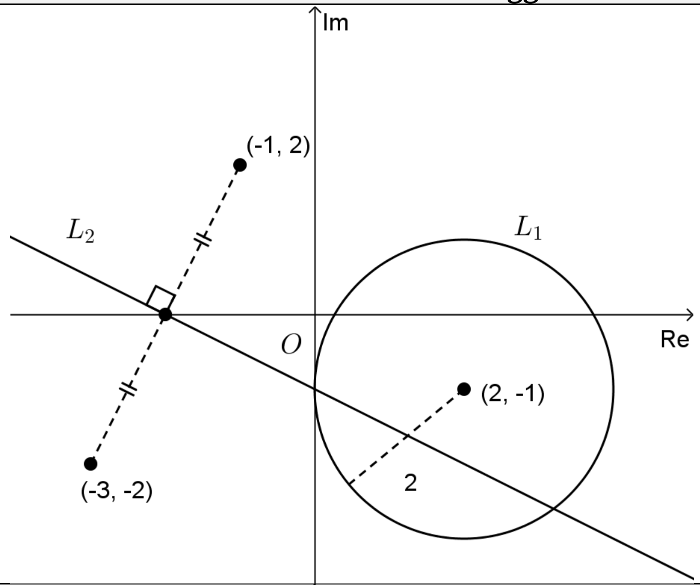
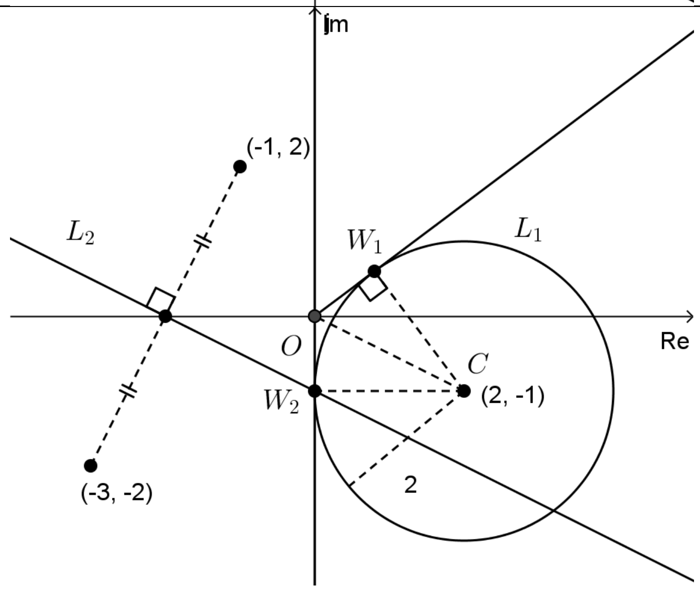
**2017 NJC H2 Further Mathematics Preliminary Examination (Paper 2) Solution**

| **1** | **Suggested Solution** |
|---|---|
| **(i)** | <br> |
| **(b)** | <br><br>$\arg\left(\frac{w_1}{w_2}\right)=\arg w_1-\arg w_2$, so we need to maximise $\arg(w_1)$ and minimize $\arg(w_2)$.<br>$a=\cos(\angle W_1OW_2)$<br>$=\cos(2\angle COW_2)$<br>$=2\cos^2(\angle COW_2)-1$<br>$=2\left(\frac{1}{\sqrt{2^2+1^2}}\right)^2-1$<br>$=-\frac{3}{5}$<br>The greatest possible value of $\arg\left(\frac{w_1}{w_2}\right)$ is $\cos^{-1}\left(-\frac{3}{5}\right)$. |

| 2 | Suggested Solution |
|---|---|

Let $\mathrm{P}_n$ be the proposition $\dfrac{\mathrm{d}^n}{\mathrm{d}x^n}\left(x^n \ln x\right) = n!\left(\ln x + \displaystyle\sum_{r=1}^{n}\frac{1}{r}\right)$ for every positive integer *n*.

