H2 Mathematics 2017 Prelim Exam Paper 2 Question
Answer all questions [100 marks].

1

(i)

$$\frac{\sin(A-B)}{\cos A\cos B}=\frac{\sin A\cos B-\cos A\sin B}{\cos A\cos B}=\frac{\sin A}{\cos A}-\frac{\sin B}{\cos B}=\tan A-\tan B$$

(ii)

$$\sum_{r=1}^{N}\frac{\sin x}{\cos(r+1)x\cos rx}=\frac{\sin(2x-x)}{\cos 2x\cos x}+\frac{\sin(3x-2x)}{\cos 3x\cos 2x}+\frac{\sin(4x-3x)}{\cos 4x\cos 3x}+\ldots+\frac{\sin((N+1)x-Nx)}{\cos(N+1)x\cos Nx}$$

$$\begin{aligned}
&= (\tan 2x-\tan x)\\
&+ (\tan 3x-\tan 2x)\\
&+ (\tan 4x-\tan 3x)\\
&\quad\vdots\\
&+ (\tan(N-1)x-\tan(N-2)x\\
&+ (\tan Nx-\tan(N-1)x\\
&+ (\tan(N+1)x-\tan Nx\\
&= \tan(N+1)x-\tan x
\end{aligned}$$

(iii)

When $x=\dfrac{\pi}{3}$, $\displaystyle\sum_{r=1}^{N}\frac{\sin x}{\cos(r+1)x\cos rx}=\sum_{r=1}^{N}\left(\frac{\sqrt{3}}{2\cos\frac{r\pi}{3}\cos\frac{(r+1)\pi}{3}}\right)$

Thus, required sum $=\tan\left[(N+1)\left(\dfrac{\pi}{3}\right)\right]-\tan\left(\dfrac{\pi}{3}\right)=\tan\left[\dfrac{(N+1)\pi}{3}\right]-\sqrt{3}$

2

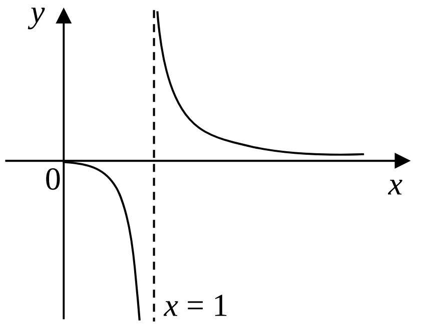
(i)

$$\begin{aligned}
\int_2^n\frac{9x}{(x^2-1)^3}\,dx &= \frac{9}{2}\int_2^n\frac{2x}{(x^2-1)^3}\,dx\\
&= \frac{9}{2}\left[-\frac{1}{2}(x^2-1)^{-2}\right]_2^n\\
&= \frac{9}{2}\left[-\frac{1}{2(n^2-1)^2}+\frac{1}{18}\right]\\
&= \frac{1}{4}-\frac{9}{4(n^2-1)^2}
\end{aligned}$$

$$\lim_{n\to\infty}\left[\int_2^n \frac{9x}{\left(x^2-1\right)^3}\,\mathrm{d}x\right]=\lim_{n\to\infty}\left[\frac{1}{4}-\frac{9}{4\left(n^2-1\right)^2}\right]$$

$$=\frac{1}{4}$$

(ii)



(iii) The equation of the transformed curve is $y=\dfrac{9x}{\left(x^2-1\right)^3}-\dfrac{2}{3}$.

Volume of revolution $=\pi\displaystyle\int_2^5\left(\frac{9x}{\left((x^2-1\right)^3}-\frac{2}{3}\right)^2\,\mathrm{d}x\ =3.385\text{ units}^3$ (to 3 d.p.)

