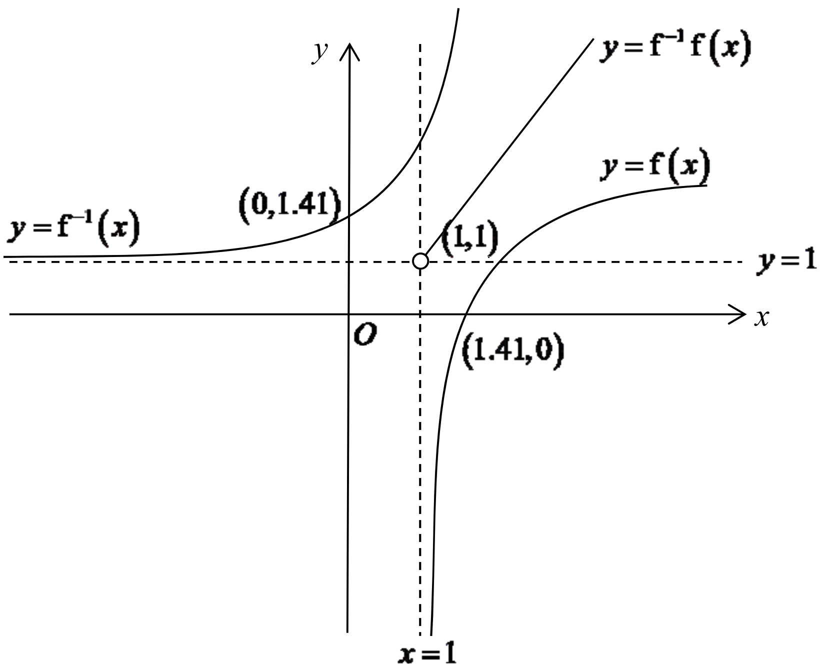
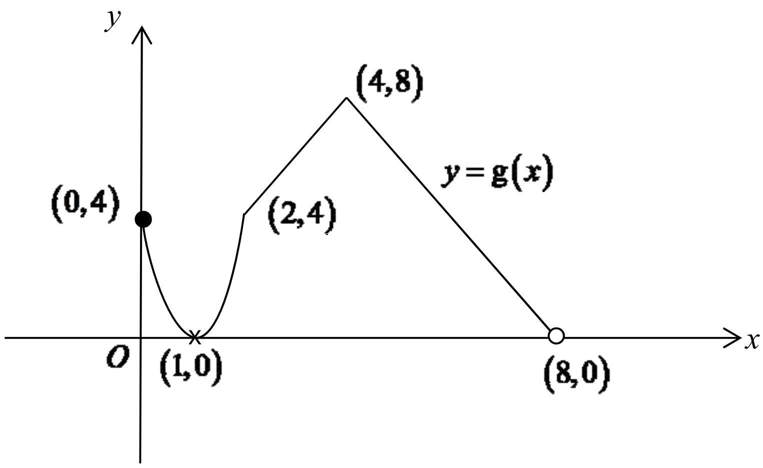
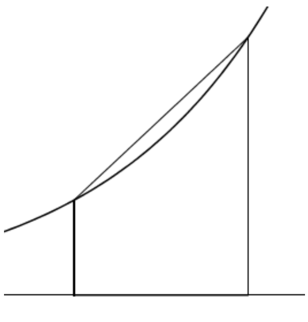
2017 H2 MATH (9758/02) JC 2 PRELIM – SOLUTIONS

