2017 Year 6 FM Prelim II P1 Solution

Question 1 [5 Marks] Mathematical Induction

Let P(n) be the statement that $8^{2n} - 3(7^n) + 2$ is divisible by 9 for all positive integers n .

When $n = 1$,

$8^{2n} - 3(7^n) + 2$

$= 8^2 - 3(7) + 2$

$= 45 = 5(9)$

Therefore P(1) is true.

Assume that P(k) is true for some positive integer k.

i.e. $8^{2k} - 3(7^k) + 2$ is divisible by 9

Need to show that P(k+1) is true

i.e. $8^{2k+2} - 3(7^{k+1}) + 2$ is divisible by 9

When $n = k + 1$, consider

$8^{2k+2} - 3(7^{k+1}) + 2$

$= 64\left(8^{2k}\right) - 21(7^k) + 2$

$= 64\left(8^{2k} - 3(7^k) + 2\right) + 171(7^k) - 126$

$= 64\left(9m\right) + 9(19)(7^k) - 9(14)$

where $9m = 8^{2k} - 3(7^k) + 2$ since $8^{2k} - 3(7^k) + 2$ divisible by 9

$= 9\left[64m + 19(7^k) - 14\right]$ where $64m + 19(7^k) - 14 \in \mathbb{Z}$

$\Rightarrow 8^{2k+2} - 3(7^{k+1}) + 2$ is divisible by 9.

Therefore P(k+1) is true.

Since P(1) is true and P(k) is true implies P(k+1) is true, P(n) is true for all positive integers n .

<table>
<tr><td colspan="3">Question 2 [6 Marks] application Of Integration</td></tr>
<tr><td>i</td><td>

$x=\sin 2t\,,\quad y=\cos 2t-\ln\cot t\quad$ where $0<t<\dfrac{\pi}{2}$

$\dfrac{dy}{dt}=-2\sin 2t+\dfrac{\operatorname{cosec}^2 t}{\cot t}$

$=-2\sin 2t+\left(\dfrac{1}{\sin^2 t}\right)\left(\dfrac{\sin t}{\cos t}\right)\left(\dfrac{2}{2}\right)$

$=-2\sin 2t+\dfrac{2}{\sin 2t}$

$\dfrac{dx}{dt}=2\cos 2t$

Therefore

$\left(\dfrac{dx}{dt}\right)^2+\left(\dfrac{dy}{dt}\right)^2$

$=\left(2\cos 2t\right)^2+\left(-2\sin 2t+\dfrac{2}{\sin 2t}\right)^2$

$=4\cos^2 2t+4\sin^2 2t+\dfrac{4}{\sin^2 2t}-8$

$=\dfrac{4}{\sin^2 2t}-4$

$=4\left(\operatorname{cosec}^2 2t-1\right)$

$=4\cot^2 2t$

</td><td></td></tr>
<tr><td>ii</td><td>

surface area generated

$=\displaystyle\int_{t=\frac{\pi}{12}}^{t=\frac{\pi}{6}} 2\pi x\ dS$

$=2\pi\displaystyle\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} x\sqrt{\left(\dfrac{dx}{dt}\right)^2+\left(\dfrac{dy}{dt}\right)^2}\ dt$

$=2\pi\displaystyle\int_{\frac{\pi}{12}}^{\frac{\pi}{6}}\left(\sin 2t\right)\left(2\cot 2t\right)\ dt$

