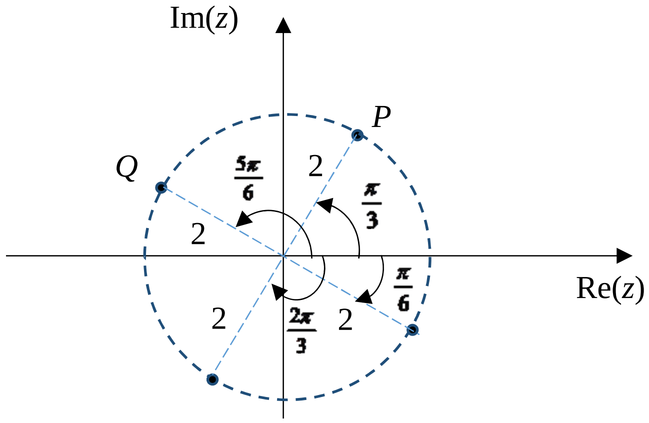
2017 Year 6 FM Prelim Solutions (Paper 2)

Qn	Solution
1(a)	Let $\begin{pmatrix}1\\0\\0\end{pmatrix}$ and $\begin{pmatrix}0\\2\\0\end{pmatrix} \in \mathbb{R}^3$, $S\left(\begin{pmatrix}1\\0\\0\end{pmatrix}+\begin{pmatrix}0\\2\\0\end{pmatrix}\right)=S\left(\begin{pmatrix}1\\2\\0\end{pmatrix}\right)=\sqrt{\begin{pmatrix}1\\2\\0\end{pmatrix}^T\begin{pmatrix}1\\2\\0\end{pmatrix}}=\sqrt{5}$ $S\begin{pmatrix}1\\0\\0\end{pmatrix}+S\begin{pmatrix}0\\2\\0\end{pmatrix}=1+2=3\neq\sqrt{5}$ Hence S is not a linear transformation.
(b)	$T(\alpha \mathrm{f}(x))=\alpha \mathrm{f}'(x)$ which is a polynomial of degree n-1 or less for any real value α. $T(\mathrm{f}(x)+\mathrm{g}(x))=\mathrm{f}'(x)+\mathrm{g}'(x)=T(\mathrm{f}(x))+T(\mathrm{g}(x))$ Therefore T is a linear transformation.

Qn	Solution
2(i)	$\mathrm{RREF}(\mathbf{A})=\begin{pmatrix}1&0&1&1&3\\0&1&1&-1&2\\0&0&0&0&0\end{pmatrix}$, $\quad$ $\mathrm{Rank}(\mathbf{A})=2$ From GC, the solution of $a+2b+3c-d+7e=0$ $-3a+4b+c-7d-e=0$ $5a-6b-c+11d+3e=0$ is $\begin{pmatrix}a\\b\\c\\d\\e\end{pmatrix}=s\begin{pmatrix}-1\\-1\\1\\0\\0\end{pmatrix}+t\begin{pmatrix}-1\\1\\0\\1\\0\end{pmatrix}+u\begin{pmatrix}-3\\-2\\0\\0\\1\end{pmatrix}$

	Basis of the null space of V is $\left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ -2 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$.
(ii)	Basis of range space is $\left\{ \begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ -6 \end{pmatrix} \right\}$. Any vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ in the range space is of the form $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix} + \gamma \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}, \qquad \lambda, \gamma \in \mathbb{R}$ which is an equation of a plane in $\mathbb{R}^3$, in parametric form. $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ is in this plane by observation, and the normal to this plane is parallel to $\begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} -1 \\ 8 \\ 5 \end{pmatrix}$ Therefore the cartesian equation of the plane required is $-x + 8y + 5z = 0$.