Qn	Solution
1	**Complex 2**
(i)	Given $\lvert z\rvert = 3,\ \arg(z) = \dfrac{2\pi}{3}$, $\left\lvert \dfrac{-2\mathrm{i}}{z^*} \right\rvert = \dfrac{\lvert -2i\rvert}{\lvert z^*\rvert} = \dfrac{2}{3}\ \ (\because \lvert z\rvert = \lvert z^*\rvert)$ $\arg\left(\dfrac{-2\mathrm{i}}{z^*}\right) = \arg(-2\mathrm{i}) - \arg(z^*)$ $= -\dfrac{\pi}{2} - \left(-\dfrac{2\pi}{3}\right)\ \ (\because \arg(z^*) = -\arg(z))$ $= \dfrac{\pi}{6}$
(ii)	$\dfrac{-2\mathrm{i}}{z^*} = \dfrac{2}{3}\left[\cos\left(\dfrac{\pi}{6}\right) + \mathrm{i}\sin\left(\dfrac{\pi}{6}\right)\right]$ $= \dfrac{2}{3}\left(\dfrac{\sqrt{3}}{2} + \dfrac{1}{2}\mathrm{i}\right)$ $= \dfrac{\sqrt{3}}{3} + \dfrac{1}{3}\mathrm{i}$
(iii)	$\dfrac{-2\mathrm{i}w}{z^*} = \left(\dfrac{\sqrt{3}}{3} + \dfrac{1}{3}\mathrm{i}\right)(1+\mathrm{i}k)$ $= \dfrac{\sqrt{3}}{3} + \dfrac{\sqrt{3}}{3}k\mathrm{i} + \dfrac{1}{3}\mathrm{i} - \dfrac{1}{3}k$ Since $\dfrac{-2\mathrm{i}w}{z^*}$ is purely imaginary, $\dfrac{\sqrt{3}}{3} - \dfrac{1}{3}k = 0$ $k = \sqrt{3}$

Qn	Solution
2	**Differential Equations**
(i)	$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}=\frac{2}{(t+1)^3}$ $\frac{\mathrm{d}x}{\mathrm{d}t}=\int 2(t+1)^{-3}\ \mathrm{d}t$ $\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{2(t+1)^{-2}}{-2}+a$, where $a\in\mathbb{R}$ $=-(t+1)^{-2}+a$ $x=\int -(t+1)^{-2}+a\ \mathrm{d}t$ $=(t+1)^{-1}+at+b$, where $b\in\mathbb{R}$ $=\frac{1}{t+1}+at+b$, where $b\in\mathbb{R}$
(ii)	$\frac{\mathrm{d}x}{\mathrm{d}t}=k-cx^2,\quad k,c>0$ When $x=0.5$, $\frac{\mathrm{d}x}{\mathrm{d}t}=0$ $k=c(0.5)^2$ $c=4k$ $\frac{\mathrm{d}x}{\mathrm{d}t}=k-4kx^2=k(1-4x^2)$ (shown)
(iii)	$\frac{\mathrm{d}x}{\mathrm{d}t}=k-4kx^2=k\left(1-4x^2\right)$ $\int\frac{1}{1-4x^2}\mathrm{d}x=\int k\ \mathrm{d}t,\quad 1-4x^2\neq 0$ $\frac{1}{2\left(2\right)}\ln\left\|\frac{1+2x}{1-2x}\right\|=kt+d$, $d\in\mathbb{R}$ $\frac{1}{4}\ln\left\|\frac{1+2x}{1-2x}\right\|=kt+d$ $\ln\left\|\frac{1+2x}{1-2x}\right\|=4kt+4d$ $\frac{1+2x}{1-2x}=\pm\mathrm{e}^{4kt+4d}=A\mathrm{e}^{4kt}$ where $A=\pm\,\mathrm{e}^{4d}$ When $t=0, x=1$, $\frac{1+2}{1-2}=A$ $A=-3$ $\frac{1+2x}{1-2x}=-3\mathrm{e}^{4kt}$ $1+2x=-3\mathrm{e}^{4kt}+6x\mathrm{e}^{4kt}$ $x\left(2-6\mathrm{e}^{4kt}\right)=-3\mathrm{e}^{4kt}-1$ $x=\frac{-3\mathrm{e}^{4kt}-1}{2-6\mathrm{e}^{4kt}}=\frac{1+3\mathrm{e}^{4kt}}{6\mathrm{e}^{4kt}-2}$ or $\frac{1}{3\mathrm{e}^{4kt}-1}+\frac{1}{2}$