When $n=1$,

$$\text{LHS} = \frac{\mathrm{d}}{\mathrm{d}x}\left(x\ln x\right) = x\left(\frac{1}{x}\right) + (1)\ln x = 1 + \ln x.$$

$\text{RHS} = 1!\left(\ln x + \displaystyle\sum_{r=1}^{1}\frac{1}{r}\right) = \ln x + 1$. LHS = RHS, so $\mathrm{P}_1$ is true.

Assume $\mathrm{P}_k$ is true for some $k \in \mathbb{Z}^+$, i.e.

$$\frac{\mathrm{d}^k}{\mathrm{d}x^k}\left(x^k \ln x\right) = k!\left(\ln x + \sum_{r=1}^{k}\frac{1}{r}\right).$$

For $\mathrm{P}_{k+1}$:

$$\begin{aligned}
\text{LHS} &= \frac{\mathrm{d}^{k+1}}{\mathrm{d}x^{k+1}}\left(x^{k+1}\ln x\right) \\
&= \frac{\mathrm{d}^k}{\mathrm{d}x^k}\left[\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{k+1}\ln x\right)\right] \\
&= \frac{\mathrm{d}^k}{\mathrm{d}x^k}\left[(k+1)x^k \ln x + x^k\right] \\
&= (k+1)k!\left(\ln x + \sum_{r=1}^{k}\frac{1}{r}\right) \\
&= (k+1)!\left(\ln x + \sum_{r=1}^{k}\frac{1}{r} + \frac{1}{k+1}\right) \\
&= (k+1)!\left(\ln x + \sum_{r=1}^{k+1}\frac{1}{r}\right) = \text{RHS}
\end{aligned}$$

Thus, $\mathrm{P}_{k+1}$ is true.

Since $\mathrm{P}_1$ is true and $\mathrm{P}_k$ implies $\mathrm{P}_{k+1}$ is true, $\mathrm{P}_n$ is true for all positive integers *n* by Mathematical Induction.

**(b)**

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12}.$$

$$\iiiint \ln x \,\mathrm{d}x\,\mathrm{d}x\,\mathrm{d}x\,\mathrm{d}x$$

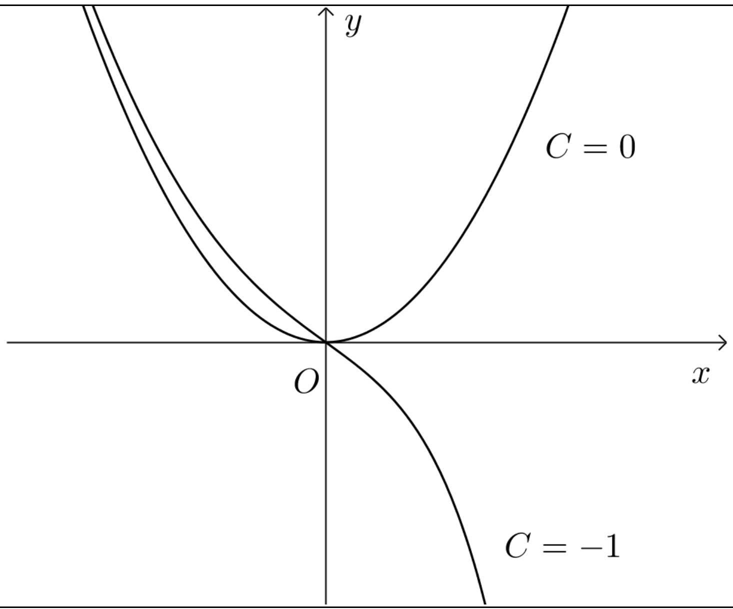
$$= \frac{1}{4!}\int\!\!\int\!\!\int\!\!\int 4!\left(\ln x + \frac{25}{12}\right)\mathrm{d}x\,\mathrm{d}x\,\mathrm{d}x\,\mathrm{d}x - \int\!\!\int\!\!\int\!\!\int\left(\frac{25}{12}\right)\mathrm{d}x\,\mathrm{d}x\,\mathrm{d}x\,\mathrm{d}x$$