Qn	Solution
3(i)	

(ii)	P represents $2\mathrm{e}^{\mathrm{i}\frac{\pi}{3}}$ and Q represents $2\mathrm{e}^{\mathrm{i}\frac{5\pi}{6}}$. $\overrightarrow{QP}$ represents $2\mathrm{e}^{\mathrm{i}\frac{\pi}{3}}-2\mathrm{e}^{\mathrm{i}\frac{5\pi}{6}}=2\left(\mathrm{e}^{\mathrm{i}\frac{\pi}{3}}-\mathrm{e}^{\mathrm{i}\frac{5\pi}{6}}\right)$ $\overrightarrow{QR}$ represents $2\mathrm{e}^{i\frac{\pi}{3}}\left(\mathrm{e}^{i\frac{\pi}{3}}-\mathrm{e}^{i\frac{5\pi}{6}}\right)=2\left(\mathrm{e}^{i\frac{2\pi}{3}}-\mathrm{e}^{-i\frac{5\pi}{6}}\right)$ R represents $2\left(\mathrm{e}^{i\frac{2\pi}{3}}-\mathrm{e}^{-i\frac{5\pi}{6}}\right)+2\mathrm{e}^{i\frac{5\pi}{6}}$ $=2\mathrm{e}^{i\frac{2\pi}{3}}+2\left(\mathrm{e}^{i\frac{5\pi}{6}}-\mathrm{e}^{-i\frac{5\pi}{6}}\right)$ $=-1+\mathrm{i}\left(2+\sqrt{3}\right)$ Or $\overrightarrow{QR}$ represents $2\mathrm{e}^{-i\frac{\pi}{3}}\left(\mathrm{e}^{i\frac{\pi}{3}}-\mathrm{e}^{i\frac{5\pi}{6}}\right)=2\left(1-\mathrm{e}^{i\frac{\pi}{2}}\right)=2-2i$ R represents $2-2i+2\mathrm{e}^{i\frac{5\pi}{6}}$ $=2-2i+2\cos\left(\frac{5\pi}{6}\right)+2\sin\left(\frac{5\pi}{6}\right)i$ $=2-\sqrt{3}-i$ Alternatively, Students may use the perpendicular bisector of P and Q and proceed to identify the 3rd pt to get the equilateral triangle. $\left(\sqrt{2}+\sqrt{6}\right)\mathrm{e}^{\mathrm{i}\frac{7}{12}\pi}$ or $\left(\sqrt{6}-\sqrt{2}\right)\mathrm{e}^{-\mathrm{i}\frac{5}{12}\pi}$

Qn	Suggested Solution
4(i)	$u_1=3,\quad u_2=\frac{4}{3},\quad u_3=1$
(ii)	Rewriting: $u_0=\frac{2}{-1}$, $u_1=\frac{3}{1}$, $u_2=\frac{4}{3}$, $u_3=\frac{5}{5}$ and given: $u_4=\frac{6}{7}$ and $u_5=\frac{7}{9}$, Conjecture: $u_n=\frac{n+2}{2n-1}$ for all integers $n\geq 0$

	Let P(n) be the proposition $u_n = \dfrac{n+2}{2n-1}$. When n = 0, LHS of P(0) = $u_0 = -2$ (given) RHS of P(0) = $\dfrac{2}{-1} = -2$ $\therefore$ P(0) is true. Assume P(k) is true for some $k \in \mathbb{Z}_0^+$ i.e. $u_k = \dfrac{k+2}{2k-1}$. To show that P(k+1) is true i.e. $u_{k+1} = \dfrac{k+3}{2k+1}$. When $n = k + 1$, LHS of P(k+1) = $u_{k+1} = \dfrac{(k+3)u_k}{2u_k + k + 2}$ $= \dfrac{\dfrac{(k+3)(k+2)}{2k-1}}{2\left(\dfrac{k+2}{2k-1}\right) + k + 2}$ $= \left[\dfrac{(k+3)(k+2)}{(2k+1)(k+2)}\right]$ $= \dfrac{k+3}{2k+1}$ = RHS of P(k+1) Since P(0) is true & P(k) is true $\Rightarrow$ P(k + 1) is also true, hence by mathematical induction P(n) is true for all $n \in \mathbb{Z}^+ \cup \{0\}$.

