1. Find the general solution of the differential equation

$$(x+1)\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{y}{\ln(x+1)}=x^2+x\text{, where } x>0\text{.}$$ [6]

2. The sequence $\{u_n\}$ has the recurrence relation $u_{n+1}=(-1)^n|1-u_n|-u_n$, $n\in\square$, $n\geq 0$ and $u_0=1$.

(i) Write down u_1, u_2 and u_3. [1]

The sequence $\{v_n\}$ has the recurrence relation $v_{n+1}=(-1)^n|1-v_n|-v_n$, $n\in\square$, $n\geq 0$.

(ii) Show that $v_n=u_n$ for all $n\in\square$, $n\geq 1$ when $v_0\geq 1$. [3]

(iii) Find the set of values of v_0 such that $v_1\neq u_1$ but $v_n=u_n$ for all $n\in\square$, $n\geq 2$.

[4]

3. **(a)** Kepler's equation of motion relates the mean anomaly, *M*, to the eccentric anomaly, *E*, of an elliptic orbit in the following way:

$$M+e\sin E-E=0\text{.}$$

Given $e=0.0167$ and $M=1.5$. Show that the equation has a root, α, in the interval (0, 2). [2]

Use the Newton-Raphson method to find this root, correct to 3 decimal places.

[3]

(b) If the velocity of a body at time *t* (in seconds) is given by

$$v(t)=\mathrm{e}^{-t}(5\cos t-10\sin t)\text{,}$$

the displacement of the body from the initial position at time *t* (in seconds) can be calculated using the integral

$$s(t)=\int_0^t \mathrm{e}^{-x}(5\cos x-10\sin x)\,\mathrm{d}x\text{.}$$

Use Simpson's rule with five ordinates to find an approximation to $s(0.4)$. Give your answer to 3 decimal places. [3]

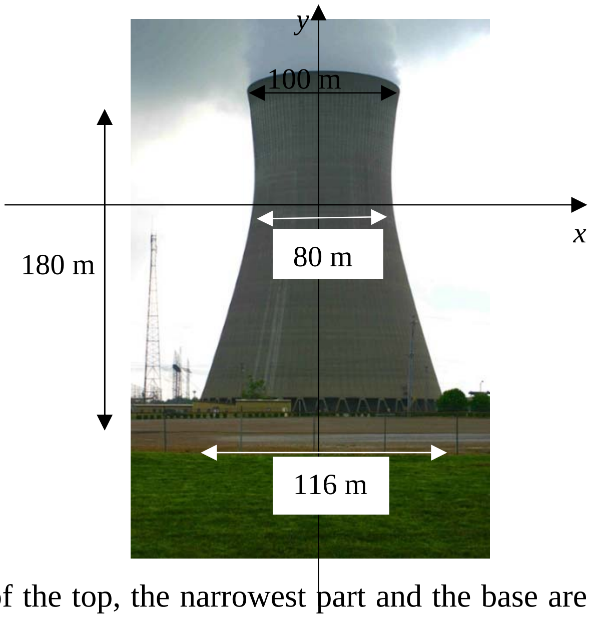
Turn Over

4. **(i)** Sketch the curves $C_1 : r = \sin\theta$ and $C_2 : r = \sqrt{\sin\theta\cos\theta}$, where $-\pi < \theta \le \pi$ on the same diagram. Find the exact polar coordinates of the point(s) of intersection of C_1 and C_2 . [6]

(ii) Show that the length of C_2 is given by $k\int_0^{\pi/2}\sqrt{\operatorname{cosec}2\theta}\,d\theta$, where k is a constant to be determined. [3]

(iii) Find the exact area of the common region of C_1 and C_2 . [3]

5. Cooling towers of nuclear power plants and large coal-fired power plants are commonly designed to have hyperbolic outlines because of their structural strength and minimum usage of material. The base has to be broad to provide stability. The narrowest part helps to enhance the speed of flow of vapor (referred to as laminar flow). The top widens so that the hot vapor can mix with the cool air outside quickly. The natural draft cooling tower in Niederaussem, Germany has the following dimensions:



The diameter of the top, the narrowest part and the base are 100 m, 80 m and 116 m respectively. The height of the tower is 180 m.

(i) Find the cartesian equation of the hyperbola that forms the outline of the tower. [4]

(ii) Find the volume of the tower, assuming that the thickness of the wall is negligible. [3]

(iii) For such a tower to be stable, the eccentricity e of the hyperbola must be such that $2.5 \le e \le 4$. For another tower with the same top and bottom diameters and the same height, but with a diameter of $2a$ m at the narrowest part (may not be along the x-axis), express e in terms of a. Find the possible range of values of a. [5]

6. The graph of $y = \mathrm{f}(x)$ has a stationary point at $(0,5)$. It is given that

$$\mathrm{f}''(x) + 2\mathrm{f}'(x) + 10\mathrm{f}(x) = 13e^{-3x} + 10 .$$

Find the solution for the above differential equation and its limit when $x \to \infty$. [12]

