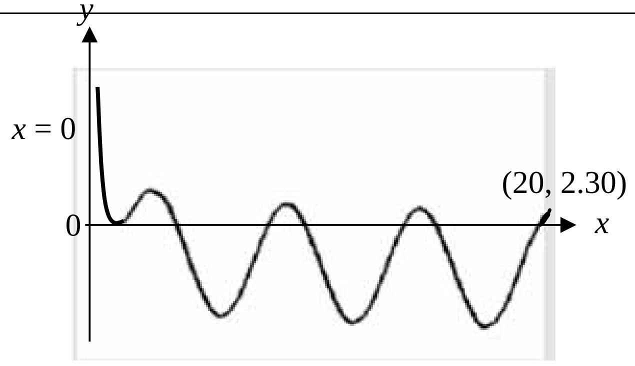
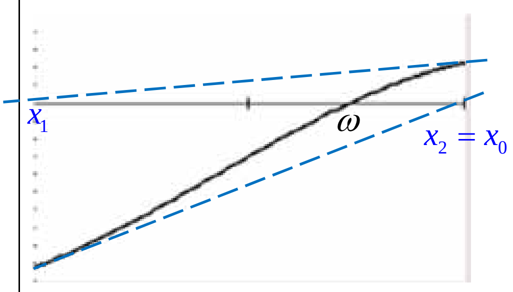
Paper 1 Solutions

1(i)	From GC, $\mathbf{A}=\begin{pmatrix}1&2&2&6\\1&1&2&4\\3&7&6&20\end{pmatrix}\sim\begin{pmatrix}1&0&2&2\\0&1&0&2\\0&0&0&0\end{pmatrix}$ $\text{Dim}(R_{\text{T}})=\text{Rank}(A)=2$ By the Rank-Nullity Theorem, $\text{Dim}(K_{\text{T}})=4-2=2$
(ii)	Basis for $R_{\text{T}}=\left\{\begin{pmatrix}1\\1\\3\end{pmatrix},\begin{pmatrix}2\\1\\7\end{pmatrix}\right\}$ Let $\begin{pmatrix}1&0&2&2\\0&1&0&2\\0&0&0&0\end{pmatrix}\begin{pmatrix}w\\x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$. Then $\begin{matrix}w+2y+2z=0\\x+2z=0\end{matrix}\Rightarrow\begin{matrix}w=-2y-2z\\x=-2z\end{matrix}$ $\begin{pmatrix}w\\x\\y\\z\end{pmatrix}=\begin{pmatrix}-2y-2z\\-2z\\y\\z\end{pmatrix}=y\begin{pmatrix}-2\\0\\1\\0\end{pmatrix}+z\begin{pmatrix}-2\\-2\\0\\1\end{pmatrix}$ Basis for $K_{\text{T}}=\left\{\begin{pmatrix}-2\\0\\1\\0\end{pmatrix},\begin{pmatrix}-2\\-2\\0\\1\end{pmatrix}\right\}$ As $\begin{pmatrix}3\\1\\11\end{pmatrix}\in R_{\text{T}}$, $\mathbf{Ax}=\begin{pmatrix}3\\1\\11\end{pmatrix}$ for some $\mathbf{x}\in\mathbb{R}^4$. Observe $\begin{pmatrix}3\\1\\11\end{pmatrix}=-1\begin{pmatrix}1\\1\\3\end{pmatrix}+2\begin{pmatrix}2\\1\\7\end{pmatrix}$, so a possible solution $\mathbf{x}=\begin{pmatrix}-1\\2\\0\\0\end{pmatrix}$ $\therefore$ The general solution of $\text{T}(\mathbf{x})=\begin{pmatrix}3\\1\\11\end{pmatrix}$ is $\mathbf{x}=\begin{pmatrix}-1\\2\\0\\0\end{pmatrix}+\lambda\begin{pmatrix}-2\\0\\1\\0\end{pmatrix}+\mu\begin{pmatrix}-2\\-2\\0\\1\end{pmatrix},\ \lambda,\mu\in\mathbb{R}$

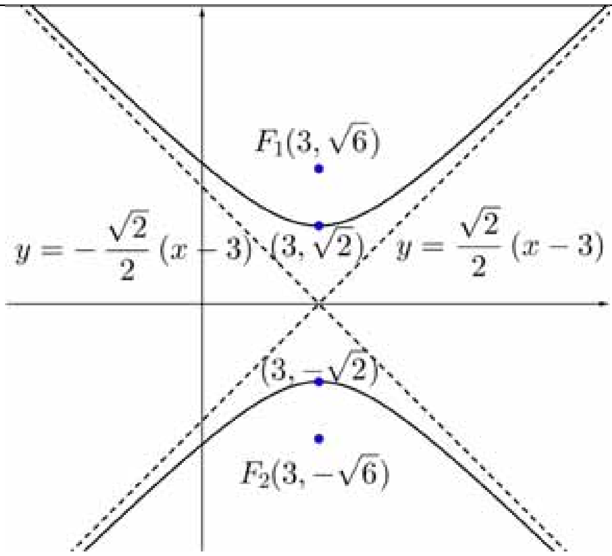
2(i)		**B1** shape and asymptote with label **B1** end point
(ii)	$n=19$ Let $\mathrm{f}(x)=7\sin x-\ln 3x$. $x_1=\dfrac{19\mathrm{f}(20)-20\mathrm{f}(19)}{\mathrm{f}(20)-\mathrm{f}(19)}=19.6\ \ (1\text{ d.p.})$	**B1** **MA1**
(iii)	$\mathrm{f}'(x)=7\cos x-\dfrac{1}{x}$ $x_{n+1}=x_n-\dfrac{\mathrm{f}(x_n)}{\mathrm{f}'(x_n)}=x_n-\dfrac{7\sin x_n-\ln 3x_n}{7\cos x_n-\dfrac{1}{x_n}}$ From GC, $x_1=19.6$ $x_2=19.4620$ $x_3=19.4696$ $x_4=19.4696$ $\omega=19.470\ \ (3\text{ d.p.})$ $\mathrm{f}(19.4695)=-5.3\times10^{-4}<0$ whereas $\mathrm{f}(19.4705)=0.0051>0$ As $\mathrm{f}(x)$ is a continuous function, $19.4695<\omega<19.4705$. $\therefore\ \omega=19.470\ \ (3\text{ d.p.})$ (verified)	**B1** correct equation **B1** answer **B1** verification