Qn	Suggested Solution
5	$\dfrac{dy}{dx} + 2xy = x^3$ IF: $e^{\int 2x\,dx} = e^{x^2}$ $e^{x^2}\dfrac{dy}{dx} + 2xe^{x^2}y = x^3 e^{x^2}$ $ye^{x^2} = \int x^3 e^{x^2}\,dx$ $= \dfrac{1}{2}x^2 e^{x^2} - \dfrac{1}{2}\int 2xe^{x^2}\,dx$ $= \dfrac{1}{2}\left(x^2 - 1\right)e^{x^2} + c$ $\therefore y = \dfrac{1}{2}\left(x^2 - 1\right) + ce^{-x^2}$ The solution curves are symmetrical about the y-axis.

$$y=\frac{1}{2}\left(x^2-1\right)+ce^{-x^2}$$

$$\frac{dy}{dx}=x-2xce^{-x^2}=x\left(1-2ce^{-x^2}\right)$$

$$\frac{d^2y}{dx^2}=\left(1-2ce^{-x^2}\right)+x\left(1+4cxe^{-x^2}\right)$$

For,

$$1-2ce^{-x^2}=0 \quad \Rightarrow \quad 2c=e^{x^2}$$

To have solutions for this equation, $c\geq\frac{1}{2}$

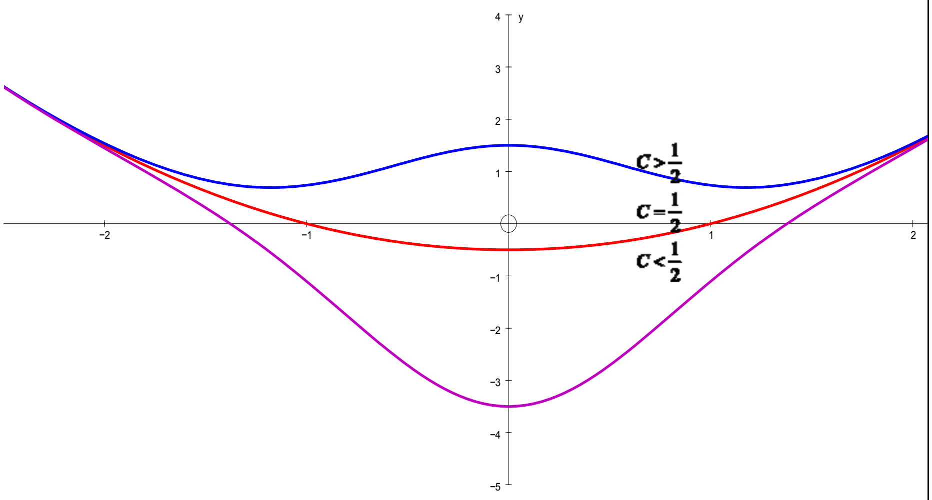
$c=\frac{1}{2}$: $\frac{dy}{dx}=0$ for $x=0$ $\therefore$ there is only 1 stationary point

$c<\frac{1}{2}$: $\frac{dy}{dx}=x\left(1-2ce^{-x^2}\right)=0$ for $x=0$ as $1-2ce^{-x^2}>0$

$\therefore$ there is only 1 stationary point

$c>\frac{1}{2}$: $\frac{dy}{dx}=x\left(1-2ce^{-x^2}\right)=0$ for $x=0$ and $x=\pm\ln\sqrt{2c}$

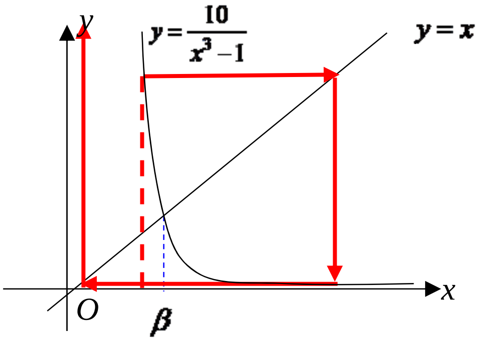
$\therefore$ there are 3 stationary points



