## 2017 NYJC JC2 Prelim 9758/1 Solution

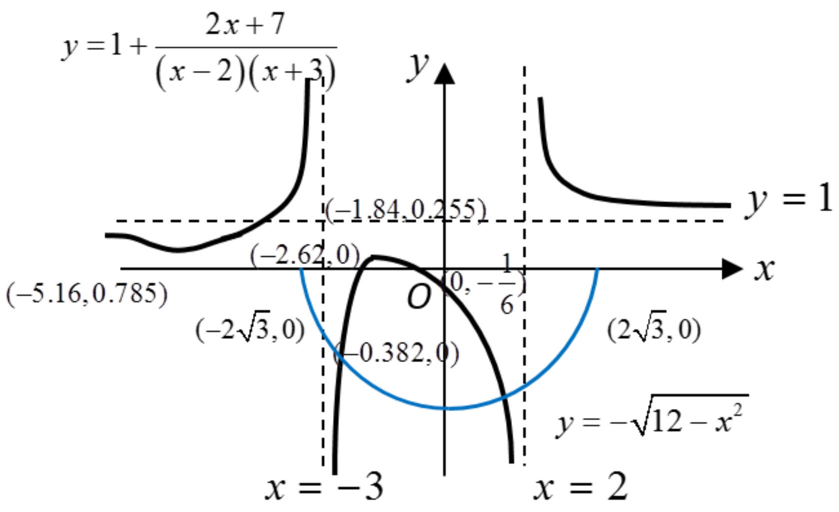
| Qn | |
|---|---|
| 1 | Sum of numbers in $k$th row = $\sum_{r=1}^{k} r = \frac{1}{2}k(k+1)$<br>Required sum = $\sum_{k=1}^{n} \frac{k(k+1)}{2}$<br>$= \frac{1}{2}\sum_{k=1}^{n}(k^2+k)$<br>$= \frac{1}{12}n(n+1)(2n+1) + \frac{1}{4}n(n+1)$<br>$= \frac{1}{12}n(n+1)(2n+1+3)$<br>$= \frac{1}{6}n(n+1)(n+2)$ |
| 2(i) | Differentiating $2x - y^2 = (x+y)^2$ ______(1)<br>implicitly with respect to $x$,<br>$2 - 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 2(x+y)\left(1+\frac{\mathrm{d}y}{\mathrm{d}x}\right)$<br>Where tangent is parallel to the $x$-axis, $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$.<br>$2 = 2(x+y)$<br>$y = 1 - x$ ______(2)<br>Sub (2) in (1),<br>$2x - (1-x)^2 = (x+1-x)^2$<br>$x^2 - 4x + 2 = 0$<br>$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(2)}}{2} = 2 \pm \sqrt{2}$<br>When $x = 2-\sqrt{2}$, $y = 1-(2-\sqrt{2}) = -1+\sqrt{2}$<br>When $x = 2+\sqrt{2}$, $y = 1-(2+\sqrt{2}) = -1-\sqrt{2}$ |