Qn	Solution
3	**Functions**
(i)	Let $y=\ln\left(x^2-1\right)$. $x=\pm\sqrt{1+\mathrm{e}^y}$ Since $x>1>0$, $\therefore x=\sqrt{1+\mathrm{e}^y}$. $\mathrm{D}_{\mathrm{f}^{-1}}=\mathrm{R}_{\mathrm{f}}$ $=\mathbb{R}$ $\mathrm{f}^{-1}:x\mapsto\sqrt{1+\mathrm{e}^x},\ x\in\mathbb{R}$
(ii)	
(iii)	
(iv)	Since $\mathrm{R}_{\mathrm{h}}=[0,3]\subseteq[0,8)=\mathrm{D}_{\mathrm{g}}$, therefore the function gh exists. Restrict D_{g} to be $[0,3]$ From the graph in **(iii)**, $\mathrm{R}_{\mathrm{gh}}=[0,6]$.

Qn	Solution
4	**Techniques of Integration and Summation**
(i)	$A_n = \frac{1}{n}\left(e^0 + e^{\frac{1}{n}} + e^{\frac{2}{n}} + e^{\frac{3}{n}} + \cdots + e^{\frac{n-2}{n}} + e^{\frac{n-1}{n}}\right)$ $= \frac{1}{n}\cdot\frac{e^0\left(1-\left(e^{\frac{1}{n}}\right)^n\right)}{1-e^{\frac{1}{n}}}$ $= \frac{1}{n}\cdot\frac{1-e}{1-e^{\frac{1}{n}}} = \frac{e-1}{n\left(e^{\frac{1}{n}}-1\right)}$ $\therefore c = e-1$
(ii)	$e^x - 1 = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right) - 1 = x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$ $\lim_{x\to 0}\frac{1}{x}\left(e^x - 1\right) = \lim_{x\to 0}\left[\frac{1}{x}\left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right)\right]$ $= \lim_{x\to 0}\left(1 + \frac{x}{2!} + \frac{x^2}{3!} + \ldots\right)$ $= 1$
(iii)	$\lim_{n\to\infty}\frac{e-1}{n\left(e^{\frac{1}{n}}-1\right)} = \lim_{x\to 0}\frac{e-1}{\frac{1}{x}\left(e^x-1\right)}$ $= e-1$
(iv)	e – 1 is the exact area under the graph of $y = e^x$ from $x = 0$ to $x = 1$. area $= \int_0^1 e^x \mathrm{d}x = e-1$
	Since the graph of $y = e^x$ is concave upwards, and $\frac{A_n + B_n}{2}$ is the sum of the area of n trapeziums each of width $\frac{1}{n}$, the area of all trapeziums will be greater than the exact area under the graph, which is $\int_0^1 e^x \mathrm{d}x$. 

<table>
<tr><th>Qn</th><th>Solution</th></tr>
<tr><td>5</td><td>Permutations and Combinations</td></tr>
<tr><td>(a)
(i)</td><td>Total number of possible passcodes
$=26\times25\times24\times10\times9\times8\times7=78624000$
or
$={}^{26}C_3\times3!\ \times\ {}^{10}C_4\times4!=78624000$
or
$={}^{26}P_3\times{}^{10}P_4=78624000$</td></tr>
<tr><td>(ii)</td><td><u>Case 1:</u> 1st digit 4, 6, 8

<table><tr><td>4, 6, 8
(3 choices)</td><td>8 choices</td><td>7 choices</td><td>1, 3, 5, 7, 9
(5 choices)</td></tr></table>

Number of possible passcodes
$=26\times25\times24\times3\times8\times7\times5=13104000$

