Section A: Pure Mathematics [50 marks]

1 Let w_n denote the number of possible codewords of length n using the letters from the set $\{A, J, C\}$ in which the letter A and the letter J are not adjacent.

(i) Find w_1 and w_2. [1]

(ii) Show that the number w_n satisfy the recurrence relation

$$w_n = 2w_{n-1} + w_{n-2},\ n = 3, 4, 5, \ldots$$ [4]

(iii) Hence find the explicit formula of w_n in exact form. [4]

2 On the Argand diagram, the complex number a representing the point A is such that $\arg(a) = \frac{\pi}{4}$ and $|a| = \sqrt{2}$. The complex number z satisfies both $0 \le \arg(z-a) \le \frac{\pi}{4}$ and $|z - a - 2\mathrm{i}| = 2$. Sketch, on the Argand diagram, the locus of points L representing the complex number z. [3]

Given that $\arg(z)$ attains a minimum value, determine the exact expression of z, giving your answers in the form $\left(\frac{3+p\sqrt{6}}{q}\right) + \mathrm{i}\left(\frac{r+s\sqrt{6}}{5}\right)$ where p, q, r, and s are integers to be determined. [5]

3 Farmer Tay is using high-technology farming technique to grow abalone. The population, A (in hundreds), of abalone in Farmer Tay's farm satisfies the differential equation

$$\frac{\mathrm{d}A}{\mathrm{d}t} = rA\left(1 - \frac{A}{k}\right) - H(A).$$

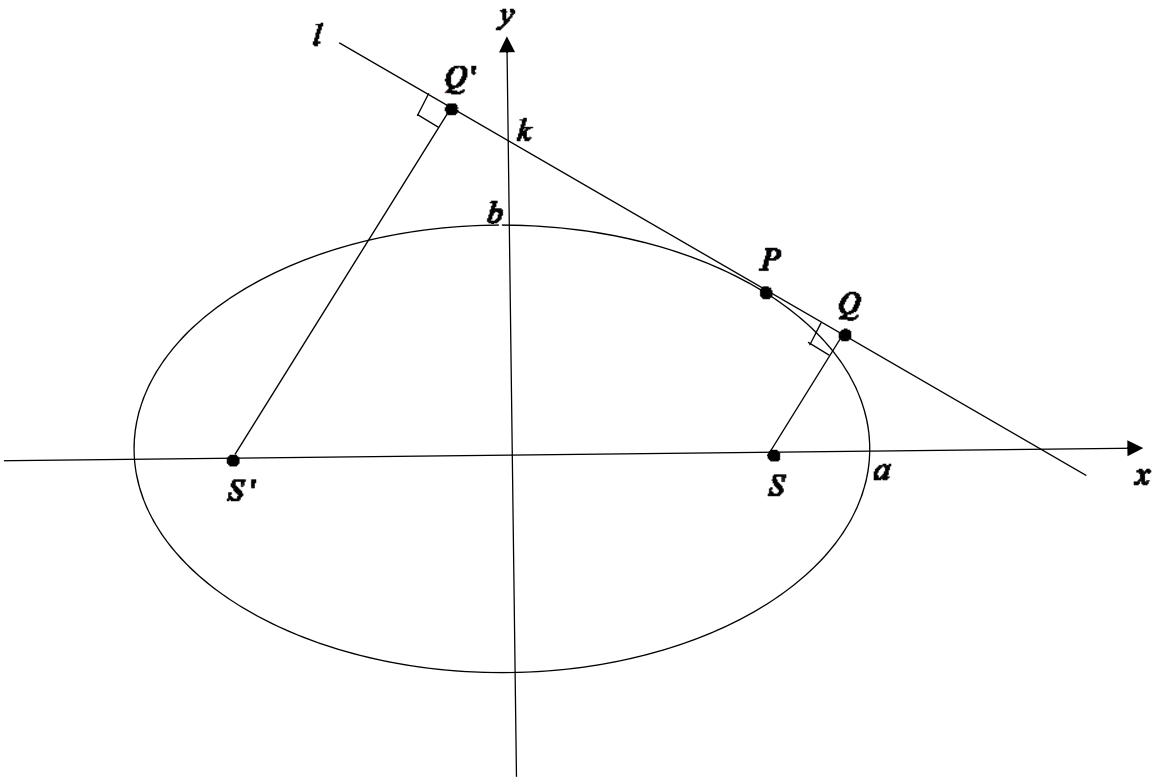
where t represents the time in years. The positive constants r and k respectively denote the intrinsic growth rate and the carrying capacity of the population without harvesting. The harvesting function is given by $H(A)$.

