ST ANDREW'S JUNIOR COLLEGE

PRELIMINARY EXAMINATION

FURTHER MATHEMATICS
Higher 2 **9649/02**

Friday **15 September 2017** **3 hours**

READ THESE INSTRUCTIONS FIRST

Write your name, civics group and index number on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.

Answer **all** the questions. Total marks is **100**.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphic calculator.
Unsupported answers from a graphic calculator are allowed unless a question specifically state otherwise.
Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematic steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.
At the end of the examination, fasten all your work securely together.

This document consists of **6** printed pages including this page.

[Turn Over

Section A: Pure Mathematics [50 marks]

1 The sequence $\{u_n\}$ of positive integers is defined by $u_{n+2} = u_{n+1} + 2u_n$, and $u_1 = u_2 = 1$.

(i) Show that for any $n \in \mathbb{Z}^+$,

$$u_{4n+4} = 5u_{4n+1} + 6u_{4n} \ .$$ **[2]**

(ii) Prove by mathematical induction that u_{4n} is divisible by 5 for all $n \in \mathbb{Z}^+$. **[5]**

2 Let *P* be the set of all points in $\mathbb{R}^2$ with coordinates (x, y), where $x \in \mathbb{R}$ and $y \in \mathbb{R} \setminus \{0\}$. A special operation called *transcaling addition*, represented by the symbol $\oplus$, is defined on *P*, such that for any two elements (x_1, y_1) and (x_2, y_2) of *P*,

$$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2, y_1 \times y_2)$$

(i) Show that *P* satisfies all the addition axioms of a vector space. **[5]**

(ii) Explain why *P* would not be a vector space under the usual scalar multiplication $k(x, y) = (kx, ky)$, where $k \in \mathbb{R}$. **[1]**

(iii) Suggest another scalar multiplication $\otimes$, such that *P* is a vector space under $\oplus$ and $\otimes$. (You do not have to prove the multiplication axioms are satisfied.) **[1]**

3 **(i)** Sketch, on the same diagram, the graphs of $y = x^2$ and $y = 1 - \cos^{-1}\sqrt{x+1}$ for $-1 \le x \le 0$, labelling the coordinates of the end points clearly. **[2]**

A student was challenged by his friend to solve the equation

$$x^2 = 1 - \cos^{-1}\sqrt{x+1} \qquad (*)$$

without using his graphing calculator. He decides he will use a programme that is capable of executing a recurrence relation of the form $x_n = \mathrm{f}(x_{n-1})$.

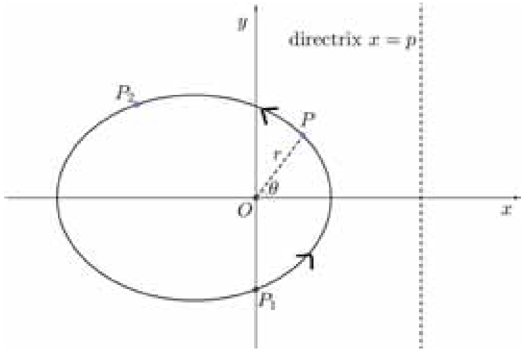
(ii) **(a)** He first tries the obvious step of using $\mathrm{f}(x) = \sqrt{1 - \cos^{-1}\sqrt{x+1}}$. Explain why this will not be able to give the root of (*). **[1]**

(b) State the corrected function $\mathrm{f}(x)$ and calculate the programme's output for x_1 using $x_0 = 0$. Explain why this corrected function will still fail to solve (*). **[2]**

(c) **(i)** Find another function $\mathrm{f}(x)$ that might be used to carry out the iterative procedure. **[2]**

(ii) Calculate the programme's output for x_1, x_2, x_3 and x_4 using the initial values $x_0 = 0$ and $x_0 = -0.5$. Analyse how the choice of x_0 affects the programme's ability to solve (*). **[3]**

4 **(a)** The diagram shows the elliptical orbit of a planet P around its sun, O, which is at a focus of the ellipse. Taking O as the pole, the eccentricity of the ellipse is e, where $0<e<1$, and a directrix of the ellipse is $x=p$, where p is a positive constant.



