INNOVA JUNIOR COLLEGE
JC 2 PRELIMINARY EXAMINATION
in preparation for General Certificate of Education Advanced Level
**Higher 2**

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|---|---|---|---|
| CANDIDATE NAME | | | |
| CLASS | | INDEX NUMBER | |

# FURTHER MATHEMATICS 9649/02

**11 Sep 2017**

Paper 2

**3 hours**

Additional Materials: Answer Paper
Cover Page
MF26

**READ THESE INSTRUCTIONS FIRST**

**Do not open this booklet until you are told to do so.**

Write your name, class and index number on all the work you hand in.

Write in dark blue or black pen on both sides of the paper. You may use a soft pencil for any diagrams or graphs.
**Do not use staples, paper clips, highlighters, glue or correction fluid.**

Answer **all** the questions.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

You are expected to use a graphic calculator.

Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.

You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.

The number of marks is given in brackets [ ] at the end of each question or part question.

## Section A: Pure Mathematics [50 marks]

**1** By using differentiation, show that the function $\mathrm{f}(x) = \cos x - x$ has exactly one root in the interval $(0, 1)$.

Using the iteration formula $x_{n+1} = \cos x_n$, where $x_1 = 1$, find the root to 2 decimal places.

By using the graphs of $y = \cos x$ and $y = x$, illustrate the behavior of the sequence $x_n$. [8]

**2** The parametric equations of the curve $C$ are

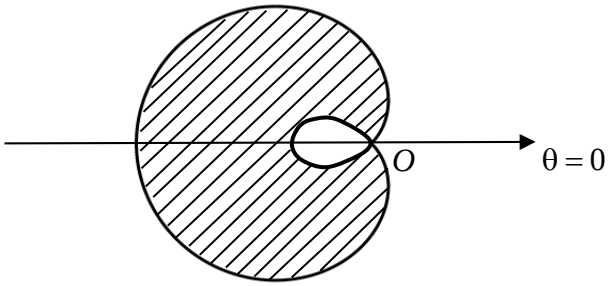
$$x = a\cos t - 2a\cos^2 t, \qquad y = a\sin t - 2a\sin t\cos t,$$

where $t \in \mathbb{R}$ and $a > 0$.

Show that the polar equation of $C$ is

$$r = a(1 - 2\cos\theta), \qquad \text{where } -\pi \le \theta \le \pi. \qquad [4]$$

The outline of a swimming pool is in the shape of $C$, as shown in the diagram below. The shaded area represents the pool. Find the exact area of the surface of the pool. [5]




**3** **(a)** In a computer simulation, the velocity, $\frac{\mathrm{d}y}{\mathrm{d}t}$, of a virtual object is represented by the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \mathrm{e}^{\sin t} + \mathrm{e}^{\cos y},$$

where $y$ (in metres) is the displacement of the object from a fixed point $O$ at time $t$ (in seconds). It is given that the object is initially at $O$. Estimate, to 3 decimal places, the displacement of the object after 1 second using

**(i)** the Euler method with $h = 0.5$. [3]

**(ii)** the improved Euler method with $h = 0.5$. [3]

**(b)** In another computer simulation, the velocity, $v$ (in metres per second) of a virtual object at time $t$ (in seconds) is represented by the equation

$$v = \mathrm{e}^{\sin t}.$$

The distance travelled by the object is the area under the curve of the velocity-time graph. Using the Simpson's rule with 5 ordinates, estimate the distance travelled by the object in the first 2 seconds, to 3 decimal places. [3]

**4** A newly constructed warehouse is $p$ m wide and 6 m long as shown in **Fig. 1**. The roof is designed such that its cross-section is an ellipse with eccentricity 0.85 and minor axis of length 2 m. The focus of the ellipse is at the pole as shown in **Fig. 2**.

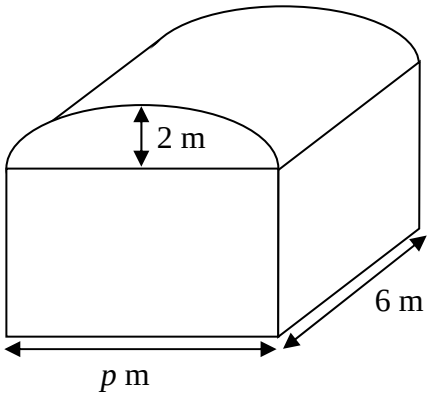



**Fig. 1**

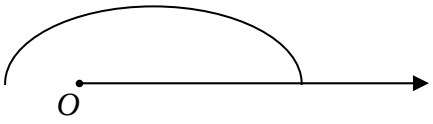



**Fig. 2**

**(i)** Find the value of $p$. [2]

**(ii)** Show that the polar equation of the ellipse is given by $\dfrac{k}{1-0.85\cos\theta}$, where $k$ is a constant to be found. [4]

**(iii)** The curved surface of the roof is to be covered with a coat of highly reflective paint. Given that it costs \$28 per square metre to paint the roof, find the total cost of painting the roof. Give your answer to the nearest dollar. [3]

**[Turn over**

**5** It is given that $\lambda = -3$ is an eigenvalue of the matrix $\mathbf{M} = \begin{pmatrix} a & -2 & -2 \\ -2 & a & -2 \\ -2 & -2 & a \end{pmatrix}$, where $a > 0$, with corresponding eigenvector $\begin{pmatrix} b \\ c \\ d \end{pmatrix}$. The matrix $\mathbf{Q} = \begin{pmatrix} b & l & p \\ c & m & q \\ d & n & r \end{pmatrix}$ is such that $\mathbf{Q}^{-1}\mathbf{M}\mathbf{Q}$ is a diagonal matrix.