<u>Case 2:</u> 1st digit 3, 5, 7, 9

<table><tr><td>3, 5, 7, 9
(4 choices)</td><td>8 choices</td><td>7 choices</td><td>4 choices</td></tr></table>

Number of possible passcodes
$=26\times25\times24\times4\times8\times7\times4=13977600$

Total number of possible passcodes
$=13104000+13977600=27081600$</td></tr>
<tr><td>(b)</td><td>Total number of possible passcodes
$=26^3\times10^4\times\dfrac{7!}{4!3!}=6151600000$</td></tr>
</table>

Qn	Solution
6	**Discrete Random Variable and Probability**
(i)	

x	1	2	3	4
$\mathrm{P}(X=x)$	θ	3θ	5θ	k

Since $\sum_{\text{all } x} \mathrm{P}(X=x)=1$,

$\theta+3\theta+5\theta+k=1$

$\therefore k=1-9\theta$

Probability distribution of X is

x	1	2	3	4
$\mathrm{P}(X=x)$	θ	3θ	5θ	$1-9\theta$

(ii)

$$\begin{aligned}\mathrm{E}(X)&=1(\theta)+2(3\theta)+3(5\theta)+4(1-9\theta)\\&=\theta+6\theta+15\theta+4-36\theta\\&=4-14\theta\end{aligned}$$

$$\begin{aligned}\mathrm{E}(X^2)&=1^2(\theta)+2^2(3\theta)+3^2(5\theta)+4^2(1-9\theta)\\&=\theta+12\theta+45\theta+16-144\theta\\&=16-86\theta\end{aligned}$$

$$\begin{aligned}\mathrm{Var}(X)&=\mathrm{E}(X^2)-\left[\mathrm{E}(X)\right]^2\\&=16-86\theta-(4-14\theta)^2\\&=16-86\theta-(16-112\theta+196\theta^2)\\&=26\theta-196\theta^2\end{aligned}$$

(iii)

$Y=a+bX$

$\mathrm{Var}(Y)=\mathrm{Var}(a+bX)$

$\mathrm{Var}(Y)=b^2\mathrm{Var}(X)$

$\frac{1}{3}b^2=b^2\left(26\theta-196\theta^2\right)$

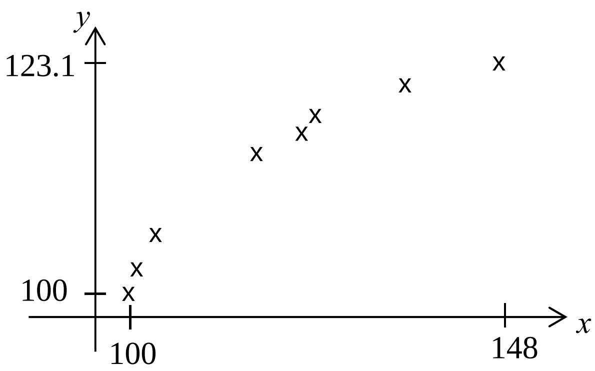
$196\theta^2-26\theta+\frac{1}{3}=0 \quad (\because \mathrm{b}\neq 0)$

Using GC,

$\theta=0.0144$ or $\theta=0.118$ (rejected $\because 0<\theta<\frac{1}{9}$)

