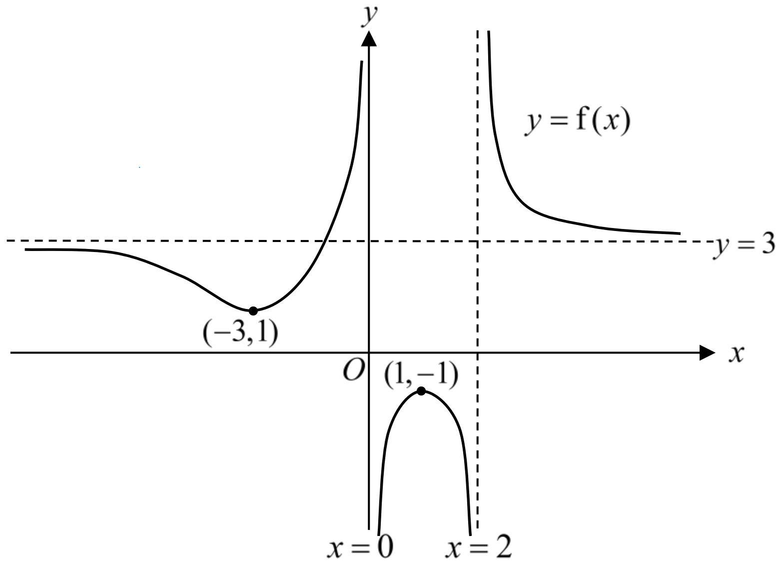
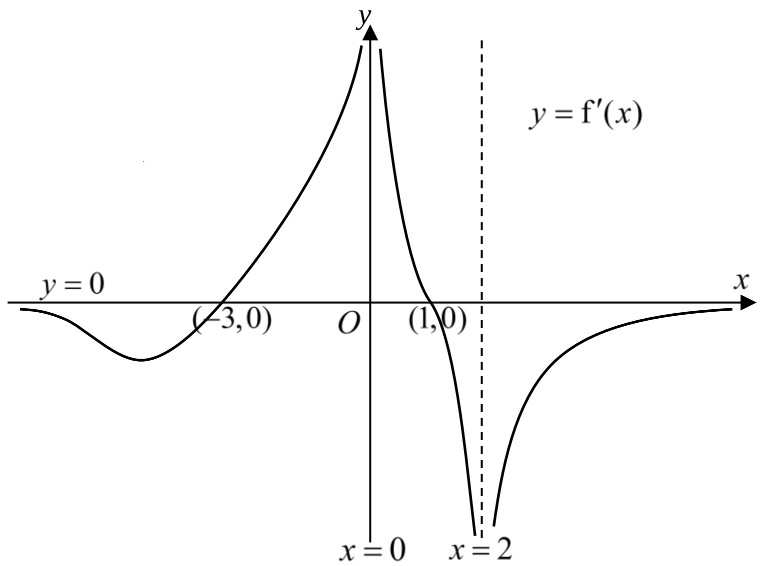
H2 Mathematics 2017 Preliminary Exam Paper 1 Solutions

1	The graph of $y = \mathrm{f}(x)$ is shown below.  **(a)** The graph of $y = \mathrm{f}(2 - x)$ is obtained when the graph of $y = \mathrm{f}(x)$ undergoes a sequence of transformations. Describe the sequence of transformations. [2] **(b)** Sketch the graph of $y = \mathrm{f}'(x)$, stating the equations of any asymptotes and the coordinates of any points of intersection with the axes. [3]
	(a) **Translation** of 2 **units** in the negative x-**direction**, followed by **reflection about** the y-axis. (b) 

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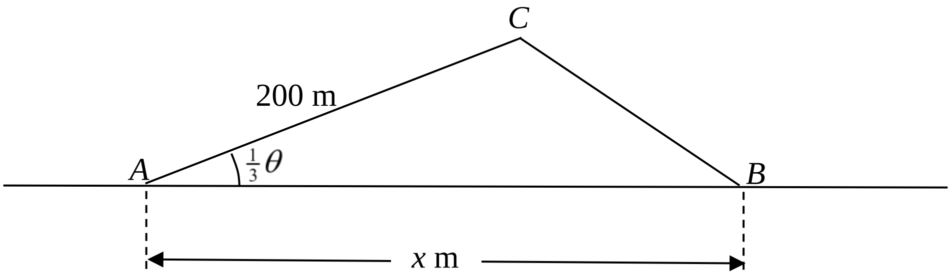


The diagram shows two points at ground level, A and B. The distance in metres between A and B is denoted by x. The angle of elevation of C from B is twice the angle of elevation of C from A. The distance AC is 200 m and $\angle BAC = \frac{1}{3}\theta$ radians. Show that

$$x = \frac{200\sin\theta}{\sin\frac{2}{3}\theta}.$$ [2]

It is given that θ is a small angle such that θ^4 and higher powers of θ are negligible. By using appropriate expansions from the List of Formulae (MF26), show that

$$x \approx \frac{2700-250\theta^2}{9}.$$ [4]