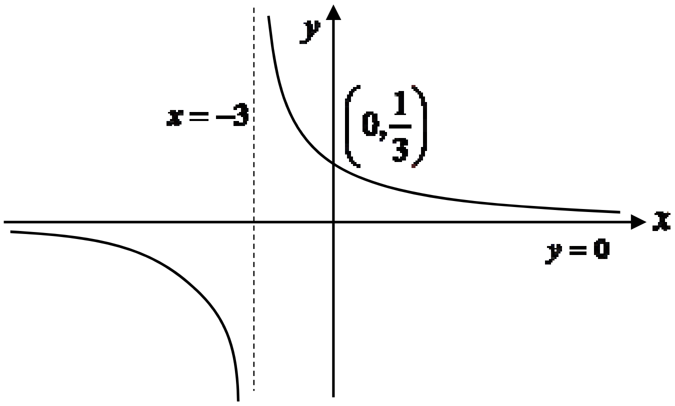
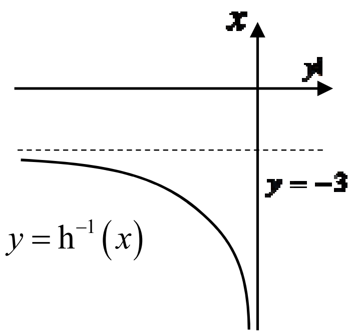
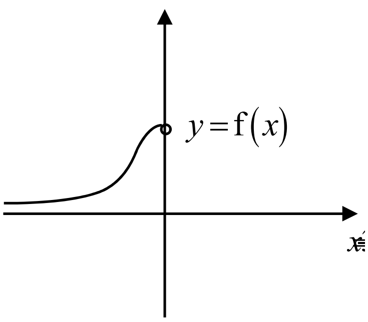
| Qn | |
|---|---|
| **2(ii)** | $2-2y\frac{\mathrm{d}y}{\mathrm{d}x}=2(x+y)\left(1+\frac{\mathrm{d}y}{\mathrm{d}x}\right)$<br>$2=2(x+y)+2(x+2y)\frac{\mathrm{d}y}{\mathrm{d}x}$<br>$2=2(x+y)+2(x+2y)\frac{\mathrm{d}y}{\mathrm{d}x}$<br>$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1-(x+y)}{x+2y}$<br>When $x=0,\ y=0$, $-\frac{1}{\frac{\mathrm{d}y}{\mathrm{d}x}}=0$.<br>Hence normal to $C$ at the origin is $y=0$.<br>When $x=2,\ y=-2$, $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{-2}$<br>Tangent to $C$ at $A(2,-2)$, $y-(-2)=-\frac{1}{2}(x-2)$<br>Where the normal and the tangent intersect,<br>$2=-\frac{1}{2}(x-2)$<br>$x=-2$<br><br>Area of triangle $OAB=\frac{1}{2}(2)(2)=2\text{ units}^2$ |
| **3(i)** | Since $a$ is real, the polynomial equation has real coefficients, and thus all non-real roots must be in conjugate pairs. Since the degree of the polynomial is three, there will be 3 roots. The highest even number below 3 is 2. |
| **3(ii)** | $z^3+az^2+az+7=0$<br>$(-7)^3+a(-7)^2+a(-7)+7=0$<br>$a=8$<br>$z^3+8z^2+8z+7=0$<br>$(z+7)(z^2+z+1)=0$<br>$z=-7$ or $z=\frac{-1\pm\mathrm{i}\sqrt{3}}{2}$<br>$z=7e^{i\pi},\ \mathrm{e}^{\frac{i2\pi}{3}},\mathrm{e}^{-\frac{i2\pi}{3}}$ |

