# 2017 Year 6 FM Prelim Solutions (Paper 1)

| Qn | Solution |
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| **1(a)** | Let $\mathbf{u}, \mathbf{v} \in A$.<br>$(\mathbf{u}+\mathbf{v}) \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$<br>$= \mathbf{u} \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + \mathbf{v} \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$<br>$= 0 + 0 = 0$<br><br>Therefore $\mathbf{u}+\mathbf{v} \in A$ and $A$ is closed under vector addition.<br><br>$(a\mathbf{u}) \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = a\left(\mathbf{u} \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}\right) = 0$ where $a$ is a scalar.<br>Therefore $a\mathbf{u} \in A$ and $A$ is closed under scalar multiplication.<br>In particular, the zero vector is in $A$.<br><br>Therefore $A$ is a subspace of $\mathbb{R}^3$.<br><br>The zero vector is not in $B$ since<br>$2(0)+3(0)-5(0)=0 \neq 1$<br>Therefore $B$ is a not a subspace of $\mathbb{R}^3$. |
| **(b)(i)** | Both $\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \in B' \subseteq A \cup B'$<br>However, $\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$<br>and $\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \notin A$, $\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \in B$.<br>Therefore $\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \notin A \cup B'$ and<br>since $A \cup B'$ is not closed under vector addition,<br>it is not a subspace of $\mathbb{R}^3$. |

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| **(ii)** | $\begin{pmatrix}-2\\2\\0\end{pmatrix} \in A \cap B'$<br>However, $\frac{1}{2}\begin{pmatrix}-2\\2\\0\end{pmatrix} = \begin{pmatrix}-1\\1\\0\end{pmatrix} \in B$<br>Therefore $\begin{pmatrix}-1\\1\\0\end{pmatrix} \notin A \cap B'$ and<br>since $A \cap B'$ is not closed under scalar multiplication,<br>it is not a subspace of $\mathbb{R}^3$. |

| Qn | Solution |
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| 2 | Since $x = \sqrt{t}$,<br>$x^2 = t$<br>Differentiating wrt $y$,<br>$2x\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{\mathrm{d}t}{\mathrm{d}y} \quad \Rightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{1}{2x}\frac{\mathrm{d}y}{\mathrm{d}x}$ -----(1)<br>Differentiating wrt $x$,<br>$\frac{\mathrm{d}^2y}{\mathrm{d}t^2}\frac{\mathrm{d}t}{\mathrm{d}x} = \left(\frac{1}{2x}\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - \frac{1}{2x^2}\frac{\mathrm{d}y}{\mathrm{d}x}$<br>Since $\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{2x}$,<br>$2x\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = \left(\frac{1}{2x}\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - \frac{1}{2x^2}\frac{\mathrm{d}y}{\mathrm{d}x}$<br>$4x^2\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = \frac{\mathrm{d}^2y}{\mathrm{d}x^2} - \frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}x}$ ---- (2)<br><br>Subst (1) and (2) into the DE,<br>$\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - \frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}x}\right) + 2x\frac{\mathrm{d}y}{\mathrm{d}x} + 24x^2 = 0$<br>$4x^2\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 4x^2\frac{\mathrm{d}y}{\mathrm{d}t} + 24x^2 = 0$<br>$4x^2\left(\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + \frac{\mathrm{d}y}{\mathrm{d}t} + 6\right) = 0$<br>Since $x > 0$, $\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + \frac{\mathrm{d}y}{\mathrm{d}t} + 6 = 0$. Thus $a = 1$ and $b = 6$.<br>$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + \frac{\mathrm{d}y}{\mathrm{d}t} = -6$<br>Characteristic eqn: |

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| | $m^2+m=0$<br>$m(m+1)=0 \Rightarrow m=0,-1$<br>Complementary solution: $y_c=A+B\mathrm{e}^{-t}$<br>Particular integral: $y_p=-6t$<br>$y=A-6t+B\mathrm{e}^{-t}$<br>$\therefore$ General solution is $y=A-6x^2+B\mathrm{e}^{-x^2}$ . |

