## Section A: Pure Mathematics [50 marks]

**1** Giant otters have no predators except for humans, who hunt them widely for their fur. The population of the giant otters in the Amazon River at time *t* years, after the start of a study, is modelled by

$$\frac{\mathrm{d}P}{\mathrm{d}t} = -\frac{1}{5}P\left(1-\frac{P}{3}\right)-h,$$

where $P$ (in thousands) is the population of giant otters and $h$ (in thousands) is the number of giant otters hunted per year.

**(i)** Find the stable population $P_S$ in terms of $h$, [2]

**(ii)** Describe the trend in population when
**(a)** $0 < P_0 < P_S$,
**(b)** $P_0 > P_S$,
where $P_0$ is the initial population of the otter. [3]

**(iii)** Given that $P_0$ is 6, find the maximum number of otters that can be hunted per year such that the otter will not become extinct. [3]

**2** Three $n\times 1$ column vectors are denoted by $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$, and $\mathbf{M}$ is an $n\times n$ matrix. Show that if $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ are linearly dependent then the vectors $\mathbf{Mx}_1, \mathbf{Mx}_2, \mathbf{Mx}_3$ are also linearly dependent. [2]

The vectors $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3$ and the matrix $\mathbf{A}$ are defined as follows:

$$\mathbf{y}_1 = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}, \quad \mathbf{y}_2 = \begin{pmatrix} -4 \\ 2 \\ 1 \end{pmatrix}, \quad \mathbf{y}_3 = \begin{pmatrix} 23 \\ 27 \\ 41 \end{pmatrix},$$

$$\mathbf{A} = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 4 & 2 \\ 3 & 5 & 4 \end{pmatrix}.$$

**(i)** Show that $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3$ are linearly dependent. [1]

**(ii)** Find a basis for the linear space spanned by the vectors $\mathbf{Ay}_1, \mathbf{Ay}_2, \mathbf{Ay}_3$. [2]

A linear transformation $\mathrm{T}: \mathbb{R}^3 \to \mathbb{R}^3$ is represented by the matrix $\mathbf{A}$.

**(iii)** Find a basis for the null space of T. [1]

**(iv)** Determine, with a reason, if the set of solutions of the equation $\mathrm{T}(\mathbf{x}) = \begin{pmatrix} 1 \\ 6 \\ 9 \end{pmatrix}$ is a vector space. By first writing down the general solution of the equation, determine the solution $\mathbf{x}$ which gives the least possible value of $|\mathbf{x}|$. [4]

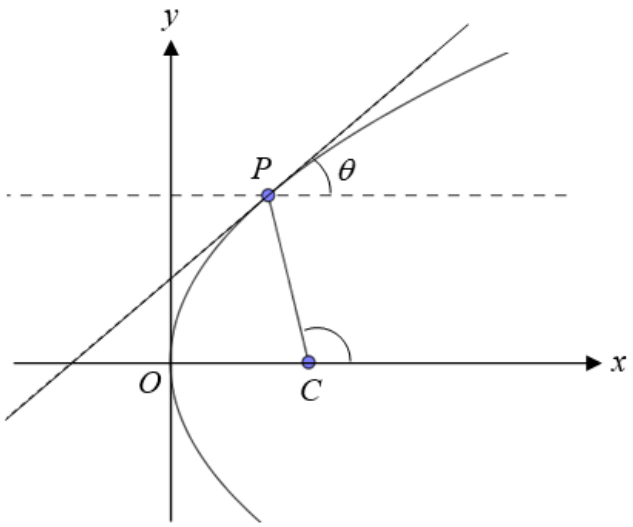
**3** The complex number $z$, satisfies the relations $\left|z-\sqrt{2}+\sqrt{2}\mathrm{i}\right|=2$ and $\arg\left(z-\sqrt{2}+\sqrt{2}\mathrm{i}\right)\leq-\frac{\pi}{5}$.

