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Mathematics Department

PRELIM SOLUTION

Subject : JC2 H2 MATHEMATICS 9758 P2 **Date** :

Qn	Solution
1(i)	$\sum_{r=1}^{n} a_r = \sum_{r=1}^{n}\left(T_r - T_{r-1}\right)$ $= \cancel{T_1} - T_0$ $+ \cancel{T_2} - \cancel{T_1}$ $+ \cancel{T_3} - \cancel{T_2}$ $\vdots$ $+ \cancel{T_{n-1}} - \cancel{T_{n-2}}$ $+ T_n - \cancel{T_{n-1}}$ $= T_n - T_0$ $= T_n$
(ii)	Let $T_r = r^2\pi^{-r}$ Note $T_0 = 0$ $T_r - T_{r-1} = r^2\pi^{-r} - (r-1)^2\pi^{-r+1}$ $= \pi^{-r}\left[r^2 - (r^2 - 2r + 1)\pi\right]$ $= \pi^{-r}\left[(1-\pi)r^2 + 2\pi r - \pi\right]$ $= a_r$ $\therefore$ From (i), $\sum_{r=1}^{n}\pi^{-r}\left[(1-\pi)r^2 + 2\pi r - \pi\right] = \sum_{r=1}^{n} a_r$ $= T_n = n^2\pi^{-n}$
(iii)	$\sum_{r=4}^{20}\pi^{-r}\left[(1-\pi)r^2 + 2\pi r - \pi\right]$ $= \sum_{r=1}^{20}\pi^{-r}\left[(1-\pi)r^2 + 2\pi r - \pi\right] - \sum_{r=1}^{3}\pi^{-r}\left[(1-\pi)r^2 + 2\pi r - \pi\right]$ $= 400\pi^{-20} - 9\pi^{-3}$
2(i)	Largest $\alpha = -3$ Let $y = \mathrm{g}(x) = x^2 + 6x + 8$ $= (x+3)^2 - 1$ $(x+3)^2 = y + 1$ $x + 3 = \pm\sqrt{y+1}$

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	$x=-3\ \pm\sqrt{y+1}$ Since $x\le -3$, $x=-3\ -\sqrt{y+1}$ $g^{-1}: x\ \mapsto\ -3\ -\sqrt{x+1},\ \ x\in[-1,\infty)$ A reflection about the line $y=x$ will transform the curve $y=g(x)$ onto the curve $y=g^{-1}(x)$.
(ii)	Since $R_g=[-1,\infty)\subseteq(-2,\infty)=D_h$, the composite function hg exists. $R_{hg}=\left(-\infty,-\frac{1}{e}\right]$
3	By Conjugate Root Theorem, $z=1-i$ is also a root. $\left[z-(1+i)\right]\left[z-(1-i)\right]=\left[(z-1)-i\right]\left[(z-1)+i\right]$ $=(z-1)^2-i^2$ $=z^2-2z+2$ $\left(z^2-2z+2\right)\left(Az^2+Bz+C\right)=2z^4+az^3+7z^2+bz+2$ By observation, $A=2$, $C=1$. i.e. $\left(z^2-2z+2\right)\left(2z^2+Bz+1\right)=2z^4+az^3+7z^2+bz+2$ Coeff. of z^2 : $1-2B+4=7\Rightarrow B=-1$ Coeff. of z^3 : $B-4=a\Rightarrow a=-5$ Coeff. of z : $-2+2B=b\Rightarrow b=-4$ $2z^2-z+1=0$ $z=\frac{1\pm\sqrt{1-4(2)}}{2(2)}$ $z=\frac{1\pm\sqrt{7}i}{4}$ Hence other roots are $1-i$, $\frac{1\pm\sqrt{7}i}{4}$. $2z^4+bz^3+7z^2+az+2=0$ $2+b\frac{1}{z}+7\frac{1}{z^2}+a\frac{1}{z^3}+2\frac{1}{z^4}=0$ Hence $z=\frac{1}{1+i},\qquad \frac{1}{1-i},\quad \frac{4}{1+\sqrt{7}i},\qquad \frac{4}{1-\sqrt{7}i}$ $z=\frac{1-i}{2},\qquad \frac{1+i}{2},\quad \frac{1-\sqrt{7}i}{2},\qquad \frac{1+\sqrt{7}i}{2}$