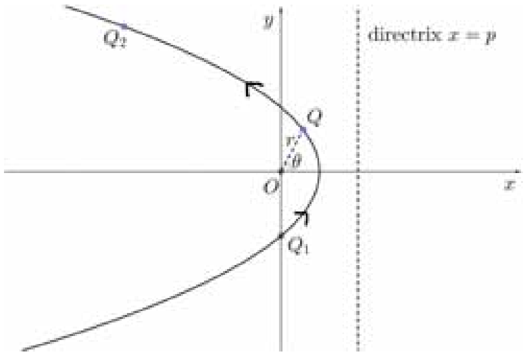
(i) By using the directrix property of an ellipse, show that the polar equation of the orbit of P is $r=\dfrac{ep}{1+e\cos\theta}$, where θ is in radians. **[3]**

It is now given that $e=\dfrac{1}{4}$, and the planet moves in an anti-clockwise direction.

(ii) Use your graphing calculator to evaluate the area swept out by OP as P moves from the point P_1, where $\theta=-\dfrac{\pi}{2}$ to the point P_2, where $\theta=\dfrac{2\pi}{3}$. **[2]**

(iii) **Kepler's Second Law of Planetary Motion** states that the line segment OP sweeps out equal areas in equal times. If P takes time T to complete the entire orbit, find, in terms of T, the time P takes to travel from P_1 to P_2. **[2]**

(b) The diagram shows the parabolic path of an asteroid Q around the sun, O, which is at the focus of the parabola. The directrix of the parabola is $x=p$, and the polar equation of the orbit of Q is $r=\dfrac{p}{1+\cos\theta}$, where θ is in radians.



(i) By using the substitution $u=\tan\dfrac{\theta}{2}$, show that

$$\int\frac{1}{(1+\cos\theta)^2}\,\mathrm{d}\theta=\frac{1}{2}\left(\tan\frac{\theta}{2}+\frac{1}{3}\tan^3\frac{\theta}{2}\right)+C\ . \qquad \textbf{[5]}$$

(ii) Hence, find the exact area swept out by OQ as Q moves from the point Q_1, where $\theta=-\dfrac{\pi}{2}$ to the point Q_2, where $\theta=\dfrac{2\pi}{3}$. **[2]**

5 A manufacturer's pricing policy is defined by the differential equation

$$\frac{dP}{dt} = k\left[I_0 - I(t)\right] \text{ --- (1)}$$

where t is the time in decades (1 decade = 10 years) from the current year, P is the price of goods at time t; k is a positive constant; I_0 and $I(t)$ represent the optimum inventory level and the actual inventory level at time t respectively.

The rate of change of the inventory level can be expressed in the form $R(t) - S(t)$, where $R(t)$ is the manufacturer's production level and $S(t)$ is his anticipated sales.

It is given that $R(t) = 250 - 5P$ while $S(t) = 500 - 40P - 10\frac{dP}{dt}$.

(i) Show, via differentiation, that equation (1) can be reduced to a differential equation of the form $\frac{d^2P}{dt^2} + 10k\frac{dP}{dt} + 35kP = 250k$. **[2]**

(ii) Let $k = 1$. Solve the differential equation in part **(i)** and state, in the context of the question, the significance of this solution as t grows large. **[6]**

(iii) The goods are priced at \$5.50 currently and increase to \$6.50 in 5 years' time. Find the price of the goods in 10 years' time. **[4]**

Section B: Probability and Statistics [50 marks]

6 To examine the association between two given factors, Factor A and Factor B, a χ^2-test of independence is performed based on the 2×2 contingency table as shown, which records the frequencies of observations from a random sample based on whether each factor is present or absent. You may assume all entries of the table are non-negative integers.