**(i)** Indicate, in an Argand diagram, the locus of the points which represent $z$. [3]

**(ii)** Given that $|z|$ is as large as possible, find the exact value of $|z|$. [1]

**(iii)** It is given instead that $\left|\arg\left(z+\sqrt{2}-\sqrt{2}\mathrm{i}\right)\right|$ is as large as possible, find $\mathrm{Re}\,(z)$. [4]

**4**




The diagram above shows a parabola with focus $C(c,0)$ and a tangent at the point $P\left(cp^2,2cp\right)$. The tangent makes an acute angle $\theta$ with the horizontal. By considering the gradients of suitable lines, determine, in terms of $\theta$, the angle $CP$ makes with the positive $x$-axis. Hence prove the reflection property of the parabola, stating the property clearly. [5]

If the normal at $P$ meets the $x$-axis at $Q$, find the coordinates of the mid-point $M$ of $PQ$. [2]

If $P$ is a variable point on the parabola, show that the locus of $M$ is also a parabola, stating its cartesian equation, and find the equation of its directrix. [4]

**5** The curve $C$ is defined by the polar equation $r=4\cos\theta\sin^2\theta$, where $0\leq\theta\leq\pi$.

**(i)** Sketch $C$. [2]

**(ii)** Express in terms of $\theta$ the $y$-coordinate of a general point on $C$ and determine the exact values of $\theta$ for which this coordinate has a stationary value. [5]

**(iii)** The points $P$, $Q$ and $R$ correspond to the three points on $C$ where the $y$-coordinate has a stationary value. Find the exact area of triangle $PQR$. [3]

**(iv)** Find the arc length of $C$. [3]

## Section B: Probability and Statistics [50 marks]

**6** The continuous random variable $Y$ denotes the side of a square in centimetres. The area of the square denoted by the random variable $W$ is uniformly distributed over the interval $[a, 4a]$, $a > 0$. Find the probability density function of $Y$. [3]

A set of trials consists of making single random observations of $Y$. A 'success' occurs when $Y > \frac{3}{2}\sqrt{a}$. Show that the probability of a success at any one trial is $\frac{7}{12}$. [2]

Calculate the exact probability that the first success occurs on even number of trials. [2]

**7** The number of system failures at an industrial facility is approximately Poisson distributed with mean 0.25 per hour. If the system start-up begins at 7 am on a typical work day, show that the probability that at least one failure occurs in a period of $t$ hours is

$$1-\mathrm{e}^{-0.25t}.$$ [1]

The time between successive failures is denoted by $T$ hours. Show that the probability density function of $T$ is given by

$$\mathrm{f}(t)=\begin{cases}0.25\mathrm{e}^{-0.25t} & t \geq 0,\\ 0 & t<0.\end{cases}$$ [2]

**(i)** It is known that there are less than 2 system failures in a one-hour period, find the probability that there are no system failures in that one hour. [2]

**(ii)** A technician tests the operations of the system at the industrial facility daily. He needs 2 hours of uninterrupted time to complete a full test. If a system failure is detected, he will have to restart the testing procedure after addressing the fault. One morning, he starts the testing after a system failure has been resolved, find the probability that he cannot complete the test before at least 3 more system failures occur. [3]

**8** Peter grows cherry tomatoes in his green house. A new brand of fertilizer, *Sweetgro*, has recently come on to the market. Peter decides to compare its effectiveness for growing cherry tomatoes with *Fruitplus*, the brand that he has used previously. He grows two sets of plants of the same variety of cherry tomatoes under the same conditions in his green house, fertilizing one set of ten plants with *Sweetgro* and the other set of eight plants with *Friutplus*.

The yield, in grams, from the each plant is shown in the table.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| *Sweetgro* | 1383 | 1397 | 1373 | 1404 | 1369 | 1357 | 1343 | 1375 | 1392 | 1381 |
| *Fruitplus* | 1375 | 1345 | 1315 | 1350 | 1348 | 1334 | 1327 | 1343 | | |

It may be assumed that, for each fertilizer, the yields are normally distributed and have the same variance.