Qn	Solution
4(i)	$\frac{\mathrm{d}x}{\mathrm{d}t}=2t, \quad \frac{\mathrm{d}y}{\mathrm{d}t}=1-3t^2$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}t}\div\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{1-3t^2}{2t}$ At P, $x=p^2, \quad y=p-p^3, \quad \frac{\mathrm{d}y}{\mathrm{d}t}=\frac{1-3p^2}{2p}$ Equation of tangent at P, $\frac{y-(p-p^3)}{x-p^2}=\frac{1-3p^2}{2p}$ $\Rightarrow 2py-2p(p-p^3)=\left(x-p^2\right)\left(1-3p^2\right)$ $\Rightarrow 2py-2p^2+2p^4=x\left(1-3p^2\right)-p^2+3p^4$ $\Rightarrow 2py=x\left(1-3p^2\right)+p^2+p^4$ (shown) -------(1)
(ii)	At A, substitute $x=6, y=5$ into eqn (1) $2p(5)=6\left(1-3p^2\right)+p^2+p^4$ $10p=6-18p^2+p^2+p^4$ $p^4-17p^2-10p+6=0$ From GC, $p=4.35$ (rejected) or $p=-3.7261$ or $p=-1$ or $p=0.370$ (rejected) Hence coordinates of P: (1,0) and (13.9, 48.0)
(iii)	Required area = $-\int_0^1 y \ \mathrm{d}x$ $=-\int_0^{-1}\left(t-t^3\right)\left(2t\right)\ \mathrm{d}t$ $=0.267$ unit^2
5(i)	If p_1 and p_2 meet at l, then $\mathbf{m}$ is perpendicular to $\mathbf{n}_2$. $\mathbf{m}\cdot\mathbf{n}_2=0\Rightarrow\begin{pmatrix}-4\\1\\1\end{pmatrix}\cdot\begin{pmatrix}m\\n\\2\end{pmatrix}=0$ $-4m+n=-2$ Since $\left(2,-0.5,0\right)$ lies on p_2, $2m-0.5n=1$ $m=0$ $n=-2$
(ii)	Let θ be the acute angle between p_1 and p_2. $\cos\theta=\frac{\left\|\begin{pmatrix}1\\-4\\8\end{pmatrix}\cdot\begin{pmatrix}1\\2\\2\end{pmatrix}\right\|}{\sqrt{1+16+64}\sqrt{1+4+4}}$ $=\frac{1}{3}$ Therefore $\theta=70.5°$