| Qn | Suggested Solutions |
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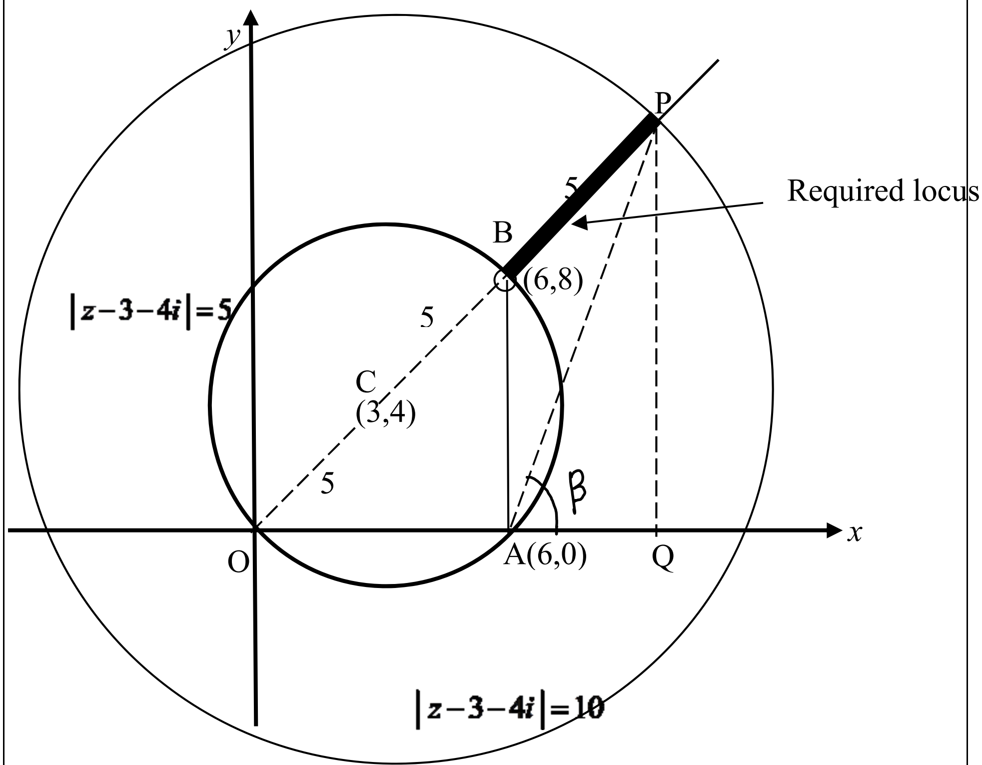
| 3 | $5 \le \lvert z-(3+4i) \rvert \le 10$ , $\arg\left(z-(6+8i)\right)=\arg(3+4i)$<br><br> |
| (i) | Max value of $\lvert z-6 \rvert$ = AP<br>By similar triangles,<br>$\dfrac{OP}{OB}=\dfrac{PQ}{BA} \Rightarrow \dfrac{15}{10}=\dfrac{PQ}{8}$<br>$PQ=12 \Rightarrow OQ=9 \quad \Rightarrow AQ=3$<br><br>Max value of $\lvert z-6 \rvert$ = AP = $\sqrt{3^2+12^2}=\sqrt{153}$ |
| (ii) | Range of values of arg (z – 6) is $\beta \le \arg(z-6) < \dfrac{\pi}{2}$<br>$\beta=\tan^{-1}\left(\dfrac{12}{3}\right)=1.33$<br>$\Rightarrow 1.33 \le \arg(z-6) < \dfrac{\pi}{2}$ |