(iv)	A cyclical pattern could occur e.g. where $x_2=x_0$. 	**B1** correct diagram

3(i)	$\frac{dy}{dx} = y^2 \sec x - y\tan x + \cos x \quad ---(*)$ When $y = \sin x$, $\text{LHS} = \frac{dy}{dx} = \cos x$ $\text{RHS} = \sin^2 x \sec x - \sin x \tan x + \cos x$ $= \frac{\sin^2 x}{\cos x} - \frac{\sin^2 x}{\cos x} + \cos x$ $= \cos x = \text{LHS}$ $\therefore y = \sin x$ is a particular solution of (*).	**M1** find $\frac{dy}{dx}$ and attempt to substitute y and $\frac{dy}{dx}$ into equation **A1**
(ii)	$y = \sin x + \frac{1}{u} \Rightarrow \frac{dy}{dx} = \cos x - \frac{1}{u^2}\frac{du}{dx}$ Sub into (*): $\cos x - \frac{1}{u^2}\frac{du}{dx} = \left(\sin x + \frac{1}{u}\right)^2 \sec x - \left(\sin x + \frac{1}{u}\right)\tan x + \cos x$ $-\frac{1}{u^2}\frac{du}{dx} = \left(\sin x + \frac{1}{u}\right)^2 \sec x - \left(\sin x + \frac{1}{u}\right)\tan x$ $\frac{du}{dx} = -u^2\left(\sin^2 x + \frac{2}{u}\sin x + \frac{1}{u^2}\right)\sec x + u^2\left(\sin x + \frac{1}{u}\right)\tan x$ $= -u^2\frac{\sin^2 x}{\cos x} - 2u\frac{\sin x}{\cos x} - \frac{1}{\cos x} + u^2\frac{\sin^2 x}{\cos x} + u\frac{\sin x}{\cos x}$ $= -u\frac{\sin x}{\cos x} - \frac{1}{\cos x}$ $= -u\tan x - \sec x$	**M1** **M1** simplify equation **A1** AG
(iii)	$\frac{du}{dx} + u\tan x = -\sec x$ Integrating Factor = $e^{\int \tan x \, dx}$ $= e^{\int \tan x \, dx} = e^{\int \frac{\sin x}{\cos x} dx} = e^{-\ln \cos x} = e^{\ln(\cos x)^{-1}} = \sec x$ Multiplying throughout by the I.F., $\sec x \frac{du}{dx} + u\frac{\sin x}{\cos x} \cdot \sec x = -\sec^2 x$ $\frac{d}{dx}(u\sec x) = -\sec^2 x$ $u\sec x = -\tan x + C$ where C is an arbitrary constant $y = \sin x + \frac{1}{u} \Rightarrow u = \frac{1}{y - \sin x}$ $\frac{1}{y - \sin x} \cdot \sec x = -\tan x + C$ $y = \sin x + \frac{\sec x}{-\tan x + C} \quad \left(\text{or} \quad \sin x + \frac{1}{-\sin x + C\cos x}\right)$	**M1** rewriting in standard form and using Integrating Factor method **A1** correct I.F. **M1** reversing product rule **A1** **M1** substituting back for y **A1**

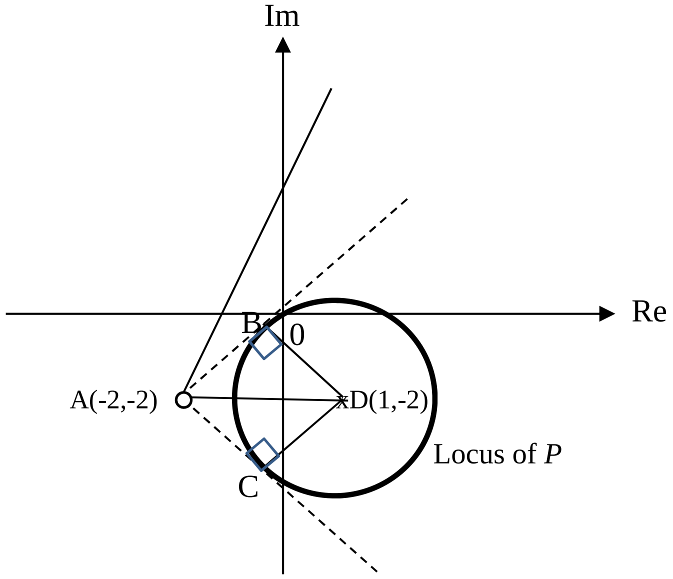
