NANYANG JUNIOR COLLEGE

JC2 PRELIMINARY EXAMINATION

Higher 2

FURTHER MATHEMATICS **9649/01**

Paper 1 **14th September 2017**

3 Hours

Additional Materials: Answer Paper

List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name and class on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphic calculator.
Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **5** printed pages.

1 An astroid, A, is a curve traced out by a point on a circle as it rolls inside a larger, fixed circle with four times the radius. It is defined parametrically by the equations

$$x = a\cos^3\theta, \qquad y = a\sin^3\theta$$

where $a > 0$ and $0 \le \theta \le 2\pi$.

(i) Sketch A. [1]

(ii) Show that $\left(\dfrac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\dfrac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2 = 9a^2\sin^2\theta\cos^2\theta$. [2]

(iii) Find the exact area of the curved surface generated when A is rotated through one complete revolution about the x-axis. [3]

2 Prove that if $x > 1$, then

$$\ln x > \frac{(x-1)^2}{x^2}. \quad [3]$$

The curves C_1 and C_2 have respective cartesian equations

$$y = \ln x \text{ and } y = \frac{(x-1)^2}{x^2}.$$

Use the 'shell method' to find the exact volume swept out when the region enclosed between C_1 and C_2 and the ordinates $x = 2$ and $x = 4$, is rotated completely about the y-axis. [4]

3 Let $f_1,\ f_2,\ f_3, \ldots$ be a sequence of positive numbers that satisfies the recurrence relation

$$f_n = f_{n-1} + f_{n-2} \text{ for } n \ge 2,$$

where $f_0 = 0$ and $f_1 = a$, $a \in \mathbb{R}^+$.

Let $u_n = \left(f_{n-1}\right)^2 - f_{n-2}f_n$ for $n \ge 2$.

(i) Write down the values of u_2, u_3 and u_4. [2]

(ii) Hence make a conjecture for u_n in terms of a and n. [1]

(iii) Use the method of mathematical induction to prove the conjecture in **(ii)**. [5]

4 Let O denote the origin of cartesian coordinates and let P and Q be two points on the rectangular hyperbola with cartesian equation

$$xy = c^2$$

where c^2 is a positive real number.

Straight lines through Q parallel to the asymptotes meet the line through O perpendicular to OP in H and K.

Prove that the circle with HK as a diameter passes through P. [8]

5 The complex number z satisfies the inequalities

$$|4-z| \le |z|, \quad \text{and} \quad \left|\mathrm{i}z+\sqrt{3}\right| \le \sqrt{7}.$$

(i) On an Argand diagram, sketch the region in which the point representing z can lie. [3]

(ii) Find the possible values of $\arg z$, leaving your answer in exact form. [2]

(iii) Find the area of the region in (i). [3]

(iv) Given further that z satisfies the equation $(z-2)^5 = 1$, find z, leaving your answer in the form $x+\mathrm{i}y$. [2]

6 Show that the equation

$$2x^3 + x - 1 = 0 \qquad \text{(E)}$$

has a root r in the interval $(0,1)$. [1]

(i) Use the bisection method to obtain an approximation to r correct to 1 decimal place, justifying your accuracy.
Using the bisection method, a sequence of approximations $a_1, a_2, \ldots, a_n, \ldots$ is produced to approximate the value of r such that a_1 is the midpoint of the first interval $(0,1)$ which contains r, a_2 is the midpoint of the second interval which contains r and in general, a_n is the midpoint of the n^{th} interval which contains r. Find the least value of n in order to ensure that a_n is within 0.001 units of r. [4]

(ii) Use one application of the Newton-Raphson method with a starting value of 0.8 to determine an approximation to r correct to 2 decimal places. [2]

(iii) An attempt to use the fixed-point iteration method to find an approximation for r is undertaken by student X.
He rewrites equation (E) in the form

$$x = 1 - 2x^3$$

and chooses the first iterate $x_1 = 1$.
Explain why his attempt will result in failure.
In a separate attempt, student X rewrites equation (E) in the form

$$x = \left(\frac{1-x}{2}\right)^{\frac{1}{3}}$$

and chooses the first iterate $x_1 = 1$.
Show that his attempt will be successful and use the graphing calculator to obtain the value of x_{10} correct to 4 decimal places. Draw a sketch to illustrate the iterative process. [6]

7 **(i)** Find the general solution of the differential equation

$$(x+1)\frac{\mathrm{d}y}{\mathrm{d}x}+2y=\mathrm{e}^{x}\ .$$ [4]

(ii) Let $\mathrm{a}(x)$, $\mathrm{b}(x)$ and $\mathrm{c}(x)$ be functions of x such that $\mathrm{a}''(x)-\mathrm{b}'(x)+\mathrm{c}(x)=0$. Find the functions $\mathrm{f}(x)$ and $\mathrm{g}(x)$ such that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\mathrm{f}(x)\frac{\mathrm{d}y}{\mathrm{d}x}+\mathrm{g}(x)y\right\}=\mathrm{a}(x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\mathrm{b}(x)\frac{\mathrm{d}y}{\mathrm{d}x}+\mathrm{c}(x)y.$$