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(iii)	Let $B \equiv (1, b, 5)$. Observe $A_1(4, 0, 0)$ lies on p_1 $\overrightarrow{A_1B} = \begin{pmatrix} 1 \\ b \\ 5 \end{pmatrix} - \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ b \\ 5 \end{pmatrix}$ Shortest distance of B from p_1 $= \dfrac{\left\| \begin{pmatrix} -3 \\ b \\ 5 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -4 \\ 8 \end{pmatrix} \right\|}{\sqrt{1+16+64}} = \dfrac{\|37-4b\|}{9}$ Observe $A_2(1, 0, 0)$ lies on p_2 $\overrightarrow{A_2B} = \begin{pmatrix} 1 \\ b \\ 5 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ b \\ 5 \end{pmatrix}$ Shortest distance of B from p_1 $= \dfrac{\left\| \begin{pmatrix} 0 \\ b \\ 5 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \right\|}{\sqrt{1+4+4}} = \dfrac{\|10+2b\|}{3}$ $\dfrac{37-4b}{9} = \dfrac{2b+10}{3}$ or $\dfrac{37-4b}{9} = -\dfrac{2b+10}{3}$ $b = \dfrac{7}{10}$ $\quad$ $b = -\dfrac{67}{2}$
6(a)(i)	No. of ways = $\dfrac{4!}{2!} = 12$
(ii)	No. of ways = $\dfrac{3! \times {}^4C_3}{2!} = 12$
(iii)	Case 1: Ending with "T" No. of ways $= \dfrac{5!}{3!} = 20$ Case 2: Ending with "R" No. of ways $= \dfrac{5!}{3!2!} = 10$ Total no. of ways $= 20 + 10 = 30$
(b)	Choose the two sizes that have the same colour: ${}^3C_2 = 3$ Choose colour that is same for two sizes: ${}^4C_1 = 4$ Choose colour of remaining size: ${}^3C_1 = 3$ No. of ways = ${}^3C_2 \times {}^4C_1 \times {}^3C_1 = 36$

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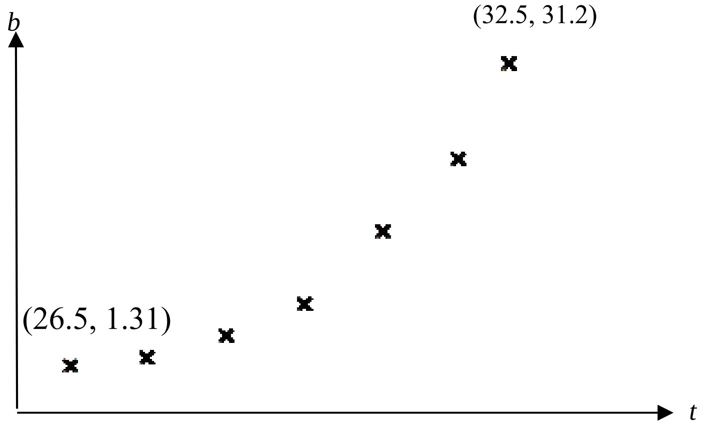
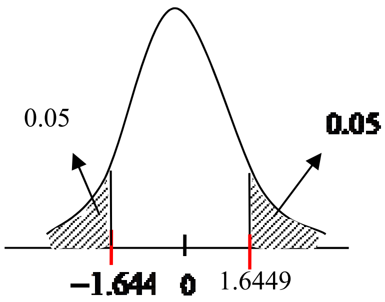
7(i)	P(score of 4) $=$P(obtain 1 and 3 for the first 2 cards, and obtain 2 for the third card) $=\left(\frac{1}{4}\times\frac{1}{3}\times 2\right)\times\frac{1}{2}=\frac{1}{12}$
(ii)	(see table below) Expected score $=1\left(\frac{1}{12}\right)+3\left(\frac{1}{6}\right)+4\left(\frac{1}{12}\right)+5\left(\frac{4}{12}\right)+7\left(\frac{1}{6}\right)+9\left(\frac{1}{12}\right)+16\left(\frac{1}{12}\right)$ $=\frac{35}{6}$
(iii)	P(score < 5 | draws three cards) $=\frac{\text{P(score <5 and draws three cards)}}{\text{P(draws three cards)}}$ $=\frac{\text{P(score 4 and 3 cards)+P(score 1 and 3 cards)}}{\text{P(obtain 1,3 or 2,4 for first two cards)}}$ $=\frac{\frac{1}{12}+\frac{1}{12}}{\left(\frac{1}{4}\times\frac{1}{3}\times 2\right)\times 2}$ $=\frac{1}{2}$
8(i)	Let X be the random variable 'number of rocks that contain fossils out of 25 rocks' $X\sim \mathrm{B}(25, 0.1)$ $\mathrm{P}(4<X\le 10) = \mathrm{P}(X\le 10)-\mathrm{P}(X\le 4)$ ≈ 0.0979819403 ≈ 0.0980 (3 sig fig)
(ii)	$\frac{\mathrm{P}(X=k+1)}{\mathrm{P}(X=k)}$ $=\frac{{}^{25}C_{k+1}(0.1)^{k+1}(0.9)^{25-k-1}}{{}^{25}C_{k}(0.1)^{k}(0.9)^{25-k}}$ $=\frac{\frac{25!}{(k+1)!(25-k-1)!}(0.1)^{k+1}(0.9)^{25-k-1}}{\frac{25!}{k!(25-k)!}(0.1)^{k}(0.9)^{25-k}}$ $=\frac{(25-k)(0.1)}{(k+1)(0.9)}$ $=\frac{25-k}{9(k+1)}$ for $k = 0, 1, 2 \ldots, 24$ (shown)

