Name:		Index Number:		Class:	

DUNMAN HIGH SCHOOL
Preliminary Examination
Year 6

FURTHER MATHEMATICS (Higher 2) 9649/01

Paper 1

September 2017

3 hours

Additional Materials: Answer Paper
List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your Name, Index Number and Class on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

For teachers' use:

Qn	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	Total
Score											
Max Score	**7**	**7**	**8**	**10**	**10**	**11**	**11**	**11**	**12**	**13**	**100**

1 The sets A and B are defined as follows:

$$A = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : x + y - z = 0 \right\},$$

$$B = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : 2x + 3y - 5z = 1 \right\}.$$

(a) Show that A is a subspace of $\mathbb{R}^3$ but B is not a subspace of $\mathbb{R}^3$. [3]

(b) Show that the following are not a subspace of $\mathbb{R}^3$:

(i) $A \cup B'$, [2]

(ii) $A \cap B'$. [2]

2 By letting $x = \sqrt{t}$, show that the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \left(2x - \frac{1}{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} + 24x^2 = 0$$

where $x > 0$, may be transformed to

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + a\frac{\mathrm{d}y}{\mathrm{d}t} + b = 0$$

where a and b are constants to be determined. [4]

Hence find the general solution of y in terms of x. [3]

3 The complex number z satisfies both the conditions

$$5 \le |z - 3 - 4\mathrm{i}| \le 10 \text{ and } \arg\left(\frac{3 + 4\mathrm{i}}{z - 6 - 8\mathrm{i}}\right) = 0.$$

Sketch the locus of the points representing the complex number z on an Argand diagram. [4]

Hence find

(i) the maximum value of $|z - 6|$, giving your answer in exact form, [2]

(ii) the range of values of $\arg(z - 6)$. [2]

4 The motion of the tip of a tuning fork can be modelled by the differential equation

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+k\frac{\mathrm{d}x}{\mathrm{d}t}+m\omega^2x=0$$

where x is the displacement of the tip from its equilibrium position at time t and m, k and ω are positive constants. It is known that k is so small that k^2 can be ignored as k models the slight damping due to the resistance of the air. It is given that the tip of the fork is initially in its equilibrium position and moving with speed v in the positive x-direction.

(i) Solve the differential equation. [4]

The amplitude of a vibration is the maximum displacement of the tip from its equilibrium position and one period of a vibration is the time interval between the occurrences of two consecutive amplitudes.

(ii) Comment on the period of the vibrations over time and show that the amplitude of successive vibrations follows a geometric progression. [3]

(iii) Given that k is no longer small and $k^2>4m^2\omega^2$, describe the behaviour of x as time progresses and sketch a possible graph of x vs t. Justify your answer. [3]

5 **(i)** Use de Moivre's theorem to express $\cos 6\theta$ in terms of $\cos\theta$ and $\sin\theta$ and express $\sin 6\theta$ in a similar form. Deduce that

$$\tan 6\theta=\frac{6t-20t^3+6t^5}{1-15t^2+15t^4-t^6},$$

where $t=\tan\theta$ and $0<\theta<\frac{\pi}{2}$. [4]

(ii) By giving θ a suitable value, prove that one of the roots of the cubic equation

$$x^3-15x^2+15x-1=0$$

is $\tan^2\left(\frac{1}{12}\pi\right)$. State the other two roots of this equation. [4]

Deduce that $\tan^2\left(\frac{1}{12}\pi\right)=7-4\sqrt{3}$. [2]

6 It is given that **A** and **C** are 3×3 matrices where **C** is invertible. Show that if $\mathbf{x}$ is an eigenvector of **A** with corresponding eigenvalue λ, then $\mathbf{C}^{-1}\mathbf{x}$ is an eigenvector of $\mathbf{C}^{-1}\mathbf{AC}$ with corresponding eigenvalue λ. [2]

Hence find the eigenvalues and the corresponding eigenvectors of

$$\mathbf{S}=\begin{pmatrix}1&0&-1\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&-3&-3\\-8&6&-3\\8&-2&7\end{pmatrix}\begin{pmatrix}1&0&1\\0&1&0\\0&0&1\end{pmatrix}.$$ [7]