| Qn | Suggested Solutions |
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| 4(i) | $m\dfrac{\mathrm{d}^2x}{\mathrm{d}t^2}+k\dfrac{\mathrm{d}x}{\mathrm{d}t}+m\omega^2x=0$ <br> Characteristic equation: <br> $ms^2+ks+m\omega^2=0$ <br> $s=\dfrac{-k\pm\sqrt{k^2-4(m)(m\omega^2)}}{2m}=\dfrac{-k}{2m}\pm\mathrm{i}\omega \quad \because k^2$ can be ignored <br> General solution of $x$ is $x=\mathrm{e}^{-\frac{k}{2m}t}\left(A\cos(\omega t)+B\sin(\omega t)\right)$. <br> At $t=0$, $x=0$, $A=0$. <br> Since $x=B\mathrm{e}^{-\frac{k}{2m}t}\sin(\omega t)$, <br> $\dfrac{\mathrm{d}x}{\mathrm{d}t}=-\dfrac{Bk}{2m}\mathrm{e}^{-\frac{k}{2m}t}\sin(\omega t)+B\omega\mathrm{e}^{-\frac{k}{2m}t}\cos(\omega t)$ <br> At $t=0$, $\dfrac{\mathrm{d}x}{\mathrm{d}t}=v$, $B=\dfrac{v}{\omega}$. <br> $\therefore x=\dfrac{v}{\omega}\mathrm{e}^{-\frac{k}{2m}t}\sin(\omega t)$ |
| (ii) | Period of vibrations, $T=\dfrac{2\pi}{\omega}$. <br> Let $t_0=\dfrac{\pi}{2\omega}$, <br> Amplitude at time $t_0$ after the $n$th period, $A_n=\dfrac{v}{\omega}\mathrm{e}^{-\frac{k}{2m}(t_0+nT)}$ <br> Amplitude at time $t_0$ after the $(n+1)$th $A_{n+1}=\dfrac{v}{\omega}\mathrm{e}^{-\frac{k}{2m}(t_0+(n+1)T)}$ <br> $\dfrac{A_{n+1}}{A_n}=\dfrac{\frac{v}{\omega}\mathrm{e}^{-\frac{k}{2m}(t_0+(n+1)T)}}{\frac{v}{\omega}\mathrm{e}^{-\frac{k}{2m}(t_0+nT)}}=\mathrm{e}^{-\frac{k}{2m}(T)}$ (constant) <br> Thus, amplitude of successive vibrations follows a geometric progression. |

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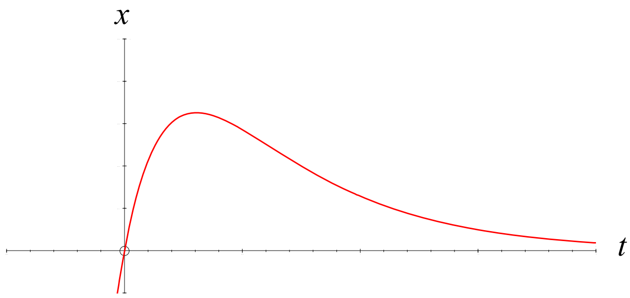
| (iii) | Roots to the characteristic will be real, distinct and negative. Thus the general soln will be $x = A\left(e^{-m_1 t} - e^{-m_2 t}\right)$ where $m_1$ and $m_2$ are positive constants.<br><br> |

| Qn | Suggested Solution |
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| **5(i)** | $\cos 6\theta + \mathrm{i}\sin 6\theta$<br>$= (c + \mathrm{i}s)^6, c = \cos\theta, s = \sin\theta$<br>$= c^6 + 6c^5(\mathrm{i}s) + 15c^4(\mathrm{i}s)^2 + 20c^3(\mathrm{i}s)^3 + 15c^2(\mathrm{i}s)^4 + 6c(\mathrm{i}s)^5 + (\mathrm{i}s)^6$<br>Comparing real and imaginary parts<br>$\cos 6\theta = \cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta$<br>$\sin 6\theta = 6\cos^5\theta\sin\theta - 20\cos^3\theta\sin^3\theta + 6\cos\theta\sin^5\theta$<br>$\tan 6\theta = \dfrac{\sin 6\theta}{\cos 6\theta}$<br>$= \dfrac{6\cos^5\theta\sin\theta - 20\cos^3\theta\sin^3\theta + 6\cos\theta\sin^5\theta}{\cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta}$<br>$= \dfrac{6t - 20t^3 + 6t^5}{1 - 15t^2 + 15t^4 - t^6}$ |
| **(ii)** | Letting $\theta = \dfrac{\pi}{12}$, $\tan 6\theta$ is undefined.<br>Therefore $1 - 15x + 15x^2 - x^3 = 0, x = t^2$<br>Rearranging $x^3 - 15x^2 + 15x - 1 = 0$ which has a root $x = \tan^2\left(\dfrac{\pi}{12}\right)$<br>2 other values of $\theta = \dfrac{3\pi}{12} = \dfrac{\pi}{4}, \dfrac{5\pi}{12}$ will cause $\tan 6\theta$ to be undefined.<br>The other 2 roots are 1 and $x = \tan^2\left(\dfrac{5\pi}{12}\right)$.<br>$x^3 - 15x^2 + 15x - 1 = (x-1)\left(x^2 - 14x + 1\right) = 0$<br>$x = 1$ or $x^2 - 14x + 1 = 0$<br>$\Rightarrow x = 1$ or $x = \dfrac{14 \pm \sqrt{196 - 4}}{2} = 7 \pm 4\sqrt{3}$<br>Since $\tan^2\left(\dfrac{\pi}{4}\right) = 1$ and $\tan^2\left(\dfrac{\pi}{12}\right) < \tan^2\left(\dfrac{5\pi}{12}\right)$, $\tan^2\left(\dfrac{1}{12}\pi\right) = 7 - 4\sqrt{3}$. |

