Jurong Junior College
2017 JC2 H2 Mathematics Preliminary Examination Paper 2 Solutions

1 (i)

$$y = \ln(1+\sin x) \quad \Rightarrow \mathrm{e}^y = 1+\sin x$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\cos x}{1+\sin x}$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{(1+\sin x)(-\sin x)-(\cos x)(\cos x)}{(1+\sin x)^2}$$

$$= \frac{-\sin x-\sin^2 x-\cos^2 x}{(1+\sin x)^2}$$

$$= \frac{-\sin x-1}{(1+\sin x)^2}$$

$$= \frac{-(\sin x+1)}{(1+\sin x)^2}$$

$$= \frac{-1}{1+\sin x}$$

$$= \frac{-1}{\mathrm{e}^y}$$

$$= -\mathrm{e}^{-y} \qquad \text{(Shown)}$$

(ii)

$$\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = -\mathrm{e}^{-y}\left(-\frac{\mathrm{d}y}{\mathrm{d}x}\right)$$

$$= \mathrm{e}^{-y}\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$\frac{\mathrm{d}^4y}{\mathrm{d}x^4} = \mathrm{e}^{-y}\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\mathrm{e}^{-y}\left(-\frac{\mathrm{d}y}{\mathrm{d}x}\right)\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)$$

$$= \mathrm{e}^{-y}\left(-\mathrm{e}^{-y}\right)-\mathrm{e}^{-y}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 \qquad \text{(from } \textbf{(i)}\text{)}$$

$$= \underline{\underline{-\left(\mathrm{e}^{-y}\right)^2-\mathrm{e}^{-y}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}} \text{ or } \underline{\underline{-\mathrm{e}^{-y}\left[\mathrm{e}^{-y}+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right]}}$$

(iii) When $x=0$,

$$y = \ln 1 = 0$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\cos 0}{1+\sin 0} = 1$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\mathrm{e}^0 = -1$$

$$\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = 1$$

$$\frac{\mathrm{d}^4y}{\mathrm{d}x^4} = -1-1 = -2$$

$$\therefore \ln(1+\sin x) = 0+x+\frac{(-1)}{2}x^2+\frac{1}{3!}x^3+\frac{(-2)}{4!}x^4+\ldots$$

$$= \underline{\underline{x-\frac{1}{2}x^2+\frac{1}{6}x^3-\frac{1}{12}x^4+\ldots}}$$

2(i)

$$x = ut\cos\theta, \qquad y = ut\sin\theta - 5t^2$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = u\cos\theta \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = u\sin\theta - 10t$$

$$\begin{aligned}
\therefore \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{u\sin\theta - 10t}{u\cos\theta} \\
&= \tan\theta - \frac{10t}{u\cos\theta} \\
&= \tan\theta - \frac{10}{u}t\sec\theta \quad \text{(Shown)}
\end{aligned}$$

2(ii) When $u = 30$ and $t = \frac{1}{2}$,

$x = 15\cos\theta,\ y = 15\sin\theta - \frac{5}{4},\ \frac{\mathrm{d}y}{\mathrm{d}x} = \tan\theta - \frac{1}{6}\sec\theta$

Equation of tangent is

$$\begin{aligned}
y - 15\sin\theta + \frac{5}{4} &= \left(\tan\theta - \frac{1}{6}\sec\theta\right)(x - 15\cos\theta) \\
&= \left(\tan\theta - \frac{1}{6}\sec\theta\right)x - 15\sin\theta + \frac{5}{2} \\
\therefore y &= \underline{\underline{\left(\tan\theta - \frac{1}{6}\sec\theta\right)x + \frac{5}{4}}}
\end{aligned}$$