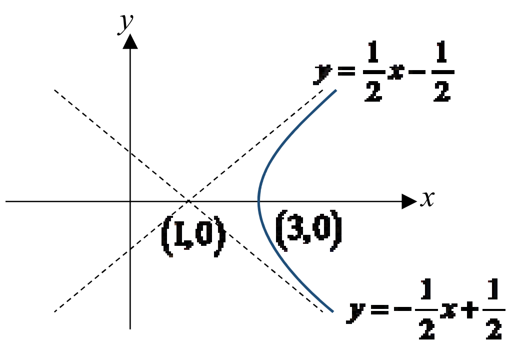
| Qn | |
|---|---|
| 3(iii) | $-\mathrm{i}z^3-8z^2+8\mathrm{i}z+7=0$<br>$(\mathrm{i}z)^3+8(\mathrm{i}z)^2+8(\mathrm{i}z)+7=0$<br><br>From (ii), replace $z$ with $\mathrm{i}z$<br>$\mathrm{i}z=7e^{i\pi},\ \mathrm{e}^{\frac{i2\pi}{3}},\mathrm{e}^{-\frac{i2\pi}{3}}$<br>$\Rightarrow z=-\mathrm{i}7e^{i\pi},\ -\mathrm{i}\mathrm{e}^{\frac{i2\pi}{3}},-\mathrm{i}\mathrm{e}^{-\frac{i2\pi}{3}}$<br>$\Rightarrow z=e^{-\frac{\mathrm{i}\pi}{2}}7e^{i\pi},\ e^{-\frac{\mathrm{i}\pi}{2}}\mathrm{e}^{\frac{i2\pi}{3}},e^{-\frac{\mathrm{i}\pi}{2}}\mathrm{e}^{-\frac{i2\pi}{3}}$<br>$\Rightarrow z=7e^{\frac{i\pi}{2}},\ \mathrm{e}^{\frac{i\pi}{6}},\mathrm{e}^{\frac{i5\pi}{6}}$ |
| 4(i) | $x-1=3\tan\theta$<br>$\frac{\mathrm{d}x}{\mathrm{d}\theta}=3\sec^2\theta$<br>$\int\frac{1}{\sqrt{x^2-2x+10}}\,\mathrm{d}x=\int\frac{1}{\sqrt{(x-1)^2+3^2}}\,\mathrm{d}x$<br>$=\int\frac{1}{\sqrt{(3\tan\theta)^2+3^2}}\cdot3\sec^2\theta\ \mathrm{d}\theta$<br>$=\int\frac{1}{3\sec\theta}\cdot3\sec^2\theta\ \mathrm{d}\theta$<br>$=\int\sec\theta\ \mathrm{d}\theta$<br>$=\ln\lvert\sec\theta+\tan\theta\rvert+C$<br>$=\ln\left\lvert\frac{\sqrt{x^2-2x+10}}{3}+\frac{x-1}{3}\right\rvert+C$ |

| Qn | |
|---|---|
| 4(ii) | $x+3=\frac{1}{2}(2x-2)+4$ <br> $\int \frac{x+3}{\sqrt{x^2-2x+10}}\,\mathrm{d}x$ <br> $=\int \frac{\frac{1}{2}(2x-2)+4}{\sqrt{x^2-2x+10}}\,\mathrm{d}x$ <br> $=\frac{1}{2}\int \frac{2x-2}{\sqrt{x^2-2x+10}}\,\mathrm{d}x+\int \frac{4}{\sqrt{(x-1)^2+3^2}}\,\mathrm{d}x$ <br> $=\frac{1}{2}\frac{\sqrt{x^2-2x+10}}{\frac{1}{2}}+4\int \frac{1}{\sqrt{(x-1)^2+3^2}}\,\mathrm{d}x$ <br> $=\sqrt{x^2-2x+10}+4\ln\left\lvert \frac{\sqrt{x^2-2x+10}}{3}+\frac{x-1}{3}\right\rvert+C$ |
| 5(i) | $\mathrm{f}(r)-\mathrm{f}(r+1)=\frac{\sqrt{r}}{2\sqrt{r}+1}-\frac{\sqrt{r+1}}{2\sqrt{r+1}+1}$ <br> $=\frac{\left(2\sqrt{r}\sqrt{r+1}+\sqrt{r}\right)-\left(2\sqrt{r+1}\sqrt{r}+\sqrt{r+1}\right)}{\left(2\sqrt{r}+1\right)\left(2\sqrt{r+1}+1\right)}$ <br> $=\frac{\sqrt{r}-\sqrt{r+1}}{\left(2\sqrt{r}+1\right)\left(2\sqrt{r+1}+1\right)}$ <br> $\sum_{r=1}^{n}\frac{\sqrt{r}-\sqrt{r+1}}{\left(2\sqrt{r}+1\right)\left(2\sqrt{r+1}+1\right)}=\sum_{r=1}^{n}\left[\mathrm{f}(r)-\mathrm{f}(r+1)\right]$ <br> $=\mathrm{f}(1)-\mathrm{f}(2)$ <br> $+\mathrm{f}(2)-\mathrm{f}(3)$ <br> $+\ldots$ <br> $+\mathrm{f}(n)-\mathrm{f}(n+1)$ <br> $=\mathrm{f}(1)-\mathrm{f}(n+1)$ <br> $=\frac{1}{3}-\frac{\sqrt{n+1}}{2\sqrt{n+1}+1}$ |