$\angle ABC = 2\times\angle BAC = \dfrac{2\theta}{3} \quad \Rightarrow \quad \angle ACB = \pi-\theta$

Using Sine rule, $\dfrac{x}{\sin(\pi-\theta)} = \dfrac{200}{\sin\left(\dfrac{2\theta}{3}\right)}$

Since $\sin(\pi-\theta) = \sin\pi\cos\theta - \cos\pi\sin\theta = \sin\theta$, we have

$$\frac{x}{\sin\theta} = \frac{200}{\sin\left(\dfrac{2\theta}{3}\right)} \quad \Rightarrow \quad x = \frac{200\sin\theta}{\sin\left(\dfrac{2}{3}\theta\right)} \qquad \text{(Shown)}$$

$$x = \frac{200\sin\theta}{\sin\left(\dfrac{2}{3}\theta\right)} \approx \frac{200\left(\theta-\dfrac{\theta^3}{3!}\right)}{\left(\dfrac{2}{3}\theta-\dfrac{\left(\frac{2}{3}\theta\right)^3}{3!}\right)}$$

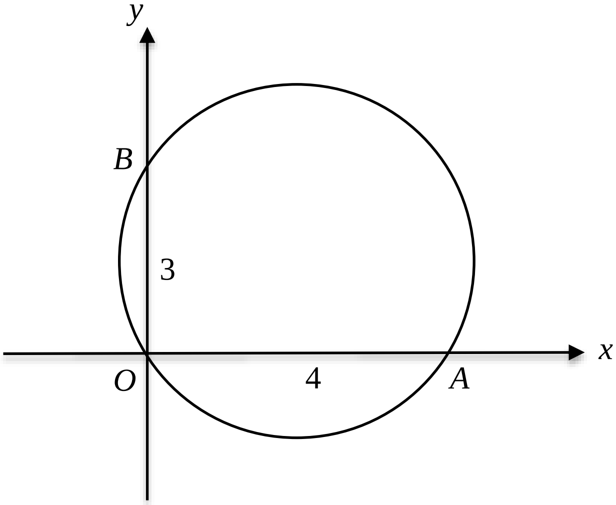
$$= 300\left(1-\frac{\theta^2}{6}\right)\left(1-\frac{2\theta^2}{27}\right)^{-1}$$

$$= 300\left(1-\frac{\theta^2}{6}\right)\left(1+\frac{2\theta^2}{27}+\ldots\right)$$

$$= 300\left(1-\frac{5\theta^2}{54}+\ldots\right)$$

$$\approx \frac{2700-250\theta^2}{9}$$

3



The diagram above shows a circle C which passes through the origin O and the points A and B. It is given that $OA = 4$ units and $OB = 3$ units.

(i) Show that the coordinates of the centre of C is $\left(2, \frac{3}{2}\right)$. Hence write down the equation of C in the form $(x-2)^2 + \left(y - \frac{3}{2}\right)^2 = r^2$, where r is a constant to be determined. [2]

(ii) By adding a suitable line to the diagram above, find the range of values of m for which the equation $mx - \frac{3}{2} = \sqrt{\frac{25}{4} - (x-2)^2}$ has a solution. [4]

(i) Since ΔAOB is a right-angle in a semi-circle, AB forms the diameter of the circle.

Hence, centre of circle is at the mid point of AB, i.e., $\left(2, \frac{3}{2}\right)$. Since $AB = 5$, $r = \frac{5}{2}$

Therefore, equation of C is $(x-2)^2 + \left(y - \frac{3}{2}\right)^2 = \left(\frac{5}{2}\right)^2$

(ii) C: $y - \frac{3}{2} = \pm\sqrt{\frac{25}{4} - (x-2)^2}$

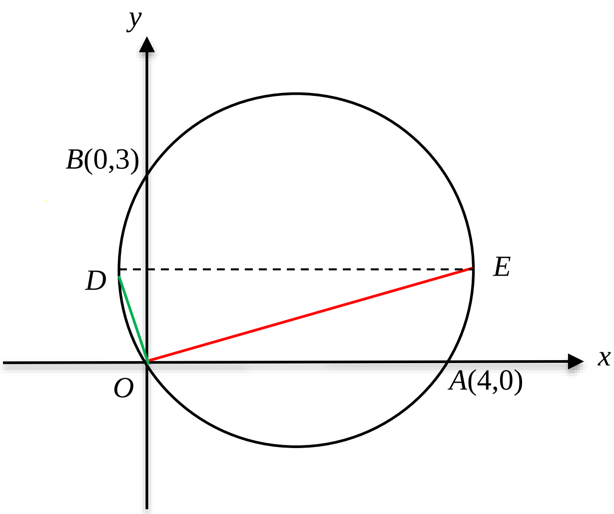
Suitable line to add: $y = mx$

$D = \left(-\frac{1}{2}, \frac{3}{2}\right)$ and $E = \left(\frac{9}{2}, \frac{3}{2}\right)$