$=4\pi\displaystyle\int_{\frac{\pi}{12}}^{\frac{\pi}{6}}\cos 2t\ dt$

$=2\pi\left[\sin 2t\right]_{\frac{\pi}{12}}^{\frac{\pi}{6}}$

$=\pi\left[\sqrt{3}-1\right]$ units^2

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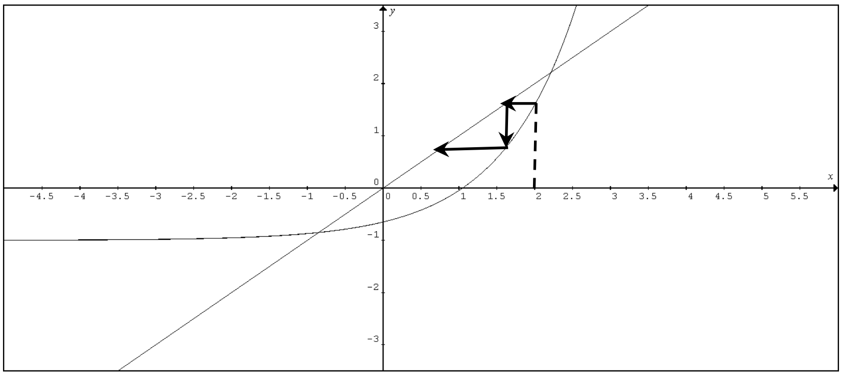
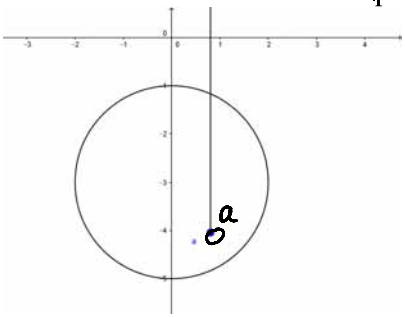
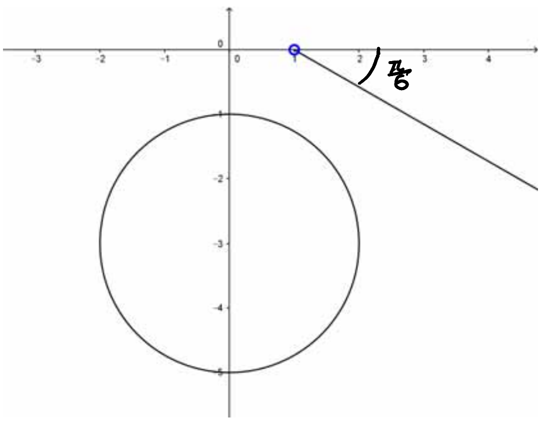
<table>
<tr><td colspan="3">Question 3 [7 marks] 1st Order DE</td></tr>
<tr><td></td><td>$\frac{\mathrm{d}y}{\mathrm{d}x}+2xy=2x\left(x^2+1\right)$
Integrating factor: $e^{\int 2x\,dx}=e^{x^2}$
$e^{x^2}\frac{\mathrm{d}y}{\mathrm{d}x}+2xe^{x^2}y=2xe^{x^2}\left(x^2+1\right)$
$\frac{d}{dx}\left(e^{x^2}y\right)=x^2 2xe^{x^2}+2xe^{x^2}$
$e^{x^2}y=\int x^2 2xe^{x^2}+2xe^{x^2}\ dx$
$=e^{x^2}x^2-\int e^{x^2}2x\ dx+\int 2xe^{x^2}\ dx$
$=e^{x^2}x^2+C$
$\therefore y=x^2+Ce^{-x^2}$ (shown)</td><td></td></tr>
<tr><td>(i)</td><td>$y=x^2+Ce^{-x^2}$
$\frac{dy}{dx}=2x-2xCe^{-x^2}=0$
$x\left(1-Ce^{-x^2}\right)=0$
$\Rightarrow x=0$ or $e^{-x^2}=\frac{1}{C}$ (*)

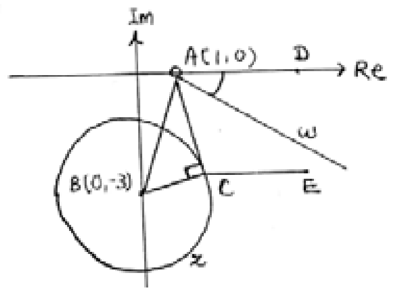
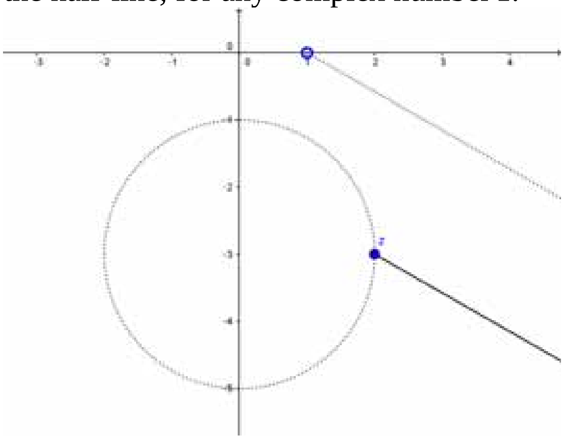
When $C<1$, $\frac{1}{C}\geq 1$, then (*) has no solution.
When $C=1$, $\frac{1}{C}\geq 1$, then (*) has a solution at $x=0$.
Thus, there is only one stationary point.</td><td></td></tr>
<tr><td>(ii)</td><td>When $C>1$, $0<\frac{1}{C}<1$, then (*) has 2 solutions since $y=e^{-x^2}$ is symmetrical about the y-axis. Thus, there are 3 stationary points.</td><td></td></tr>
</table>

