Class | Adm No

Candidate Name: ______________________________

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millennia institute

2017 Preliminary Examination II
Pre-University 3

MATHEMATICS **9740/01**

Paper 1 **11 September 2017**

3 hours

Additional Materials: Answer Paper
List of Formulae (MF15)

READ THESE INSTRUCTIONS FIRST

Write your admission number, name and class on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

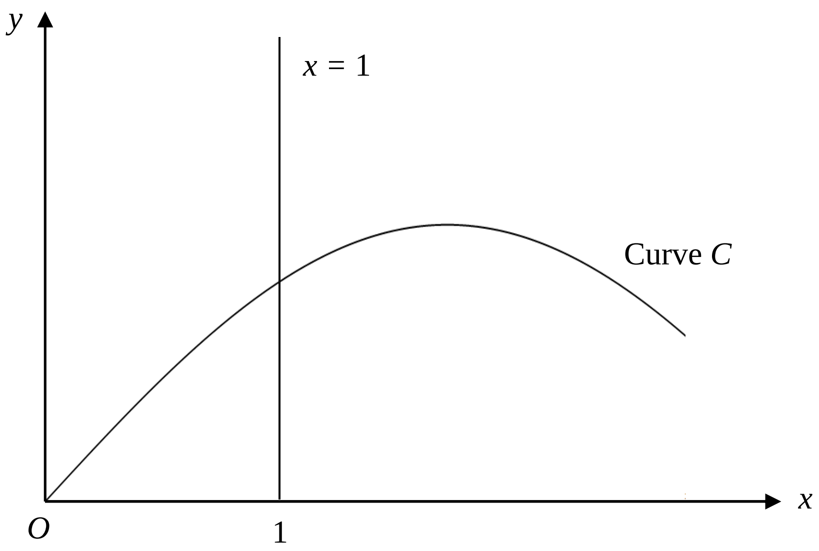
At the end of the examination, arrange your answers in NUMERICAL ORDER and fasten all your work securely together.

The number of marks is given in brackets [ ] at the end of each question or part question.

1 The sum of the first n terms of a sequence is denoted by S_n. The first term of the sequence is 3 and it is known that $S_3 = 21$ and $S_{10} = 210$. Given that S_n is a quadratic polynomial in n, find S_n in terms of n. [3]

2 Using the substitution $v = \sqrt{x} + 1$, find $\int \frac{1}{x + \sqrt{x}}\, \mathrm{d}x$, where $x > 0$. [3]

3



The diagram shows the curve C with equation $y = \sin x$ and the line $x = 1$. With reference to the diagram, a student wrote down the following series

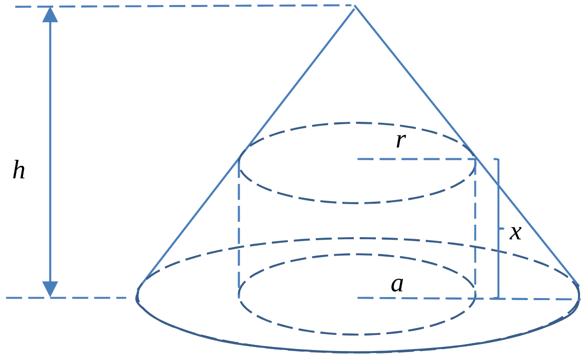
$$S = \frac{1}{n}\left[\sin\left(\frac{1}{n}\right) + \sin\left(\frac{2}{n}\right) + \sin\left(\frac{3}{n}\right) + \ldots + \sin\left(\frac{n}{n}\right)\right].$$

(i) State what the series represents. [2]

(ii) When $n \to \infty$, $S \to L$. State the geometrical meaning of L. Determine the exact value of L, leaving your answer in the form $a - \cos b$, where a and b are constants to be determined. [3]

(iii) What can be said about the value of S in relation to the value of L? [1]

4 [It is given that the volume of a circular cone with base radius r and height h is $\frac{1}{3}\pi r^2 h$.]



The diagram above shows a right circular cone with fixed radius a and fixed height h. A cylinder of radius r and height x is removed from the cone.

(i) Show that the volume of the remaining shape, V, is $\frac{\pi h}{3}\left(a^2 - 3r^2 + \frac{3r^3}{a}\right)$. [2]

(ii) As r varies, use differentiation to find the value of r that gives the minimum value of V, leaving your answer in terms of a. [4]

5 A line L passes through the points $A(3, -1,\ 0)$ and $B(11,\ 11,\ 4)$.

(i) Find the angle between L and the y-axis. [2]

(ii) State the geometrical meaning of $\left|\overrightarrow{OB} \square \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right|$. [1]

The point $F(2a+1,\ a,\ a-1)$ is a point on L, where a is a positive constant. The point P is such that $\overrightarrow{PF} = \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}$ and the area of the triangle AFP is $\sqrt{\frac{59}{2}}$ units^2.