Observed data	Factor A present	Factor A absent
Factor B present	$a + x$	$a - x$
Factor B absent	$b - x$	$b + x$

(i) Show that the expected frequency of observations where both Factors A and B are present is a, explaining clearly why the formula you use is true. **[2]**

(ii) Find the test statistic in terms of a, b and x, and hence write down an inequality if the conclusion of the test at the 5% significance level is that the factors are not independent of each other. State any required assumptions on the values of a and b for the test to be successfully carried out. **[5]**

7 A call center receives calls randomly and at a constant average rate of 3 per minute.

(i) By using an appropriate Poisson distribution, find, in terms of t, the probability that the call center receives at least 1 call in a period of t minutes. **[2]**

(ii) Let T be the time, in minutes, that passes between two consecutive received calls. By considering $\mathrm{P}(T<t)$ and finding the probability density function of T, show that T follows an exponential distribution with parameter 3. **[2]**

(iii) Find the exact value of the interquartile range of T, simplifying your answer. **[3]**

8 To track the improvement of the mathematical ability of students in a school, it is required to test whether students in a school have performed better, on average, for their Block Test 2 (BT2) as compared to their Block Test 1 (BT1). Principal Ella selects 12 students at random and their results for both assessments are listed below:

Student	A	B	C	D	E	F	G	H	I	J	K	L
BT1	60.0	34.5	45.5	41.5	45.5	51.5	38.5	56.0	36.5	48.0	77.5	73.0
BT2	57.0	25.5	54.0	52.5	47.5	55.5	62.5	76.5	42.0	57.5	77.5	71.5

(i) Explain why it is generally better to use a Wilcoxon matched pair signed rank test than a sign test. **[1]**

(ii) Perform a Wilcoxon matched pair signed rank test at 5% level of significance, clearly stating your hypotheses. **[5]**

(iii) It is now known that the results of the 12 students were taken from normal distributions. State a more appropriate test to perform (you do not need to perform the test). **[1]**

9 Farmer Xin raises a batch of chickens with Feed A and a batch of chickens with Feed B. It is known that the masses of each batch of chickens are normally distributed with the same variance.

Xin takes a random sample of 10 chickens from the batch of chickens raised with Feed A and measures their masses, x kg, where $\Sigma x = 21.4$ and $\Sigma x^2 = 46.3$.

(i) Find a 95% confidence interval for the mean mass of the batch of chickens raised with Feed A, showing your working. **[3]**

(ii) Give a possible reason why the interval you have found may not contain the population mean. **[1]**

(iii) Xin performs a test at a 5% significance level on whether the mean mass of the batch of chickens raised with Feed A differs from 2.35kg. State, in context, the conclusion of this test and explain how you derived this conclusion from the answer in part **(i)**. **[2]**

Xin also takes a random sample of 15 chickens from the batch of chickens raised with Feed B and measures their masses, y kg, where $\Sigma y = 34.8$, $\Sigma y^2 = 81.72$. He wishes to test his belief, at 10% significance level, that the mean mass of the batch of chickens raised with Feed B exceeds the mean mass of the batch of chickens raised with Feed A by more than k kg, where k is a positive constant.

(iv) State appropriate hypotheses for the tests, defining any symbols that you use. **[2]**

(v) Obtain a pooled estimate of the common population variance, leaving your answer correct to five significant figures. **[2]**

(vi) Find the range of values of k such that there is sufficient evidence to support Xin's belief. **[4]**

10 It is known that $\sum_{r=1}^{\infty} rx^{r-1} = \dfrac{1}{(1-x)^2}$ and $\sum_{r=1}^{\infty} r(r-1)x^{r-2} = \dfrac{2}{(1-x)^3}$.

The random variable X follows a geometric distribution with parameter p.

(i) Show that $\mathrm{E}(X) = \dfrac{1}{p}$. **[2]**

(ii) By considering $\mathrm{E}(X(X-1))$, show that $\mathrm{Var}(X) = \dfrac{q}{p^2}$, where $q = 1-p$. **[4]**

Professor Vince conducts a study on the number of attempts a group of 100 students take to solve a mathematical puzzle. The results are shown in the table.

Number of attempts	1	2	3	4	5	6	7 or more
Observed frequency	49	19	6	3	5	18	0

(iii) Test, at the 1% level of significance, the goodness of fit of a geometric distribution with parameter $p = 0.4$ to the given data, showing your working. **[6]**

(iv) Evaluate the mean and variance of the given data. By comparing these values with the results of parts (i) and (ii) when $p = 0.4$, what can Professor Vince conclude with reference to the results of the test in part (iii)? **[3]**

End of Paper