ST ANDREW'S JUNIOR COLLEGE

PRELIMINARY EXAMINATION

FURTHER MATHEMATICS
Higher 2 **9649/01**

Tuesday **29 August 2017** **3 hours**

READ THESE INSTRUCTIONS FIRST

Write your name, civics group and index number on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.

Answer **all** the questions. Total marks is **100**.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphic calculator.
Unsupported answers from a graphic calculator are allowed unless a question specifically state otherwise.
Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematic steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.
At the end of the examination, fasten all your work securely together.

This document consists of **7** printed pages including this page.

1 The linear transformation $\mathrm{T}:\mathbb{R}^4 \to \mathbb{R}^3$ is represented by the matrix $\mathbf{A}$ where

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 2 & 6 \\ 1 & 1 & 2 & 4 \\ 3 & 7 & 6 & 20 \end{pmatrix}$$

(i) Let R_T and K_T be the range space and null space of T respectively. Find the dimension of R_T and deduce the dimension of K_T. **[3]**

(ii) By finding the bases for R_T and K_T, find the general solution of $\mathrm{T}(\mathbf{x}) = \begin{pmatrix} 3 \\ 1 \\ 11 \end{pmatrix}$. **[5]**

2 **(i)** Sketch the graph of $y = 7\sin x - \ln 3x$ for the interval $0 < x \le 20$, labelling the coordinates of the end point(s) and the equations of any asymptotes clearly. **[2]**

(ii) The largest positive root, ω, of the equation $7\sin x = \ln 3x$, where $0 < x \le 20$, lies in the interval $[n, n+1]$ for some integer n. State the value of n and use linear interpolation once on this interval to obtain an estimate of ω correct to 1 decimal place. **[2]**

(iii) With the estimate in part **(ii)** as the starting value x_1, perform three iterations of the Newton-Raphson method to find the value of ω correct to 3 decimal places. Verify the correctness of this new estimate. **[3]**

(iv) Explain with a diagram why using a starting value very close to ω might not necessarily yield the root ω eventually. **[1]**

3 **(i)** Verify that $y = \sin x$ is a particular solution of the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = y^2 \sec x - y\tan x + \cos x$ (*). **[2]**

(ii) Show that the substitution $y = \sin x + \frac{1}{u}$ reduces (*) to $\frac{\mathrm{d}u}{\mathrm{d}x} = -u\tan x - \sec x$. **[3]**

(iii) Hence find the general solution of (*). **[6]**

4 **(a)** Figure 1 shows a fixed line l and a fixed point A which does not lie on l. For a point P on l, let l_P be the line passing through P which is perpendicular to l. Let R be the point of intersection of l_P and the perpendicular bisector of the line segment PA.

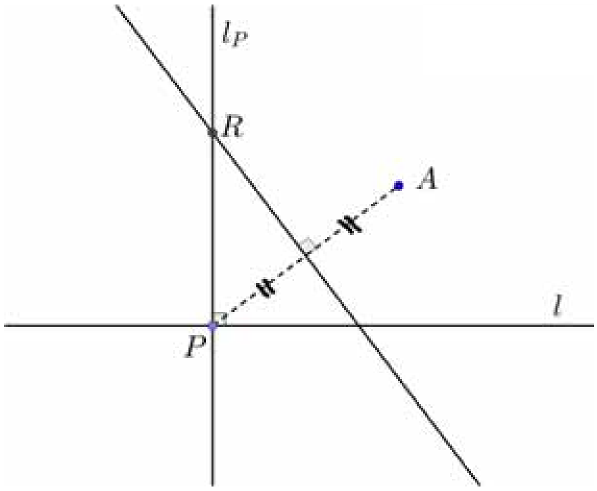

Figure 1

Given that P varies along the line l, by considering PR and AR, describe the locus of R. **[2]**

(b) Figure 2 shows two circles centred at the origin O: circle C_1 with radius a and circle C_2 with radius b, where $a > b$. For a point Q on C_1, let N be the foot of perpendicular of Q onto the horizontal axis, let S be the point of intersection of the line segment OQ and C_2, and let θ be the angle that OQ makes with the positive x-axis. The point T is the foot of perpendicular of S onto the line segment QN.