Qn	Suggested Solution
6	Let $y = x^4 - x - 10$ When $x = -1.7$, $y = (-1.7)^4 - (-1.7) - 10 = 0.0521 > 0$. When $x = -1.6$, $y = (-1.6)^4 - (-1.6) - 10 = -1.8464 < 0$. When $x = 1.8$, $y = 1.8^4 - 1.8 - 10 = -1.3024 < 0$. When $x = 1.9$, $y = 1.9^4 - 1.9 - 10 = 1.1321 > 0$. Since there is a change in sign for y and it is continuous for all x, there exists a root each in the intervals $(-1.7, -1.6)$ and $(1.8, 1.9)$. $\frac{\mathrm{d}y}{\mathrm{d}x} = 4x^3 - 1$ $\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \Rightarrow x = \frac{1}{\sqrt[3]{4}}$ Since there is only one minimum point at $x = \frac{1}{\sqrt[3]{4}}$ and y is quartic function with positive coefficient of x^4 which implies $x \to \pm\infty$, $y \to \infty$. Thus the graph of $y = x^4 - x - 10$ cuts the x-axis at either 0 or 2 times. Thus $x^4 - x - 10 = 0$ has exactly two real roots.
(ii)	See table and graph below.

x	(A)	(B)	(C)
x_1	1.8100	1.8100	1.8100
x_2	2.0285	1.8538	1.8987
x_3	1.3611	1.8555	1.8168
x_4	6.5720	1.8556	1.8921
x_5	0.0354	1.8556	1.8226
x_6	-10.0004	1.8556	1.8866

(a) From the table, arrangement A will not result in convergence to β.



(b) Arrangement B will converge faster than arrangement C.

	Since $\left\|\frac{\mathrm{d}}{\mathrm{d}x}\left(\left(x+10\right)^{\frac{1}{4}}\right)\right\| < \left\|\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\sqrt{(x+10)}}{x}\right)\right\| < 1$ for $x \in (1.8100, 1.8556)$, arrangement B will converge faster than arrangement C $\beta = 1.8556$ Let $\mathrm{f}(x) = x^4 - x - 10$, $\mathrm{f}(1.85555) = -0.000848 < 0$ $\mathrm{f}(1.85565) = 0.00161 > 0$ Since there is a change in sign, $\beta = 1.8556$ is correct to 4 decimal places.

Qn	**Suggested Solution**
7(i)	H_0 : Eye colour and reaction to ultraviolet light are independent H_1 : Eye colour and reaction to ultraviolet light are associated Under H_0, the expected frequencies, E_i, are as follows

	Blue	Grey or green	Brown	Total
no reaction	15.675	7.425	9.9	33
slight reaction	26.125	12.375	16.5	55
strong reaction	15.2	7.2	9.6	32
Total	57	27	36	120

Degrees of freedom $= (3-1)(3-1) = 4$

Test statistic: $\sum \frac{(O_i - E_i)^2}{E_i} \sim \chi^2(4)$ where O_i and E_i are observed and expected frequencies.

From GC, p-value $= 0.000325 < 0.01$, we reject the null hypothesis and conclude that there is sufficient evidence, at 1% significance level, that eye colour and reaction to ultraviolet light are associated.

Qn	**Suggested Solution**

8(i)	From GC, the unbiased estimate of population mean and variances are $\bar{x} = 55$ and $s^2 = 17.9355^2 = 321.68 = 322$ (3 s.f.)
(ii)	Let X be the survival times (in days) of a randomly chosen guinea pig in the farm, with population mean μ. $\mathrm{H}_0 : \mu = 60$ $\mathrm{H}_1 : \mu < 60$ To perform a one-tail test at 10% significance level. Under H_0 $T = \dfrac{\bar{X} - 60}{S / \sqrt{20}} \sim t(19)$ From sample, $\bar{x} = 55$, $s = 17.9355$ and $n = 20$ Using t-test, p-value $= 0.11382$ (from GC) Since p-value $= 0.114 > 0.1$, we do not reject H_0 and conclude that there is insufficient evidence, at 10% significance level, that the **average/mean** survival times of guinea pigs from the farm is not at least 60 days.
(iii)	The data seems to be skewed to the right, and hence does not seem to come from a normal distribution. The data seems to be not symmetrical around the mean which suggest that a normal distribution may not be appropriate.
(iv)	Sign Test
(v)	Let m be the median survival times of the guinea pigs from the farm. Let K_+ be the number of plus signs. $\mathrm{H}_0 : m = 60$ $\mathrm{H}_1 : m < 60$ Subtracting the observed data from postulated median $m = 60$ and writing down the signs, $-\quad -\quad -\quad -\quad -\quad -\quad -\quad -\quad -\quad -$ $-\quad -\quad -\quad -\quad -\quad 0\quad +\quad +\quad +\quad +$ Under H_0, $K_+ \sim \mathrm{B}(19, 0.5)$. From sample, $k_+ = 4$ p-value $= \mathrm{P}(K_+ \leq 4)$ $= 0.0096054$