Qn	Solutions
7	**Binomial Distribution**
	A binomial distribution is appropriate as there is a large number of lego pieces with constant probability 0.15 of them being red suggests independence in selection. Moreover, there are only two possible outcomes (red or non red). (Answer provided by SEAB.)
(i)	Let X be the number of lego pieces, out of 20, that are red. $X \sim \mathrm{B}(20, 0.15)$ $\mathrm{P}(X \geq 4) = 1 - \mathrm{P}(X \leq 3)$ $= 0.35227$ $= 0.352 \quad (3\text{s.f.})$
(ii)	Let Y be the number of boxes of lego pieces, out of 50, that contain at least 4 red lego pieces. $Y \sim \mathrm{B}(50, 0.35227)$ $\mathrm{P}(Y \leq 19) = 0.71498$ $= 0.715 \quad (3\text{ s.f.})$
(iii)	Let A be the number of lego pieces, out of 20, that are red. $A \sim \mathrm{B}(20, p)$ $\mathrm{P}(1 \leq A < 4) = 0.22198$ $\mathrm{P}(A=1) + \mathrm{P}(A=2) + \mathrm{P}(A=3) = 0.22198$ $\binom{20}{1} p(1-p)^{19} + \binom{20}{2} p^2(1-p)^{18} + \binom{20}{3} p^3(1-p)^{17} = 0.22198$ $20p(1-p)^{19} + 190p^2(1-p)^{18} + 1140p^3(1-p)^{17} = 0.22198$ Since $0.2 < p < 1$, $p = 0.250$ (3 s.f)

Qn	Solution
8	**Hypothesis Testing**
(i)	Using GC, Unbiased estimate of population mean is $\bar{x} = 325.58 \ (2 \text{ d.p.})$ Unbiased estimate of population variance is $s^2 = 1.5326^2 = 2.35 \ (2 \text{ d.p.})$
	Let μ denote the population mean volume of shampoo dispensed by the machine. Given $X \sim \mathrm{N}\left(\mu, \sigma^2\right) \ \therefore \bar{X} \sim \mathrm{N}\left(\mu, \frac{\sigma^2}{n}\right)$ $\mathrm{H_0}$: $\mu = 325$ $\mathrm{H_1}$: $\mu > 325$ Test statistic: $Z = \dfrac{\bar{X} - \mu}{S/\sqrt{n}}$ Level of significance: 5% Reject $\mathrm{H_0}$ if $p-\text{value} < 0.05$ Alternatively, Reject $\mathrm{H_0}$ if z-value > 1.6449 Under $\mathrm{H_0}$, using GC, z-value = 2.9314 or 2.9482 Under $\mathrm{H_0}$, using GC, $p-\text{value} = 0.00160$ (3 s.f) or 0.00169 (3 s.f) Conclusion: Since $p-\text{value} = 0.00169 < 0.05$, we **reject $\mathrm{H_0}$** and conclude that there is **sufficient evidence**, at the 5% significance level, that the mean volume dispensed is more than 325 ml. Thus, the assembly manager's suspicion is valid at 5% level of significance.
	There is a probability of 0.05 of concluding that the mean volume of shampoo dispensed is more than 325 ml when in fact, it is 325 ml.
(ii)	$\mathrm{H_0}$: $\mu = 325$ $\mathrm{H_1}$: $\mu \neq 325$ Given $X \sim \mathrm{N}\left(\mu, \sigma^2\right) \ \therefore \bar{X} \sim \mathrm{N}\left(\mu, \frac{\sigma^2}{n}\right)$ Test statistic: $Z = \dfrac{\bar{X} - \mu}{\sigma/\sqrt{n}}$ Level of significance: 5% Since $\mathrm{H_0}$ is not rejected, $-1.9600 < z-\text{value} < 1.9600$ $-1.9600 < \dfrac{\bar{x} - 325}{\frac{1.2}{\sqrt{50}}} < 1.9600$ $324.67 < \bar{x} < 325.33 \quad (2 \text{ d.p.})$ $\left\{\bar{x} \in \mathbb{R} : 324.67 < \bar{x} < 325.33\right\}$

