1 The position (x, y) of a moving object at time t is given by

$$x = \frac{2}{3a}\left(at + \frac{3a^2}{4}\right)^{\frac{3}{2}}, \; y = \frac{1}{2}\left(t^2 + at\right),$$

where a is a positive constant.

Given that the object travelled a distance of $\frac{21}{2}$ units from $t = 0$ to $t = 3$, find the value of a. [5]

2 Use de Moivre's theorem to express $\cos 3\theta$ in terms of $\cos\theta$. [2]

Hence find the roots of $8x^3 - 6x + \sqrt{3} = 0$, leaving your answers in a trigonometric form. [4]

3 **(a)** Sketch the graph of $y = \mathrm{e}^{-\frac{x^2}{2}}$, stating the equation of any asymptote and the coordinates of any point of intersection with the axes. [2]

(b) A normal distribution with mean $\mu = 0$ and standard deviation $\sigma = 1$ is called a standard normal distribution. Let Z be the random variable that follows a standard normal distribution. Then, the probability $\mathrm{P}(0 \leqslant Z \leqslant z)$ is given by the integral $\int_0^z \mathrm{f}(x)\,\mathrm{d}x$, where $\mathrm{f}(x) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{x^2}{2}}$.

Given that the total area between the curve $y = \mathrm{f}(x)$ and the x-axis is equal to one, use Simpson's rule with 5 ordinates to approximate the probability $\mathrm{P}(Z \leqslant 1.6)$, leaving your answer to 5 decimal places. [4]

4 The elliptical orbit of a satellite around a planet can be modelled by the polar equation

$$r = \frac{a}{1 - e\cos\theta},$$

where the pole represents the centre of the planet.

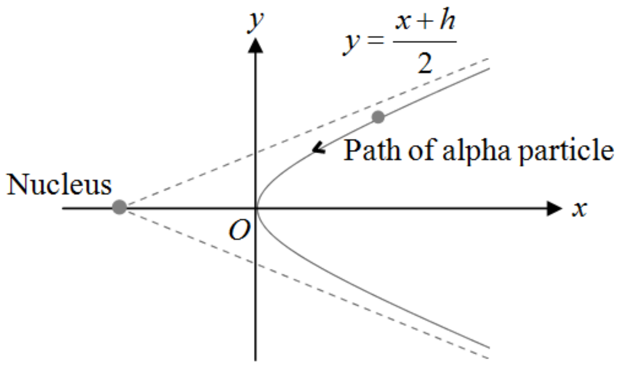
Using appropriate units of measurement, the greatest distance and least distance of the satellite from the centre of the planet are 6 units and 4 units respectively. Find the values of a and e. [2]

Assume that the satellite's path also follows Kepler's Second Law, which states that a line joining the satellite to the centre of the planet sweeps out equal areas in equal intervals of time. If the satellite takes 92.3 Earth days to make one complete orbit, determine how long it takes to travel from $\theta = \frac{\pi}{2}$ to $\theta = \pi$. [2]

Determine the cartesian equations of the two directrices of the orbit. [3]

5 The hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ has eccentricity e and its foci have coordinates $(c, 0)$ and $(-c, 0)$, where a, b, c are positive numbers. By considering the definition of eccentricity and without quoting other results for a hyperbola, show that $c = ae$. [2]

The physicist Ernest Rutherford discovered that when alpha particles were shot towards the nucleus of an atom, they were eventually repelled away from the nucleus along hyperbolic paths. The figures shows the path of an alpha particle, where one of the asymptotes of the hyperbolic path has equation $y=\frac{x+h}{2}$, with the alpha particle subsequently passing through the origin, when it is nearest the nucleus. Determine the eccentricity of the hyperbolic path. [3]



If the focus of the hyperbolic path has coordinates $(3\sqrt{5}-6, 0)$, determine the cartesian equation of the hyperbola. [2]

6 It is given that $u_n = 9\left(4^n\right)+2$ is a particular solution to the recurrence relation

$$u_n = au_{n-1} + bu_{n-2},$$

for all positive integers n.