3

(a) (i) $\dfrac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta-\dfrac{1}{3}\sin^3\theta\right)$

$$=\cos\theta-\sin^2\theta\cos\theta$$

$$=\cos\theta\left(1-\sin^2\theta\right)$$

$$=\cos\theta\left(\cos^2\theta\right)=\cos^3\theta$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-\sin x\cos^2 x$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\left(-\sin x\right)\left(\cos x\right)^2$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\left(\cos x\right)^3}{3}+C$$

$$=\frac{1}{3}\left(\cos x\cdot\cos^2 x\right)+C$$

$$=\frac{1}{3}\left(\cos x\cdot\left(1-\sin^2 x\right)\right)+C$$

$$=\frac{1}{3}\left(\cos x-\cos x\cdot\sin^2 x\right)+C$$

$$y=\frac{1}{3}\left(\sin x-\frac{\sin^3 x}{3}\right)+Cx+D$$

When $x = 0$ and $y = 0$, $D = 0$

When $x = 0$ and $\frac{dy}{dx}=\frac{1}{3}+\frac{2}{\pi}$, $C=\frac{2}{\pi}$

$$y=\frac{1}{3}\left(\sin x-\frac{\sin^3 x}{3}\right)+\frac{2}{\pi}x$$

(b) $v=x^2y$ --------------------(1)

$\frac{dv}{dx}=2xy+x^2\frac{dy}{dx}$ -------- (2)

$x\frac{dy}{dx}+2y+4x^2y=0$ ----- (3)

(3) × x, $x^2\frac{dy}{dx}+2xy+4x^2y(x)=0$ -------- (4)

$\frac{dv}{dx}+4x\left(x^2y\right)=0$

$\frac{dv}{dx}+4vx=0$

$\frac{dv}{dx}=-4vx$ (Shown)

$\frac{dv}{dx}=-4vx$

$\int\frac{1}{v}\,dv=-4\int x\,dx$

$\ln|v|=-2x^2+c$

$v=\pm e^{-2x^2+c}$

$v=Ae^{-2x^2}$, where $A=\pm e^c$

$x^2y=Ae^{-2x^2}$

Given that $y=\frac{1}{3}$ when $x=-3$,

$(-3)^2\left(\frac{1}{3}\right)=Ae^{-18}$

$A=3e^{18}$

	$y=\dfrac{3\mathrm{e}^{18-2x^2}}{x^2}$
4	**(i)** $l_1 : \mathbf{r}=\begin{pmatrix}1\\1\\0\end{pmatrix}+\lambda\begin{pmatrix}2\\0\\1\end{pmatrix}, \lambda\in\mathbb{R}$ $\overrightarrow{AB}=\begin{pmatrix}0\\0\\2\end{pmatrix}-\begin{pmatrix}-1\\1\\0\end{pmatrix}=\begin{pmatrix}1\\-1\\2\end{pmatrix}$; $\overrightarrow{AC}=\begin{pmatrix}0\\3\\-3\end{pmatrix}-\begin{pmatrix}-1\\1\\0\end{pmatrix}=\begin{pmatrix}1\\2\\-3\end{pmatrix}$; $\overrightarrow{BC}=\begin{pmatrix}0\\3\\-3\end{pmatrix}-\begin{pmatrix}0\\0\\2\end{pmatrix}=\begin{pmatrix}0\\3\\-5\end{pmatrix}$ A normal to the plane is: $\begin{pmatrix}1\\-1\\2\end{pmatrix}\times\begin{pmatrix}1\\2\\-3\end{pmatrix}=\begin{pmatrix}-1\\5\\3\end{pmatrix}$ $\mathbf{r}\cdot\begin{pmatrix}-1\\5\\3\end{pmatrix}=\begin{pmatrix}0\\0\\2\end{pmatrix}\cdot\begin{pmatrix}-1\\5\\3\end{pmatrix}=6$ Thus an equation for Π_1 is $-x+5y+3z=6$. (shown) **(ii)** Let *N* be the point of intersection between the line and the plane. $\overrightarrow{ON}=\begin{pmatrix}1+2\lambda\\1\\\lambda\end{pmatrix}$ for some $\lambda\in\mathbb{R}$ Since *N* lies on the plane, $\begin{pmatrix}1+2\lambda\\1\\\lambda\end{pmatrix}\cdot\begin{pmatrix}-1\\5\\3\end{pmatrix}=6\Rightarrow\lambda=2$ Thus, coordinates of *N* are (5, 1, 2). **(iii)** Let the foot of the perpendicular from *P* to the plane be denoted by *F*. $l_{PF} : \mathbf{r}=\begin{pmatrix}1\\1\\0\end{pmatrix}+\mu\begin{pmatrix}-1\\5\\3\end{pmatrix}, \mu\in\mathbb{R}$ Since *F* lies on l_{PF}, $\overrightarrow{OF}=\begin{pmatrix}1-\mu\\1+5\mu\\3\mu\end{pmatrix}$ for some $\mu\in\mathbb{R}$ Since *F* lies on the plane, $\begin{pmatrix}1-\mu\\1+5\mu\\3\mu\end{pmatrix}\cdot\begin{pmatrix}-1\\5\\3\end{pmatrix}=6$ Solving, $\mu=\dfrac{2}{35}$

$$\overrightarrow{OF} = \begin{pmatrix} 33/35 \\ 9/7 \\ 6/35 \end{pmatrix}$$

Let the reflection of point P in the mirror be P'.