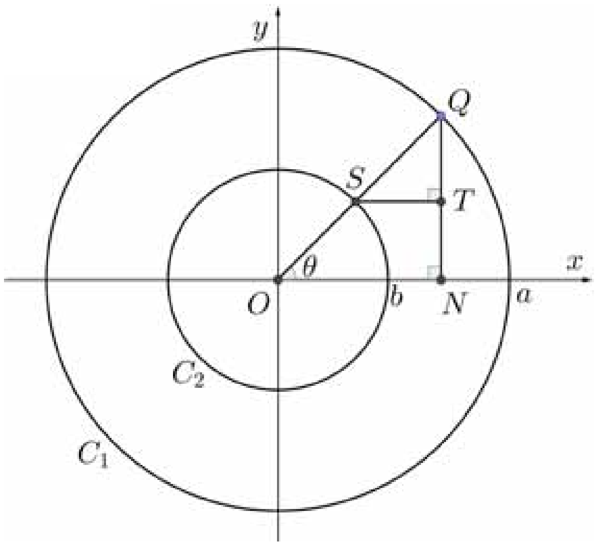

Figure 2

Given that Q varies along the circle C_1, by expressing the coordinates of T in terms of a, b and θ, describe the locus of T. **[3]**

[Turn over

5 It is known that each hour after an antibiotic injection, 12% of the antibiotic units remaining in the bloodstream is lost.

A doctor considers prescribing a course of treatment which involves a patient taking an injection of 50 units of antibiotic medicine every 6 hours over a long period of time. However, the doctor is aware that more than 100 units of this antibiotic in the bloodstream is regarded as dangerous to the patient.

Let a_n be the number of antibiotic units in the patient's bloodstream upon the *n*th injection under this course of treatment.

(i) Show that $a_2 = 73.22$ correct to 2 decimal places. **[1]**

(ii) Write down a recurrence relation for a_n, stating clearly the initial condition. **[1]**

(iii) Solve the recurrence relation and hence determine whether the doctor should prescribe this antibiotic course. **[5]**

6 The matrix $\mathbf{M} = \begin{pmatrix} m-1 & m-1 & -m+1 \\ m+1 & m-3 & -m-1 \\ 2 & -2 & -2 \end{pmatrix}$ has eigenvalues $0, -4$ and $2m-2$.

(i) State a pair of linearly dependent column vectors of **M**. Hence write down an eigenvector of **M** corresponding to the eigenvalue 0. **[2]**

(ii) Using a similar method as part **(i)**, or otherwise, find the eigenvectors of **M** corresponding to the other eigenvalues. **[3]**

(iii) The mapping $\mathrm{f}: \mathbb{R}^3 \to \mathbb{R}^3$ is defined by $\mathrm{f}: \begin{pmatrix} x \\ y \\ z \end{pmatrix} \to \mathbf{M} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

Given that one of the eigenvectors of **M** is mapped onto itself, state the value of *m*. Describe the effect of the transformation f on the other eigenvectors of **M**. **[3]**

(iv) By making use of parts **(i)** and **(ii)**, express the matrix $\mathbf{A}^{50}$, where $\mathbf{A} = \begin{pmatrix} 2017 & 2017 & -2017 \\ 2019 & 2015 & -2019 \\ 2 & -2 & -2 \end{pmatrix}$, in the form $\mathbf{PBP}^{-1}$, where **P** is a non-singular matrix and **B** is a diagonal matrix. **[3]**

7 **(a)** Sketch the conic section with equation $2y^2 - x^2 + 6x - 13 = 0$, indicating the exact coordinates of the foci and vertices, as well as the exact equations of any asymptotes. **[4]**

(b) **(i)** The line with equation $y = mx + c$ is tangent to the hyperbola with equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where a and b are real constants. Show that $b^2 + c^2 = a^2m^2$. **[4]**

(ii) One of the tangents to the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$ has positive gradient and intersects the x-axis at $(2,0)$. Using part **(b)(i)**, find the equation of this tangent. **[3]**

8 **(a)** (Give all answers in this part in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leq \pi$.)
Find the fourth roots of $-4 - 4\sqrt{3}i$. **[3]**
Hence find all the roots of the equation $z^8 + 8z^4 + 64 = 0$. **[3]**

(b) Sketch, on an Argand diagram, the locus of P representing the complex number z satisfying the equation $|z - 1 + 2i| = \sqrt{5}$. **[1]**

The locus of Q represents another complex number w satisfying the equation $\arg(w + 2 + 2i) = \theta$ where $-\pi < \theta \leq \pi$.