| Qn | Suggested Solution |
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| **6** | $\mathbf{Ax} = \lambda\mathbf{x}$<br>$\Rightarrow \mathrm{C}^{-1}(\mathbf{Ax}) = \mathrm{C}^{-1}(\lambda\mathbf{x})$<br>$\Rightarrow \left(\mathrm{C}^{-1}\mathbf{A}\mathrm{C}\right)(\mathrm{C}^{-1}\mathbf{x}) = \lambda(\mathrm{C}^{-1}\mathbf{x})$<br>$\therefore \mathrm{C}^{-1}\mathbf{x}$ is an eigenvector of $\mathrm{C}^{-1}\mathbf{A}\mathrm{C}$.<br><br>Alternatively,<br>$\mathbf{C^{-1}AC}\left(\mathbf{C}^{-1}\mathbf{x}\right) = \mathbf{C^{-1}Ax} = \mathbf{C}^{-1}\lambda\mathbf{x} = \lambda\mathbf{C}^{-1}\mathbf{x}$ |

To verify $C=\begin{pmatrix}1&0&1\\0&1&0\\0&0&1\end{pmatrix}$ and $C^{-1}=\begin{pmatrix}1&0&-1\\0&1&0\\0&0&1\end{pmatrix}$

$$CC^{-1}=\begin{pmatrix}1&0&1\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&-1\\0&1&0\\0&0&1\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}=I$$

Ie, $\mathbf{S}=\mathbf{C}^{-1}\mathbf{A}\,\mathbf{C}$ where $A=\begin{pmatrix}1&-3&-3\\-8&6&-3\\8&-2&7\end{pmatrix}$

<u>Find eigenvalues and eigenvectors of **A**:</u>

$$\det(\mathbf{A}-\lambda\mathbf{I})=0$$
$$(\lambda-1)(\lambda-4)(\lambda-9)=0 \qquad [1+1]$$
$$\Rightarrow \lambda=1,\ 4,\ 9$$

When $\lambda=1$, $\mathbf{x}=\begin{pmatrix}1\\1\\-1\end{pmatrix}$

When $\lambda=4$, $\mathbf{x}=\begin{pmatrix}1\\1\\-2\end{pmatrix}$

When $\lambda=9$, $\mathbf{x}=\begin{pmatrix}0\\1\\-1\end{pmatrix}$

Hence, **S** has eigenvalues and corresponding eigenvectors given by $\mathbf{C}^{-1}\,\mathbf{x}$ :

$$\lambda=1,\quad C^{-1}\mathbf{x}=\begin{pmatrix}1&0&-1\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1\\1\\-1\end{pmatrix}=\begin{pmatrix}2\\1\\-1\end{pmatrix}$$

$$\lambda=4,\quad C^{-1}\mathbf{x}=\begin{pmatrix}1&0&-1\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1\\1\\-2\end{pmatrix}=\begin{pmatrix}3\\1\\-2\end{pmatrix}$$

$$\lambda=9,\quad C^{-1}\mathbf{x}=\begin{pmatrix}1&0&-1\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}0\\1\\-1\end{pmatrix}=\begin{pmatrix}1\\1\\-1\end{pmatrix}$$

