $y$ value of 10 is within data range of the available $y$ values. | |

| Question 10 [12 marks] | | |
|---|---|---|
| (i) | $X \sim \mathrm{N}(950,\ \sigma^2)$<br>Given that $\mathrm{P}(X < 960) = 0.65$,<br>then $\mathrm{P}(Z < \frac{960-950}{\sigma}) = 0.65$<br>$\Rightarrow \frac{960-950}{\sigma} = 0.3853204726$<br>$\Rightarrow \sigma = 25.95242327 = 26.0$ (1 decimal place) | |
| (ii) | Let $X_1$ and $X_2$ be the amount of electricity used by the 2 randomly chosen household in a particular month.<br>Then $X_1 - X_2 \sim \mathrm{N}(0,\ 26.0^2 + 26.0^2)$<br>Thus, $\mathrm{P}(\lvert X_1 - X_2 \rvert \le 30)$<br>$= \mathrm{P}(-30 \le X_1 - X_2 \le 30)$<br>$= 0.585$ | |
| (iii) | Let $N_1$, $N_2$ and $S$ be the amount of electricity used by the 2 randomly chosen households in the North District and household in the South district respectively in August.<br><br>Then their total electricity bill = \$ T<br>= \$ $0.5 \times 0.22 \times (N_1 + N_2) + 0.7 \times 0.22 \times S$<br>= \$ $0.11N_1 + 0.11N_2 + 0.154S$<br>Then<br>$\mathrm{E}(T) = 0.11 \times 950 \times 2 + 0.154 \times 950 = 355.3$<br>$\mathrm{Var}(T) = 0.11^2 \times 26.0^2 \times 2 + 0.154^2 \times 26.0^2 = 32.391216$<br>So, $T \sim \mathrm{N}(355.3,\ 32.391216)$<br>Hence, $\mathrm{P}(T < 360) = 0.796$ | |
| (iv) | Let $\overline{X} = \frac{X_1 + X_2 + X_3 + \dots\dots + X_n}{n} \sim \mathrm{N}\left(950, \frac{26.0^2}{n}\right)$<br>where $X_i$ :electricity usage for each of the randomly selected household in the month of December<br>Then, we have<br>$\mathrm{P}\left(\overline{X} < 955\right) \ge 0.9$<br>$\Rightarrow \mathrm{P}\left(Z < \frac{955-950}{26.0/\sqrt{n}}\right) \ge 0.9$<br>$\Rightarrow \frac{955-950}{26.0/\sqrt{n}} \ge 1.281551567$<br>$\Rightarrow \frac{26.0}{\sqrt{n}} \le \frac{5}{1.281551567} = 3.901520726$<br><br>$\Rightarrow \sqrt{n} \ge \frac{26}{3.901520726}$<br>$\Rightarrow n \ge 44.40980429$<br><br>Thus, the least value of $n$ is 45. | |

| Question 11 [12 marks] | | |
|---|---|---|
| (i) | Let $\mu$ denote the population mean amount of energy released in the collisions.<br>Test H0: $\mu = 1860$<br>Against H1: $\mu > 1860$<br>Using a one-tail test at 1% significance level.<br><br>Under H0, $\overline{X} \sim \mathrm{N}\left(1860, \frac{40^2}{n}\right)$ approx<br>Test statistic: $Z = \frac{\overline{X} - 1860}{40/\sqrt{n}} \sim \mathrm{N}(0,1)$<br>$z_{calc} = \frac{1864-1860}{40/\sqrt{n}} = \frac{\sqrt{n}}{10}$<br>To reject H0 at 1% level of significance, the critical region is:<br>$z_{calc} > 2.32635$<br><br>Hence,<br>$\frac{\sqrt{n}}{10} > 2.32635$<br>$n > 541.189$<br><br>Thus, the least value of $n$ is 542. | |
| (ii) | Test H0: $\mu = 1860$<br>Against H1: $\mu > 1860$<br>Using a one-tailed test at 1% significance level.<br><br>Under H0, $\overline{X} \sim \mathrm{N}\left(1860, \frac{40^2}{600}\right)$ approx<br>Test statistic: $Z = \frac{\overline{X} - 1860}{40/\sqrt{600}} \sim \mathrm{N}(0,1)$<br><br>From GC,<br>$p\text{-value} = 0.00715$<br><br>The $p$-value means that the lowest level of significance at which we would reject the hypothesis that the mean amount of energy released is 1860 MeV in favour of the hypothesis that the amount is greater than 1860 MeV is 0.715 %. | |
| (iii) | No assumption needed. This is because the sample size of 600 is large and thus by Central Limit Theorem, $\overline{X}$ follows a normal distribution. | |

| | | |
|---|---|---|
| (iv) | Let $Z \sim N(0, 1)$<br>$P(Z \geq 2) = 0.0228$<br><br>Hence, lowest level of significance for which the experiment meets the "two sigma" threshold is 2.28%.<br><br>Since $p$-value = 0.00715 < 0.0228, the result meets the "two sigma" threshold.<br><br><u>Alternative</u>:<br>Under $H_0$, $\overline{X} \sim N\left(1860, \frac{40^2}{600}\right)$ approx<br>$2\sigma = 2\sqrt{\frac{1600}{600}} = 3.265986$<br>So $\mu + 2\sigma = 1863.265986$<br><br>$P\left(\overline{X} > 1863.265986\right) \approx 0.0228$<br>Hence, lowest level of significance for which the experiment meets the "two sigma" threshold is 2.28%.<br><br>Since $\overline{x} = 1864 > 1863.265986$ the test meets the "two sigma" threshold. | |