Gradient of line $OD = -\frac{3/2}{1/2} = -3$

Gradient of line $OE = \frac{3/2}{9/2} = \frac{1}{3}$

$\therefore\ m \le -3$ or $m \ge \frac{1}{3}$



4 The curve C has equation $y = \sin 2x + 2\cos x$, $0 \le x \le 2\pi$.

(i) Using an algebraic method, find the exact x-coordinates of the stationary points. [You do not need to determine the nature of the stationary points.] [3]

(ii) Sketch the curve C, indicating clearly the coordinates of the turning points and the intersection with the axes. [1]

(iii) Find the area bounded by the curve C and the line $y = \dfrac{1}{\pi}x$. [3]

(i) $y = \sin 2x + 2\cos x$

$$\frac{dy}{dx} = 2\cos 2x - 2\sin x$$

For stationary points, $\dfrac{dy}{dx} = 0$

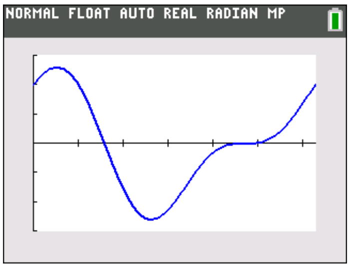
$$2(1 - 2\sin^2 x) - 2\sin x = 0 \;\Rightarrow\; 2\sin^2 x + \sin x - 1 = 0$$

$$\Rightarrow (2\sin x - 1)(\sin x + 1) = 0$$

$$\Rightarrow \sin x = 0.5 \quad \text{or} \quad \sin x = -1$$

$$\Rightarrow x = \frac{\pi}{6},\; x = \frac{5\pi}{6} \quad \text{or} \quad x = \frac{3\pi}{2}$$

(ii)



(iii) From GC, the line $y = \dfrac{1}{\pi}x$ intersects the curve C at $x = 1.4544031$ and at $x = 2\pi$

Required area $= \displaystyle\int_{1..4544031}^{2\pi} \left[\frac{1}{\pi}x - (\sin 2x + 2\cos x)\right] dx = 8.92$ (to 3 sig figs)

5 The curve C has equation $y = kx^3$. The tangent at the point P on C meets the curve again at point Q. The tangent at point Q meets the curve again at point R. It is given that the x-coordinates of P, Q and R are p, q, and r respectively, where $p \ne 0$.

(i) Show that p and q satisfy the equation $\left(\dfrac{q}{p}\right)^2 + \left(\dfrac{q}{p}\right) - 2 = 0$. [4]

(ii) Show that p, q and r are three consecutive terms of a geometric progression. Hence determine if this geometric series is convergent. [4]

[You may use the identity $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$ for $a, b \in$ □]

	(i) $y = kx^3 \quad \Rightarrow \quad \dfrac{\mathrm{d}y}{\mathrm{d}x} = 3kx^2$ Point $P = (p,\ kp^3)$, Point $Q = (q,\ kq^3)$, Point $R = (r,\ kr^3)$ Equation of tangent at point P: $y - kp^3 = 3kp^2\left(x - p\right)$ When tangent meets the curve again at Q: $kq^3 - kp^3 = 3kp^2(q-p)$ $\Rightarrow \quad q^3 - p^3 = 3p^2(q-p)$ $\Rightarrow \quad (q-p)(q^2+pq+p^2) = 3p^2(q-p)$ $\Rightarrow \quad (q-p)(q^2+pq-2p^2) = 0$ $\Rightarrow \quad q^2+pq-2p^2 = 0 \quad$ since $p \neq q$ Dividing by p^2: $\left(\dfrac{q}{p}\right)^2 + \left(\dfrac{q}{p}\right) - 2 = 0$ (Shown) (ii) $\left(\dfrac{q}{p}\right)^2 + \left(\dfrac{q}{p}\right) - 2 = 0 \quad \Rightarrow \quad \left(\dfrac{q}{p}+2\right)\left(\dfrac{q}{p}-1\right) = 0 \quad \Rightarrow \quad \dfrac{q}{p} = -2$ or $\dfrac{q}{p} = 1$ (rejected) Similarly for the other case, $\dfrac{r}{q} = -2$ $\therefore \dfrac{q}{p} = \dfrac{r}{q} = -2$ Since the common ratio is the same, p, q and r are 3 consecutive terms of a geometric progression. As $\lvert \text{common ratio} \rvert = 2 > 1$, the geometric series is not convergent.

6 **(a)** The vectors **a** and **b** are the position vector of points A and B respectively. It is given that $OA = 2\sqrt{7}$, $\mathbf{b} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ and $\mathbf{a} \cdot \mathbf{b} = -14$.