| Qn | |
|---|---|
| **5(ii)** | $\sum_{r=1}^{\infty}\frac{\sqrt{r}-\sqrt{r+1}}{\left(2\sqrt{r}+1\right)\left(2\sqrt{r+1}+1\right)}=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{\sqrt{r}-\sqrt{r+1}}{\left(2\sqrt{r}+1\right)\left(2\sqrt{r+1}+1\right)}$<br>$=\lim_{n\to\infty}\left(\frac{1}{3}-\frac{\sqrt{n+1}}{2\sqrt{n+1}+1}\right)$<br>$=\lim_{n\to\infty}\left(\frac{1}{3}-\frac{1}{2+\frac{1}{\sqrt{n+1}}}\right)$<br>$=-\frac{1}{6}$ |
| **5(iii)** | $\sum_{r=1}^{n}\frac{\sqrt{r+1}-\sqrt{r+2}}{\left(2\sqrt{r+1}+1\right)\left(2\sqrt{r+2}+1\right)}=\sum_{r=2}^{n+1}\frac{\sqrt{r}-\sqrt{r+1}}{\left(2\sqrt{r}+1\right)\left(2\sqrt{r+1}+1\right)}$<br>$=\sum_{r=1}^{n+1}\frac{\sqrt{r}-\sqrt{r+1}}{\left(2\sqrt{r}+1\right)\left(2\sqrt{r+1}+1\right)}-\frac{1-\sqrt{2}}{3(2\sqrt{2}+1)}$<br>$=\frac{1}{3}-\frac{\sqrt{n+2}}{2\sqrt{n+2}+1}-\frac{1-\sqrt{2}}{3(2\sqrt{2}+1)}$<br>$=\frac{\sqrt{2}}{2\sqrt{2}+1}-\frac{\sqrt{n+2}}{2\sqrt{n+2}+1}$<br>Need $\frac{\sqrt{2}}{2\sqrt{2}+1}-\frac{\sqrt{n+2}}{2\sqrt{n+2}+1}<-0.1$<br><table><tr><td>$n$</td><td>$\frac{\sqrt{2}}{2\sqrt{2}+1}-\frac{\sqrt{n+2}}{2\sqrt{n+2}+1}$</td></tr><tr><td>56</td><td>-0.099797</td></tr><tr><td>57</td><td>-0.100043</td></tr></table><br>Using GC, least $n = 57$ |

| Qn | |
|---|---|
| 6(i) | $y=1+\dfrac{2x+p}{x^2+x-6}$ <br> $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{2\left(x^2+x-6\right)-(2x+p)(2x+1)}{\left(x^2+x-6\right)^2}$ <br> $\dfrac{\mathrm{d}y}{\mathrm{d}x}=0 \Rightarrow 2\left(x^2+x-6\right)-(2x+p)(2x+1)=0$ <br> $2x^2+2px+12+p=0$ <br> $4p^2-4(2)(12+p)>0$ <br> $p^2-2p-24>0$ <br> $(p+4)(p-6)>0$ <br> $p<-4$ or $p>6$ |
| 6(ii) |   |
| 6(iii) | $1+\sqrt{12-x^2}\geq\dfrac{2x+7}{(2-x)(x+3)}$ <br> $1+\dfrac{2x+7}{(x-2)(x+3)}\geq-\sqrt{12-x^2}$ <br> Sketch $y=-\sqrt{12-x^2}$ (as above) |

| Qn | |
|---|---|
| | $-2\sqrt{3}\le x<-3$ OR $-2.92\le x\le 1.46$ OR $2<x\le 2\sqrt{3}$ |
| 7(i) | <br>Every horizontal line $y=k$ cuts the graph at most once. This implies g is one-one. Therefore $g^{-1}$ exists<br>$g^{-1}: x \mapsto \frac{1}{x}-3, x\in\mathbf{R},\ x\neq 0$ |
| 7(ii) | $\{x\in\mathbf{R} \mid x\neq 0\}$ |
| 7(iii) | $R_{g^{-1}} = D_g = \mathbf{R}\setminus\{-3\}, D_f = \mathbf{R}^-$.<br>Since $R_{g^{-1}} \not\subset D_f$, $fg^{-1}$ does not exist. |
| 7(iv) | $k=-3$ |
| 7(v) (a) |  <br>$R_{fh^{-1}} = \left(0, e^{-9}\right)$ |