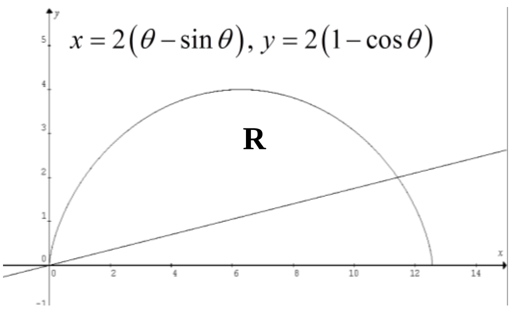
$$= \frac{1}{24}x^4\ln x - \frac{25}{12(4!)}x^4 + C_3x^3 + C_2x^2 + C_1x + C_0$$

$$= \frac{1}{24}x^4\ln x - \frac{25}{288}x^4 + C_3x^3 + C_2x^2 + C_1x + C_0$$

, where $C_3, C_2, C_1, C_0$ are arbitrary constants.

| 3 | **Suggested Solution** |
|---|---|
| | $\left(\begin{array}{ccc\|c} 1 & 2 & -1 & 1 \\ 2 & 5 & a & 3 \\ 3 & a & b & a-3 \end{array}\right) \xrightarrow[R_3-3R_1]{R_2-2R_1} \left(\begin{array}{ccc\|c} 1 & 2 & -1 & 1 \\ 0 & 1 & a+2 & 1 \\ 0 & a-6 & b+3 & a-6 \end{array}\right)$ <br> $\xrightarrow[R_3-(a-6)R_2]{R_1-2R_2} \left(\begin{array}{ccc\|c} 1 & 0 & -2a-5 & -1 \\ 0 & 1 & a+2 & 1 \\ 0 & 0 & b+3-(a+2)(a-6) & 0 \end{array}\right)$ <br> For the three planes not to have a unique point of intersection, $b+3-(a+2)(a-6)=0 \Rightarrow b=a^2-4a-15$. <br><br> In this case, since the last row of the augmented matrix is zero row, the system is consistent. Therefore, the three planes meet in a line. |
| | From the row-echelon form, <br> $c+(-2a-5)(5)=-1 \Rightarrow c=10a+24$ $b+(a+2)(5)=1 \Rightarrow b=-5a-9$ <br> Solving simultaneously, <br> $a^2-4a-15=-5a-9$ <br> $a^2+a-6=0$ <br> $(a-2)(a+3)=0$ <br> $a=2$ or $a=-3$ <br><br> When $a=2$, $b=-5\times2-9=-19$ and $c=10\times2+24=44$ <br> The row echelon form of the augmented matrix then becomes $\left(\begin{array}{ccc\|c} 1 & 0 & -9 & -1 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$. Hence, <br> $x-9z=-1$ and $y+4z=1$. <br> Let $z=t$. Then $x=-1+9t$ and $y=1-4t$. <br> So, equation of the line of intersection is <br> $\mathbf{r}=\begin{pmatrix}-1\\1\\0\end{pmatrix}+t\begin{pmatrix}9\\-4\\1\end{pmatrix},\ t\in\mathbf{R}$ <br> When $a=-3$, $b=-5\times(-3)-9=6$ and $c=10\times(-3)+24=-6$ <br> The row echelon form of the augmented matrix then becomes $\left(\begin{array}{ccc\|c} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$. Hence, <br> $x+z=-1$ and $y-z=1$. <br> Let $z=s$. Then $x=-1-s$ and $y=1+s$. <br> So, equation of the line of intersection is <br> $\mathbf{r}=\begin{pmatrix}-1\\1\\0\end{pmatrix}+s\begin{pmatrix}-1\\1\\1\end{pmatrix},\ s\in\mathbf{R}$. |