(i) Find angle AOB. [2]

(ii) State the geometrical meaning of $|\hat{\mathbf{a}} \cdot \mathbf{b}|$, where $\hat{\mathbf{a}}$ is the unit vector of **a**. [1]

(iii) Hence or otherwise, find the position vector of the foot of perpendicular from B to line OA in terms of **a**. [2]

(b) The non-zero vectors **p** and **q** are such that $|\mathbf{p} \times \mathbf{q}| = 2$. Given that **p** is a unit vector and $\mathbf{q} \cdot \mathbf{q} = 4$, show that **p** and **q** are perpendicular to each other. [3]

(a) (i) Given: $|\mathbf{a}| = 2\sqrt{7}$, $\mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$ and $\mathbf{a} \cdot \mathbf{b} = -14$

$$\Rightarrow \quad |\mathbf{a}||\mathbf{b}|\cos AOB = -14$$

$$\Rightarrow \quad (2\sqrt{7})\sqrt{1+4+9}\cos AOB = -14$$

$$\Rightarrow \quad \cos AOB = -\frac{7}{\sqrt{7}\sqrt{14}} = -\frac{1}{\sqrt{2}}$$

$$AOB = 135^\circ$$

(ii) Length of projection of **b** on **a**

(iii) Let N be the foot of perpendicular from B to line OA.

Length of projection, $ON = |\hat{\mathbf{a}} \cdot \mathbf{b}| = \dfrac{|\mathbf{a} \cdot \mathbf{b}|}{|\mathbf{a}|} = \dfrac{|-14|}{2\sqrt{7}} = \sqrt{7} = \dfrac{|\underset{\sim}{a} \cdot \underset{\sim}{b}|}{|\underset{\sim}{a}|} = \dfrac{|-14|}{2\sqrt{7}} = \sqrt{7}$

Since $\angle AOB$ is an obtuse angle, $\overrightarrow{ON} = -\sqrt{7}\dfrac{\mathbf{a}}{|\mathbf{a}|} = -\dfrac{1}{2}\mathbf{a}$

(b) Given: $|\mathbf{p}| = 1$, $\mathbf{q} \cdot \mathbf{q} = 4 \quad \Rightarrow \quad |\mathbf{q}|^2 = 4 \quad \Rightarrow \quad |\mathbf{q}| = 2$

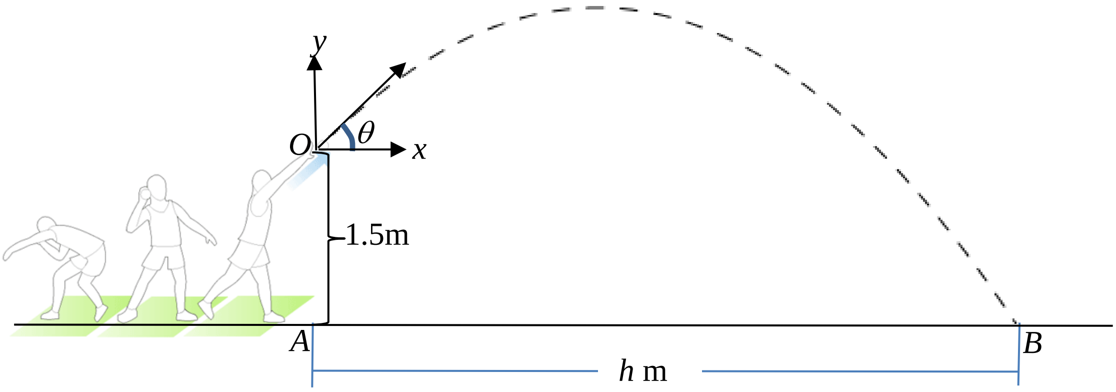
$|\mathbf{p} \times \mathbf{q}| = 2$

$\Rightarrow \quad |\mathbf{p}||\mathbf{q}|\sin\theta = 2$ where θ is the angle between **p** and **q**

$\Rightarrow \quad \sin\theta = \dfrac{2}{2} = 1$

$\Rightarrow \quad \theta = 90^\circ$

Thus, **p** and **q** are perpendicular to each other (shown)