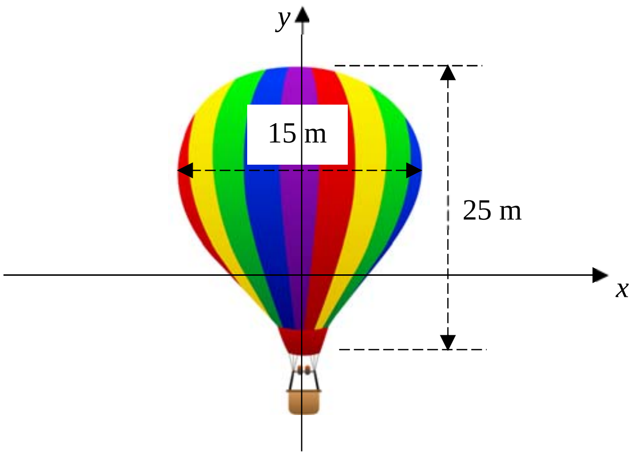
7. The outline of a hot air balloon can be modeled by the parametric equations:

$$x = a\left(\sin t + \frac{\sin 2t}{2}\right),$$

$$y = 2b\cos t ,$$

where a and b are constants, $0 \le t \le \frac{3\pi}{4}$.

The height of a hot air balloon is 25 m and the widest cross-sectional diameter is 15 m.



(i) Find the value of t when x (in metres) is maximum. Show that

$$a = \frac{10}{\sqrt{3}} \text{ and } b = \frac{25}{2+\sqrt{2}} .$$ [5]

Suppose that the thickness of the balloon is negligible.

(ii) Assuming that the surface of the balloon is smooth, find the surface area of the balloon to 1 decimal place. [4]

(iii) Show that the volume of the balloon is given by

$$k\int_{3\pi/4}^{0} (1 + 2\cos t - 2\cos^3 t - \cos^4 t)(-\sin t)\, dt ,$$

where k is a constant to be found. Find the exact volume of the balloon. [4]

8. **(i)** The matrix **A** is given by

$$\mathbf{A} = \begin{pmatrix} 1 & c & 3 \\ 4 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}.$$

It is given that **A** has an eigenvalue of 6. Find the value of c and the remaining eigenvalues. [4]

(ii) Hence, find matrices **P** and **D** such that $\mathbf{A} = \mathbf{PDP}^{-1}$, where **D** is a diagonal matrix. [4]

(iii) It is given that three function y_1, y_2, y_3 are the solutions of the following system of differential equations,

$$\frac{\mathrm{d}y_1}{\mathrm{d}x} = y_1 + cy_2 + 3y_3,$$
$$\frac{\mathrm{d}y_2}{\mathrm{d}x} = 4y_1 + y_2 \quad \text{and}$$
$$\frac{\mathrm{d}y_3}{\mathrm{d}x} = 3y_1 + y_3,$$

where c equals to the value found in part **(i)**.

By considering $\mathbf{U} = \mathbf{P}^{-1}\mathbf{Y}$, where $\mathbf{U} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ and $\mathbf{Y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$, show that the above system can be rewritten as $\mathbf{U}' = \mathbf{DU}$ where $\mathbf{U}' = \begin{pmatrix} \frac{\mathrm{d}u_1}{\mathrm{d}x} \\ \frac{\mathrm{d}u_2}{\mathrm{d}x} \\ \frac{\mathrm{d}u_3}{\mathrm{d}x} \end{pmatrix}$. [3]

[You may assume that $\mathbf{U}' = \mathbf{P}^{-1}\mathbf{Y}'$, where $\mathbf{Y}' = \begin{pmatrix} \frac{\mathrm{d}y_1}{\mathrm{d}x} \\ \frac{\mathrm{d}y_2}{\mathrm{d}x} \\ \frac{\mathrm{d}y_3}{\mathrm{d}x} \end{pmatrix}$.]

Hence, or otherwise, find the general solution of the functions y_1, y_2, y_3 in terms of x. [3]

9. Amy and Sheldon have a fishing pond and they have different models on the growth of the fish population. Let P denote the population in thousands at t years.

(a) Amy suggests the differential equation $\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1 - \frac{P}{a}\right) - H$, where a, k, H are positive constants.

(i) What is the condition of H if there exists two equilibrium population values? Leave your answer in terms of a and k. [3]

(ii) Given that $H = \frac{ak}{4}$ and the initial population $P_0 = a$, describe the behaviour of P. [2]

(b) Sheldon suggests that the population can be modelled by

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1 - \frac{P}{a}\right)\left(1 - \frac{P}{b}\right)$$

where a and k are the same constants that Amy uses, b is a positive constant and $b < a$.

(i) Sketch a graph of P against time t, for various initial population P_0. [3]

(ii) Hence, find the set of values of P_0 such that the population stabilizes in the long run. [1]

(iii) State the significance of the constants a and b. [1]

(iv) Given that $a = 60,\ b = 20,\ k = \frac{8}{5}$, use one iteration of the Euler method to estimate the value of P_0, for which the population halves after one year. Justify if the population will stabilize in the long run. [5]

End of Paper