Find a matrix **P** and a diagonal matrix **E** such that $\mathbf{S}^4=\mathbf{PEP}^{-1}$. [2]

7 **(a)** A curve C is defined parametrically by

$$x=3t^5-15t,\quad y=-10t^3,\quad 1\le t\le\sqrt{2}.$$

It is given that C is rotated completely about the x-axis. Show that the area of the surface generated is $\frac{1575}{2}\pi$. [5]

(b) A curve Γ is defined parametrically by

$$x=\tan^{-1}t,\quad y=1+t^2,\quad t\ge0.$$

The region R is bounded by the curves Γ and $y=5-x^2$, the line $x=\frac{\pi}{4}$ and the y-axis. A solid of revolution is generated when R is rotated completely about the y-axis.

Find the exact volume of the solid. [6]

8 A curve C has equation, in polar coordinates,

$$r=\frac{3}{1-2\cos\theta},\qquad \text{where } \frac{\pi}{3}<\theta<\frac{5\pi}{3}.$$

(i) By finding the Cartesian equation of the curve, show that C is part of a hyperbola. State the eccentricity and the cartesian equation of the directrix of this hyperbola. [5]

The points P and P' on curve C are such that their shortest distance to the directrix is $\frac{3}{4}$ units.

(ii) Find the polar coordinates of P and P'. [2]

The region R is bounded by the curve, the line passing through P and the pole and the line passing through P' and the pole.

(iii) Find the area and perimeter of R. [4]

9 Mr Ma opened an account with an Emerging Market (EM) fund and invested an amount of \$30,000 at the beginning of 2013. The interest rate was 5% per year, so that on the last day of each year the amount in the account on that day was increased by 5%.

The EM fund allowed Mr Ma to choose one of the following schemes for the entire duration of his investment period:

Scheme *A*: He withdraws \$1000 from his account on the last day of each year after interest has been credited into it.

Scheme *B*: He does not withdraw any amount and a bonus will be credited into his account on the last day of each year. This bonus will be 11.5% of the amount in his account at the start of the previous year.

The amount of money Mr Ma has in the EM fund on first day of the *n*th year is denoted by u_n.

(i) For Scheme *A*, find the recurrence relation in terms of u_n and u_{n-1}.

For Scheme *B*, the recurrence relation for u_n is given by $u_n = au_{n-1} + bu_{n-2}$.
State the values of *a* and *b*. [3]

(ii) Express u_n in terms of *n* for each scheme. [6]

(iii) In 2013, Mr Ma chose scheme *B*. Four years later, Mr Ma invested another \$80,000 in the EM fund through Scheme *A*. Find the year when Mr Ma's amount of money in the first investment first exceeds that of his second investment with the EM fund. [3]

10 A tank, with capacity of 20000 litres, initially contains 10000 litres of brine with a salt concentration of 1 kg salt per 100 litres. Brine with 2 kg salt per 100 litres enters the tank at a rate of 20 litres per second. The well-stirred mixture leaves at 10 litres per second. It is given that x is the amount, in kg, of salt in the tank at time t seconds.

(i) Show that before the tank overflows, a differential equation involving x and t is

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 0.4 - \frac{x}{1000+t}.$$

[3]

(ii) Given that the amount of salt in the tank at time 960 seconds is 341 kg, use the Euler's method with step size of 20 seconds to estimate the amount of salt when the brine in the tank begins to overflow, giving your answer to two decimal places. [4]

Another tank with a different way of mixing brine follows the differential equation of the form $\frac{\mathrm{d}y}{\mathrm{d}t} = a + b\mathrm{e}^{-\frac{(t+1000)^2}{1000000}}$, where a and b are real constants and y is the amount of salt at time t seconds. It is known that the two tanks has the same rate of change in the amount of salt at $t = 0$ and the same amount of salt added from time 960 seconds to the time the first tank overflows.

(iii) Show that the amount of salt, in kg, added in the second tank from time 960 seconds to the time the first tank begins to overflow is

$$\int_{960}^{1000} a + (0.3-a)\mathrm{e}^{1-\frac{(t+1000)^2}{1000000}}\,\mathrm{d}t.$$

Use Simpson's rule with 5 ordinates to estimate the definite integral in terms of a.

Hence use the result in part **(ii)** to estimate the value of a. [6]