(iii) Determine the value of a. [3]

(iv) The point C on L is such that the ratio of the area of triangle AFP to the area of triangle FCP is $2:1$. State the ratio $AF : CF$, justifying your answer. [2]

[Turn over

6 **(i)** Show that $\int e^{2x}\cos x \, dx = \frac{2}{5}e^{2x}\cos x + \frac{1}{5}e^{2x}\sin x + C$. [3]

(ii) Find the volume of the solid generated when the region bounded by $y = e^{x}\sqrt{(\cos x)}$ and $y = -\frac{2}{\pi}x + 1$ between $x = 0$ and $x = \frac{\pi}{2}$ is rotated through 2π radians about the x-axis, leaving your answer in exact form. [4]

7 **(i)** Prove by the method of mathematical induction that

$$\sum_{r=1}^{n} \frac{2}{r(r+2)} = \frac{3}{2} - \frac{2n+3}{(n+1)(n+2)}$$

for all positive integers of n. [5]

(ii) Explain why $\sum_{r=1}^{n} \frac{2}{r(r+2)}$ is a convergent series, and state the value of the sum to infinity. [2]

(iii) Using the result in part **(i)**, find $\sum_{r=5}^{N} \frac{2}{(r-2)(r-4)}$. [2]

8 Using the substitution $y = ux$, show that the differential equation

$$x\frac{dy}{dx} = 3x + y - 2$$

can be reduced to the form

$$x^2\frac{du}{dx} = 3x - 2 .$$

Hence, find the general solution to the differential equation $x\frac{dy}{dx} = 3x + y - 2$. [5]

(i) State the equation of the locus where the stationary points of the solution curves lie. [1]

(ii) Sketch, on a single diagram, the graph of the locus found in part **(i)** and two members of the family of solution curves, where the arbitrary constant in the general solution is equal to 1 and -1. [3]

9 It is given that

$$\mathrm{f}(x)=\begin{cases}(x-1)^2+4 & , \quad k\le x<3,\\ 3x-1 & , \quad 3\le x\le 4,\end{cases}$$

where $k\in\mathbb{R}$, $k<3$.

(i) Sketch, for $k=0$, the graph of $y=\mathrm{f}(x)$, stating the coordinates of the turning point. Write down the range of f. [3]

(ii) Explain why f^{-1} does not exist. State the smallest value of k for f^{-1} to exist. [2]

(iii) Using the value of k in part **(ii)**, find f^{-1} in similar form. [4]

(iv) State the geometrical relationship between f and f^{-1}. The point $P(a,\ b)$, where a and b are constants, lies on the graph $y=\mathrm{f}(x)$. The point Q on the graph $y=\mathrm{f}^{-1}(x)$ is the point corresponding to P. State the coordinates of Q. [2]

10 **(a)** It is given that $-1+\mathrm{i}$ is a root of the equation $2z^3+az^2+bz+(3+\mathrm{i})=0$.

(i) Find the values of the real numbers a and b. [4]

(ii) Using these values of a and b, find the other roots of this equation. [3]

(b) It is given that $w=-1+(\sqrt{3})\mathrm{i}$.

(i) Without using a calculator, find an exact expression for w^5. Give your answer in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r>0$ and $0\le\theta\le 2\pi$. [3]

(ii) Without using a calculator, find the three smallest positive whole number values of n for which $\dfrac{w^*}{w^n}$ is a real number. [4]

11 A curve C_1 is defined parametrically by the equations $x = t - \dfrac{1}{t}$, $\quad y = t + \dfrac{1}{t}$, $\ t \neq 0$.

(i) Sketch C_1, stating the equation of the asymptotes and coordinates of any points of intersection with the y-axis. [2]

(ii) Show that the equation of the normal to C_1 at the point with parameter p is given by $y = -\dfrac{p^2+1}{p^2-1}x + \dfrac{2(p^2+1)}{p}$. [4]

(iii) The normal in part **(ii)** intersects the x-axis at the point A and the y-axis at the point B. Find, in terms of p, an expression for the area of the triangle OAB. [4]

The line l is the normal to C_1 when $p = 2$.

(iv) Find the equation of l. [1]

A curve C_2 is defined parametrically by the equations $x = 3at$, $\quad y = -t^2 + a$, $\ t \in \mathbb{R}$ where a is a non-zero constant.

(v) Given that l intersects C_2, show that the parameter q of the point(s) of intersection satisfies the equation

$$q^2 - 5aq + 5 - a = 0\,.$$

Hence, determine the range of values of a such that l intersects C_2 at two distinct points. [3]

12 As part of a project, a group of engineering students design two robots for a game. One robot is called 'Prey' and the other robot is called 'Predator'. The two robots are designed with the following specifications.

'Prey': It is designed to leap 1 m forward for the first leap. Subsequently, it leaps 2.5 cm less than the previous leap distance. 'Prey' shuts down when the leap distance is 0 or when it is caught by 'Predator'.

'Predator': It is designed to leap 2 m forward for the first leap. Subsequently, it leaps 90% of the previous leap distance. 'Predator' shuts down when 'Prey' shuts down or when it catches 'Prey'.

Both robots take each leap at the same time and the number of leaps taken is given by n. 'Predator' starts the game from the starting line while 'Prey' starts the game 7 m in front of 'Predator'.

(i) Find the distance of 'Prey' and of 'Predator' from the starting line after n leaps, leaving your answers in terms of n. [2]

(ii) Explain why 'Predator' has to catch 'Prey' before 'Predator's distance from the starting line reaches 20 m. [2]

(iii) Using a graphical method, explain why 'Predator' will not catch 'Prey'. [3]

(iv) 'Prey' now starts the game 4 m in front of 'Predator'. 'Predator' catches 'Prey' on the k-th leap. Find the value of k.

Calculate the distance of 'Predator' from the starting line after completing the k-th leap. [3]

End of Paper