| 4 | **Suggested Solution** |
|---|---|
| **(a)** | $\left(1+x\right)y - x\dfrac{\mathrm{d}y}{\mathrm{d}x} = x^3 - x^2$<br>$\dfrac{\mathrm{d}y}{\mathrm{d}x} - \left(\dfrac{1+x}{x}\right)y = -x^2 + x$<br>Integrating factor $= \mathrm{e}^{\int -\left(\frac{1+x}{x}\right)\mathrm{d}x}$<br>$= \mathrm{e}^{-\ln x - x}$<br>$= \dfrac{1}{x}\mathrm{e}^{-x}$<br>$\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{y}{x}\mathrm{e}^{-x}\right) = \dfrac{1}{x}\mathrm{e}^{-x}\left(-x^2 + x\right)$<br>$\dfrac{y}{x}\mathrm{e}^{-x} = \int \mathrm{e}^{-x}\left(1-x\right)\mathrm{d}x$<br>$= -\mathrm{e}^{-x}\left(1-x\right) - \int \mathrm{e}^{-x}\,\mathrm{d}x$<br>$= -\mathrm{e}^{-x}\left(1-x\right) + \mathrm{e}^{-x} + C$<br>$= x\mathrm{e}^{-x} + C$<br>$y = x^2 + Cx\mathrm{e}^{x}$ |
| **(a)** |  |
| **(b)** | Step size = 0.2<br>$N_1^* = 1 + (0.2)\cdot\left[3(0) - 1^2\right] = 0.8$<br>$N_1 = 1 + (0.2)\cdot\left[\dfrac{[3(0)-1^2] + [3(0.2) - 0.8^2]}{2}\right] = 0.896$<br>$N_2^* = 0.896 + (0.2)\cdot\left[3(0.2) - 0.896^2\right] = 0.8554368$<br>$N_2 = 0.896 + (0.2)\cdot\left[\dfrac{[3(0.2) - 0.896^2] + [3(0.4) - 0.8554368^2]}{2}\right]$<br>$= 0.9225$ (4 d.p.) |

| 5 | **Suggested Solution** |
|---|---|
| **(i)** | $x = r(\theta - \sin\theta),\ y = r(1-\cos\theta) \Rightarrow \dfrac{\mathrm{d}x}{\mathrm{d}\theta} = r(1-\cos\theta),\ \dfrac{\mathrm{d}y}{\mathrm{d}\theta} = r\sin\theta$<br>Surface area<br>$= \displaystyle\int_0^{2\pi} 2\pi y\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2}\ \mathrm{d}\theta$<br>$= 2\pi r\displaystyle\int_0^{2\pi} (1-\cos\theta)\sqrt{r^2(1-\cos\theta)^2 + r^2\sin^2\theta}\ \mathrm{d}\theta$<br>$= 2\pi r^2\displaystyle\int_0^{2\pi} (1-\cos\theta)\sqrt{1-2\cos\theta+\cos^2\theta+\sin^2\theta}\ \mathrm{d}\theta$<br>$= 2\pi r^2\displaystyle\int_0^{2\pi} (1-\cos\theta)\sqrt{2-2\cos\theta}\ \mathrm{d}\theta$<br>$= 2\pi r^2\displaystyle\int_0^{2\pi} (1-\cos\theta)\sqrt{2\left(1-1+2\sin^2\frac{\theta}{2}\right)}\ \mathrm{d}$<br>$= 8\pi r^2\displaystyle\int_0^{2\pi} \left(1-\cos^2\frac{\theta}{2}\right)\sin\frac{\theta}{2}\ \mathrm{d}\theta$<br>$= 8\pi r^2\displaystyle\int_0^{2\pi} \left(\sin\frac{\theta}{2} - \cos^2\frac{\theta}{2}\sin\frac{\theta}{2}\right)\ \mathrm{d}\theta$<br>$= 8\pi r^2\left[-2\cos\frac{\theta}{2} + \frac{2}{3}\cos^3\frac{\theta}{2}\right]_0^{2\pi}$<br>$= 8\pi r^2\left(2-\frac{2}{3}+2-\frac{2}{3}\right) = \dfrac{64\pi r^2}{3}$<br>Given $\dfrac{64\pi r^2}{3} = 432\pi.$<br>$r^2 = \dfrac{81}{4} \Rightarrow r = \dfrac{9}{2}$ |
| **(ii)** | At $(3\pi+2, 2)$, $2 = 2(1-\cos\theta) \Rightarrow \cos\theta = 0 \Rightarrow \theta = \dfrac{3\pi}{2}$<br>Equation of $L$: $y = \dfrac{2}{3\pi+2}x$<br><br><br>Volume of solid generated<br>$= 2\pi\displaystyle\int_0^{1.5\pi} 2(\theta-\sin\theta)\cdot 2(1-\cos\theta)\cdot 2(1-\cos\theta)\ \mathrm{d}\theta - \left(\pi - \frac{\pi}{3}\right)(3\pi+2)^2(2)$<br>$= 16\pi\displaystyle\int_0^{1.5\pi} (\theta-\sin\theta)\cdot(1-\cos\theta)^2\ \mathrm{d}\theta - \frac{4\pi}{3}(3\pi+2)^2$<br>$= 835$ (3 s.f.) |