Question 4 [7 marks] Numerical Analysis		
	$\mathrm{P}(X \le 1) = 0.35093$ $\mathrm{P}(X = 0) + \mathrm{P}(X = 1) = 0.35093$ $e^{-\lambda} + e^{-\lambda}\lambda = 0.35093$ $e^{-\lambda}(1+\lambda) = 0.35093$	
	$e^{-\lambda}(1+\lambda) = 0.35093 \Rightarrow e^{-\lambda}(1+\lambda) - 0.35093 = 0$ Let $f(\lambda) = e^{-\lambda}(1+\lambda) - 0.35093$. $f(2) = 3e^{-2} - 0.35093 = 0.0551 > 0$ $f(2.5) = 3.5e^{-2.5} - 0.35093 = -0.0636 < 0$ Thus since $f(x)$ is continuous over the interval (2,2.5), the equation has a root in the interval (2, 2.5).	
(i)	Sketching $y = x$ and $\mathrm{F}(x) = 0.35093\mathrm{e}^{x} - 1$ on the same diagram.  From the diagram, we see that the sequence of approximations approaches the negative root which is not relevant here since $\lambda >$ 0. Thus, $\mathrm{F}(\lambda) = 0.35093\mathrm{e}^{\lambda} - 1$ would not be appropriate in finding α.	
(ii)	$e^{-\lambda}(1+\lambda) = 0.35093$ $-\lambda + \ln(1+\lambda) = \ln 0.35093$ $\lambda = \ln(1+\lambda) - \ln 0.35093 = \ln\left(\dfrac{1+\lambda}{0.35093}\right)$ That is, c = 1 and d = 0.35093	

	$\lambda_{n+1} = \ln\left(\dfrac{1+\lambda_n}{0.35093}\right)$ $\lambda_0 = 2$ $\lambda_1 = 2.145780794$ $\lambda_2 = 2.19323063$ $\lambda_3 = 2.208201647$ $\lambda_4 = 2.212879051$ $\lambda_5 = 2.214335941$ $\lambda_6 = 2.214789292 \approx 2.215$ $\lambda_7 = 2.214930322 \approx 2.215$ Thus, $\alpha = 2.215$ (correct to 3 dec pl) Let $f(\lambda) = e^{-\lambda}(1+\lambda) - 0.35093$. $f(2.2145) = 3.2145e^{-2.2145} - 0.35093 = 0.000119 > 0$ $f(2.2155) = 3.2155e^{-2.2155} - 0.35093 = -0.000122 < 0$ $\therefore 2.2145 < \alpha < 2.2155$ $\alpha = 2.215$ (correct to 3 dec pl)	