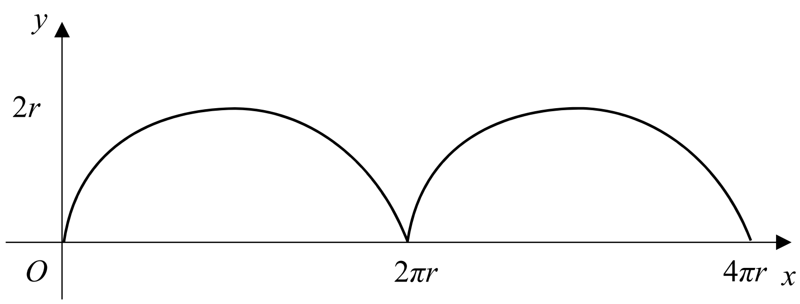
| Qn | |
|---|---|
| **7(iv) (b)** | $\mathrm{h}(x)=\mathrm{h}^{-1}(x)$<br>$\mathrm{h}(x)=x$<br>$\frac{1}{x+3}=x$<br>$x^2+3x-1=0$<br>Since $x<-3$, $x=-3.30$ (3sf) |
| **8(i)** | $(x-1)^2=\left(\mathrm{e}^t+\mathrm{e}^{-t}\right)^2=\mathrm{e}^{2t}+2+\mathrm{e}^{-2t}$<br>$(2y)^2=\left(\mathrm{e}^t-\mathrm{e}^{-t}\right)^2=\mathrm{e}^{2t}-2+\mathrm{e}^{-2t}$<br>Hence $(x-1)^2-(2y)^2=4$<br>$\frac{(x-1)^2}{2^2}-y^2=1$<br><br><u>Alternative solution by students</u>:<br>$(x-1)+2y=2\mathrm{e}^t$ ___(1)<br>$(x-1)-2y=2\mathrm{e}^{-t}$ ___(2)<br>(1)×(2):<br>$(x-1)^2-(2y)^2=4\mathrm{e}^t\mathrm{e}^{-t}=4$ |
| **8(ii)** |  |

| Qn | |
|---|---|
| 8(iii) | When $x=3$, $3=1+e^{t}+e^{-t}$<br>$e^{t}+e^{-t}=2$<br>$t=0$<br>When $x=1+e+e^{-1}$, $t=\pm 1$ ($t=1:\ y>0,\ t=-1:\ y<0$)<br><br>$x=1+e^{t}+e^{-t}$<br>$\frac{dx}{dt}=e^{t}-e^{-t}$<br><br>Area of required region $=2\int_{3}^{1+e+e^{-1}} y\,dx$<br>$=2\int_{0}^{1}\frac{e^{t}-e^{-t}}{2}\left(e^{t}-e^{-t}\right)dt$<br>$=\int_{0}^{1}\left(e^{t}-e^{-t}\right)^{2}dt$<br>$=\int_{0}^{1}\left(e^{2t}-2+e^{-2t}\right)dt$<br>$=\left[\frac{1}{2}e^{2t}-2t-\frac{1}{2}e^{-2t}\right]_{0}^{1}$<br>$=\left[\frac{1}{2}e^{2}-2-\frac{1}{2}e^{-2}\right]-0$<br>$=\frac{1}{2}\left(e^{2}-e^{-2}\right)-2$ |
| 8(iv) | $\frac{(x-1)^{2}}{2^{2}}-y^{2}=1$<br>$(x-1)^{2}=2^{2}\left(1+y^{2}\right)$<br>$x=1+2\sqrt{1+y^{2}}$ since $x>1$<br><br>Volume $=\pi\int_{0}^{4}x^{2}dy-\pi\left(3^{2}\right)(4)$<br>$=\pi\int_{0}^{4}\left(1+2\sqrt{1+y^{2}}\right)^{2}dy-36\pi$<br>$=335$ units$^{3}$ (3 s.f.) |