3 (i) $A(3, 0, 2)$, $B(1, 0, 3)$, $C(2, -3, 5)$ on H_1

$$\overrightarrow{AB} = \begin{pmatrix}1\\0\\3\end{pmatrix} - \begin{pmatrix}3\\0\\2\end{pmatrix} = \begin{pmatrix}-2\\0\\1\end{pmatrix} \qquad \overrightarrow{AC} = \begin{pmatrix}2\\-3\\5\end{pmatrix} - \begin{pmatrix}3\\0\\2\end{pmatrix} = \begin{pmatrix}-1\\-3\\3\end{pmatrix}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix}-2\\0\\1\end{pmatrix} \times \begin{pmatrix}-1\\-3\\3\end{pmatrix} = \begin{pmatrix}3\\5\\6\end{pmatrix}$$

Take $\mathbf{n}_1 = \begin{pmatrix}3\\5\\6\end{pmatrix}$, $\mathbf{a} \cdot \mathbf{n}_1 = \begin{pmatrix}1\\0\\3\end{pmatrix} \cdot \begin{pmatrix}3\\5\\6\end{pmatrix} = 3+0+18 = 21$

A vector equation of H_1 is $\mathbf{r} \cdot \begin{pmatrix}3\\5\\6\end{pmatrix} = 21$.

3(ii) Equation of H_2 is $2x - y + kz = 14$.

Sub. $A(3, 0, 2)$ into equation of H_2,

$$2(3) - 0 + k(2) = 14$$
$$\therefore k = 4 \quad \text{(Shown)}$$

Sub. $B(1, 0, 3)$ into LHS of equation of H_2,

$$\text{LHS} = 2x - y + 4z = 2(1) - 0 + 4(3) = 14 = \text{RHS}$$

$\therefore$ B is also in H_2.

Since B is in both H_1 and H_2, $\therefore$ B is on the river. (Deduced)

3 (iii) Recall $\overrightarrow{AB} = \begin{pmatrix}-2\\0\\1\end{pmatrix}$, using $A(3, 0, 2)$ or $B(1, 0, 3)$,

a cartesian equation of the river (line AB) is

$$\frac{x-3}{-2} = z-2,\ y=0 \quad \text{or} \quad \frac{x-1}{-2} = z-3,\ y=0$$

3 (iv) Since

$$\overrightarrow{BC} \cdot \overrightarrow{AB} = \begin{pmatrix}1\\-3\\2\end{pmatrix} \cdot \begin{pmatrix}-2\\0\\1\end{pmatrix} = 1(-2) + (-3)(0) + 2(1) = 0,$$

BC is perpendicular to AB.

$\therefore$ B is the point on the river that is nearest to C.

Exact distance from C to the river

$$= \left|\overrightarrow{BC}\right| = \left|\begin{pmatrix}1\\-3\\2\end{pmatrix}\right| = \sqrt{1+9+4} = \sqrt{14}$$

3 (v) Acute angle between BC and H_2

$$\theta = \sin^{-1}\frac{\left|\begin{pmatrix}1\\-3\\2\end{pmatrix} \cdot \begin{pmatrix}2\\-1\\4\end{pmatrix}\right|}{\sqrt{14}\sqrt{21}} = \sin^{-1}\frac{13}{\sqrt{14}\sqrt{21}}$$

$$= 49.3^\circ \text{ or } 0.861 \text{ rad}$$

4(i)

$$\frac{\mathrm{d}A}{\mathrm{d}t} = kA(100-A)$$

(ii)

$$\int \frac{1}{A(100-A)}\,\mathrm{d}A = \int k\,\mathrm{d}t$$

$$\int \left[\frac{1}{100A} + \frac{1}{100(100-A)}\right]\mathrm{d}A = kt + c \qquad \text{(by partial fractions)}$$

$$\frac{1}{100}\int \left[\frac{1}{A} + \frac{1}{(100-A)}\right]\mathrm{d}A = kt + c$$

$$\frac{1}{100}\left(\ln|A| - \ln|100-A|\right) = kt + c$$

$$\frac{1}{100}\left[\ln A - \ln(100-A)\right] = kt + c$$

$(\because A > 0 \text{ and } 100 - A > 0)$

$$\ln\frac{A}{100-A} = 100(kt+c)$$

$$\frac{A}{100-A} = \mathrm{e}^{100(kt+c)} = \mathrm{e}^{100kt}\mathrm{e}^{100c} = D\mathrm{e}^{k_1 t}$$

where $k_1 = 100k$ and $D = \mathrm{e}^{100c}$.