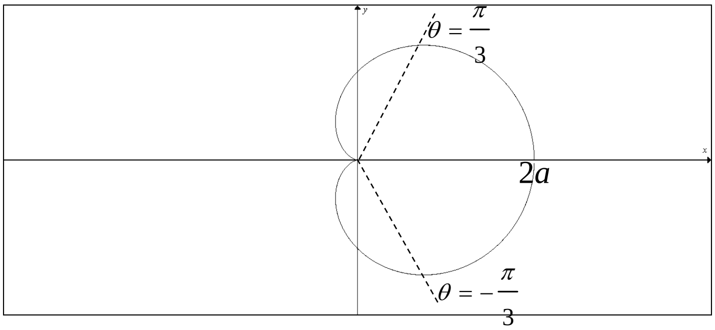
Question 5 [9 Marks] Complex Numbers		
a	$\frac{1-z^{2n}}{1+z^{2n}}\mathrm{i}=\frac{1-\left(\mathrm{e}^{2\mathrm{i}n\theta}\right)}{1+\left(\mathrm{e}^{2\mathrm{i}n\theta}\right)}\mathrm{i}$ $=\frac{\mathrm{e}^{\mathrm{i}n\theta}\left[\mathrm{e}^{-\mathrm{i}n\theta}-\mathrm{e}^{\mathrm{i}n\theta}\right]}{\mathrm{e}^{\mathrm{i}n\theta}\left[\mathrm{e}^{-\mathrm{i}n\theta}+\mathrm{e}^{\mathrm{i}n\theta}\right]}\mathrm{i}$ $=\frac{-2\mathrm{i}\,\mathrm{Im}\left[\mathrm{e}^{\mathrm{i}n\theta}\right]}{2\,\mathrm{Re}\left[\mathrm{e}^{\mathrm{i}n\theta}\right]}\mathrm{i}$ $=\frac{\sin(n\theta)}{\cos(n\theta)}$ $=\tan(n\theta)$ $\frac{2\tan(\theta)}{1-\tan^2(\theta)}=\frac{2\mathrm{i}\left(\frac{1-z^2}{1+z^2}\right)}{1-\left(\frac{1-z^2}{1+z^2}\mathrm{i}\right)^2}$ $=\frac{2\mathrm{i}\left(\frac{1-z^2}{1+z^2}\right)}{\left(\frac{\left(1+z^2\right)^2-\mathrm{i}^2\left(1-z^2\right)^2}{\left(1+z^2\right)^2}\right)}$ $=\frac{2\mathrm{i}\left(1-z^2\right)\left(1+z^2\right)}{2+2z^4}$ $=\frac{1-z^4}{1+z^4}\mathrm{i}$ $=\tan(2\theta)$	
bi	$w^3=-1$ $=\mathrm{e}^{\mathrm{i}\,(\pi+2k\pi)},k=-1,0,1$ $w=\mathrm{e}^{\mathrm{i}\left(\frac{(2k+1)\pi}{3}\right)},k=-1,0,1$ $w=\mathrm{e}^{-\frac{\pi}{3}\mathrm{i}},\ \mathrm{e}^{\frac{\pi}{3}\mathrm{i}},\ \mathrm{e}^{\mathrm{i}\pi}$	
bi	$w+w^2+w^3+\ldots+w^{99}=\frac{w\left(1-w^{99}\right)}{1-w}$ $=\frac{w\left(1-(-1)\right)}{1-w}=\frac{2w}{1-w}$ When $w=\mathrm{e}^{-\frac{\pi}{3}i}$, $\frac{2w}{1-w}=\frac{2\,\mathrm{e}^{-\frac{\pi}{3}i}}{1-\mathrm{e}^{-\frac{\pi}{3}i}}=-1-\mathrm{i}\sqrt{3}$	

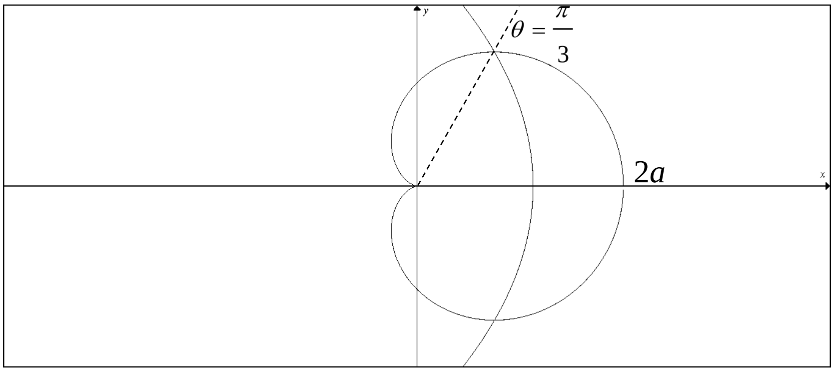
Question 6 [8 Marks] Complex Numbers Loci		
i	The locus of *z* is a circle centre -3i, radius 2. The locus of *w* is a half-line from *a* in the positive imaginary direction.  There are three possible cases whereby the two loci have exactly one point of intersection. Case 1: *a* is completely inside the circle Then $\{a \in \mathbb{C} : \lvert a+3i \rvert < 2\}$ Or $\left\{a = x+iy : x, y \in \mathbb{R}, x^2 + (y+3)^2 < 4\right\}$ Case 2: *a* lies on the circumference of the lower half of circle Then $\{a \in \mathbb{C} : \lvert a+3i \rvert = 2 \text{ and } -\pi < \arg(a+3i) < 0\}$ Or $\left\{a = x+iy : x, y \in \mathbb{R},\ x^2 + (y+3)^2 = 2 \text{ and } y < -3\right\}$ Case 3: The locus of *w* is a tangent to the circle Then $\{a \in \mathbb{C} : \operatorname{Im}(a) < -3 \text{ and } \operatorname{Re}(a) = \pm 2\}$ Or $\{a = x+iy : x, y \in \mathbb{R},\ y < -3,\ x = \pm 2\}$	
ii	$\lvert z+3i \rvert = 2$ is a circle, centre -3i, radius 2 $\arg(w-1) = -\dfrac{\pi}{6}$ is a half-line from 1, making an angle of $-\dfrac{\pi}{6}$ with the positive real axis.  arg(*w* – *z*) is the argument of any vector connecting the	