The initial abalone population is at near its carrying capacity. Suppose Farmer Tay decides to harvest abalone at a constant rate such that $H(A) = h$, find the maximum value of h in order to ensure the abalone population does not go extinct in the long run. [3]

Suppose instead Farmer Tay decides that to harvest the abalone at a rate proportional to population such that $H(A) = qA$, where q is a positive constant. Find the set of values of q for which the farming technique is sustainable in the long run, giving your answers in terms of r. [3]

[Turn over

4 The diagram shows the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$ where $a>b>0$. Let e be the eccentricity of the ellipse.



The line l is the tangent to the ellipse at the point P. The line l has equation $y=mx+k$ where m is the slope and k is the y-intercept. The points S and S' are the focal points of the ellipse, where S is on the positive x-axis. The perpendiculars to l through S and S' intersect l at Q and Q' respectively.

(i) Show that $a^2m^2+b^2=k^2$. [3]

(ii) Show that the perpendicular distance from S to l is given by $QS=\frac{|mae+k|}{\sqrt{1+m^2}}$. [2]

(iii) Given that $Q'S'=\frac{|mae-k|}{\sqrt{1+m^2}}$, prove that $QS\times Q'S'=b^2$. [3]

The lines SQ and $S'P$ meet at the point R.

(iv) Show that $SQ=RQ$. [2]

(v) Explain why $S'R=2a$. [1]

(vi) Prove that Q lies on the auxiliary circle given by the equation $x^2+y^2=a^2$. [2]

It is given that Q' also lies on the auxiliary circle. Without finding the length of QS and of $Q'S'$ explicitly, prove that $QS\times Q'S'=b^2$. [4]

[Turn over

5 Let P_2 be the set of polynomials of degree two represented by the form $a_0 + a_1x + a_2x^2$, where a_0, a_1 and a_2 are real numbers.

The transformation $T : P_2 \to P_2$ is such that

$$T\left(a_0 + a_1x + a_2x^2\right) = \left(a_0 + 3a_1 + 2a_2\right) + \left(a_0 - a_1 - a_2\right)x + \left(2a_0 + 2a_1 + \lambda a_2\right)x^2.$$

Show that *T* is a linear transformation for all real values of λ.

The range space and null space of *T* is denoted by *R* and *N* respectively. Find the rank of *R*, distinguishing the cases $\lambda = 1$ and $\lambda \neq 1$.

For the case where $\lambda = 1$, find a basis for *R* and for *N*.

Determine whether $R \cup N$ is a vector space, justifying your answer. [10]

Section B: Probability and Statistics [50 marks]

6 Seven darts players believe that their scores improve after drinking one pint of beer. To test the hypothesis, the seven players each throws three darts before and after drinking one pint of beer. Their total individual scores are recorded as shown in the table.

Player	A	B	C	D	E	F	G
Score before drink	101	85	140	100	100	65	85
Score after one pint of beer	141	81	180	125	101	60	100

(i) Carry out a Wilcoxon matched-pairs signed rank test at 5% level of significance to determine whether there is any evidence that the median score has improved after drinking one pint of beer. [5]

(ii) Explain why a sign test is likely to produce a misleading result with this particular data set. [1]

It is decided to conduct the same trial with a bigger sample size of 40 players using the Wilcoxon matched-pairs signed rank test. Determine the range of values of *T* (as defined in MF26) that will lead to the acceptance of the null hypothesis. [3]

[Turn over

7 A gardener is planning to grow a certain variety of flower from seed. The seeds are sold in packets of 20. It is not possible to know the colour of the flower until it blooms. The label on the packets carries this message:

Flower blooms in the ratio red : blue : white 6 : 3 : 1

(i) Explain why the gardener could not carry out a χ^2 test using only one packet of seeds. [1]

(ii) Find the least number of packets he must use in order to apply a χ^2 test. [1]

The gardener buys five packets of seeds and sow them. After some time, the flowers bloom and he observes 51 red, 38 blue and 11 white flowers. Carry out a χ^2 test at the 5% level of significance to determine whether the observed colours are consistent with the message on the packet. [4]

8 The number of vehicles, X passing a given point on Ang Mo Kio Avenue 6 in any one minute may be modelled by a Poisson distribution. The unknown parameter λ of this distribution may be estimated by keeping count of the number of vehicles that pass in several one-minute intervals. In a traffic survey, 60 such observations were made and their mean was found to be 20.1 vehicles per minute.