When $t = 0, A = 20$,

$$\frac{20}{100-20} = D$$

$$D = \frac{1}{4}$$

When $t = 5, A = 40$,

$$\frac{40}{100-40} = \frac{1}{4}\mathrm{e}^{5k_1}$$

$$\frac{1}{4}\mathrm{e}^{5k_1} = \frac{2}{3}$$

$$\mathrm{e}^{5k_1} = \frac{8}{3}$$

$$5k_1 = \ln\frac{8}{3}$$

$$k_1 = \frac{1}{5}\ln\frac{8}{3}$$

$$\therefore \frac{A}{100-A} = \frac{1}{4}\mathrm{e}^{\left(\frac{1}{5}\ln\frac{8}{3}\right)t}$$

When $A = 0.5\times 100 = 50$,

$$\begin{aligned}
\frac{50}{100-50} &= \frac{1}{4}e^{\left(\frac{1}{5}\ln\frac{8}{3}\right)t} \\
1 &= \frac{1}{4}e^{\left(\frac{1}{5}\ln\frac{8}{3}\right)t} \\
e^{\left(\frac{1}{5}\ln\frac{8}{3}\right)t} &= 4 \\
\left(\frac{1}{5}\ln\frac{8}{3}\right)t &= \ln 4 \\
t &= \frac{\ln 4}{\frac{1}{5}\ln\frac{8}{3}} = 7.07 \text{ (2 dp)}
\end{aligned}$$

The required time is 7.07 days.

(iii) When $t = 14$ (days),

$$\frac{A}{100-A} = \frac{1}{4}e^{\left(\frac{1}{5}\ln\frac{8}{3}\right)(14)}$$

Method 1 Solve algebraically

$$\begin{aligned}
\frac{A}{100-A} &= 3.8963 \text{ (5 sf)} \\
A &= (100-A)(3.8963) \\
&= 389.63 - 3.8963A \\
4.8963A &= 389.63 \\
A &= 79.58 \text{ (2 dp)}
\end{aligned}$$

For the bread to be deemed safe for human consumption in terms of the amount of preservatives present, $79.58 \le A \le 100$

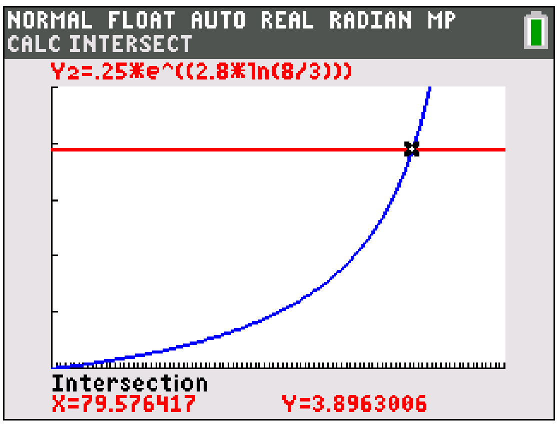
Method 2 Use GC to plot graphs

Use GC to plot $y = \dfrac{A}{100-A}$ and $y = \dfrac{1}{4}e^{\left(\frac{1}{5}\ln\frac{8}{3}\right)(14)}$ (≈ 3.8963)

Look for the point of intersection (adjust window).

$$A = 79.58 \text{ (2 dp)}$$

For the bread to be deemed safe for human consumption in terms of the amount of preservatives present, $79.58 \le A \le 100$