4(a)	Let B be the midpoint of AP . Triangles ABR and PBR are congruent (SAS) Hence, $AR = PR$. R has the same distance to A as to the line l. Hence, the locus of R is a parabola with A as the focus and l as the directrix.	**B1** for working to show that $AR = PR$ **B1** for description of the locus of R
(b)	x-coordinate of T $= x$-coordinate of Q $= a\cos\theta$ y-coordinate of T $= y$-coordinate of S $= b\sin\theta$ $\therefore T\left(a\cos\theta, b\sin\theta\right)$ Let $x = a\cos\theta$, $y = b\sin\theta$ $\cos^2\theta + \sin^2\theta = 1$ $\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ Hence, the locus of T is an ellipse with major axis of length $2a$ and minor axis of length $2b$, centred at the origin O.	**B1** for either x or y coordinates of T **B1** for use of Pythagorean formula **B1** for description of locus of T

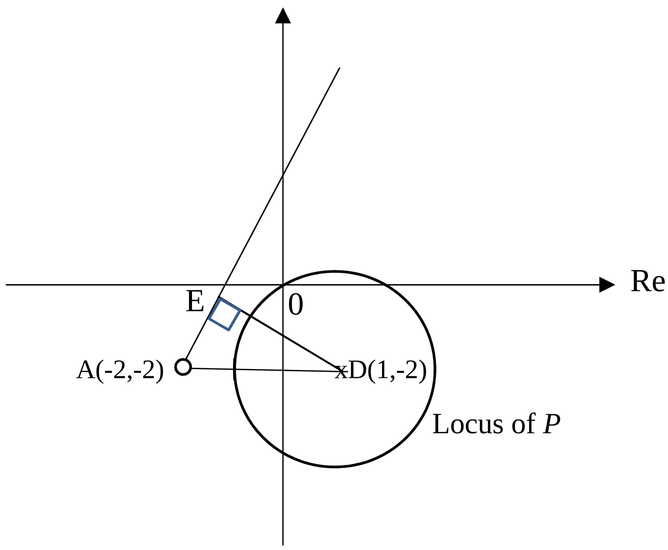
5(i)	$a_2 = (0.88)^6(50) + 50$ $= 73.22$ (2 d.p.)	**M1** AG
(ii)	$a_n = (0.88)^6 a_{n-1} + 50$, $a_1 = 50$	**B1**
(iii)	Let $a_n = C(0.88)^{6n} + D, \quad n \geq 1$ $n = 1: \quad (0.88)^6 C + D = 50$ --- (1) $n = 2: \quad (0.88)^{12} C + D = 73.22$ --- (2) Solving (1) & (2), $C = -93.4, \ D = 93.4$ $\therefore a_n = -93.4(0.88)^{6n} + 93.4, \quad n \geq 1$ The number of antibiotic units is highest upon each injection (i.e. the local maxima occur at these points). As $a_n \leq 93.4 < 100$ for all $n \geq 1$, it is safe for the doctor to administer this medicine.	**M1** for the general solution for the RR **M1** form simultaneous equations **A1** C and D **A1** **A1** accurate explanation with reference to G.S.
6(i)	$\begin{pmatrix} m-1 \\ m+1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} -m+1 \\ -m-1 \\ -2 \end{pmatrix}$ are linearly dependent column vectors. $1\begin{pmatrix} m-1 \\ m+1 \\ 2 \end{pmatrix} + 1\begin{pmatrix} -m+1 \\ -m-1 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \mathbf{M}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.	**B1**

	An eigenvector corresponding to the eigenvalue 0 is $\mathbf{v_1} = \begin{pmatrix}1\\0\\1\end{pmatrix}$.	**B1** (accept any non-zero multiple)