(i) Show that

$$P\left(-1.96\sqrt{\frac{\lambda}{60}} < \bar{X} - \lambda < 1.96\sqrt{\frac{\lambda}{60}}\right) = 0.95$$ [3]

(ii) Show that the approximate 95% confidence limits for λ are given by the roots of the equation

$$\lambda^2 - 40.264\lambda + 404.01 = 0.$$

Hence, obtain the confidence limits, giving your answers to 2 decimal places. [4]

(iii) The discipline master at a college in the vicinity claims that average number of car passing through in one minute interval is 22. What would be your conclusion for a hypothesis test done at 5% level of significance? [1]

[Turn over

9 Twenty plots of field, each of area $100\,m^2$, were randomly selected. Each plot was treated with one of the two fertilizers *A* and *B*. Each fertilizer was used on 10 randomly selected plots. The yields (in *kg*) of blackcurrants per year, a_i and b_i, due to fertilizers *A* and *B* are shown below respectively.

a	34	48	72	35	52	83	24	42	44	37
b	28	49	66	31	42	75	26	33	42	33

It may be assumed that the yield due to each type of fertiliser is normally distributed.

Test, at the 10% significance level, whether fertiliser *A* produces a greater yield than that by fertiliser *B*. State any other assumptions made in carrying out the test. [5]

10 Let X be a continuous random variable whose probability density function and cumulative distribution function are given by $\mathrm{f}(x)$ and $\mathrm{F}(x)$ respectively. The random variable *Y* is defined such that

$$Y = \begin{cases} X & \text{if} \quad X \geq c \\ 0 & \text{if} \quad X < c \end{cases}$$

The probability density function and cumulative distribution function of *Y* are given by $\mathrm{g}(y)$ and $\mathrm{G}(y)$ respectively. By considering $\mathrm{G}(y)$ or otherwise, show that

$$\mathrm{g}(y) = \begin{cases} \dfrac{\mathrm{f}(y)}{1-\mathrm{F}(c)} & , \quad y \geq c \\ 0 & , \quad y < c \end{cases} \qquad [4]$$

Jersina makes exactly one telephone call to her hometown in China per day. If the call is less than one minute, no charge will be levied. Otherwise, she will be charged for the time exceeding one minute at a rate of 30 cents per minute, with part thereof being charged proportionally. For example, she will be charged 45 cents if the length of the call is 2.5 minutes.

The duration, *T* of her calls has the probability density function

$$\mathrm{f}(t) = \begin{cases} \dfrac{2}{3} t \mathrm{e}^{-\frac{1}{3}t^2} & , \quad t \geq 0 \\ 0 & , \quad t < 0 \end{cases}$$

(i) Show that probability that she will be charged for a telephone call is $\mathrm{e}^{-\frac{1}{3}}$. [1]

(ii) Find the expected amount that she is being charged per call made. [3]

Jersina allocates a monthly budget of \$18 for her telephone bills. Any unused budget from previous month will not be carried over to the following month. Find the expected number of months that elapse before she first exceeds her budget, assuming that one month consists of exactly 30 days. [4]

[Turn over

11 The number of customers *X*, arriving at a bank in any one-minute interval follows a Poisson distribution with mean λ. In this bank, there exists two different counters, each providing a different banking service – namely, corporate banking and personal banking. There are two separate queues, each leading to each counter.

When a customer arrives at the bank, the probability that he joins the queue for the corporate and the personal banking service are given by p and $1-p$ respectively. In any one-minute interval, let *Y* denote the number of customers joining the queue for the corporate banking service.

(i) Show that $P(Y=r)=\sum_{s=r}^{\infty}\left(\frac{e^{-\lambda}\lambda^{s}}{s!}\binom{s}{r}p^{r}(1-p)^{s-r}\right)$, where $r \geq 0$. [1]

(ii) Hence or otherwise, show that the number of customers joining the queue for corporate banking service in any one-minute interval follows a Poisson distribution with mean λp. [4]

(iii) In a particular one-minute interval, the number of customers joining the queue for corporate banking service is *k*. Find the probability that no customer joins the queue for the personal banking service during this time interval. [3]

At any instance during the bank business hours, let *T* be the time (in minutes) that has since elapsed before a customer arrives and joins the queue for corporate banking services. Show that *T* follows an exponential distribution with parameter λp. [3]

It is given that $\lambda = 0.3,\ p = 0.15$.

There is only one customer service officer at the counter providing corporate banking service. At a particular instance, he has finished serving all his customers and wants to take a toilet break. There is no other customer waiting in this queue. Find the maximal time duration that he can go for his break such that he will be at least 90% confident that there will not be any customer waiting in this queue when he comes back from his break. [2]

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