7	 The diagram shows a shot put being projected with a velocity v ms^{-1} from the point O at an angle θ made with the horizontal. The point O is 1.5m above the point A on the ground. The x-y plane is taken to be the plane that contains the trajectory of this projectile motion with x-axis parallel to the horizontal and O being the origin. The equation of the trajectory of this projectile motion is known to be $$y = x\tan\theta - \frac{gx^2}{2v^2\cos^2\theta},$$ where g ms^{-2} is the acceleration due to gravity. The constant g is taken to be 10 and the distance between A and B is denoted by h m. Given that $v = 10$, show that h satisfies the equation $$h^2 - 10h\sin 2\theta - 15\cos 2\theta - 15 = 0 \qquad [3]$$ As θ varies, h varies. Show that stationary value of h occurs when θ satisfies the following equation $$3\tan^2 2\theta - 20\sin 2\theta\tan 2\theta - 20\cos 2\theta - 20 = 0. \qquad [5]$$ Hence find the stationary value of h. [2]
	$y = x\tan\theta - \frac{10x^2}{2(10)^2\cos^2\theta} \quad \Rightarrow \quad y = x\tan\theta - \frac{x^2}{20\cos^2\theta}$ When $x = h$ $y = -1.5$ $\therefore \quad -1.5 = h\tan\theta - \frac{h^2}{20\cos^2\theta}$ $\Rightarrow \quad -30\cos^2\theta = 20h\tan\theta\cos^2\theta - h^2$ $\Rightarrow \quad h^2 - 20h\sin\theta\cos\theta - 30\cos^2\theta = 0$ □ $\Rightarrow \quad h^2 - 10h\sin 2\theta - 15(1+\cos 2\theta) = 0$ $\Rightarrow \quad h^2 - 10h\sin 2\theta - 15\cos 2\theta - 15 = 0 \quad (*)$ □ Differentiate both sides w.r.t. θ, we have $2h\frac{dh}{d\theta} - 10\frac{dh}{d\theta}\sin 2\theta - 20h\cos 2\theta + 30\sin 2\theta = 0$ At stationary value, $\frac{dh}{d\theta} = 0 \Rightarrow -20h\cos 2\theta + 30\sin 2\theta = 0 \Rightarrow h = \frac{3}{2}\tan 2\theta$

Sub into (*), we have

$$\left(\frac{3}{2}\tan 2\theta\right)^2 - 10\left(\frac{3}{2}\tan 2\theta\right)\sin 2\theta - 15\cos 2\theta - 15 = 0$$

$$\Rightarrow \quad \frac{9}{4}\tan^2 2\theta - 15\tan 2\theta \sin 2\theta - 15\cos 2\theta - 15 = 0$$

$$\Rightarrow \quad 3\tan^2 2\theta - 20\sin 2\theta \tan 2\theta - 20\cos 2\theta - 20 = 0 \quad \text{(shown)}$$

Using GC, $\theta = 0.71999$ (5 sig fig)

Therefore, max $h = \frac{3}{2}\tan 2(0.71999) = 11.4$ (3 sig fig) □□□ □□

8 **(a)** In an Argand diagram, points P and Q represent the complex numbers $z_1 = 2+3\mathrm{i}$ and $z_2 = \mathrm{i}z_1$.

(i) Find the area of the triangle OPQ, where O is the origin. [2]

(ii) z_1 and z_2 are roots of the equation $(z^2+az+b)(z^2+cz+d)=0$, where $a, b, c, d \in \mathbf{R}$. Find a, b, c and d. [4]

(b) Without using the graphing calculator, find in exact form, the modulus and argument of $v^* = \left(\frac{\sqrt{3}+\mathrm{i}}{-1+\mathrm{i}}\right)^{14}$. Hence express v in exponential form. [5]

(a) (i) Since $w = \mathrm{i}z$, then $OP \perp OQ$, i.e. $\angle POQ = 90^\circ$

Area of triangle OPQ $= \frac{1}{2}|z||w| = \frac{1}{2}|2+3\mathrm{i}|^2 = \frac{13}{2}$ units^2

(ii) Since $(z^2+az+b)(z^2+cz+d)=0$ is a polynomial with constant coefficients, complex roots occur in conjugate pairs. Therefore, the four roots are $2+3\mathrm{i}$, $2-3\mathrm{i}$, $-3+2\mathrm{i}$ and $-3-2\mathrm{i}$.

$$\left[z-(2+3\mathrm{i})\right]\left[z-(2-3\mathrm{i})\right]\left[z-(-3+2\mathrm{i})\right]\left[z-(-3-2\mathrm{i})\right]$$

$$=(z^2-4z+13)(z^2+6z+13)$$

$$\therefore \quad a=-4,\ b=13,\ c=6,\ d=13$$

(b) $$|v^*| = \frac{|\sqrt{3}+\mathrm{i}|^{14}}{|-1+\mathrm{i}|^{14}} = \frac{2^{14}}{(\sqrt{2})^{14}} = 2^7$$

$$\arg(v^*) = \arg\left(\frac{\sqrt{3}+\mathrm{i}}{-1+\mathrm{i}}\right)^{14} = 14\left[\arg(\sqrt{3}+\mathrm{i}) - \arg(-1+\mathrm{i})\right] = 14\left[\frac{\pi}{6} - \frac{3\pi}{4}\right] = -\frac{49\pi}{6}$$

$$\therefore \quad \arg(v^*) = -\frac{\pi}{6} \quad \Rightarrow \quad \arg(v) = \frac{\pi}{6}$$