## 2017 NYJC JC2 Prelim 9758/1 Solution

| Qn | |
|---|---|
| **9(i)** | $\overrightarrow{OP} = \begin{pmatrix} \lambda - \mu \\ 1 + 2\mu \\ 2 - 3\lambda \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$<br>Locus of $P$ is the plane with equation<br>$\mathbf{r} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}, \quad \lambda, \mu \in \mathbb{R}$ |
| **9(ii)** | Normal of the locus of $P$ (plane), $\begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} \times \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix}$<br>$\begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -2 \\ -3 \end{pmatrix} = 0$<br>Hence the line $l$ and the plane are parallel.<br>Equation of the plane, $\mathbf{r} \cdot \begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix} = 7$<br>$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 6 \\ 2 \\ 3 \end{pmatrix} = 8 \neq 7$<br>Hence $l$ is parallel to the plane and does not lie in the plane.<br>Hence points $P$ and $Q$ will never meet.<br>[Note that it is not sufficient just to show that $l$ is parallel to the plane as it may actually lie on it. One must still need to show that there is a point on $l$ that is not on the plane]<br>Alternatively, one can check that<br>$\left[ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2 \\ -2 \\ -3 \end{pmatrix} \right] \cdot \begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix} \neq 7$ for all $t \in \mathbb{R}$. |
| **9(iii)** | Shortest distance between $P$ and $Q$ is the distance between the line and the parallel plane.<br>$\dfrac{\left\| \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix} - 7 \right\|}{\left\| \begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix} \right\|} = \dfrac{2}{7}$ |

| Qn | |
|---|---|
| **9(iv)** | Lines $l$ and $m$ are non-parallel. Hence $k \neq 1$.<br>If the two lines intersect,<br>$\begin{pmatrix}1\\k\\0\end{pmatrix}+s\begin{pmatrix}2\\-2\\-3k\end{pmatrix}=\begin{pmatrix}1\\1\\0\end{pmatrix}+t\begin{pmatrix}2\\-2\\-3\end{pmatrix}$ for some $s,t \in \mathbb{R}$<br>$1+2s=1+2t$ ___(1)<br>$k-2s=1-2t$ ___(2)<br>$-3sk=-3t$ ___(3)<br>From (1), $s=t$<br>Substituting $s=t$ in (2), $k=1$<br>$s=t$ and $k=1$ satisfies (3)<br>Thus for the system of linear equations to be inconsistent, $k \neq 1$.<br>Hence lines $l$ and $m$ are skew when $k \neq 1$. |
| **9(v)** | Let $F$ be the foot of perpendicular from $X$ to line $l$.<br>$\overrightarrow{OF}=\begin{pmatrix}1\\1\\0\end{pmatrix}+t\begin{pmatrix}2\\-2\\-3\end{pmatrix}$ for some $t \in \mathbb{R}$<br>$\overrightarrow{XF}=\begin{pmatrix}1\\2\\5\end{pmatrix}+t\begin{pmatrix}2\\-2\\-3\end{pmatrix}$<br>$\left[\begin{pmatrix}1\\2\\5\end{pmatrix}+t\begin{pmatrix}2\\-2\\-3\end{pmatrix}\right]\cdot\begin{pmatrix}2\\-2\\-3\end{pmatrix}=0$<br>$-17+17t=0$<br>$t=1$<br>$\overrightarrow{OF}=\begin{pmatrix}1\\1\\0\end{pmatrix}+\begin{pmatrix}2\\-2\\-3\end{pmatrix}=\begin{pmatrix}3\\-1\\-3\end{pmatrix}$<br>$F \equiv (3,-1,-3)$ |
| **10** | Let $s$ be the distance between the front of the car and the train at time $t$.<br>$s=\sqrt{x^2+30^2}$ and $x^2=(40-30t)^2+(20t)^2$<br>$s=\sqrt{(40-30t)^2+(20t)^2+30^2}=\sqrt{1300t^2-2400t+2500}$ |
| **10(i)** | $\frac{ds}{dt}=\frac{1}{2}(1300t^2-2400t+2500)^{-\frac{1}{2}}(2600t-2400)$<br>When $t=1$, $\frac{ds}{dt}=2.67$ |