**(i)** Test, at the 5% significance level, the claim that the mean yield from plants grown using *Sweetgro* is greater than that from plants grown using *Fruitplus* by more than 20 grams. [5]

**(ii)** It is given that the common variance of the yields is 285.75 grams$^2$. Without calculations, state what changes in the test statistic and its distribution there would be in carrying out the test in part **(i)**. [1]

**(iii)** The total cost of *Sweetgro* for Peter's 10 plants was \$15 whereas the total cost of *Fruitplus* for his 8 plants was \$10. Peter sells all his cherry tomatoes at \$3.50 per kilogram.

Use this information, together with your findings in part **(i)**, to advise Peter on which of the two brands of fertilizer he should use for the next season in order to maximise his profit per plant. [3]

**9** **(a)** A random sample of 400 students from a large institution was classified, from their score on a personality test, as having either an introvert personality or an extrovert personality. The students were also each asked to state their colour preferences – red, yellow, green or blue – with the purpose of investigating whether there is an association between personality type and colour preference. Data collected are summarised in the table shown below.

| | | Colour preference | | | | |
|---|---|---|---|---|---|---|
| | | Red | Yellow | Green | Blue | Total |
| Personality type | Introvert | 36 | 8 | 24 | 26 | 94 |
| | Extrovert | 164 | 32 | 56 | 54 | 306 |
| | Total | 200 | 40 | 80 | 80 | 400 |

**(i)** Carry out a $\chi^2$ - test, at the 5% level of significance, to investigate whether colour preference is independent of personality type. [5]

**(ii)** Describe the difference, if any, in colour preference for the two personality types as indicated by your solution in part **(a)(i)**. Give a reason for your answer. [2]

**(b)** The institution wanted to find out the opinion of students regarding whether the mid-year examination currently conducted in the new semester after the mid-year vacation should be moved to the end of the semester before the start of mid-year vacation. The intention is to let the students have a stress-free mid-year vacation. The institution is prepared to grant 3 full days of study leave just before the start of the mid-year examination if the move is implemented. A random sample of 1250 students took part in an on-line survey regarding the move and 458 thought that the mid-year examination should be moved.

**(i)** Calculate an approximate 95% confidence interval for the actual proportion of students who are in favour of the move. [2]

**(ii)** Does the interval calculated in part **(b)(i)** support a claim that more than a third of the students in the institution is in favour of the move? Justify your answer. [2]

**10** An investigation was carried out into the effectiveness of two well-known drugs, A and B, in relieving pain for adult arthritis sufferers.

Ten adult arthritis sufferers each volunteered to record the number of hours of relief from pain gained when taking drug A and when taking drug B.

The adults agreed to take one of the drugs, A or B, on one day and the other drug on another day. The order of taking the drugs was randomly assigned.

The recorded times, in hours, are given in the table.

| | Adult | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| Drug A | 2.0 | 3.6 | 2.6 | 2.6 | 7.2 | 3.4 | 6.5 | 2.5 | 6.1 | 8.5 |
| Drug B | 3.5 | 5.7 | 2.8 | 2.3 | 9.8 | 3.3 | 5.9 | 3.7 | 9.1 | 11.9 |

**(i)** Explain the purpose of randomly assigning the order of taking drug A or drug B. [2]

**(ii)** It was believed that drug B is more effective in relieving pain gained by arthritis sufferers than drug A. Determine if the data support this belief, at the 1% level of significance, using

**(a)** the Wilcoxon matched-pairs signed rank test, [4]

**(b)** a *t*-test, given that the corresponding populations are normally distributed. [4]

State, giving a reason, which of the two tests is more appropriate. [1]

**(iii)** Give a reason why your conclusions in part **(ii)** might not apply to all adult arthritis sufferers. [1]

**(iv)** The investigator claims that the average number of hours of relief from pain gained by arthritis sufferers when taking drug A is 4.5 hours. Use a non-parametric test, at the 10% significance level, to test this claim against the alternative hypothesis that the average number of hours is less than 4.5 hours. [3]