| 6 | Suggested Solution |
|---|---|
| **(i)** | $X \sim \text{Geo}(0.5)$ |

**(ii)** To test at the 5% level of significance,

$H_0$: $X \sim \text{Geo}(0.5)$

$H_1$: $X \not\sim \text{Geo}(0.5)$

Under $H_0$, the corresponding expected frequencies are

| $x$ | 1 | 2 | 3 | 4 | 5 | 6 | $\geq 7$ |
|---|---|---|---|---|---|---|---|
| $f_i$ | 45 | 31 | 11 | 7 | 4 | 5 | 0 |
| $e_i$ | 50 | 25 | 12.5 | 6.25 | 3.125 | 1.5625 | 1.5625 |

After merging,

| $x$ | 1 | 2 | 3 | 4 | $\geq 5$ |
|---|---|---|---|---|---|
| $f_i$ | 45 | 31 | 11 | 4 | 9 |
| $e_i$ | 50 | 25 | 12.5 | 6.25 | 6.25 |

$v = 5 - 1 = 4$, $\chi^2 = \sum_{i=1}^{5} \frac{(e_i - F_i)^2}{e_i} \sim \chi^2(4)$.

By GC, $p$-value $= 0.387 > 0.1$. Thus, we do not reject $H_0$ at 10% significance level and conclude that $\text{Geo}(0.5)$ is a good fit for the distribution of $X$ evidently.

No. **(ii)** does not suggest there is an equal chance for each number to appear, as it only tests for whether there is an equal chance for odd number or even number to appear.

| 7 | **Suggested Solution** |
|---|---|
| **(i)** | $S=\frac{1}{2}\times1\times\left(1-\tan\theta\right)=0.5-0.5\tan\theta,\ 0\le\theta\le\frac{\pi}{4}$<br><br>Let $\mathrm{F}$ be the cumulative distribution function of *S*. Then<br>$\mathrm{F}\left(s\right)=\mathrm{P}\left(S\le s\right)=\begin{cases}0, & s\le0\\1, & s\ge0.5\end{cases}$<br>For $0<s<0.5$,<br>$\mathrm{F}\left(s\right)=\mathrm{F}\left(0\right)+\mathrm{P}\left(0\le0.5-0.5\tan\theta\le s\right)$<br>$=\mathrm{P}\left(1-2\mathrm{s}\le\tan\theta\le1\right)$<br>$=\mathrm{P}\left(\tan^{-1}\left(1-2s\right)\le\theta\le\frac{\pi}{4}\right)$<br>$=\dfrac{\frac{\pi}{4}-\tan^{-1}\left(1-2s\right)}{\frac{\pi}{4}-0}$<br>$=1-\frac{4}{\pi}\tan^{-1}\left(1-2s\right)$<br>$\frac{\mathrm{d}}{\mathrm{d}s}\left[1-\frac{4}{\pi}\tan^{-1}\left(1-2s\right)\right]=-\frac{4}{\pi}\left(\frac{-2}{1+\left(1-2s\right)^2}\right)$<br>$=\frac{4}{\pi}\left(\frac{2}{2-4s+4s^2}\right)$<br>$=\frac{4}{\pi\left(1-2s+2s^2\right)}$ |
| **(ii)** | $\mathrm{E}\left(S\right)=\int_0^{\frac{1}{2}}\frac{4s}{\pi\left(1-2s+2s^2\right)}\,\mathrm{d}s$<br>$=\int_0^{\frac{1}{2}}\frac{4s-2}{\pi\left(1-2s+2s^2\right)}\,\mathrm{d}s+\int_0^{\frac{1}{2}}\frac{2}{\pi\left(1-2s+2s^2\right)}\,\mathrm{d}s$<br>$=\left(\frac{1}{\pi}\ln\left\|1-2s+2s^2\right\|\right)_0^{\frac{1}{2}}+\frac{1}{2}\left[-\frac{4}{\pi}\tan^{-1}\left(1-2s\right)\right]_0^{\frac{1}{2}}$<br>$=\frac{1}{\pi}\ln\left(\frac{1}{2}\right)-\frac{1}{\pi}\ln1-\frac{2}{\pi}\tan^{-1}0+\frac{2}{\pi}\tan^{-1}1$<br>$=-\frac{\ln2}{\pi}-0-0+\frac{2}{\pi}\left(\frac{\pi}{4}\right)$<br>$=\frac{1}{2}-\frac{\ln2}{\pi}$ |

