NANYANG JUNIOR COLLEGE

JC2 PRELIMINARY EXAMINATION

Higher 2

FURTHER MATHEMATICS **9649/02**

Paper 2 **20th September 2017**

3 Hours

Additional Materials: Answer Paper

List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name and class on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphic calculator.
Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

Section A: Pure Mathematics [50 marks]

1 A sequence of real numbers $\{x_n\}$ is defined by the recurrence relation

$$x_n = 1 + \tan\left(\frac{1}{4}x_{n+1}\right), \text{ for } n \geq 1.$$

It is given that if the sequence converges, then it converges to either α, β or γ, where $-2\pi < \alpha < \beta < \gamma < 2\pi$.

(i) Find the values of α and β, each correct to 3 decimal places. [1]

(ii) Prove algebraically that if $\alpha < x_n < \beta$, then $\alpha < x_{n+1} < \beta$. [3]

(iii) Show, with the aid of a diagram, that if $\alpha < x_n < \beta$, then $x_{n+1} < x_n$. Hence state, with a reason, the value the sequence converges to. [3]

2 An amateur geographer wants to map a particular path leading from the base of Mount Pakoko to its summit. The path is smooth and can be contained in a vertical plane. The geographer lost her altimeter, and decides to use an inclinometer to measure the inclination. She starts from the base of the mountain, and records the inclination at intervals of 0.5 km in horizontal distance. By using the Euler's Method, the geographer was able to estimate the altitude at every interval. The following table shows her findings.

Horizontal distance from start, x km	0	0.5	1.0	1.5	2.0	2.5	3.0
Inclination, $\theta°$	5.0	6.3	7.8	9.5	11.2	16.0	21.3
Altitude, y m	10	53.744	a	177.43	261.10	360.10	b

(i) Find the values of a and b. [2]

(ii) Assuming that the geographer's horizontal speed is the same throughout her journey, use the Simpson's Rule to calculate her mean altitude during the journey. [3]

(iii) Suppose the concavity did not change during the geographer's travel, would using the Trapezium Rule yield an overestimate or underestimate to the mean altitude? [1]

When the geographer finally reaches the summit at an altitude of 6.5 km, she takes another path down towards a camp. The altitude, A km, and the horizontal distance from the summit, w km, are related by the equation

$$\frac{\mathrm{d}A}{\mathrm{d}w} = -\frac{A}{8(w+1)^2}.$$

(iv) Use the Improved Euler's Method with step size of 0.5 km to estimate the geographer's altitude when she travels horizontally by 0.5 km. [3]

3 **(a)** Let M_{nn} be the space of all $n \times n$ matrices. Let $\mathbf{A}$ be an $n \times n$ invertible matrix. Define $T : \mathrm{M}_{nn} \to \mathrm{M}_{nn}$ by $T(\mathbf{B}) = \mathbf{A}^{-1}\mathbf{B}\mathbf{A}$. Show that T is a linear transformation. What can be said about $\mathbf{B}$ and $T(\mathbf{B})$? [4]

(b) Let $\mathbf{P}$ be the following matrix:

$$\mathbf{P} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & t \\ 1 & 4 & t^2 \end{pmatrix}.$$

(i) Evaluate the determinant of $\mathbf{P}$ in terms of t. [2]

(ii) Find the values of t for $\mathbf{P}$ to be singular. [2]

(iii) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation with $\mathbf{P}$ as its matrix representation. Find the possible values for the dimension of the null space of T. [2]

4 **(a)** The Archimedean spiral, *S*, which was first studied by the Greek mathematician Archimedes in the 3rd century BC, has polar equation given by

$$r = a + b\theta$$

where *a* and *b* are non-negative real constants and $\theta \geq 0$.

The Archimedean spiral has the property that any ray from the pole intersects successive turnings of the spiral at points with a constant separation distance *d*, hence also the name "arithmetic spiral".

(i) Prove the above property and state the value of *d*. [2]

(ii) For the case when *a* = *b*, prove that the angle which the tangent to *S* at the point (r, θ) makes with the initial line is given by

$$\tan^{-1}(1+\theta) + \theta$$

and hence write down the cartesian equation of the tangent to *S* at the point where $\theta = 0$. [5]

(b) Let *A* and *B* be two points on a polar curve corresponding to $\theta = \alpha$ and $\theta = \beta$ respectively. The area of the curved surface generated when the arc *AB* is rotated completely about the initial line is given by the integral

$$2\pi \int_{\alpha}^{\beta} r \sin\theta \left\{ r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta} \right)^2 \right\}^{\frac{1}{2}} \mathrm{d}\theta .$$

Use the above integral to derive the formula for the surface area of a sphere of radius *a*, explaining your working clearly. [3]

5 A logistic growth model for the population $P(t)$ of a certain species of bears in a habitat is given by the differential equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = 2P - \frac{1}{150}P^2 .$$

(i) State the carrying capacity of the habitat. **[1]**

(ii) Find the general solution of the above differential equation, expressing P explicitly in terms of t. **[3]**

The population of bears is then subjected to constant poaching rate h and the population is now modelled by the differential equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = 2P - \frac{1}{150}P^2 - h .$$

(iii) Determine the maximum sustainable poaching rate and show that it occurs when the population is at half its carrying capacity. **[3]**

For the rest of the question, assume $h = 100$.