| Qn | |
|---|---|
| **10(ii)** | At stationary point $\frac{\mathrm{d}s}{\mathrm{d}t}=0$<br>$\frac{1}{2}(1300t^2-2400t+2500)^{-\frac{1}{2}}(2600t-2400)=0$<br>$2600t-2400=0 \Rightarrow t=\frac{12}{13}$<br>$\frac{\mathrm{d}^2s}{\mathrm{d}t^2}=\frac{1}{2}(1300t^2-2400t+2500)^{-\frac{1}{2}}(2600)$<br>$+(-\frac{1}{2})\frac{1}{2}(1300t^2-2400t+2500)^{-\frac{3}{2}}(2600t-2400)$<br>When $t=\frac{12}{13},\frac{\mathrm{d}^2s}{dt^2}>0$<br>$s=\sqrt{1300(\frac{12}{13})^2-2400(\frac{12}{13})+2500}=$ 19.884=19.9 |
| **10(iii)** | Let the angle of elevation be $\theta$<br>$\sin\theta=\frac{30}{\sqrt{1300t^2-2400t+2500}}$<br>$\cos\theta\frac{\mathrm{d}\theta}{\mathrm{d}t}=-\frac{1}{2}(30)(1300t^2-2400t+2500)^{-\frac{3}{2}}(2600t-2400)$<br>When $t=1$, $\cos\theta=\frac{\sqrt{(40-30)^2+20^2}}{\sqrt{1300-2400+2500}}=\sqrt{\frac{500}{1400}}$<br>$\therefore\frac{\mathrm{d}\theta}{\mathrm{d}t}=-\frac{1}{2}(30)(1300-2400+2500)^{-\frac{3}{2}}(200)\div\sqrt{\frac{500}{1400}}$ $=-0.0958$ rad/s (or $-5.5^\circ$/s ) |
| **11(i)** |  |

| Qn | |
|---|---|
| 11(ii) | $\frac{dx}{d\theta}=r(1-\cos\theta),\ \frac{dy}{d\theta}=r\sin\theta$<br>$\frac{dy}{dx}=\frac{\sin\theta}{1-\cos\theta}$<br>$=\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}}$<br>$=\frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}=\cot\frac{\theta}{2}$ |
| 11(iii) | $\left(\frac{dy}{dx}\right)^2=\cot^2\frac{\theta}{2}=\operatorname{cosec}^2\frac{\theta}{2}-1$<br>$=\frac{1}{\sin^2\frac{\theta}{2}}-1$<br>$=\frac{2}{1-\cos\theta}-1=\frac{2r}{r(1-\cos\theta)}-1$<br>Thus $\left(\frac{dy}{dx}\right)^2=\frac{2r}{y}-1$ |
| 11(iv) | $Area=\int_0^{2\pi r} y\,dx$<br>$=\int_0^{2\pi} r(1-\cos\theta)\cdot r(1-\cos\theta)\,d\theta$<br>$=r^2\int_0^{2\pi}(1-2\cos\theta+\cos^2\theta)\,d\theta$<br>$=r^2\int_0^{2\pi}(\frac{3}{2}-2\cos\theta+\frac{1}{2}\cos 2\theta)\,d\theta$<br>$=r^2\left[\frac{3}{2}\theta-2\sin\theta+\frac{1}{4}\sin 2\theta\right]_0^{2\pi}$<br>$=3\pi r^2$ |