Qn	Solution
9	**Correlation and Regression**
(i)	Let k be the missing housing price index for 2011. $\bar{y}=111.775$ and $\bar{x}=\dfrac{815.2+k}{8}$ Since $\bar{y}$ and $\bar{x}$ lies on the regression line, $111.775=54.271+0.48363\left(\dfrac{815.2+k}{8}\right)$ $k=136$ $(3\text{ s.f.})(\text{shown})$
(ii)	 From the scatter diagram, as x increases, y increases at a decreasing rate. Thus the linear model might not be the most appropriate model.
(iv)	(Note that there is no clear independent variable.) From GC, an appropriate regression line would be $\sqrt{x}=0.0896y+0.860$ (3 s.f.) When $y=134.6$, from GC, $x=167$ (3 s.f.). The estimated housing price index in 2016 is 167. Since $y=134.6$ <u>falls outside the data range of y</u>, the linear correlation between y and $\sqrt{x}$ might no longer hold and thus, the estimate is unreliable.
(iv)	From GC, $r=0.979$ (3 s.f.).
(v)	The product moment correlation coefficient between $\sqrt{\dfrac{x}{100}}$ and $\dfrac{y}{100}$ <u>**does not differ**</u> from the value obtained in part **(iv)** as the r-value is <u>**independent of the scale of measurement.**</u> **Note that:** $\sqrt{\dfrac{x}{100}}=\dfrac{\sqrt{x}}{10}$ means that the values of $\sqrt{x}$ undergo a scaling of 10 units and $\dfrac{y}{100}$ means that the values of y undergo a scaling of 100 units.

Qn	Solution
10	**Normal and Sampling Distribution**
(i)	Let X be the mass of a randomly chosen nut in grams. $X \sim \mathrm{N}\left(\mu, \sigma^2\right)$ $\overline{X} \sim \mathrm{N}\left(\mu, \frac{\sigma^2}{50}\right)$ and $X_1 + \ldots + X_{50} \sim \mathrm{N}\left(50\mu, 50\sigma^2\right)$ Given $\mathrm{P}\left(\overline{X} < 247\right) = 0.018079$ and $\mathrm{P}\left(X_1 + \ldots + X_{50} > 12600\right) = 0.78397$ Standardizing, $Z \sim \mathrm{N}(0,1)$ $\mathrm{P}\left(Z < \frac{247-\mu}{\sigma/\sqrt{50}}\right) = 0.018079$ $\frac{247-\mu}{\sigma/\sqrt{50}} = -2.095146$ $\mu - 0.2962984\sigma = 247 \ldots(1)$ $\mathrm{P}\left(Z > \frac{12600-50\mu}{\sqrt{50}\sigma}\right) = 0.78397$ $\mathrm{P}\left(Z < \frac{12600-50\mu}{\sqrt{50}\sigma}\right) = 0.21603$ $\frac{12600-50\mu}{\sqrt{50}\sigma} = -0.7856714$ $50\mu - 0.7856714\left(\sqrt{50}\sigma\right) = 12600 \ldots(2)$ Solving equation (1) and (2), using GC, $\mu = 255$ (nearest gram) $\sigma = 27$ (nearest gram)
(ii)(a)	Let Y be the mass of a randomly chosen nut in grams. $Y \sim \mathrm{N}\left(250, 5^2\right)$ Let W be the mass of a randomly chosen bolt in grams. $W \sim \mathrm{N}\left(745, 7.3^2\right)$ $W - 3Y \sim \mathrm{N}\left(745 - 3\times 250, 7.3^2 + 3^2 \times 5^2\right)$ i.e. $W - 3Y \sim \mathrm{N}\left(-5, 278.29\right)$ $\mathrm{P}\left(\lvert W - 3Y\rvert \geq 40\right) = \mathrm{P}\left(W - 3Y < -40\right) + \mathrm{P}\left(W - 3Y > 40\right)$ $= 0.0214$ (3s.f.)
(b)	Let T be total mass of 10 randomly chosen nut, made using new material, in grams. $T = 0.9Y_1 + 0.9Y_2 + \ldots + 0.9Y_{10} \sim \mathrm{N}\left(10\times 0.9\times 250, 10\times 0.9^2 \times 5^2\right)$ $T \sim \mathrm{N}\left(2250, 202.5\right)$ $\mathrm{P}\left(T < 2240\right) = 0.241$ (3s.f.)