Since $|v| = |v^*| = 2^7$, then $v = 2^7 e^{\mathrm{i}\frac{\pi}{6}}$

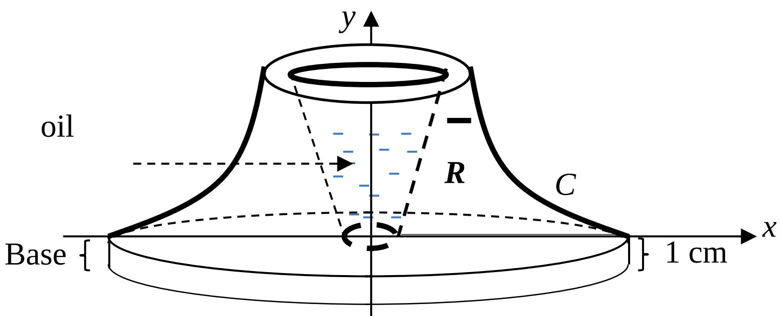
9 A curve C has parametric equation defined by

$$x = 4\sec t \text{ and } y = 8(1-\tan t)\text{, where } -\tfrac{1}{4}\pi \le t \le \tfrac{1}{4}\pi.$$

(i) Find $\dfrac{dy}{dx}$ in terms of t and hence show that the equation of tangent at the point $t = -\frac{1}{6}\pi$ is $y = 4x + 8(1-\sqrt{3})$. [3]

(ii) Find the Cartesian equation of C. [2]

R is the region bounded by C, the tangent in **(i)**, the normal to C at $t = 0$ and the x-axis. Part of an oil burner is formed by rotating R completely about the y-axis as shown in the diagram below (not drawn to scale). The base of the burner is a solid cylinder of thickness 1 cm. [You may assume each unit along the x and y axis to be 1 cm]



(iii) Find the volume of the material required to make the burner. [6]

(i) $x = 4\sec t$ and $y = 8(1-\tan t)$

$\dfrac{dx}{dt} = 4\sec t\tan t \quad \dfrac{dy}{dt} = -8\sec^2 t$

$\dfrac{dy}{dx} = -\dfrac{2}{\sin t}$

At $t = -\dfrac{\pi}{6}$, gradient of tangent $= 4$, $x = \dfrac{8}{3}\sqrt{3}$ and $y = 8\left(1+\dfrac{\sqrt{3}}{3}\right)$

Equation of tangent is $y - 8\left(1+\dfrac{\sqrt{3}}{3}\right) = 4\left(x - \dfrac{8\sqrt{3}}{3}\right)$, i.e. $y = 4x + 8(1-\sqrt{3})$

(ii) $x = 4\sec t \quad \Rightarrow \quad \sec^2 t = \dfrac{x^2}{16}$

$y = 8(1-\tan t) \quad \Rightarrow \quad \tan^2 t = \left(1-\dfrac{y}{8}\right)^2$

Since $1+\tan^2 x = \sec^2 x$, $1+\left(1-\dfrac{y}{8}\right)^2 = \dfrac{x^2}{16} \quad \Rightarrow \quad \dfrac{x^2}{16} - \dfrac{(y-8)^2}{64} = 1$

where $4 \le x \le 4\sqrt{2}$ and $0 \le y \le 16$

When C intersects x-axis, $y = 0$,

$$\frac{x^2}{16}-\frac{(0-8)^2}{64}=1 \quad \Rightarrow \quad x^2=32 \quad \Rightarrow \quad x=4\sqrt{2} \quad (\because \text{radius}>0)$$

Volume of cylindrical base $=\pi(4\sqrt{2})^2(1)=32\pi$

<u>Method 1</u>

Volume of the solid that made the burner

$$= \frac{\pi}{4}\int_0^8 64+(y-8)^2 \, \mathrm{d}y-\frac{\pi}{16}\int_0^8\left(y-8\left(1-\sqrt{3}\right)\right)^2 \mathrm{d}y+32\pi$$

$\approx 475.718=476$ units^3 (using GC)

<u>Method 2</u>

Volume of solid that made the burner

$$= \pi\int_{\frac{\pi}{4}}^{0}(4\sec t)^2(-8\sec^2 t)\,\mathrm{d}t-\frac{\pi}{16}\int_0^8\left(y-8\left(1-\sqrt{3}\right)\right)^2 \mathrm{d}y+32\pi \approx 476$$

10

The point A has coordinates (3, 1, 1). The line l has equation $\mathbf{r}=\begin{pmatrix}1\\1\\-1\end{pmatrix}+\lambda\begin{pmatrix}2\\1\\1\end{pmatrix}$, where λ is a parameter. P is a point on l when $\lambda = t$.

(i) Find cosine of the acute angle between AP and l in terms of t. Hence or otherwise, find the position vector of the point N on l such that N is the closest point to A. [6]

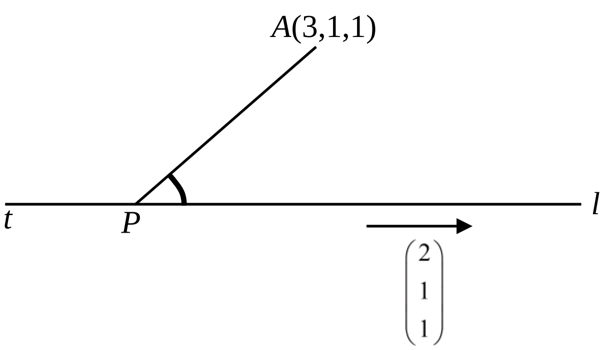
(ii) Find the coordinates of the point of reflection of A in l. [2]

The line L has equation $x=-1,\ 2y=z+2$.