	circle to the half-line.  C is the point such that AC is tangential to the circle. D and E are points such that AD and CE are horizontal lines. $\angle DAC$ $= \angle DAB - \angle CAB$ $= \arg(-3\mathrm{i}-1) - \sin^{-1}\dfrac{2}{\sqrt{10}}$ $\min\ \arg(w-z)$ $= \angle ACE$ $= \pi - \angle DAC$ $= \pi - \left[\arg(-3\mathrm{i}-1) - \sin^{-1}\dfrac{2}{\sqrt{10}}\right]$ $= 1.933764$ The minimum argument occurs as *w* extends further along the half-line, for any complex number *z*.  Hence, minimum argument limits towards $-\dfrac{\pi}{6}$. Thus, $-\dfrac{\pi}{6} < \arg(w-z) < 1.93$.	

Question 7 [9 marks]		
(i)	Let equation of *H* be $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$ since difference in distances is 2*a*. *b* is to be determined. $c^2=a^2+b^2 \Rightarrow b^2=c^2-a^2$ Thus, $\frac{y^2}{a^2}-\frac{x^2}{c^2-a^2}=1$.	
(ii)	*y* = *mx* + *d* is a tangent to *H*, then $\frac{(mx+d)^2}{a^2}-\frac{x^2}{c^2-a^2}=1$ $\left(\frac{m^2}{a^2}-\frac{1}{c^2-a^2}\right)x^2+\frac{2md}{a^2}x+\frac{d^2}{a^2}-1=0$ Discriminant = 0 $\left(\frac{2md}{a^2}\right)^2-4\left(\frac{m^2}{a^2}-\frac{1}{c^2-a^2}\right)\left(\frac{d^2}{a^2}-1\right)=0$ $\frac{m^2}{a^2}+\frac{d^2}{(c^2-a^2)a^2}-\frac{1}{c^2-a^2}=0$ $\frac{m^2(c^2-a^2)+d^2-a^2}{(c^2-a^2)a^2}=0$ $m^2(c^2-a^2)+d^2-a^2=0$	
(iii)	$(c^2-a^2)m^2+d^2-a^2=0$ $(c^2-a^2)m^2+(y-mx)^2-a^2=0$ $(c^2-a^2)m^2+y^2-2xym+m^2x^2-a^2=0$ $(c^2-a^2+x^2)m^2-2xym+y^2-a^2=0$ For perpendicular tangents, $m_1m_2=-1$. $\frac{y^2-a^2}{c^2-a^2+x^2}=-1$ $y^2-a^2=-c^2+a^2-x^2$ $x^2+y^2=2a^2-c^2$ the orthoptic of H	

	It exists only if $2a^2 - c^2 > 0$ $c^2 < 2a^2$ $\lvert c\rvert < \sqrt{2}a$ $-\sqrt{2}a < c < \sqrt{2}a$	

Question 8 [10 marks] Polar Coordinates		
(i)	$C_1:\ r = a(1+\cos\theta)$ $y = r\sin\theta = a(1+\cos\theta)\sin\theta$ $\frac{dy}{d\theta} = a(1+\cos\theta)\cos\theta - a\sin^2\theta$ $= a(\cos^2\theta + \cos\theta - \sin^2\theta)$ $= a(2\cos^2\theta + \cos\theta - 1)$ $= a(2\cos\theta - 1)(\cos\theta + 1)$ $\frac{dy}{d\theta} = 0$ $\Rightarrow \cos\theta = \frac{1}{2}$ or $\cos\theta = -1$ $\Rightarrow \theta = \pm\frac{\pi}{3}$ $\theta = \pi$ $\Rightarrow r = \frac{3}{2}a$ $r = 0$ Thus, the coordinates are $\left(\frac{3}{2}a, -\frac{\pi}{3}\right)$, $\left(\frac{3}{2}a, \frac{\pi}{3}\right)$ and $(0, \pi)$.	
(ii)	 Arc length $= 2\int_0^{\frac{\pi}{3}} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\ d\theta$ $= 2\int_0^{\frac{\pi}{3}} \sqrt{a^2(1+\cos\theta)^2 + a^2\sin^2\theta}\ d\theta$ $= 2a\int_0^{\frac{\pi}{3}} \sqrt{1 + 2\cos\theta + \cos^2\theta + \sin^2\theta}\ d\theta$ $= 2a\int_0^{\frac{\pi}{3}} \sqrt{2 + 2\cos\theta}\ d\theta$ $= 2\sqrt{2}a\int_0^{\frac{\pi}{3}} \sqrt{1+\cos\theta}\ d\theta$	