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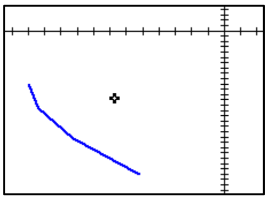
| | Let $\mathbf{S} = \mathbf{QDQ}^{-1}$ where<br><br>$\mathrm{Q}=\begin{pmatrix} 2 & 3 & 1 \\ 1 & 1 & 1 \\ -1 & -2 & -1 \end{pmatrix}$ , $\mathrm{D}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{pmatrix}$ (M1)<br><br>$\mathbf{S}^4 = \left(\mathbf{QDQ}^{-1}\right)\left(\mathbf{QDQ}^{-1}\right)\left(\mathbf{QDQ}^{-1}\right)\left(\mathbf{QDQ}^{-1}\right) = \mathbf{QD}^4\mathbf{Q}^{-1}$<br><br>$\therefore\ \mathbf{E} = \mathbf{D}^4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4^4 & 0 \\ 0 & 0 & 9^4 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 256 & 0 \\ 0 & 0 & 6561 \end{pmatrix}$ and $\mathbf{P} = \mathbf{Q}$ |

**7a**

$x = 3t^5 - 15t,\quad y = -10t^3,\qquad 1 \le t \le \sqrt{2}.$

$\dfrac{dx}{dt} = 15t^4 - 15,\quad \dfrac{dy}{dt} = -30t^2$



Surface area generated

$$= 2\pi\int_1^{\sqrt{2}} (-y)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

$$= 2\pi\int_1^{\sqrt{2}} 10t^3\sqrt{(15t^4 - 15)^2 + (-30t^2)^2}\ dt$$

$$= 300\pi\int_1^{\sqrt{2}} (t^3)\sqrt{(t^4 - 1)^2 + 4t^4}\ dt$$

$$= 300\pi\int_1^{\sqrt{2}} (t^3)\sqrt{(t^4 + 1)^2}\ dt$$

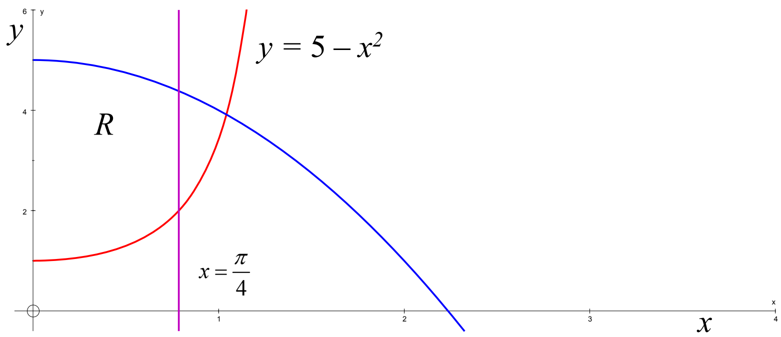
$$= 300\pi\int_1^{\sqrt{2}} (t^3)(t^4 + 1)\ dt$$

$$= 300\pi\left[\frac{t^8}{8} + \frac{t^4}{4}\right]_1^{\sqrt{2}}$$

$$= 300\pi\left[(2+1) - \frac{3}{8}\right]$$

$$= \tfrac{1575}{2}\pi$$

**b**




Volume required

$$= 2\pi\int_0^{\frac{\pi}{4}} x(y_2 - y_1)\ dx$$

$$= 2\pi\left[\int_0^{\frac{\pi}{4}} x(5 - x^2)\ dx - \int_0^1 (\tan^{-1} t)(1+t^2)\frac{1}{1+t^2}\,dt\right]$$

$$= 2\pi\left[\left[-\tfrac{1}{4}(5 - x^2)^2\right]_0^{\frac{\pi}{4}} - \left[t\tan^{-1} t\right]_0^1 + \int_0^1 t\frac{1}{1+t^2}\,dt\right]$$

$$= 2\pi\left[\tfrac{5}{4} - \tfrac{1}{4}\left(5 - \frac{\pi^2}{16}\right)^2 - \frac{\pi}{4} + \left[\tfrac{1}{2}\ln(1+t^2)\right]_0^1\right]$$

$$= 2\pi\left[\frac{5\pi^2}{32} - \frac{\pi^4}{1024} - \frac{\pi}{4} + \tfrac{1}{2}\ln(2)\right]$$