(iii) Determine whether L and l are skew lines. [2]

(iv) Find the shortest distance from A to L. [3]

(i)



P is a point on l with parameter t

□ $\overrightarrow{OP}=\begin{pmatrix}1\\1\\-1\end{pmatrix}+t\begin{pmatrix}2\\1\\1\end{pmatrix},\quad t\in$□

$$\overrightarrow{AP}=\begin{pmatrix}1\\1\\-1\end{pmatrix}+t\begin{pmatrix}2\\1\\1\end{pmatrix}-\begin{pmatrix}3\\1\\1\end{pmatrix}=\begin{pmatrix}-2\\0\\-2\end{pmatrix}+t\begin{pmatrix}2\\1\\1\end{pmatrix}$$

Let θ be the acute angle between BP and l

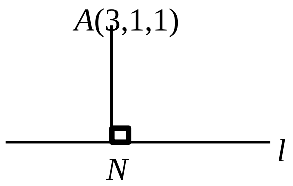
Then, $$\cos\theta=\frac{\left|\overrightarrow{AP}\cdot\begin{pmatrix}2\\1\\1\end{pmatrix}\right|}{\left|\overrightarrow{AP}\right|\left|\begin{pmatrix}2\\1\\1\end{pmatrix}\right|}=\frac{\left|\left[\begin{pmatrix}-2\\0\\-2\end{pmatrix}+t\begin{pmatrix}2\\1\\1\end{pmatrix}\right]\cdot\begin{pmatrix}2\\1\\1\end{pmatrix}\right|}{\sqrt{(-2+2t)^2+t^2+(-2+t)^2}\sqrt{4+1+1}}$$

$$=\frac{\left|(-4-2)+t(4+1+1)\right|}{\sqrt{4t^2-8t+4+t^2+t^2-4t+4}\sqrt{4+1+1}}=\frac{6\left|t-1\right|}{\sqrt{6}\sqrt{6t^2-12t+8}}$$

N is the closest point to A

when $\theta=90^\circ$.

$$\cos 90^\circ=0=\frac{6\left|t-1\right|}{\sqrt{6}\sqrt{6t^2-12t+8}}\quad\Rightarrow\quad t=1$$



Thus, $$\overrightarrow{ON}=\begin{pmatrix}1\\1\\-1\end{pmatrix}+\begin{pmatrix}2\\1\\1\end{pmatrix}=\begin{pmatrix}3\\2\\0\end{pmatrix}$$

(ii) Let A' be the point of reflection of A in l

Using ratio theorem, $\overrightarrow{ON}=\frac{1}{2}(\overrightarrow{OA}+\overrightarrow{OA'})$

$$\Rightarrow\quad \overrightarrow{OA'}=2\overrightarrow{ON}-\overrightarrow{OA}=2\begin{pmatrix}3\\2\\0\end{pmatrix}-\begin{pmatrix}3\\1\\1\end{pmatrix}=\begin{pmatrix}3\\3\\-1\end{pmatrix}$$

Thus, the coordinate of A' are $(3,3,-1)$

(iii) $l:\ \mathbf{r}=\begin{pmatrix}1\\1\\-1\end{pmatrix}+t\begin{pmatrix}2\\1\\1\end{pmatrix},\ t\in\mathbb{R}$

$L:\ x=-1,\ 2y=z+2=\lambda$ i.e., $L:\ \mathbf{r}=\begin{pmatrix}-1\\0\\-2\end{pmatrix}+\lambda\begin{pmatrix}0\\\frac{1}{2}\\1\end{pmatrix}=\begin{pmatrix}-1\\0\\-2\end{pmatrix}+m\begin{pmatrix}0\\1\\2\end{pmatrix},\ m\in\mathbb{R}$

At point of intersection of lines l and L:

$$\begin{pmatrix}1\\1\\-1\end{pmatrix}+t\begin{pmatrix}2\\1\\1\end{pmatrix}=\begin{pmatrix}-1\\0\\-2\end{pmatrix}+m\begin{pmatrix}0\\1\\2\end{pmatrix}\Rightarrow\ t=-1,\ m=0$$

Since the point $(-1, 0, -2)$ lies on both l and L, the two lines intersect and thus cannot be skew lines.

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(iv) $L:\ \mathbf{r}=\begin{pmatrix}-1\\0\\-2\end{pmatrix}+m\begin{pmatrix}0\\1\\2\end{pmatrix}$

Let B be the point $(-1,0,-2)$ on L.