	$= 2\sqrt{2}a\int_0^{\frac{\pi}{3}} \sqrt{2}\cos\frac{\theta}{2}\, d\theta$ $= 4a\int_0^{\frac{\pi}{3}} \cos\frac{\theta}{2}\, d\theta$ $= 4a\left[2\sin\frac{\theta}{2}\right]_0^{\frac{\pi}{3}}$ $= 4a$	
(iii)	 Area of *R* $= \int_0^{\frac{\pi}{3}} \frac{1}{2}a^2(1+\cos\theta)^2 - \frac{1}{2}\left(\frac{9a}{4(1+\cos\theta)}\right)^2 d\theta$ $= \int_0^{\frac{\pi}{3}} \frac{1}{2}a^2(1+\cos\theta)^2 - \frac{1}{2}\frac{81a^2}{16(1+\cos\theta)^2} d\theta$ $= \frac{1}{2}a^2\int_0^{\frac{\pi}{3}} (1+\cos\theta)^2 - \frac{81}{16(1+\cos\theta)^2}\, d\theta$ Let $f(\theta) = (1+\cos\theta)^2 - \frac{81}{16(1+\cos\theta)^2}$. θ: 0, $\frac{\pi}{12}$, $\frac{2\pi}{12}$, $\frac{3\pi}{12}$, $\frac{4\pi}{12}$ $f(\theta)$: 2.734375, 2.5549866, 2.0281662, 1.1770382, 0 Area of *R* $= \frac{1}{2}a^2 \times \frac{1}{2} \times \frac{\pi}{12} \times$ (2.734375 + 0 + 2(2.5549866 + 2.0281662 + 1.1770382)) $= 0.932971664a^2$ $= 0.9330a^2$ (to 4 dec pl) $k = 0.9330$	

θ	0	$\frac{\pi}{12}$	$\frac{2\pi}{12}$	$\frac{3\pi}{12}$	$\frac{4\pi}{12}$
$f(\theta)$	2.734375	2.5549866	2.0281662	1.1770382	0

Question 9 [12] System Of Linear Equations & Linear Transformations		
(a)	Augmented Matrix for the system of equations is $\begin{pmatrix} 1 & -2 & 2 & 2 \\ 4 & -7 & \lambda & 5 \\ 3 & \lambda & -7 & 3 \end{pmatrix}$ $\xrightarrow[R_3 : R_3 - 3R_1]{R_2 : R_2 - 4R_1} \begin{pmatrix} 1 & -2 & 2 & 2 \\ 0 & 1 & \lambda - 8 & -3 \\ 0 & \lambda + 6 & -13 & -3 \end{pmatrix}$ $\xrightarrow{R_3 : R_3 - (\lambda + 6)R_2} \begin{pmatrix} 1 & -2 & 2 & 2 \\ 0 & 1 & \lambda - 8 & -3 \\ 0 & 0 & (7-\lambda)(\lambda+5) & 3(\lambda+5) \end{pmatrix}$ (i): The system of equations has no solution when $(7-\lambda)(\lambda+5) = 0$ and $\lambda + 5 \neq 0$. Therefore $\lambda = 7$ (ii): The system of equations has exactly one solution when $(7-\lambda)(\lambda+5) \neq 0$. Therefore $\lambda \neq 7$ and $\lambda \neq -5$	
bi	$\mathbf{M} = \begin{pmatrix} 1 & -2 & 2 & 2 \\ 4 & -7 & -5 & 5 \\ 3 & -5 & -7 & 3 \end{pmatrix}$ rref of $\mathbf{M} = \begin{pmatrix} 1 & 0 & -24 & -4 \\ 0 & 1 & -13 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ $\therefore$ rank ($\mathbf{M}$) $= 2$. and a basis for R is $\left\{ \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}, \begin{pmatrix} -2 \\ -7 \\ -5 \end{pmatrix} \right\}$ Consider $\alpha \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix} + \beta \begin{pmatrix} -2 \\ -7 \\ -5 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ q \end{pmatrix}$ ----(1) then $\alpha = -5$, $\beta = -3$, $q = 0$. Therefore, a basis for R is $\left\{ \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$	