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| | Alternatively,<br>$y=1+\tan^2 x=\sec^2 x$<br>$=2\pi\int_0^{\frac{\pi}{4}} x(5-x^2-\sec^2 x)\,\mathrm{d}x$<br>$=2\pi\left[\frac{5}{2}x^2-\frac{1}{4}x^4\right]_0^{\frac{\pi}{4}}-2\pi\int_0^{\frac{\pi}{4}} x\sec^2 x\,\mathrm{d}x$<br>$=2\pi\left(\frac{5\pi^2}{32}-\frac{\pi^4}{1024}\right)-2\pi\left(\left[x\tan x\right]_0^{\frac{\pi}{4}}-\int_0^{\frac{\pi}{4}}\tan x\,\mathrm{d}x\right)$<br>$=2\pi\left(\frac{5\pi^2}{32}-\frac{\pi^4}{1024}-\frac{\pi}{4}-\left[\ln\lvert\cos x\rvert\right]_0^{\frac{\pi}{4}}\right)$<br>$=2\pi\left(\frac{5\pi^2}{32}-\frac{\pi^4}{1024}-\frac{\pi}{4}+\frac{1}{2}\ln 2\right)$ |
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| Qn | Suggested solutions |
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| 8 | $r=\frac{3}{1-2\cos\theta}$<br><br>$r-2r\cos\theta=3$<br>$r-2x=3$<br><br>$r^2=(2x+3)^2$<br>$x^2+y^2=(2x+3)^2$<br><br>Simplify,<br>$y^2=(2x+3)^2-x^2=3(x+1)(x+3)=3\left[(x+2)^2-1\right]$<br>$\frac{(x+2)^2}{1^2}-\frac{y^2}{\left(\sqrt{3}\right)^2}=1$ where $x\geq -1$<br><br>Curve is hyperbola, with $e=2$ and directrix $x=-\frac{3}{2}$. |
| (ii) | $e=\frac{4r}{3}$<br>$r=\frac{3}{2}=\frac{3}{1-2\cos\theta}$<br>$1-2\cos\theta=2$<br>$\cos\theta=-\frac{1}{2} \quad \Rightarrow \quad \theta=\frac{2\pi}{3},\frac{4\pi}{3}$ |

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| | Polar coordinates of $P$ and $P'$ are $\left(\frac{3}{2},\frac{2\pi}{3}\right)$ and $\left(\frac{3}{2},\frac{4\pi}{3}\right)$. |
| (iii) | Area of R = $2\left(\frac{1}{2}\int_{\frac{2\pi}{3}}^{\pi} r^2 \mathrm{d}\theta\right)=\int_{\frac{2\pi}{3}}^{\pi}\left(\frac{3}{1-2\cos\theta}\right)^2 \mathrm{d}\theta$ <br> = 1.39751 = 1.40 sq. units <br> To find arc length, <br> $r=\frac{3}{1-2\cos\theta}$ , <br> $\frac{\mathrm{d}r}{\mathrm{d}\theta}=\frac{-6\sin\theta}{(1-2\cos\theta)^2}$ <br> $L=2\int_{\frac{2\pi}{3}}^{\pi}\sqrt{r^2+\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}\ \mathrm{d}\theta$ <br> $=2\int_{\frac{2\pi}{3}}^{\pi}\sqrt{\left(\frac{3}{1-2\cos\theta}\right)^2+\left(\frac{-6\sin\theta}{(1-2\cos\theta)^2}\right)^2}\mathrm{d}\theta$ <br> = 2.65842 units <br> Thus, Perimeter of $R$ = 2.65842 + 2 (3/2) = 5.66 units (to 3 sig. fig) |

| Qn | Suggested Solution |
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| **9(i)** | **Scheme B:** <br> $u_n=1.05u_{n-1}+0.115u_{n-2}$, for $n\geq 2$, $u_0=0, u_1=30000$ <br> $a=1.05$ <br> $b=0.115$ <br><br> **Scheme A:** $u_n=1.05u_{n-1}-1000$ , for $n\geq 2$, $u_1=30000$ |
| **(ii)** | For Scheme B: <br> Characteristic equation is $m^2-1.05m-0.115=0$. <br> $\therefore m=1.15$ or $m=-0.1$ <br> Thus, general solution is <br> $u_n=A(1.15)^n+B(-0.1)^n$. <br> From initial conditions, <br> $u_0=0=A+B$ $---(1)$ <br> $u_1=30000=1.15A-0.1B$ $---(2)$ |