Find the range of values of θ such that the locus of Q does not intersect the locus of P. **[2]**

Given that $\theta = \tan^{-1} 4$, find the exact least value of $|z - w|$. **[3]**

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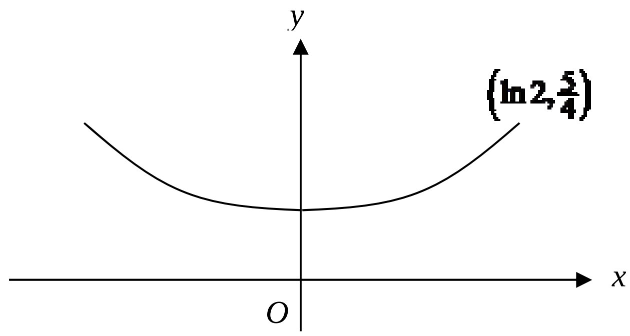


The diagram shows the curve C with equation $y = \dfrac{e^{x} + e^{-x}}{2}$ for $-\ln 2 \le x \le \ln 2$.

(i) Show that the length of C is $\frac{3}{2}$ units. **[3]**

(ii) Using the shell method, prove that the volume of the solid of revolution when the region bounded by C and the line $y = \frac{5}{4}$ rotates π radians about the y-axis is of the form

$$V = \frac{\pi}{4}\left[a(\ln 2)^2 + b(\ln 2) + c\right]$$

where a, b and c are constants to be determined. **[5]**

The y-ordinate of the centroid of a solid of revolution generated by a curve $y = f(x)$ rotating about the y-axis for $a \le y \le b$ is given by

$$\bar{y} = \frac{\pi \int_a^b yx^2 \, dy}{V}$$

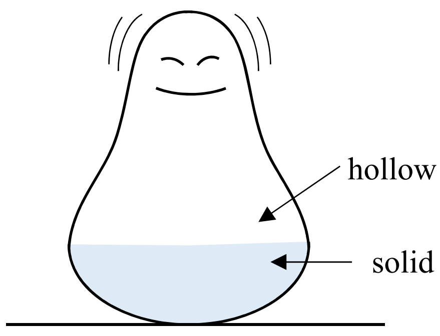
where V is the volume of the solid of revolution.

(iii) Show that the y-ordinate of the centroid of the solid of revolution in part **(ii)** can be expressed as

$$\bar{y} = \frac{\pi}{4V} \int_0^{\ln 2} x^2 \left(e^{2x} - e^{-2x}\right) dx$$

and hence evaluate the value of $\bar{y}$ to 3 decimal places. **[3]**

(iv) A tilting toy is formed by placing a tall three-dimensional hollow object of negligible weight on top of the solid of revolution in part **(ii)**. The toy is placed with its curved surface on a horizontal plane. Give a reason why the toy will not topple easily. **[1]**



10 Lalaland, like many rural communities, is in decline as her younger people migrate to the cities. In 2015, a census was carried out and the population of Lalaland was found to be 50000. Professor Rishi, an expert on migration, found a suitable model for Lalaland's future population using the differential equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1 - \frac{P}{N}\right) - I$$

where $k > 0$ is the intrinsic growth rate of Lalaland, N is the carrying capacity of Lalaland (in thousands), while $P = P(t)$ and $I = I(t)$ represent the population and emigration rates (both given in thousands) t years from 2015.

(i) Assuming a constant emigration rate, find the maximum sustainable emigration rate $I_{\max}$ in terms of N, and deduce that this rate occurs when the population is at half its carrying capacity. **[4]**

(ii) Suppose now that the emigration rate at any given time is $\frac{k}{3}P$. Given that Lalaland's estimated carrying capacity is 60000, find P in terms of t and sketch the graph of P versus t. **[7]**

(iii) With reference to the graph in part **(ii)**, explain why any approximation of the population after 2015 using the Euler Method will lead to an underestimate. **[1]**

(iv) Given $k = 0.06$, use the Improved Euler's method with a step size of 5 years to approximate the population of Lalaland in the year 2025. **[3]**

End of Paper