bii	$W = \text{span}\left\{\begin{pmatrix}1\\2\\1\end{pmatrix}\right\}$ and $R = \text{span}\left\{\begin{pmatrix}1\\4\\3\end{pmatrix},\begin{pmatrix}-2\\-7\\-5\end{pmatrix}\right\}$ Let $\mathbf{x} \in R \cap W$ $\mathbf{x} \in R$ and $\mathbf{x} \in W$ $\mathbf{x} = \lambda_1\begin{pmatrix}1\\4\\3\end{pmatrix} + \lambda_2\begin{pmatrix}-2\\-7\\-5\end{pmatrix}$ and $\mathbf{x} = \lambda_3\begin{pmatrix}1\\2\\1\end{pmatrix}$ for some $\lambda_1,\ \lambda_2,\ \lambda_3 \in \mathbb{R}$ $\lambda_1\begin{pmatrix}1\\4\\3\end{pmatrix} + \lambda_2\begin{pmatrix}-2\\-7\\-5\end{pmatrix} = \lambda_3\begin{pmatrix}1\\2\\1\end{pmatrix}$ ----(2) $\lambda_1\begin{pmatrix}1\\4\\3\end{pmatrix} + \lambda_2\begin{pmatrix}-2\\-7\\-5\end{pmatrix} + \lambda_3\begin{pmatrix}-1\\-2\\-1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$ Using GC, $\lambda_1 = -3\lambda_3,\ \lambda_2 = -2\lambda_3$ $\Rightarrow \mathbf{x} = \lambda_3\begin{pmatrix}1\\2\\1\end{pmatrix},\ \lambda_3 \in \mathbb{R}$ ----(3) $\Rightarrow R \cap W = \text{span}\left\{\begin{pmatrix}1\\2\\1\end{pmatrix}\right\}$ $W = R \cap W$ since they have the same set of basis. dim (W) < dim (R), W is a proper sub space of R i.e. $W \subset R$	
bii	Consider $\mathbf{Mx} = \begin{pmatrix}0\\0\\0\end{pmatrix}$ rref of augmented matrix is $\begin{pmatrix}1 & 0 & -24 & -4 & 0\\0 & 1 & -13 & -3 & 0\\0 & 0 & 0 & 0 & 0\end{pmatrix}$ Then $\mathbf{x} = \begin{pmatrix}24s+4t\\13s+3t\\s\\t\end{pmatrix} = s\begin{pmatrix}24\\13\\1\\0\end{pmatrix} + t\begin{pmatrix}4\\3\\0\\1\end{pmatrix}$ where $s, t \in \mathbb{R}$ Basis for null space $K = \left\{\begin{pmatrix}24\\13\\1\\0\end{pmatrix},\begin{pmatrix}4\\3\\0\\1\end{pmatrix}\right\}$	

Consider $\mathbf{Mx} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$$\mathbf{M}\begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\therefore\ \mathbf{x} = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + s\begin{pmatrix} 24 \\ 13 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 4 \\ 3 \\ 0 \\ 1 \end{pmatrix} \quad \text{where } s, t \in \mathbb{R}$$

Let $\begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + s\begin{pmatrix} 24 \\ 13 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 4 \\ 3 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

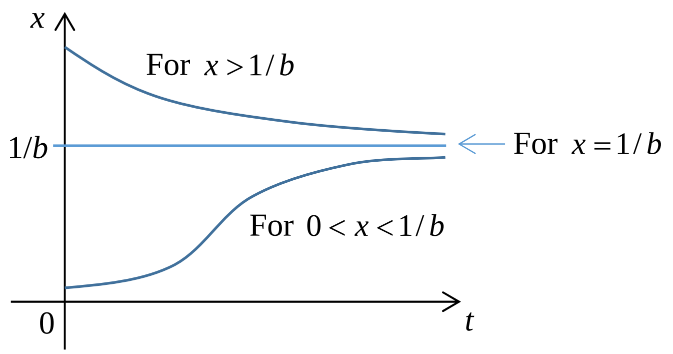
There is no consistent value for s and t.

The zero vector $\mathbf{0}$ is not in the solution set

$\Rightarrow$ the solution set is not a vector space.