(i) Find the values of a and b. [2]

(ii) Use the method of induction to show that $4^{n-1} \ldots n^2$, for all positive integers n. [5]

(iii) Hence, show $9+27+45+\ldots+(18n+9) < u_n$, [3]

7 $\begin{pmatrix} k \\ -9 \\ 2k \end{pmatrix}$ and $\begin{pmatrix} 5 \\ -6 \\ k \end{pmatrix}$ are eigenvectors of the matrix $\mathbf{M}=\begin{pmatrix} 6 & k & 1 \\ -6 & -1 & 3 \\ 8 & 8 & k \end{pmatrix}$. Find a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{M}^3 = \mathbf{P}^{-1}\mathbf{D}\mathbf{P}$. [11]

8 The Lotka-Volterra equations, also known as the predator-prey equations, are frequently used to describe the dynamics of the interaction of two species, one as a predator and the other as a prey.

In a long term observation of the population of foxes and rabbits in a certain region, a team of scientists agreed the population is well modelled by the equation

$$14 = -0.0002z + 0.8\ln z - 0.01x + 2\ln x,$$

where z and x represent the number of rabbits and foxes respectively.

(i) When the number of rabbits is 2500, show that the number of foxes, α, can be approximated by the solution of $2\ln x - 0.01x - 8.2408 = 0$. [1]

(ii) Show that there exists a value of α in the interval $[100,110]$. Use linear interpolation over this interval to obtain a first approximation, x_1, for α, leaving your answer in 2 decimal places. Determine, with a reason, if x_1 is an over-estimate or under-estimate for α. [4]

(iii) Using the Newton-Raphson iteration with the value of x_1 found in part **(ii)**, find the value of α, correct to 2 decimal places. [2]

(iv) Another method to solve for α is to consider the equation

$$100(2\ln x - 8.2408) = x.$$

By considering the graph of $y = x$ and another appropriate graph, determine if we will be able to obtain the value of α, if the value of the initial approximation x_1 is such that $x_1 < \alpha$. [3]

9 The displacement, x cm, from the origin of a particle moving on the x-axis varies with time, t s, according to the differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2\frac{\mathrm{d}x}{\mathrm{d}t} = \mathrm{e}^{-2t}. \text{ ------ (1)}$$

Find the general solution of this differential equation. [6]

The particle is initially at the origin. Sketch two members of the family of solution curves given that as $t \to \infty$, $x \to 0$ for one curve and as $t \to \infty$, $x \to -2$ for the other curve. Indicate clearly the coordinates of the stationary point on each curve. [6]

Given further that the particle is initially at rest, find the particular solution of (1) and show that after a long period of time, the particle will approach a point E. Find the displacement of E from the origin. [2]

10 **(a)** The region bounded by the curve $y = 4x^3 - 4x^2 + x - 1$, the x-axis, the y-axis and the line $y = 5$ is rotated completely about the y-axis to form a solid of revolution. Find the exact volume of the solid formed. [4]

(b) **(i)** Show that

$$\int x \ln x \, dx = \frac{x^2}{2} \ln x - \frac{x^2}{4} + C,$$

where C is an arbitrary constant. [2]

(ii) Find the curved surface area formed when the portion of the curve $y = \frac{1}{2}x^2 - \frac{1}{4}\ln x$ from $x = 2$ to $x = 4$ is rotated 2π radians about the x-axis. Give your answer in the form

$$\pi\left[A + B\ln 2 + C(\ln 2)^2\right],$$

where A, B and C are constants to be determined. [6]

11 An industrial cooler initially contains 50 litres of pure water. When the temperature of the water rises to 60 $^{\circ}$C, a coolant containing 30 grams of chemical X in every litre of water is added into the cooler at a constant rate of 4 litres per minute. The mixture is instantaneously mixed thoroughly and flows out of the cooler at a constant rate of 3 litres per minute through another outlet.

(i) If x grams is the amount of chemical X in the cooler t minutes after the coolant is added, show that x satisfies the differential equation

$$\frac{dx}{dt} = 120 - \frac{3x}{50 + t}.$$ [2]

(ii) Find the amount of chemical X in the cooler when there are 51 litres of mixture in it. [6]

The temperature of the mixture, θ $^{\circ}$C follows the differential equation

$$\frac{d\theta}{dt} = 60e^{0.04t}\left[\cos(0.01\theta) - 1\right].$$

Use the improved Euler method with step size 0.5 to estimate to 1 decimal place, the temperature of the mixture 1 minute after the start of the addition of coolant. [4]