	Since $p\text{-value} = 0.00961 < 0.1$, we reject H_0 and conclude that there is sufficient evidence at 10% significance level that the **average/median** survival times of guinea pigs from the farm is less than 60 days.

Qn	Solution
9(i)	$X \sim \text{Po}(\lambda)$ $\text{P}(M \le 1)$ $= \left(\text{P}(X \le 1)\right)^n$ $= \left(\text{P}(X = 0) + \text{P}(X = 1)\right)^n$ $= \left(\text{e}^{-\lambda} + \lambda\text{e}^{-\lambda}\right)^n$ $= \text{e}^{-n\lambda}\left(1+\lambda\right)^n$
(ii)	Using the result in (i) where M is the random variable denoting the largest number of defective components in 8 randomly chosen boxes of 100 components each. Probability that not all boxes are accepted $= 1 - \text{P}(M \le 1)$ $= 1 - \text{e}^{-800p}\left(1+100p\right)^8$
(iii)	If p = 0.001, Probability that not all boxes are accepted $= 1 - \text{e}^{-0.8}\left(1.1\right)^8 = 0.036823$ Expected profit $= 10000(0.96318) - 100000(0.036823)$ $= 5949.42$ Since the company is able to make an expected profit of \$5949.42 per batch and it is a client who purchases a large number of batches, the company will make money. If the client purchase a lot of batches of components, the average profit will converge to the expected profit per batch. If the client purchase a few batches of components, the actual profit may differ from the expected profit per batch.
(iv)	Let Y be the time (in days) to ship the goods to the client's site. Let a and b be the interval $Y \sim U(a, b)$

	$\mathrm{E}(Y)$ $= \int_a^b \frac{y}{b-a}\,\mathrm{d}y$ $= \frac{1}{b-a}\left[\frac{y^2}{2}\right]_a^b$ $= \frac{1}{2(b-a)}\left(b^2 - a^2\right)$ $= \frac{1}{2}\left(b+a\right)$ Since E(Y) = 3, $\left(b+a\right) = 6$ ----(1) $\mathrm{Var}(Y)$ $= \mathrm{E}(Y^2) - \left(\mathrm{E}(Y)\right)^2$ $= \int_a^b \frac{y^2}{b-a}\,\mathrm{d}y - \frac{(b+a)^2}{4}$ $= \frac{1}{3\left(b-a\right)}\left[b^3 - a^3\right] - \frac{(b+a)^2}{4}$ $= \frac{1}{3}\left(b^2 + ab + a^2\right) - \frac{(b+a)^2}{4}$ $= \frac{(b-a)^2}{12}$ $\frac{(b-a)^2}{12} = \frac{1}{3} \Rightarrow b = a + 2$ ----(2) $\because b > a$ Solving, a = 2 and b = 4 $\therefore P(Y > 3.5) = 0.25$