(ii)	Consider $(\mathbf{M}-\lambda\mathbf{I})\mathbf{x}=\mathbf{0}$. For $\lambda=-4$, $\mathbf{M}-\lambda\mathbf{I}=\begin{pmatrix}m+3 & m-1 & -m+1\\ m+1 & m+1 & -m-1\\ 2 & -2 & 2\end{pmatrix}$. $\begin{pmatrix}m-1\\m+1\\-2\end{pmatrix}$ and $\begin{pmatrix}-m+1\\-m-1\\2\end{pmatrix}$ are linearly dependent column vectors. $\mathbf{v_2}=\begin{pmatrix}0\\1\\1\end{pmatrix}$ is an eigenvector. For $\lambda=2m-2$, $\mathbf{M}-\lambda\mathbf{I}=\begin{pmatrix}-m+1 & m-1 & -m+1\\ m+1 & -m-1 & -m-1\\ 2 & -2 & -2m\end{pmatrix}$. $\begin{pmatrix}-m+1\\m+1\\2\end{pmatrix}$ and $\begin{pmatrix}m-1\\-m-1\\-2\end{pmatrix}$ are linearly dependent column vectors. $\mathbf{v_3}=\begin{pmatrix}1\\1\\0\end{pmatrix}$ is an eigenvector.	**M1** writing down the matrices $\mathbf{M}-\lambda\mathbf{I}$ **M1** observe column vectors or any other valid method e.g. Gaussian elimination **A1** both eigenvectors (accept any on-zero multiples of $\mathbf{v_2}$ and $\mathbf{v}_3$)
(iii)	$m=1.5$ (Note: Line of invariant points corresponds to an eigenvalue of 1) The line given by $\mathbf{r}=\alpha\mathbf{v_1}$ is mapped onto the origin. The line given by $\mathbf{r}=\beta\mathbf{v}_2$ is an invariant line.	**B1** **B1** **B1**
(iv)	$\mathbf{M}=\mathbf{PDP}^{-1}$, where $\mathbf{P}=\begin{pmatrix}1&0&1\\0&1&1\\1&1&0\end{pmatrix}$, $\mathbf{D}=\begin{pmatrix}0&0&0\\0&-4&0\\0&0&2m-2\end{pmatrix}$ Observe that matrix **A** is obtained when *m* = 2018. $\mathbf{A}^{50}=\underbrace{\left(\mathbf{PDP}^{-1}\right)\left(\mathbf{PDP}^{-1}\right)\ldots\left(\mathbf{PDP}^{-1}\right)}_{50\text{ factors}}$ $=\mathbf{PD}^{50}\mathbf{P}^{-1}$ $=\mathbf{PBP}^{-1}$ where $\mathbf{P}=\begin{pmatrix}1&0&1\\0&1&1\\1&1&0\end{pmatrix}$, $\mathbf{B}=\begin{pmatrix}0&0&0\\0&4^{50}&0\\0&0&4034^{50}\end{pmatrix}$	**M1** using eigenvectors and eigenvalues from part (ii) to form matrices **P** and **D** **M1** for $\mathbf{A}^{50}=\mathbf{PD}^{50}\mathbf{P}^{-1}$ **A1**

7(a)	 $2y^2 - x^2 + 6x - 13 = 0$ $2y^2 - \left((x-3)^2 - 9\right) - 13 = 0$ $2y^2 - (x-3)^2 = 4$ $\frac{y^2}{\left(\sqrt{2}\right)^2} - \frac{(x-3)^2}{2^2} = 1$ Centre: $(3,0)$ $a = 2, b = \sqrt{2}$, $c = \sqrt{a^2+b^2} = \sqrt{6}$ Vertices: $(3, \pm b) = \left(3, \pm\sqrt{2}\right)$ Foci: $(3, \pm c) = \left(3, \pm\sqrt{6}\right)$ Asymptotes: $y = \pm\frac{b}{a}(x-3) = \pm\frac{\sqrt{2}}{2}(x-3)$	**B1** for shape **B1** for coordinates of vertices **B1** for coordinates of foci **B1** for equations and shape of asymptotes