By the midpoint theorem, $\overrightarrow{OP'} = 2\overrightarrow{OF} - \overrightarrow{OP} = \begin{pmatrix} 31/35 \\ 11/7 \\ 12/35 \end{pmatrix}$

A direction vector for the reflected line is $\begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 31/35 \\ 11/7 \\ 12/35 \end{pmatrix} = \begin{pmatrix} 144/35 \\ -4/7 \\ 58/35 \end{pmatrix} = \frac{2}{35}\begin{pmatrix} 72 \\ -10 \\ 29 \end{pmatrix}$

Thus, an equation of the reflected line is:

$$l_1' : \mathbf{r} = \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix} + \gamma \begin{pmatrix} 72 \\ -10 \\ 29 \end{pmatrix}, \gamma \in \mathbb{R}$$

Since l_2 is parallel to Π_1, $\begin{pmatrix} 2 \\ 0 \\ \alpha \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 5 \\ 3 \end{pmatrix} = 0 \Rightarrow \alpha = \frac{2}{3}$

$$\begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ \beta \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ -\beta \\ 0 \end{pmatrix}$$

Since the distance is $\frac{14}{\sqrt{35}}$, $\left| \frac{\begin{pmatrix} -1 \\ -\beta \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 5 \\ 3 \end{pmatrix}}{\sqrt{35}} \right| = \frac{14}{\sqrt{35}}$

$$|1 - 5\beta| = 14$$

Solving, $\beta = -\frac{13}{5}$ (rejected) or $\beta = 3$

5

$$\sum_{\text{all } x} \mathrm{P}(X = x) = 1 = 0.2 + a + b + 0.45 \Rightarrow a + b = 0.35 \quad \ldots (1)$$

$$E\left(|X - 4|\right) = 1\frac{1}{10} \Rightarrow \sum_{\text{all } x} |x - 4| \mathrm{P}(X = x) = \frac{11}{10}$$

$$\Rightarrow 2(0.2) + a + 0 + 0.45 = \frac{11}{10}$$

$$\Rightarrow a = 0.25 \text{ and } b = 0.1$$

	P(required) $=\mathrm{P}(X_1=2, X_2=2)+2\left[\mathrm{P}(X_1=2, X_2=3)+\mathrm{P}(X_1=2, X_2=4)\right]$ $=0.2\times0.2+2\left[0.2\times0.25+0.2\times0.1\right]$ $=0.18$
6	**(i)** Let X be the random variable "number of damaged mangoes out of 21 mangoes". $X \sim B(21, 0.045)$ $\mathrm{P}(X \le 3)=0.98673=0.987$ (3 s.f.) **(ii)** Let Y be the random variable "number of boxes of mangoes out of 12 boxes which are of low standard". $Y \sim B(12, 1-0.98673) \Rightarrow Y \sim B(12, 0.013268)$ $\mathrm{P}(Y \ge 2)=1-\mathrm{P}(Y \le 1)$ $=1-0.98936=0.01064=0.0106$ (3 s.f.) **(iii)** P(required) = $\mathrm{P}(X \le 5 \mid \text{box is of low standard})$ $=\mathrm{P}\left(X \le 5 \mid X > 3\right)$ $=\dfrac{\mathrm{P}\left(X \le 5 \cap X > 3\right)}{\mathrm{P}\left(X > 3\right)}$ $=\dfrac{\mathrm{P}\left(X=4\right)+\mathrm{P}\left(X=5\right)}{1-\mathrm{P}\left(Y \le 3\right)}$ $=\dfrac{0.011219+0.0017975}{1-0.98673}$ $=0.981$
7	**(a)(i)** Required probability = $\dfrac{7!\times{}^{8}C_{5}\times 5!}{12!}$ $=\dfrac{7}{99}$ **(a)(ii)** Required probability = $\dfrac{7!\times4\times 5!}{12!}$ $=\dfrac{1}{198}$ Required probability = $\dfrac{\left(10-1\right)!\times2!}{\left(12-1\right)!}$ $=\dfrac{1}{55}$