Table for 7(ii):

s	1	3	4	5	7	9	16
$\mathrm{P}(S=s)$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{4}{12}$	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{12}$

Qn	Solution
	$P(X=k+1)>P(X=k)$ $\dfrac{P(X=k+1)}{P(X=k)}=\dfrac{(25-k)}{9(k+1)}>1$ $25-k>9k+9$ $10k<16$ $k<1.6$ $\Rightarrow\ k=0$ or 1 for $P(X=k+1)>P(X=k)$ Since $P(X=2)>P(X=1)>P(X=0)$, most probable value of $X=2$
	Let Y be the 'number of rocks that contain fossils out of 20 rocks in the new area' $Y\sim B(20,\ \dfrac{p}{100})$ $P(Y=2)=0.17$ Using g.c. $\dfrac{p}{100}=0.045473$ or $\dfrac{p}{100}=0.1815827$ Since $p>10$, $p=18.16$ (2 d.p)
9(i)	$b=-129.368+4.75214t$ From GC, $\bar{t}=29.5$ $\bar{b}=-129.368+4.75214\bar{t}$ $\bar{b}=-129.368+4.75214(29.5)$ $=10.82013$ $\dfrac{1.31+2.1+3.65+5.8+\alpha+19.56+31.2}{7}=10.82013$ $\alpha=12.121=12.12$ (2 dp)
(ii)	
(iii)	From (ii), the scatter diagram shows that as t increases, b increases at an increasing rate which would not be the case if the data follows a linear model. Hence the model $b=kt^3+l$ is a better model.

Qn	Solution
	$b=-37.370+0.0018516t^3$ $=-37.4+0.00185t^3\ \ (3\text{s.f.})$
(iv)	When $t=33$, $b=-37.370+0.0018516(33)^3$ $=29.171$ $=29.2(3\text{s.f.})$ The population of the bacteria is 29.2 millions. Since the estimate is obtained via extrapolation, the estimate is not reliable.
(v)	$b=-37.370+0.0018516\left(\frac{T-32}{1.8}\right)^3$ $=-37.370+\left(3.1749\times10^{-4}\right)(T-32)^3$ $=-37.4+\left(3.17\times10^{-4}\right)(T-32)^3\ \ (3\text{ s.f.})$
10(i)	Since sample size is large, by Central Limit Theorem, the sample mean time for 60 smartphones is approximately normal. Hence the assumption that the time taken by a machine to assembly a smartphone is not necessary.
(ii)	Unbiased estimate for population mean μ is $\bar{x}$ $=\frac{3129}{60}=52.15$ Unbiased estimate for population variance σ^2 is s^2 $=\frac{60}{59}(18.35)$ $=18.661$ $=18.7$ (3sf) $\mathrm{H_0}:\mu=53$ $\mathrm{H_1}:\mu<53$ Under $\mathrm{H_0}$, the test statistic $Z=\frac{\bar{X}-\mu}{S/\sqrt{n}}\sim \mathrm{N}(0,1)$ approx. by CLT, where $\mu=53, s=\sqrt{18.661}, \bar{x}=52.15,\ \ n=60$. By GC, *p*-value $=0.0637(3\text{ s.f.})$. Since *p*-value < 0.1, we reject $\mathrm{H_0}$ and conclude at 10% level that there is sufficient evidence that average time taken by a machine to assembly a smartphone has reduced.
(iii)	There is a probability of 0.1 of concluding that the average time taken by a machine to assembly a smartphone has decreased when the average time taken by a machine to assembly a smartphone is 53 minutes.