[Your answer must be given in terms of one or more of $\mathrm{a}(x)$, $\mathrm{b}(x)$ and $\mathrm{c}(x)$.] [3]

(iii) Hence use a substitution of the form

$$z=\mathrm{f}(x)\frac{\mathrm{d}y}{\mathrm{d}x}+\mathrm{g}(x)y$$

to find the general solution of the differential equation

$$\left(x^2+x\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+(4x+1)\frac{\mathrm{d}y}{\mathrm{d}x}+2y=(x+1)\mathrm{e}^{x}.$$ [5]

8 **(a)** Let $\mathbf{A}$ be a 3×3 matrix with distinct eigenvalues λ_1, λ_2 and λ_3 with corresponding eigenvectors $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$.

(i) Explain why $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ is a basis for $\mathbb{R}^3$. [1]

(ii) By writing $\mathbf{x}\in\mathbb{R}^3$ in the form $a_1\mathbf{e}_1+a_2\mathbf{e}_2+a_3\mathbf{e}_3$, express $\mathbf{A}^n\mathbf{x}$ in terms of λ_1, λ_2, λ_3, $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ a_1, a_2, a_3 and n. [2]

(iii) If $\lambda_1=1$, $|\lambda_2|<1$ and $|\lambda_3|<1$, compute $\lim\limits_{n\to\infty}\mathbf{A}^n\mathbf{x}$, explaining your reasoning clearly. [2]

(b) A linear transformation $T:\mathbb{R}^3\to\mathbb{R}^3$ has matrix representation $\mathbf{A}$. Suppose $\mathbf{A}$ is a 3×3 matrix with eigenvalues $\lambda_1=-1$, $\lambda_2=0$ and $\lambda_3=1$, and corresponding eigenvectors

$$\mathbf{e}_1=\begin{pmatrix}1\\0\\2\end{pmatrix},\ \mathbf{e}_2=\begin{pmatrix}-1\\1\\0\end{pmatrix}\text{ and }\mathbf{e}_3=\begin{pmatrix}0\\0\\1\end{pmatrix}.$$

(i) Find, with sufficient working, a basis for K, the null space of T. [2]

(ii) Find the matrix $\mathbf{A}$. [2]

(iii) Describe the range space of T geometrically. [3]

9 [In this part of the question, all distances are measured in Astronomical Units (AU for short)]

Planet X revolves around the sun in an elliptical path with the sun at one focus, assume fixed. Treating planet X and the sun as point masses at their centres, it is known that the distance between the two points on the elliptical path of planet X furthest apart is 6 AU and the closest distance from planet X to the sun is 1 AU.
State the eccentricity e of the elliptical path of planet X. [1]

The centre of the sun is the pole O of a polar coordinate system with the initial line along the major axis of the elliptical path of planet X.
Obtain the polar equation of the path of planet X in the form

$$r = \frac{ed}{1 + e\cos\theta}$$

where e is the eccentricity and d is a positive constant to be determined. [3]

The point on the path of planet X closest to the sun is denoted by P_0 and the point on the path of planet X such that the radius vector is perpendicular to the initial line is denoted by $P_{\pi/2}$.
Show that the distance traversed by planet X in moving from P_0 to $P_{\pi/2}$ is given by the integral

$$5\int_0^{\frac{\pi}{2}} \frac{\sqrt{13 + 12\cos\theta}}{(3 + 2\cos\theta)^2}\,\mathrm{d}\theta$$

and use the graphing calculator to find its value to 8 decimal places. [4]

The time taken by planet X to revolve around the sun is Y years.
Kepler's law states that for a planet orbiting around the sun, the line joining the planet to the sun sweeps out equal areas in equal intervals of time. Use Kepler's law to find the time taken for planet X to traverse the path $P_0P_{\pi/2}$, giving your answer in the form kY where k is to be determined to 4 decimal places. [4]

10 In an interrogation procedure, a captured espionage will be given a 25 milligram dose of a truth serum every 4 hours. 15% of the truth serum present in his body is lost every hour. Let u_n be the amount of the serum in his body just after the n^{th} dose.

(i) Calculate, in milligrams, the amount of truth serum remaining in his body after 4 hours and just before the second dose is administered. [1]

(ii) Find a recurrence relation that u_n satisfies in the form $u_{n+1} = au_n + b$, where a and b are constants to be determined. [2]

(iii) Solve the recurrence relation stated in **(ii)**. [4]

(iv) It is known that the level of serum in the body has to be continuously above 20 milligrams before the victim starts to confess. Determine the number of doses that are needed before the interrogation can begin. [3]

(v) It is also known that 55 milligrams of this serum in the body is fatal to the human body. Is there any maximum length of time the serum can be administered so that the espionage can be kept alive? [2]

———END OF PAPER———