	(b)(i) $\mathrm{P}(A \mid B) = \frac{13}{20}$ $\frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)} = \frac{13}{20}$ $\mathrm{P}(A \cap B) = \frac{13}{20}\left(\frac{2}{5}\right) = \frac{13}{50}$ (or 0.26) $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$ $= \frac{7}{10} + \frac{2}{5} - \frac{13}{50}$ $= \frac{21}{25}$ (or 0.84) **(b)(ii)** Since $\mathrm{P}(A \mid B) \neq \mathrm{P}(A)$, therefore events *A* and *B* are not independent. **Alternatively,** Since $\mathrm{P}(A \cap B) = \frac{13}{50}$ and $\mathrm{P}(A) \times \mathrm{P}(B) = \frac{7}{10} \times \frac{2}{5} = \frac{7}{25} \neq \mathrm{P}(A \cap B)$, therefore events *A* and *B* are not independent. **(c)** Probability of winning the game $= \frac{2}{9} + \frac{2}{9}\left(\frac{3}{9}\right) + \frac{2}{9}\left(\frac{3}{9}\right)^2 + \ldots$ $= \frac{2/9}{1 - \frac{3}{9}}$ $= \frac{1}{3}$
8	**(i)** Let *X* be the random variable denoting volume of the randomly chosen iced coffee bottle in ml from Machine *A*. $\bar{x} = \frac{24965}{50} = 499.3$ Unbiased estimate of population variance $s^2 = \frac{1}{n-1}\sum (x - \bar{x})^2 = \frac{50}{49}\left(\frac{365}{50}\right) = \frac{365}{49} = 7.4489 \approx 7.45$ $\mathrm{H}_0 : \mu = 500$ $\mathrm{H}_1 : \mu \neq 500$ Two tailed Z test at 2% level of significance Under H_0, since the sample size of 50 is large, by Central Limit Theorem

	$\overline{X} \sim \mathrm{N}(500, \frac{7.4489}{50})$ approx. From GC, p-value = 0.06974> 0.02 Conclusion: Since the p-value is more than the level of significance, we do not reject H_0 and conclude that there is insufficient evidence at 2% that the mean volume is not 500ml. **(ii)** Let Y be the random variable denoting the volume of a randomly chosen iced coffee bottle in ml from Machine B. Unbiased estimate for population variance = $\frac{70}{69}\left(4^2\right) = 16.232$ $H_0 : \mu = \mu_0$ $H_1 : \mu < \mu_0$ One tailed Z test at 2% level of significance Under H_0, since the sample size of 70 is large, by Central Limit Theorem $\overline{Y} \sim \mathrm{N}\left(\mu_0, \frac{16.232}{70}\right)$ approx. Value of test statistic, $z_{\text{test}} = \frac{489.1 - \mu_0}{\sqrt{\frac{16.232}{70}}}$ For H_0 to be rejected, p-value $\leq$ 0.02 $\frac{489.1 - \mu_0}{\sqrt{\frac{16.232}{70}}} \leq -2.053748911$ $\mu_0 \geq 490$ (to 3 s.f.)
9	Let X denote the random variable representing the amount of time a randomly chosen junior college student spends on mobile phones each day. $\therefore X \sim \mathrm{N}(3.4, \sigma^2)$ $\mathrm{P}(3 < X < 3.8) = 0.341$ $\mathrm{P}\left(\frac{3-3.4}{\sigma} < Z < \frac{3.8-3.4}{\sigma}\right) = 0.341$ $\mathrm{P}\left(\frac{-0.4}{\sigma} < Z < \frac{0.4}{\sigma}\right) = 0.341$ $\Rightarrow \mathrm{P}\left(Z < \frac{-0.4}{\sigma}\right) = \frac{1-0.341}{2} = 0.3295$ From GC, $\frac{-0.4}{\sigma} = -0.4412942379$ $\Rightarrow \sigma = 0.90642 = 0.906$ (3 dp) **(i)** Probability required $= (0.341)^4$ $= 0.0135$ (3 sf)