| 8 | **Suggested Solution** |
|---|---|
| **(a)** | To have $x$ events in the interval $(0, t+\delta t)$, we shall either have<br><br>Case 1: $x$ events in $(0,t)$ and no event in $(t, t+\delta t)$; or<br>Case 2: $(x-1)$ events in $(0,t)$ and an event in $(t, t+\delta t)$.<br><br>Since the events are independent, by addition principle,<br>$\begin{aligned} p_x(t+\delta t) &= p_x(t)\times(1-\lambda\delta t)+p_{x-1}(t)\times(\lambda\delta t) \\ &=(1-\lambda\delta t)\,p_x(t)+\lambda\delta t p_{x-1}(t) \end{aligned}$. |
| **(b)** | To have no event in the interval $(0, t+\delta t)$, we shall have no event in $(0,t)$ and no event in $(t, t+\delta t)$.<br>$p_0(t+\delta t)=p_0(t)\times(1-\lambda\delta t)=(1-\lambda\delta t)\,p_0(t)$ |
| | No more than 1 event can occur in any interval of length $\delta t$, OR<br>The probability of having two or more events in any interval of length $\delta t$ is negligible.<br><br>$p_0(t+\delta t)=\dfrac{\mathrm{e}^{-\lambda(t+\delta t)}\left[\lambda(t+\delta t)\right]^0}{0!}=\mathrm{e}^{-\lambda(t+\delta t)}=\mathrm{e}^{-\lambda t}\mathrm{e}^{-\lambda\delta t}$,<br>$(1-\lambda\delta t)\,p_0(t)=\dfrac{(1-\lambda\delta t)\mathrm{e}^{-\lambda t}\left[\lambda t\right]^0}{0!}=(1-\lambda\delta t)\mathrm{e}^{-\lambda t}$.<br>By the series expansion of $\mathrm{e}^x$, when $\delta t$ is small enough,<br>$\mathrm{e}^{-\lambda\delta t}=1+(-\lambda\delta t)+\dfrac{(-\lambda\delta t)^2}{2!}+\ldots\approx 1+(-\lambda\delta t)$.<br>Thus, $p_0(t+\delta t)\approx(1-\lambda\delta t)\,p_0(t)$ |
| | The probability that there are more than 2 earthquakes occurring in a year is<br>$\begin{aligned} 1-\left[p_0(1)+p_1(1)+p_2(1)\right] &= 1-\mathrm{e}^{-1,3}\left(1+1.3+\dfrac{1.3^2}{2}\right) \\ &=0.14289 \end{aligned}$<br><br>The desired answer is $1-(1-0.14289)^{10}=0.786\ (3\,\text{s.f.})$ |

| 9 | **Suggested Solution** |
|---|---|
| **(i)** | Let $D$ denote distance before beer – distance after beer, and $\mu_D$ denote its mean. |

| | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| Bef<br>Aft | 5.0<br>3.7 | 2.4<br>7.6 | 6.9<br>7.1 | 0.3<br>5.3 | 4.1<br>4.7 | 1.2<br>1.1 | 2.0<br>2.4 | 4.2<br>7.9 |
| $d_i$ | 1.3 | -5.2 | -0.2 | -5 | -0.6 | 0.1 | -0.4 | -3.7 |

To test at the 5% significance level:

$H_0$: $\mu_D = 0$

$H_1$: $\mu_D < 0$

Under $H_0$, $T = \dfrac{\overline{D} - 0}{S/\sqrt{8}} \sim t(7)$.