(b)(i)	Substitute $y = mx + c$ into $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ $\frac{x^2}{a^2} - \frac{(mx+c)^2}{b^2} = 1$ $b^2x^2 - a^2\left(m^2x^2 + 2mxc + c^2\right) = a^2b^2$ $\left(b^2 - a^2m^2\right)x^2 - \left(2mca^2\right)x - a^2\left(c^2+b^2\right) = 0$ Since the line and hyperbola are tangent, they meet at exactly one point, hence the above equation has exactly one solution for x.	**M1** for substituting two equations together **M1** for writing as a quadratic equation in a variable **M1** for considering discriminant = 0 and writing resulting expression out **MA1** for simplification to given answer

	Discriminant $= 0$ $\left(2mca^2\right)^2 + 4a^2\left(b^2 - a^2m^2\right)\left(c^2 + b^2\right) = 0$ $4a^4m^2c^2 + 4a^2\left(b^2c^2 + b^4 - a^2c^2m^2 - a^2b^2m^2\right) = 0$ $a^2m^2c^2 + b^2c^2 + b^4 - a^2c^2m^2 - a^2b^2m^2 = 0$ (since $a \neq 0$) $b^2c^2 + b^4 - a^2b^2m^2 = 0$ $c^2 + b^2 - a^2m^2 = 0$ (since $b \neq 0$) $b^2 + c^2 = a^2m^2$ (shown)	
(b)(ii)	When $\frac{x^2}{9} - \frac{y^2}{4} = 1$, $a = 3$ and $b = 2$, For equation of tangent $y = mx + c$, By part (i), $4 + c^2 = 9m^2$ --(1) Since line contains $(2, 0)$: $0 = 2m + c$, $c = -2m$ --(2) Sub (2) into (1): $4 + 4m^2 = 9m^2$ $m^2 = \frac{4}{5}$ $m = \frac{2}{\sqrt{5}}$ (rej neg answer $\because$ grad is positive) $c = -2\left(\frac{2}{\sqrt{5}}\right) = -\frac{4}{\sqrt{5}}$ Hence, equation of tangent is $y = \frac{2}{\sqrt{5}}x - \frac{4}{\sqrt{5}}$.	**M1** for use of part (i) to obtain an equation in c^2 and m^2 **M1** for process of solving simultaneous equation of relevant equations (eg substitution) **A1** for answer

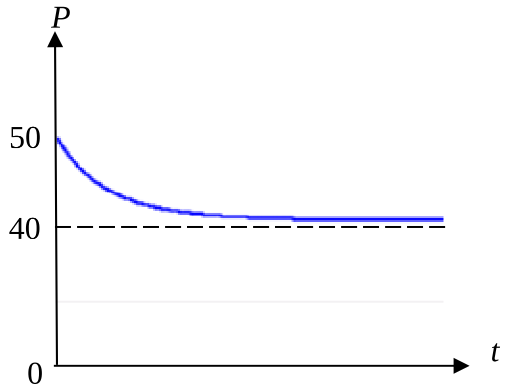
8(a)	Let $z^4 = -4-4\sqrt{3}\mathrm{i}$ $\left\|-4-4\sqrt{3}\mathrm{i}\right\| = 8$ $\arg(-4-4\sqrt{3}\mathrm{i}) = -(\pi - \tan^{-1}\sqrt{3}) = -\frac{2\pi}{3}$ $z^4 = 8e^{\mathrm{i}\left(-\frac{2\pi}{3}+2k\pi\right)}$ $z = 2^{\frac{3}{4}}e^{\mathrm{i}\left(\frac{-\pi+3k\pi}{6}\right)},\ \ k=-1,0,1,2$ $z = 2^{\frac{3}{4}}e^{\mathrm{i}\left(\frac{-2\pi}{3}\right)}, 2^{\frac{3}{4}}e^{\mathrm{i}\left(\frac{-\pi}{6}\right)}, 2^{\frac{3}{4}}e^{\mathrm{i}\left(\frac{\pi}{3}\right)}, 2^{\frac{3}{4}}e^{\mathrm{i}\left(\frac{5\pi}{6}\right)}$ $z^8+8z^4+64=0$ $\left(z^4\right)^2+8\left(z^4\right)+64=0$ $z^4 = \frac{-8\pm\sqrt{-192}}{2} = -4\pm4\sqrt{3}\mathrm{i}$ $z^4 = -4-4\sqrt{3}\mathrm{i}$ $z = 2^{\frac{3}{4}}e^{\mathrm{i}\left(\frac{-2\pi}{3}\right)}, 2^{\frac{3}{4}}e^{\mathrm{i}\left(\frac{-\pi}{6}\right)}, 2^{\frac{3}{4}}e^{\mathrm{i}\left(\frac{\pi}{3}\right)}, 2^{\frac{3}{4}}e^{\mathrm{i}\left(\frac{5\pi}{6}\right)}$ or $z^4 = -4+4\sqrt{3}\mathrm{i}$ Since coefficients of the equation are all real, the other 4 roots are conjugates of the above four complex roots. $z = 2^{\frac{3}{4}}e^{\mathrm{i}\left(\frac{2\pi}{3}\right)}, 2^{\frac{3}{4}}e^{\mathrm{i}\left(\frac{\pi}{6}\right)}, 2^{\frac{3}{4}}e^{\mathrm{i}\left(-\frac{\pi}{3}\right)}, 2^{\frac{3}{4}}e^{\mathrm{i}\left(-\frac{5\pi}{6}\right)}$	 **B1** **M1** **A1** **M1** **A1** using result from first part **A1** with explanation
(ii)		**B1** Circle with centre and passing through origin since radius is $\sqrt{5}$

	$\sin \angle BAD = \frac{\sqrt{5}}{3}$ $\angle BAD = 0.84107$ $\angle CAD = 0.84107$ For locus of Q not to intersect the locus of P, $0.842 \le \theta \le \pi$ (also accept $0.841 < \theta \le \pi$) or $-\pi < \theta \le -0.842$ (also accept $-\pi < \theta < -0.841$)	**M1** **A1**
(iii)	 Least value of $\lvert z-w \rvert$ is given by $ED-\sqrt{5}$. In ΔEAD, $\sin\theta = \frac{ED}{3}$ Given : $\theta = \tan^{-1} 4$ $\tan\theta = \frac{4}{1}$ $\therefore \sin\theta = \frac{4}{\sqrt{17}}$ $\frac{4}{\sqrt{17}} = \frac{ED}{3}$ $ED = 3\times\frac{4}{\sqrt{17}} = \frac{12}{17}\sqrt{17}$ Shortest distance is $\frac{12}{17}\sqrt{17} - \sqrt{5}$.	**M1** identify shortest distance **M1** **A1** exact ans

9(i)	$y=\dfrac{e^x+e^{-x}}{2} \quad \Rightarrow \quad \dfrac{dy}{dx}=\dfrac{e^x-e^{-x}}{2}$ Arc length $=\int_{-\ln 2}^{\ln 2}\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}\,dx$ $=2\int_0^{\ln 2}\sqrt{1+\left(\dfrac{e^x-e^{-x}}{2}\right)^2}\,dx$ $=2\int_0^{\ln 2}\sqrt{\dfrac{4+e^{2x}-2+e^{-2x}}{4}}\,dx$ $=2\int_0^{\ln 2}\sqrt{\dfrac{(e^x+e^{-x})^2}{4}}\,dx$ $=2\int_0^{\ln 2}\dfrac{e^x+e^{-x}}{2}\,dx$ $=\left[e^x-e^{-x}\right]_0^{\ln 2}=2-\dfrac{1}{2}-1+1$ $=\dfrac{3}{2}$ units	**B1** differentiation **M1** correct formula and correct limits **B1** algebraic simplification for the integrand AG