(ii) Probability required $= \mathrm{P}(X_1 + X_2 + X_3 < 2X_4)$

$= \mathrm{P}(X_1 + X_2 + X_3 - 2X_4 < 0)$

$X_1 + X_2 + X_3 - 2X_4 \sim \mathrm{N}(3.4\times3 - 2\times3.4,\ 0.90642^2\times3 + 2^2\times0.90642^2)$

i.e. $X_1 + X_2 + X_3 - 2X_4 \sim \mathrm{N}(3.4,\ 5.75118)$

$\therefore$ From GC, $(X_1 + X_2 + X_3 - 2X_4 < 0) = 0.0781$ (3 sf)

(iii) Assumption: The amount of time spent by a randomly chosen student on mobile phones is independent of the amount of time spent by another randomly chosen student.

(iv) $\overline{X} \sim \mathrm{N}\left(3.4, \dfrac{0.90642^2}{50}\right)$

From GC, $\mathrm{P}\left(\overline{X} > 3.5\right) = 0.217663$

Since expected number of samples with mean time exceeding 3.5 hours = 15, then $0.217663 \times N = 15$

$\Rightarrow N = 68.9 \approx 69$

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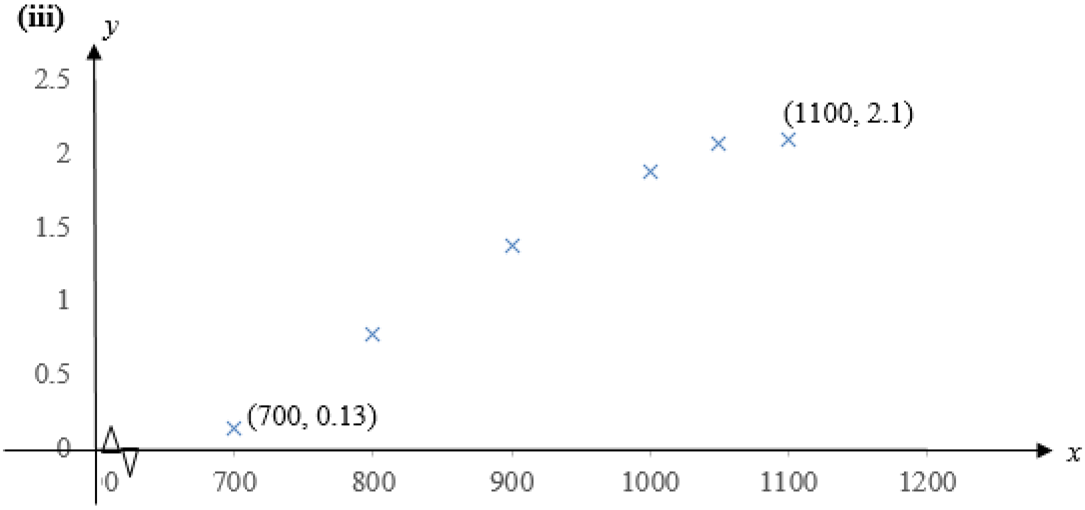
(i) The phrase ‘random sample’ means that every 50-year-old Singaporean woman has an <u>**equal probability of being included in the sample**</u>.

(ii) $r = 0.988$ (to 3 s.f.)

Although the r-value $= 0.988$ is close to 1, the value is not 1 so there may be another model with $|r|$ closer to 1.

Hence a linear model may not be the best model for the relationship between x and y.

(iii)



(iv) Using the GC, when $P = 3$, $r = -0.995337$ (to 6 d.p.)

When $P = 3$, $|r|$ is closest to 1 and thus, $P = 3$ is the most appropriate value.

(v) When $P = 3$, using the GC, $a = 3.2446 = 3.24$ (to 3 s.f.)

		$b = -0.0030988 = -0.00310$ (to 3 s.f.) When $y = 1.8$, and $P = 3$, $\ln(3-1.8) = 3.2446 - 0.0030988x$ $x = 988$ Thus, the recommended daily calcium intake is 988 mg. Since the r value is -0.995 is close to -1, there is a strong negative linear correlation between $\ln(P-y)$ and x. Also since the value of $y = 1.8$ is within the data range, thus, the estimate obtained is reliable.
	(vi)	The value of P is the maximum percentage increase in bone density achievable as the daily calcium intake increases.

− End Of Paper −