By GC, $p\text{-value} = 0.048 < 0.05$.

Therefore, we reject $H_0$ at the 5% significance level and conclude that there is sufficient evidence that consumption of alcohol causes an increase in the mean distance from the centre, thus a reduction in accuracy.

The test may not be valid as the population of the differences may not be normally distributed.

**(ii)**

To test at the 5% significance level:

$H_0$: Median of $D$ is 0.

$H_1$: Median of $D$ is less than 0.

Assign signed ranks:

| $d_i$ | 1.3 | -5.2 | -0.2 | -5 | -0.6 | 0.1 | -0.4 | -3.7 |
|---|---|---|---|---|---|---|---|---|
| signed rank | 5 | –8 | –2 | –7 | –4 | 1 | –3 | –6 |

$P = 5+1 = 6$, $Q = 8+2+7+4+3+6 = 30$, so $T = 6$.

From MF26, the critical region is $T \leq 5$.

Since $T = 6$ is not in the critical region, we do not reject $H_0$ at the 5% significance level, and conclude that there is insufficient evidence that the median of the difference is less than 0.

It is assumed that the differences have a symmetric distribution.

| 10 | Suggested Solution |
|---|---|

**(a)**

To test at 5% significance level
$H_0$: Age profile and mobile phone choice are not associated.
$H_1$: They are associated.

Calculate the expected frequencies under $H_0$:

| | | | |
|---|---|---|---|
| 45<br>$\frac{80\times170}{300}=45.33$ | 30<br>$\frac{80\times110}{300}=29.33$ | 5<br>$\frac{80\times20}{300}=5.33$ | 80 |
| 65<br>$\frac{100\times170}{300}=56.67$ | 34<br>$\frac{100\times110}{300}=36.67$ | 1<br>$\frac{100\times20}{300}=6.67$ | 100 |
| 41<br>$\frac{80\times170}{300}=45.33$ | 31<br>$\frac{80\times110}{300}=29.33$ | 8<br>$\frac{80\times20}{300}=5.33$ | 80 |
| 19<br>$\frac{40\times170}{300}=22.67$ | 15<br>$\frac{40\times110}{300}=14.67$ | 6<br>$\frac{40\times20}{300}=2.67$ | 40 |
| 170 | 110 | 20 | 300 |

After merging (the last two rows to maximize df),

| | | |
|---|---|---|
| 45<br>(45.33) | 30<br>(29.33) | 5<br>(5.33) |
| 65<br>(56.67) | 34<br>(36.67) | 1<br>(6.67) |
| 60<br>(68) | 46<br>(44) | 14<br>(8) |

$v=(3-1)(3-1)=4$, so $\chi^2=\sum_{r=1}^{9}\frac{(O_i-E_i)^2}{E_i}\sim\chi^2(4)$.
By GC, $p$-value = 0.0188 < 0.05.
(OR $\chi^2=11.81>\chi^2_{0.95,\,4}=9.488$)

Hence, we reject $H_0$ at the 5% significance level and conclude that age profile and choice of mobile phone are associated evidently.

Number of 4G mobile phone users in this age group is more than expected, number of 2G mobile phone users in this age group is fewer than expected.

**(b)**

$z_{0.99}\sqrt{\frac{0.7\times0.3}{n}}<0.05$

$\frac{0.21}{n}<(0.021493)^2$

$n>454.6$

Thus, least $n=455$

Since $p(1-p)$ is decreasing on $(0.5,1)$, we may reduce the minimum sample size required to construct such an interval.