Qn	Solution
(iv)	$H_0 : \mu = 45$ $H_1 : \mu \neq 45$ Under H_0, the test statistic $Z = \dfrac{\overline{X} - \mu}{\sigma / \sqrt{n}} \sim N(0,1)$ approx. by CLT, where $\mu = 45, \sigma = \sqrt{9},\ n = 50$.  Since H_0 is rejected, $\dfrac{\overline{x} - 45}{\sqrt{9} / \sqrt{50}} < -1.6449$ or $\dfrac{\overline{x} - 45}{\sqrt{9} / \sqrt{50}} > 1.6449$ $\overline{x} < 44.3021$ $\quad$ $\overline{x} > 45.698$ $\overline{x} < 44.3(3 \text{ s.f.})$ $\quad$ $\overline{x} > 45.7(3 \text{ s.f.})$ Range of values of $\overline{x}$: $\overline{x} < 44.3(3 \text{ s.f.})$ or $\overline{x} > 45.7(3 \text{ s.f.})$
11(i)	Let X be the random variable ‘amount (in kg) of impact modifier in a batch of resin’ $X \sim N(\mu, \sigma^2)$ $P(X < 1350) = 0.03$ $P(Z < \dfrac{1350 - \mu}{\sigma}) = 0.03$ $\dfrac{1350 - \mu}{\sigma} = -1.88079361$ $\mu - 1.88079361\sigma = 1350 --(1)$ $P(X > 1414) = 0.3$ $P(Z < \dfrac{1414 - \mu}{\sigma}) = 0.7$ $\dfrac{1414 - \mu}{\sigma} = 0.5244005101$ $\mu + 0.5244005101\sigma = 1414 ---(2)$ Solve (1) and (2), $\mu = 1400.046 = 1400$ (shown) σ = 26.609 = 26.6 (shown)

Qn	Solution
(ii)	Let Y be the random variable 'amount (in kg) of Polymer A in a batch of resin' Let W be the random variable 'amount (in kg) of Polymer B in a batch of resin' $Y \sim N(2030, 44.8^2)$, $W \sim N(1563, 22.7^2)$ Total cost of a batch, $T = 2.20Y + 2.80W + 1.50X \sim N(10942.4, 15345.9572)$ Total cost of 2 batches, $T_1 + T_2 \sim N(21884.8, 30715.9144)$ $P(T_1 + T_2 > 22000) = 0.255$ (3.s.f.)
(iii)	Let H be the r.v.' number of batches of resin with more than 1414 kg of impact modifier out of n batches.' $H \sim B(n, 0.3)$ $\mathrm{P}(H \leq 6) < 0.001$ Using GC. When n = 53, $\mathrm{P}(H \leq 6) = 0.00120 > 0.001$ When n = 54, $\mathrm{P}(H \leq 6) = 9.44 \times 10^{-4} < 0.001$ Therefore, least n = 54
(iv)	Let S be the r.v.' tensile strength (in N/m^2) of resin in a car seat' E(S) = 125, Var(S) =17^2 $\overline{S} = \dfrac{S_1 + S_2 + ... + S_{50}}{50}$ $\overline{S} \sim N\left(125, \dfrac{17^2}{50}\right)$ approx by Central Limit Thm $\mathrm{P}(\overline{S} < 130) = 0.981$ (3 s.f.)