(iv)

$$\frac{A}{100-A} = \frac{1}{4}e^{\left(\frac{1}{5}\ln\frac{8}{3}\right)t}$$

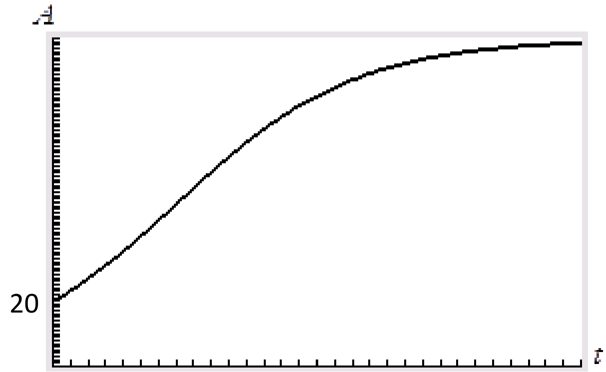
$$A = \frac{1}{4}e^{\left(\frac{1}{5}\ln\frac{8}{3}\right)t}(100-A)$$

$$4A = e^{\left(\frac{1}{5}\ln\frac{8}{3}\right)t}(100-A)$$

$$= 100e^{\left(\frac{1}{5}\ln\frac{8}{3}\right)t} - Ae^{\left(\frac{1}{5}\ln\frac{8}{3}\right)t}$$

$$\left[4+e^{\left(\frac{1}{5}\ln\frac{8}{3}\right)t}\right]A = 100e^{\left(\frac{1}{5}\ln\frac{8}{3}\right)t}$$

$$A = \underline{\underline{\frac{100e^{\left(\frac{1}{5}\ln\frac{8}{3}\right)t}}{4+e^{\left(\frac{1}{5}\ln\frac{8}{3}\right)t}}}} \text{ or } \underline{\underline{\frac{100e^{0.196t}}{4+e^{0.196t}}}} \text{ or } \underline{\underline{\frac{100\left(\frac{8}{3}\right)^{\frac{t}{5}}}{4+\left(\frac{8}{3}\right)^{\frac{t}{5}}}}}$$



(Suggested GC window: t: 0 to 30, A: 0 to 105.)

5 $b = 1-a$

x	1	2
$P(X=x)$	a	$1-a$

$$E(X) = 1(a)+2(1-a)$$

$$= \underline{\underline{2-a}}$$

$$E(X^2) = 1^2(a)+2^2(1-a)$$

$$= 4-3a$$

$$\text{Var}(X) = E(X^2)-[E(X)]^2$$

$$= 4-3a-(2-a)^2$$

$$= 4-3a-(4-4a+a^2)$$

$$= \underline{\underline{a-a^2}}$$

6 **(i)** Use 1, 2 and 3 to form 3-digit numbers

(a) $3! = \underline{\underline{6}}$

(b) $3\times3\times3 = \underline{\underline{27}}$

(c) Method 1 Consider the complement (AAA)

Number of 3-digit numbers with all 3 digits the same (AAA) = 3

Required number = $27-3 = \underline{\underline{24}}$

Method 2 Consider cases

Case 1 ABC in any order

Number of 3-digit numbers = 6 (from (i)(a))

Case 2 AAB in any order

Number of 3-digit numbers = ${}^3P_2 \times \frac{3!}{2!} = 18$

(${}^3P_2 = 3\times2$: 3 ways to select a digit to be used twice; 2 ways to select another digit)

Total number of 3-digit numbers = 6 + 18 = $\underline{\underline{24}}$

(ii) Use 1, 2 and 3 to form 4-digit numbers

Method 1 Consider cases

Case 1 AABC in any order

Number of 4-digit numbers = $3\times\frac{4!}{2!} = 36$

(3 ways to select the digit to be used twice)

Case 2 AABB in any order

Number of 4-digit numbers = ${}^3C_2 \times \frac{4!}{2!\times2!} = 18$

(${}^3C_2$ ways to select the 2 digits each to be used twice)

Total number of 4-digit numbers = 36 + 18 = $\underline{\underline{54}}$

Method 2 Consider the complement

Total number of 4-digit numbers = $3^4 = 81$

Case 1 AAAB in any order

Number of 4-digit numbers = ${}^3P_2 \times \frac{4!}{3!} = 24$

(${}^3P_2 = 3\times2$: 3 ways to select a digit to be used thrice; 2 ways to select another digit)

Case 2 AAAA

Number of 4-digit numbers = 3

Total number of 4-digit numbers = $81-(24+3) = \underline{\underline{54}}$

7 **(i)** $H_0 : \mu = 420$

$H_1 : \mu \neq 420$

$$s^2 = \frac{30}{29}(12) = 12.414$$

Under H_0, since $n = 30$ is large, by Central Limit Theorem, $\bar{X} \sim N\left(420, \frac{12.414}{30}\right)$ approximately.