Question 10 [13 Marks] Recurrence Relation		
ai	$u_n = (1.1)\left[u_{n-1} - p\right]$ where $n \geq 1,\ u_0 = b$ $u_n = (1.1)(1.1)\left[u_{n-2} - p\right] - (1.1)p$ $u_n = (1.1)^2 u_{n-2} - (1.1)^2 p - (1.1)p$ $u_n = (1.1)^2(1.1)\left[u_{n-3} - p\right] - (1.1)^2 p - (1.1)p$ $u_n = (1.1)^3 u_{n-3} - (1.1)^3 p - (1.1)^2 p - (1.1)p$ $u_n = (1.1)^n u_0 - (1.1)^n p - (1.1)^{n-1} p + ... - (1.1)p$ --(*) $u_n = (1.1)^n u_0 - \dfrac{p(1.1)\left[(1.1)^n - 1\right]}{1.1-1}$ $u_n = (1.1)^n (b) - 11p\left[(1.1)^n - 1\right]$ $u_n = (1.1)^n \left(b - 11p\right) + 11p$ where $n \geq 0$	
	Alternative Method $u_n = (1.1)\left[u_{n-1} - p\right]$ where $n \geq 1,\ u_0 = b$ $u_n - k = (1.1)\left[u_{n-1} - k\right]$ where $k = 11p$ $u_n - 11p = (1.1)\left[u_{n-1} - 11p\right]$ Let $v_n = u_n - 11p$ $v_n = (1.1)v_{n-1}$ where $n \geq 1,\ v_0 = u_0 - k = b - 11p$ $v_n = (1.1)^n v_0$ $u_n - 11p = (1.1)^n \left[b - 11p\right]$ $u_n = (1.1)^n \left(b - 11p\right) + 11p$ where $n \geq 0$	
aii	$u_n = (1.1)^n \left(b - 11p\right) + 11p$ When $b = 6600$ and $p = 100$ $u_n = (1.1)^n \left(5500\right) + 1100$ $(1.1)^n \to \infty$ as $n \to \infty$ Therefore u_n does not have a limiting value. The fish population will not stabilize in the long run.	
aiii	For the fishing activity to be sustainable, we do not want u_n to be decrease to 0, therefore $b - 11p \geq 0$	
bi	$(0.12)(6600000) + (1.1)(8700000) = 10362000$	
bii	$u_n = (1.1)u_{n-1} + (0.12)u_{n-2}$, $n \geq 2$ Let $u_n = m^n$, then $m^n = (1.1)m^{n-1} + (0.12)m^{n-2}$ $m^n - (1.1)m^{n-1} - (0.12)m^{n-2} = 0$ $m^2 - (1.1)m - (0.12) = 0$ ---(1) $m = 1.2$ or $m = -0.1$ $u_n = A(1.2)^n + B(-0.1)^n$	

	Given $u_0 = 6600$ and $u_1 = 8700$, Then $A + B = 6600$ and $(1.2)A + (-0.1)B = 8700$ $\Rightarrow A = 7200$ and $B = -600$ $u_n = 7200(1.2)^n - 600(-0.1)^n$	

<table>
<tr><td colspan="3">Question 11 [14 marks]</td></tr>
<tr><td>(i)</td><td>a is the per-capita growth rate of the population of cats on the island, and
b is the reciprocal of the carrying capacity of the system (or 1/b is the carrying capacity).</td><td></td></tr>
<tr><td>(ii)</td><td>

If the initial population is less than 1/b (i.e. $x_0 < 1/b$), the population will increase towards the carrying capacity 1/b in the long run.

If the initial population is equal to 1/b (i.e. $x_0 = 1/b$), the population will remain constant at this value.

If the initial population is greater than 1/b (i.e. $x_0 > 1/b$), the population will decrease towards the carrying capacity 1/b in the long run.</td><td></td></tr>
<tr><td>(iii)</td><td>h = 0.5
<table>
<tr><td>t</td><td>x</td><td>$\frac{dx}{dt}$</td><td>$\tilde{x}$</td><td>$\frac{\Delta x}{\Delta t}$</td></tr>
<tr><td>0</td><td>380</td><td>61.56</td><td>410.78</td><td>63.4210</td></tr>
<tr><td>0.5</td><td>411.7105</td><td>65.3915</td><td><u>444.4063</u></td><td><u>67.2615</u></td></tr>
<tr><td>1</td><td><u>445.3413</u></td><td></td><td></td><td></td></tr>
</table>

One year into the research, the population of cats is 445.</td><td></td></tr>
<tr><td>(iv)</td><td>$$\frac{dx}{dt} = 0.2x(1-0.0005x) - 0.01kx$$ $$= x[(0.2-0.01k) - 0.0001x]$$ For population of cats to become extinct, $0.2 - 0.01k < 0$, i.e. $k > 20$.</td><td></td></tr>
</table>

(v)