(iv) Determine the eventual size of the population if there are initially 80 bears, and sketch the solution curve. **[4]**

(v) Given instead that the initial population of bears is 10, determine the earliest time T, correct to 2 decimal places, that poaching can start to ensure the survival of the population. **[3]**

Section B: Probability and Statistics [50 marks]

6 A machine produces right conical containers of various sizes. The container is designed such that the base radius and height are the same. The base radius, R cm, of a randomly chosen cone has probability density function given by

$$\mathrm{f}(r) = \begin{cases} \frac{1}{6}r, & \text{if } 2 \le r \le 4, \\ 0, & \text{otherwise.} \end{cases}$$

The volume of a randomly chosen container is denoted by the random variable V. By first considering $\mathrm{P}(R \le r)$, find the probability density function of V. [4]

The containers are to be filled with a special type of fluid at a constant rate of $3\ \text{cm}^3\ \text{s}^{-1}$. Assuming that the containers are made from materials of negligible thickness, find the expected time to fill a randomly chosen container, leaving your answer to 2 decimal places. [2]

7 Students studying English at a certain college took an examination at the end of the year. Their grades are classified as “Distinction”, “Pass”, “Sub-pass” or “Fail”. A random sample of 200 students is selected and their grades are summarised in the table below.

	Distinction	Pass	Sub-pass	Fail
Male	17	86	15	2
Female	21	51	4	4

Test at the 5% level of significance whether there is any association between the grades and gender. [5]

A larger random sample of $200n$ students is now taken, where n is a positive integer. It is found that the proportions of students’ grades are identical to those in the smaller sample. State the relationship between the value of χ^2 calculated from the larger sample and that from the smaller sample. [1]

8 In a game of *odd man out*, k players each has a coin with the same probability, p, of getting a head. The players toss the coins at the same time. If there is an *odd man*, that is, a player with an outcome that is different from all other players, then the odd man is eliminated; otherwise, no player is eliminated. The game is repeated until there are only 2 players left.

(i) Show that the probability of getting an odd man when there are k players in a single round is given by

$$r_k(p) = kp(1-p)^{k-1} + kp^{k-1}(1-p),$$

where $k \geq 3$. [1]

(ii) Let N_k denotes the number of rounds needed to eliminate an odd man when there are k players. State the distribution of N_k. [1]

(iii) If M_k is the number of rounds needed to repeat the game until 2 players remain, show that $\mathrm{E}(M_k) = \sum_{j=3}^{k} \frac{1}{r_j(p)}$ and find a similar expression for the variance of M_k. [4]

9 A curriculum planning officer wishes to gather the opinion of teachers on a proposal to change the curriculum of a particular subject.

(i) Based on a random sample of 80 teachers, the symmetric $k\%$ confidence interval for p, the proportion of teachers who are in favour of the change, is $(0.676,\ 0.884)$. Find the value of k. [4]

To gather more insights about the teachers' opinions on the current curriculum, another random sample of 8 teachers is obtained. Their ratings (out of 10) for the current curriculum are as follows:

6.2 6.5 7.2 6.3 6.8 6.6 6.0 6.3

(ii) Assuming that the distribution of the ratings is normal, find a symmetric 99% confidence interval for μ, the population mean rating for the current curriculum, leaving your answers to 2 decimal places. [2]

(iii) Explain, in the context of the question, the meaning of '99% confidence interval'. [1]

(iv) Suppose the null hypothesis that $\mu = 7$ is tested against the alternative hypothesis that $\mu \neq 7$. Using your answer in **(ii)**, explain if the null hypothesis can be rejected at the 5% level of significance. [1]

10 Flaws on an electric cable from brand A can be modelled by a Poisson distribution. It is known that on average, there will be 1 flaw per 500 meter of a randomly chosen cable.

(i) Find the probability that there will be more than 4 flaws in a randomly chosen cable of length 2km. [2]

(ii) Find the probability that on a randomly chosen cable of length 1 km, there will be 3 flaws on the first 400 meters of the cable and less than 2 flaws on the remaining 600 meters of the cable. [3]

(iii) On a randomly chosen cable of length 1 km, it is found that there are exactly n flaws. Show that the number of flaws in the first 500 meters of the cable follows a binomial distribution. You are to state the parameters of the distribution clearly. [4]

(iv) Given that there is exactly 1 flaw in a randomly chosen cable of length 500 meter, find the probability that the flaw is found in the first t meters of the cable. What can you deduce about the distribution of the location of the flaw? [3]

11 The thickness of plague (measured in mm) in the carotid artery of 10 randomly selected patients with mild atheroscierotic disease were measured. Two measurements are taken: thickness before treatment with Vitamin E (x) and after two years of taking Vitamin E daily (y). The readings are given below.

x	0.66	0.72	0.85	0.62	0.54	0.63	0.64	0.67	0.73	0.68
y	0.60	0.65	0.79	0.63	0.59	0.55	0.64	0.70	0.68	0.64

The medical research team is interested on whether taking Vitamin E daily for two years will reduce the thickness of plague.

(i) Explain which test the medical research team should use. Carry out a suitable test at the 5% level of significance, stating the necessary assumptions. [7]

(ii) Give conditions for which a non-parametric test should be used. Carry out the Wilcoxin signed rank test at 5% level of significance, stating any necessary assumptions. [5]

–––––END OF PAPER–––––