(ii)	By shell method, V = volume of cylinder $-\ 2\pi\int_0^{\ln 2} xy\,dx$ $=\pi(\ln 2)^2\left(\dfrac{5}{4}\right)-2\pi\int_0^{\ln 2} x\left(\dfrac{e^x+e^{-x}}{2}\right)dx$ $=\dfrac{5}{4}\pi(\ln 2)^2-\pi\int_0^{\ln 2} x\left(e^x+e^{-x}\right)dx$ $u=x \qquad \dfrac{dv}{dx}=e^x+e^{-x}$ $\dfrac{du}{dx}=1 \qquad v=e^x-e^{-x}$ $\int_0^{\ln 2} x\left(\dfrac{e^x+e^{-x}}{2}\right)dx=\left[x(e^x-e^{-x})\right]_0^{\ln 2}-\int_0^{\ln 2}(e^x-e^{-x})\,dx$ $=\ln 2\left(2-\dfrac{1}{2}\right)-\left[e^x+e^{-x}\right]_0^{\ln 2}$ $=\dfrac{3}{2}\ln 2-\left(2+\dfrac{1}{2}-1-1\right)$ $=\dfrac{3}{2}\ln 2-\dfrac{1}{2}$ $V=\dfrac{5}{4}\pi(\ln 2)^2-\pi\left(\dfrac{3}{2}\ln 2-\dfrac{1}{2}\right)$ $=\dfrac{\pi}{4}\left[5(\ln 2)^2-6\ln 2+2\right]$ $\therefore\ a=5,\ b=-6,\ c=2$	**M1** forming the integral **M1** integration by parts **A1** **A1** **A1**

(iii)	The y-ordinate of the centroid of the solid of revolution $\bar{y} = \dfrac{\pi \int_1^{5/4} yx^2 \, \mathrm{d}y}{V}$ Consider the numerator $\int_1^{\frac{5}{4}} x^2 y \, \mathrm{d}y$ $= \int_0^{\ln 2} x^2 y \dfrac{\mathrm{d}y}{\mathrm{d}x} \mathrm{d}x$ $= \int_0^{\ln 2} x^2 \left(\dfrac{\mathrm{e}^x + \mathrm{e}^{-x}}{2}\right)\left(\dfrac{\mathrm{e}^x - \mathrm{e}^{-x}}{2}\right) \mathrm{d}x$ $= \dfrac{1}{4}\int_0^{\ln 2} x^2 \left(\mathrm{e}^{2x} - \mathrm{e}^{-2x}\right) \mathrm{d}x$ $\therefore \ \bar{y} = \dfrac{\pi}{4V}\int_0^{\ln 2} x^2 \left(\mathrm{e}^{2x} - \mathrm{e}^{-2x}\right) \mathrm{d}x$ Using GC, $\bar{y} = 1.166$ units (to 3 d.p.)	**M1** changing ‘dy’ and limits **A1** substitution leading to answer AG **A1**
(iv)	The centroid of the solid is 0.166 units above the horizontal plane which is very low compared with the height of the whole tumbler toy. Therefore the tumbler toy will not topple easily.	**B1**

10(i)	When the population is at equilibrium, $\frac{dP}{dt} = kP\left(1 - \frac{P}{N}\right) - I = 0$ $kP - k\frac{P^2}{N} - I = 0$ $kP^2 - kNP + NI = 0 \quad ---(1)$ For the population to be sustainable, equation (1) must have at least 1 positive root, so $(-kN)^2 - 4(k)(NI) \geq 0$ $k^2N^2 - 4kNI \geq 0$ $I \leq \frac{kN}{4} \quad (\because kN > 0)$ Sub $I_{max} = \frac{kN}{4}$ into (1): $kP^2 - kNP + N \cdot \frac{kN}{4} = 0$ $P^2 - NP + \frac{N^2}{4} = 0$ $\left(P - \frac{N}{2}\right)^2 = 0$ $P = \frac{N}{2}$ $\therefore$ The maximum sustainable emigration rate occurs when the population is at half its carrying capacity.	**M1** attempt to form quadratic equation in P **M1** consider discriminant (accept D = 0 instead of D ≥ 0 with proper justification) **A1** obtain $I_{max} = \frac{kN}{4}$ **MA1** substitute I_{max} back into earlier equation to arrive at conclusion