**(i)** Show that $a = 1$ and find the other eigenvalues. [5]

**(ii)** Find an eigenvector $\begin{pmatrix} b \\ c \\ d \end{pmatrix}$ for the eigenvalue of $\lambda = -3$. [2]

**(iii)** Find an equation relating $l$, $m$, $n$, and another relating $p$, $q$, $r$. Hence write down a matrix for $\mathbf{Q}$. [4]

**(iv)** Express $\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$ as a linear combination of the eigenvectors of $\mathbf{M}$. Hence if $k$ is an integer, show that $\mathbf{M}^k \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$ can be expressed as either $A\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$ or $B\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, where $A$ and $B$ are constants to be determined in terms of $k$. [4]

## Section B: Probability and Statistics [50 marks]

**6** A shopping mall manager claims that the mean daily sales of two restaurants located in the mall are equal. The table below gives the daily sales ($X_1$ and $X_2$) in dollars, as well as the differences ($D = X_1 – X_2$) in daily sales between the two restaurants, for 12 randomly selected days.

| | Restaurant 1 | Restaurant 2 | |
|---|---|---|---|
| Day | $X_1$ | $X_2$ | $D = X_1 – X_2$ |
| 1 | 1005 | 918 | 87 |
| 2 | 2073 | 2096 | –23 |
| 3 | 873 | 903 | –30 |
| 4 | 1074 | 999 | 75 |
| 5 | 1932 | 1827 | 105 |
| 6 | 1338 | 1281 | 57 |
| 7 | 1449 | 1302 | 147 |
| 8 | 759 | 678 | 81 |
| 9 | 1905 | 1792 | 113 |
| 10 | 693 | 747 | –54 |
| 11 | 2106 | 2049 | 57 |
| 12 | 981 | 949 | 32 |

**(i)** Explain why a paired-sample *t*-test instead of a two-sample *t*-test should be used to compare the mean daily sales of the two restaurants.
State two assumptions needed to carry out the test. [4]

**(ii)** Find the least value of $k$ for which the null hypothesis can be rejected at the $k$% significance level. [4]

**7** It is believed that there is an association between the colour of a person's eyes and the reaction of the person's skin to ultraviolet light. In order to investigate this, a random sample of 120 people was subjected to a standard dose of ultraviolet light. The degree of their reaction was noted, "–" indicating no reaction, "+" indicating a slight reaction and "++" indicating a strong reaction. The results are shown in the table below.

| | | Eye colour | | |
|---|---|---|---|---|
| | | Blue | Grey or Green | Brown |
| Reaction | – | 7 | 8 | 18 |
| | + | 29 | 10 | 16 |
| | ++ | 21 | 9 | 2 |

**(i)** Carry out a chi-squared test at the 5% level of significance. Based on the *p*-value and the contributions of individual cells to the test statistic, discuss what the test indicates about the association, if any, between eye colour and reaction to ultraviolet light. [7]

**(ii)** Find the proportion of people in the sample who do not suffer a reaction to ultraviolet light. Hence calculate a 95% confidence interval for the proportion of people in the population who would not suffer a reaction to ultraviolet light. [2]

**8** Suppose that the radius $X$ of a circle is uniformly distributed on the interval from 1 to 2. Denote the area of the circle by $A$.

**(i)** By considering $\mathrm{P}(A \le a)$ or otherwise, show that the probability density function of $A$ is $\mathrm{f}(a)$, where

$$\mathrm{f}(a) = \begin{cases} \dfrac{1}{2\sqrt{\pi}}\dfrac{1}{\sqrt{a}}, & \pi \le a \le 4\pi, \\ 0, & \text{otherwise} \end{cases}$$

and sketch this probability density function. [7]

**(ii)** Determine the exact value of $\mathrm{E}(A)$. [3]

**9** The random variable $X$ has a Poisson distribution with mean $\lambda$. An observation of $X$ is made. If $X = 0$, repeated observations are made until a non-zero value is obtained. The random variable $Y$ is defined to be the first non-zero value obtained for $X$. Obtain the probability distribution function of $Y$ in the form

$$\mathrm{P}(Y=r)=\begin{cases} \mathrm{g}(\lambda)\,\mathrm{P}(X=r), & r=1,2,3,\ldots \\ 0, & \text{otherwise,} \end{cases}$$

where $\mathrm{g}(\lambda)$ is a function in $\lambda$ to be determined. [3]

Hence show that $\mathrm{E}(Y) = \dfrac{\lambda}{1-\mathrm{e}^{-\lambda}}$. [2]

Given that $\lambda$ is small, show that $\mathrm{E}(Y) \approx 1+\dfrac{1}{2}\lambda$. [3]

The random variable $R$ denotes the number of observations of $X$ up to and including the first non-zero observation made. Given that $\lambda = \ln 2$,

**(i)** State the probability distribution of $R$. [1]

**(ii)** Find the expected number of observations of $X$ up to and including the second non-zero observation made. [2]

**10** A large number of candidates took two papers, Paper 1 and Paper 2, in the same subject. From a cursory examination of the papers, a teacher concluded that Paper 2 was easier than Paper 1. To verify her claim, the teacher took a random sample of 8 candidates and compared their marks, which were as follows.

| Candidate | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| Paper 1 | 34 | 54 | 42 | 66 | 74 | 66 | 18 | 50 |
| Paper 2 | 62 | 58 | 82 | 94 | 38 | 90 | 26 | 66 |

Test the teacher's claim, at the 5% level of significance, using

**(a)** the sign test, [5]

**(b)** the Wilcoxon matched-pairs signed rank test. [5]

Which test do you think is more appropriate? Justify your answer. [2]

**END OF PAPER**

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