$\overrightarrow{BA}=\begin{pmatrix}3\\1\\1\end{pmatrix}-\begin{pmatrix}-1\\0\\-2\end{pmatrix}=\begin{pmatrix}4\\1\\3\end{pmatrix}$

Shortest distance from A to L

$$=\frac{\left|\overrightarrow{BA}\times\begin{pmatrix}0\\1\\2\end{pmatrix}\right|}{\left|\begin{pmatrix}0\\1\\2\end{pmatrix}\right|}=\frac{1}{\sqrt{1+4}}\left|\begin{pmatrix}4\\1\\3\end{pmatrix}\times\begin{pmatrix}0\\1\\2\end{pmatrix}\right|=\frac{1}{\sqrt{5}}\left|\begin{pmatrix}-1\\-8\\4\end{pmatrix}\right|=\frac{\sqrt{1+64+16}}{\sqrt{5}}=\frac{9\sqrt{5}}{5}$$

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A hot air balloon rises vertically upwards from the ground as the balloon operator intermittently fires and turns off the burner. At time t minutes, the balloon ascends at a rate inversely proportional to $t+\lambda$, where λ is a positive constant. At the same time, due to atmospheric factors, the balloon descends at a rate of 2 km per minute. It is also known that initially the rate of change of the height of the balloon is 1 km per minute.

(i) Find a differential equation expressing the relation between H and t, where H km is the height of the hot air balloon above ground at time t minutes. Hence solve the differential equation and find H in terms of t and λ. [7]

Using $\lambda=15$,

(ii) Find the maximum height of the balloon above ground in exact form. [3]

(iii) Find the total vertical distance travelled by the balloon when $t=8$. [3]

(iv) Can we claim that the rate of change of the height of the balloon above the ground is decreasing? Explain your answer. [2]

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(i) Rate of increase in height $=\frac{k}{t+\lambda}$ where k is a positive constant

Rate of decrease in height = 2

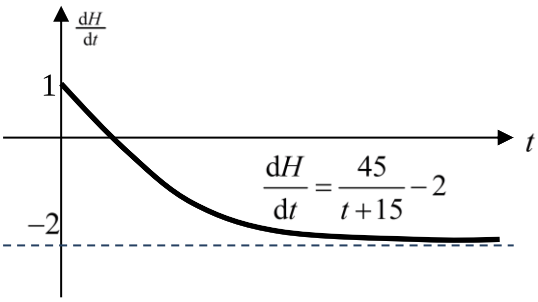
Therefore, $\frac{\mathrm{d}H}{\mathrm{d}t}=\frac{k}{t+\lambda}-2$

Since $\frac{\mathrm{d}H}{\mathrm{d}t}=1$ when $t=0$, we have $1=\frac{k}{0+\lambda}-2 \quad \Rightarrow \quad 1=\frac{k-2\lambda}{\lambda}$

$\therefore k=3\lambda$

Hence, $\frac{\mathrm{d}H}{\mathrm{d}t}=\frac{3\lambda}{t+\lambda}-2$ (Do not combine into one single fraction)

</td></tr>
</table>

	Integrating wrt t: $H=\int\left(\frac{3\lambda}{t+\lambda}-2\right)\mathrm{d}t=3\lambda\ln\lvert t+\lambda\rvert-2t+C$ Since $t+\lambda>0$, we have $H=3\lambda\ln(t+\lambda)-2t+C$ When $t=0$, $H=0$: $0=3\lambda\ln\lambda+C$ $\therefore C=-3\lambda\ln\lambda$ $H=3\lambda\ln(t+\lambda)-2t-3\lambda\ln\lambda$ $\therefore H=3\lambda\ln\left(\frac{t}{\lambda}+1\right)-2t$ (ii) Using $\lambda=15$, at maximum height $\frac{\mathrm{d}H}{\mathrm{d}t}=\frac{45}{t+15}-2=0$ $\therefore t=7.5$ $\therefore H=45\ln\left(\frac{7.5}{15}+1\right)-2(7.5)=15\left(3\ln\frac{3}{2}-1\right)$ (iii) When t = 8, $H=45\ln\left(\frac{8}{15}+1\right)-2(8)=45\ln\frac{23}{15}-16$ Total vertical distance travelled = Vertical distance travelled from $t=0$ to $t=7.5$ + Vertical distance travelled from $t=7.5$ to $t=8$ = $15\left(3\ln\frac{3}{2}-1\right)+\left[15\left(3\ln\frac{3}{2}-1\right)-45\ln\frac{23}{15}+16\right]$ = 3.26 km (correct to 3 s.f.) (iv) $\frac{\mathrm{d}^2H}{\mathrm{d}t^2}=\frac{-45}{(t+15)^2}<0$ for all real values of t, $t\geq 0$ i.e. the rate of change of the height of the balloon above ground is decreasing. Or from the graph of $\frac{\mathrm{d}H}{\mathrm{d}t}=\frac{45}{t+15}-2$, we see that $\frac{\mathrm{d}H}{\mathrm{d}t}$ decreases as t increases. 

− End Of Paper −