(ii)	$\frac{dP}{dt} = kP\left(1 - \frac{P}{60}\right) - \frac{k}{3}P$ $= kP\left(1 - \frac{P}{60} - \frac{1}{3}\right) = kP\left(\frac{2}{3} - \frac{P}{60}\right) = -\frac{kP}{60}(P - 40)$ $\int \frac{1}{P(P-40)}\, dP = \int -\frac{k}{60}\, dt$ $\int \frac{-1}{40P} + \frac{1}{40(P-40)}\, dP = \int -\frac{k}{60}\, dt$ $\frac{-1}{40}\ln P + \frac{1}{40}\ln\lvert P - 40\rvert = -\frac{k}{60}t + C_1$ where C_1 is an arbitrary cons $\frac{1}{40}\ln\frac{\lvert P-40\rvert}{P} = -\frac{k}{60}t + C_1$ $\ln\left\lvert 1 - \frac{40}{P}\right\rvert = -\frac{2k}{3}t + C_2$ where $C_2 = 40C_1$ $1 - \frac{40}{P} = \pm e^{-\frac{2k}{3}t + C_2} = Ce^{-\frac{2k}{3}t}$ where $C = \pm e^{C_2}$	**M1** use separable variables method **M1** use partial fractions or MF26 formula **A1**

	When $t = 0$, $P = 50 \Rightarrow\ 0.2 = C$ $$1-\frac{40}{P} = 0.2\mathrm{e}^{-\frac{2k}{3}t}$$ $$\frac{40}{P} = 1-0.2\mathrm{e}^{-\frac{2k}{3}t}$$ $$P = \frac{40}{1-0.2\mathrm{e}^{-\frac{2k}{3}t}}$$ 	**M1** attempt to find constant in exponential equation **A1** **B1** graph with correct shape and region **B1** intercept and asymptote
(iv)	The population-time graph is concave upwards, so the tangents (note: their gradients depend only on P) are below the graph.	**B1** mention of either "concave upwards" or "tangents below the graph"
(iv)	$k = 0.06 \Rightarrow \frac{\mathrm{d}P}{\mathrm{d}t} = \mathrm{f}\left(t, P\right) = \mathrm{f}\left(P\right) = -\frac{P}{1000}\left(P-40\right)$ Step size: $h = 5$ $t_0 = 0, P_0 = 50$ $u_1 = P_0 + h\,\mathrm{f}\left(P_0\right) = 50 + 5(-0.5) = 47.5$ $P_1 = P_0 + \frac{h}{2}\left[\mathrm{f}\left(P_0\right) + \mathrm{f}\left(u_1\right)\right] = 47.859375$ $u_2 = P_1 + h\,\mathrm{f}\left(P_1\right) = 45.97865$ $P_2 = P_1 + \frac{h}{2}\left[\mathrm{f}\left(P_1\right) + \mathrm{f}\left(u_2\right)\right] = 46.23179$ The population is approximately 46200 (3 s.f.) in 2025.	**M1** correct algorithm **A1** value of P_1 **A1**