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| | Solving : $A = 24000, B = -24000$.<br>Hence, particular solution is<br>$$u_n = 24000\left((1.15)^n - (-0.1)^n\right)$$<br>For Scheme A:<br>$u_n = 1.05u_{n-1} - 1000$<br>$= 1.05(1.05u_{n-2} - 1000) - 1000$<br>$= 1.05^2 u_{n-2} - 1000(1 + 1.05)$<br>$= 1.05^{n-1}u_1 - 1000(1 + 1.05 + 1.05^2 + ... + 1.05^{n-2})$<br>$= 1.05^{n-1}u_1 - 1000\left(\dfrac{1.05^{n-1} - 1}{0.05}\right)$<br>$= 10000(1.05^{n-1}) + 20000$ |
| **(iii)** | Let $v_m$ denote the amount of money invested in the EM fund based on Mr Ma's second investment through scheme *A*.<br>$v_m = 60000\left(1.05^{m-1}\right) + 20000$<br><br>Since $n - m = 4$,<br>$u_{m+4} = 24000\left(1.15^{m+4} - \left(-0.1\right)^{m+4}\right)$<br><br>For $u_{m+4} > v_m$,<br>From GC, $m \geq 6$<br>In year 2022, Mr Ma's amount of money from the first investment will exceed that of the second investment in the EM fund. |
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| Qn | Suggested Solutions |
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| **10i** | Volume of water at time $t$ = 10000 + 20$t$ – 10$t$ = 10000 + 10$t$ <br><br> $\frac{\mathrm{d}x}{\mathrm{d}t} =$ rate of salt entering – rate of salt leaving <br><br> $\frac{\mathrm{d}x}{\mathrm{d}t} = 20\left(\frac{2}{100}\right) - 10\left(\frac{x}{10000+10t}\right)$ <br> $= \frac{4}{10} - \frac{x}{1000+t}$ |
| **ii** | When the brine overflows, <br> 10000 + 10$t$ = 20000 $\Rightarrow t = 1000$ <br><br> Euler's formula: <br> $x_{n+1} = x_n + h \cdot \mathrm{f}(t_n, x_n), \quad \mathrm{f}(t_n, x_n) = 0.4 - \frac{x_n}{1000+t_n}$ <br> Step size $h = 20 \Rightarrow x_{n+1} = x_n + 20\left(0.4 - \frac{x_n}{1000+t_n}\right)$ <br> Using $t_0 = 960, x_0 = 341,$ <br> $\mathrm{f}(t_0, x_0) = \left(0.4 - \frac{x_0}{1000+t_0}\right) = 0.4 - \frac{341}{1000+960} = 0.22602$, <br> $x_1 = 341 + 20(0.22602) = 345.520$ <br> $\mathrm{f}(t_1, x_1) = \left(0.4 - \frac{x_1}{1000+t_1}\right) = 0.4 - \frac{345.520}{1000+980} = 0.22549$, <br> $x_2 = 345.520 + 20(0.22549) = 350.03$ (2 d.p) <br><br> The amount of salt when the brine in the tank begins to overflow is 350.03 kg. |
| **iii** | $\frac{\mathrm{d}y}{\mathrm{d}t} = a + b\mathrm{e}^{-\frac{(t+1000)^2}{1000000}}$ <br><br> At $t = 0$, $\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t} = 0.4 - \frac{10000/100}{(1000+0)} = 0.3$ <br> $\therefore 0.3 = a + \frac{b}{\mathrm{e}} \Rightarrow b = (0.3-a)\mathrm{e}$ <br> Change in $y$ <br> $= \int_{960}^{1000} a + (0.3-a)\mathrm{e}^{1-\frac{(t+1000)^2}{1000000}} \, \mathrm{d}t$ <br> $= 40a + (0.3-a)\int_{960}^{1000} \mathrm{e}^{1-\frac{(t+1000)^2}{1000000}} \, \mathrm{d}t$ |

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| | Let $\mathrm{g}(t)=\mathrm{e}^{1-\frac{(t+1000)^2}{1000000}}$<br>5 ordinates, i.e. 4 strips, each with width 10.<br>Change in $y$<br>$\approx 40a+(0.3-a)\left[\frac{10}{3}\left(\mathrm{g}(960)+4\mathrm{g}(970)+2\mathrm{g}(980)+4\mathrm{g}(990)+\mathrm{g}(1000)\right)\right]$<br>$=37.841a+0.64753$<br><br>Since the same amount of salt is added in the two tanks from 960 s to 1000 s,<br>$37.841a+0.64753=350.03-341$<br>$a=0.222$ |
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