Qn	Suggested Solution
10(i)	$\mathrm{H}_0:$ the data follow a normal distribution $\mathrm{H}_1:$ the data does not follow a normal distribution Unbiased estimates of population mean and variance:

mid-interval	frequency
–1.35	1
–1.05	6
–0.75	34
–0.45	80
–0.15	61
0.15	14
0.45	3

0.75	0
1.05	1

From GC, $\bar{x} = -0.369,\ s^2 = 0.31644^2.$

Under H_0,

Difference in wear	Expected frequency
$d < -1.2$	0.8637
$-1.2 \le d < -0.9$	8.4701
$-0.9 \le d < -0.6$	37.205
$-0.6 \le d < -0.3$	70.722
$-0.3 \le d < 0$	58.381
$0 \le d < 0.3$	20.907
$0.3 \le d < 0.6$	3.2305
$0.6 \le d < 0.9$	0.2137
$d \ge 0.9$	0.0061

Combining the first two and last four classes,
degrees of freedom $= 5 - 3 = 2$

Test statistic: $\sum \frac{(O_i - E_i)^2}{E_i} \sim \chi^2(2)$ where O_i and E_i are observed and expected frequencies.

From GC, since $p\text{-value} = 0.14560 > 0.1,$ we do not reject $\mathrm{H}_0,$ and conclude that there is insufficient evidence, at 10% significance level, that the data does not follow the normal distribution, i.e. evidence to support the statistician's suspicion.

(ii) Smallest $k = 14.6.$

(iii) Paired-sample *t*-test.
The statistician needs to further assume that the difference in wear, $D,$ is normally distributed.

(iv) Let μ_d be the mean difference in wear between the two materials.

$\mathrm{H}_0 : \mu_d = 0$

$\mathrm{H}_1 : \mu_d \ne 0$

Under $\mathrm{H}_0,$ test statistic $T = \frac{\bar{D}}{S_d / \sqrt{10}} \sim t(9)$

From GC, since $p\text{-value} = 0.00853 < 0.01,$ we reject H_0 and conclude that there is sufficient evidence, at 1% significance level, that the two materials have a difference in wear.

Qn	Solution
11(i)	$Y = 6X^2$ $\mathrm{P}(Y > 48)$ $= \mathrm{P}(6X^2 > 48)$ $= \mathrm{P}(X > 2\sqrt{2})$ $= \int_{2\sqrt{2}}^{3} \frac{1}{4} x \, \mathrm{d}x$ $= \frac{1}{8}\left[x^2\right]_{2\sqrt{2}}^{3}$ $= \frac{1}{8}$
(ii)	Let W be the number of attempts until a 'large' cube is drawn. $W \sim \mathrm{Geo}\left(\frac{1}{8}\right)$ $\mathrm{P}(W \le n) \ge 0.9$ From GC, $n \ge 18$. The least value of n is 18.
(iii)	$\mathrm{E}(Y)$ $= \mathrm{E}(6X^2)$ $= 6\mathrm{E}(X^2)$ $= 6\int_1^3 x^2 \left(\frac{1}{4}x\right) \mathrm{d}x$ $= \frac{3}{8}\left[x^4\right]_1^3$ $= 30$ $\mathrm{Var}(Y)$ $= \mathrm{Var}(6X^2)$ $= 36\mathrm{Var}(X^2)$ $= 36\left[\mathrm{E}(X^4) - \left(E(X^2)\right)^2\right]$ $= 36\left[\frac{1}{4}\int_1^3 x^5 \, \mathrm{d}x - 25\right]$ $= 36\left[\frac{1}{24}(728) - 25\right]$ $= 36\left(\frac{91}{3} - 25\right)$ $= 192$
(iv)	Since the sample size is large, by Central Limit Theorem, $\overline{Y} \sim N\left(30, \frac{192}{50}\right)$ approximately

	$P\left(\overline{Y} < 27\right) = 0.062893 = 0.0629$ (3 s.f)
(v)	Let μ be the mean surface area of cubes produced by the worker. $\mathrm{H}_0 : \quad \mu = 30$ $\mathrm{H}_1 : \quad \mu < 30$ Under H_0, since n is large, $\overline{Y} \sim N\left(30, \frac{192}{50}\right)$ by CLT approximately. From (iv), p-value = 0.0629 Since p-value > 0.05, we can conclude that there is insufficient evidence at 5% significance level to say that the training for the worker has an effect on him producing cubes of smaller sizes.