Hence it is not necessary for the volumes to have a normal distribution for the test to be valid.

Test statistic $Z = \dfrac{\bar{X} - 420}{\sqrt{\frac{12.414}{30}}} \sim N(0,1)$ approximately

$\alpha = 0.01$

From GC, $z = \dfrac{418.55 - 420}{\sqrt{\frac{12.414}{30}}} = -2.2541$

p-value = 0.0242 (3 sf)

Since p-value = 0.0242 > $\alpha = 0.01$, we do not reject H_0 at 1% level of significance and conclude that there is insufficient evidence that the population mean volume has changed.

(ii) $\alpha = 0.01 \Rightarrow \dfrac{\alpha}{2} = 0.005$

Reject H_0 if $z \leq -2.5758$ or $z \geq 2.5758$

$$\frac{\bar{x} - 420}{\sqrt{\frac{12.414}{30}}} \leq -2.5758 \quad \text{or} \quad \frac{\bar{x} - 420}{\sqrt{\frac{12.414}{30}}} \geq 2.5758$$

$$\bar{x} \leq 420 - 2.5758\sqrt{\frac{12.414}{30}} \quad \text{or} \quad \bar{x} \geq 420 + 2.5758\sqrt{\frac{12.414}{30}}$$

$$\bar{x} \leq 418.34 \quad \text{or} \quad \bar{x} \geq 421.66$$

8 Let A g be the mass of a tomato of variety A and B g be the mass of a tomato of variety B.

$A \sim \mathrm{N}\left(80, 11^2\right)$

(i) $\mathrm{P}\left(A > 90\right) = 0.18165$

P(one greater than 90 g and one less than 90 g)

$= 2 \times \mathrm{P}\left(A > 90\right) \times \mathrm{P}\left(A < 90\right)$

$= 2\left(0.18165\right)\left(1 - 0.18165\right)$

$= \underline{\underline{0.297}}$ (3 sf)

Let $B \sim \mathrm{N}\left(70, \sigma^2\right)$.

(ii) Let

$$\begin{aligned} S_B &= B_1 + B_2 + \ldots + B_6 + 15 \\ S_B &\sim \mathrm{N}\left(6 \times 70 + 15, 6\sigma^2\right) \\ &\quad \text{i.e., } \mathrm{N}\left(435, 6\sigma^2\right) \\ \mathrm{P}\left(S_B < 450\right) &= 0.8463 \\ \mathrm{P}\left(Z < \frac{450 - 435}{\sqrt{6}\,\sigma}\right) &= 0.8463 \\ \frac{15}{\sqrt{6}\,\sigma} &= 1.0207 \\ \sigma &= \frac{15}{1.0207\sqrt{6}} \\ &= 6 \text{ (nearest g)} \qquad \text{(Shown)} \end{aligned}$$

(iii)

$$\begin{aligned} S_B &\sim \mathrm{N}\left(435, 216\right) \\ \text{Let} \quad S_A &= A_1 + A_2 + \ldots + A_5 + 25 \\ S_A &\sim \mathrm{N}\left(5 \times 80 + 25, 5 \times 11^2\right) \\ &\quad \text{i.e., } \mathrm{N}\left(425, 605\right) \\ S_A - S_B &\sim \mathrm{N}\left(425 - 435, 605 + 216\right) \\ &= \mathrm{N}\left(-10, 821\right) \\ \mathrm{P}\left(S_A > S_B\right) &= \mathrm{P}\left(S_A - S_B > 0\right) \\ &= \underline{\underline{0.364}} \text{ (3 sf)} \end{aligned}$$

9(i) (1) Selection of balls is done with replacement.
(2) The balls are thoroughly mixed before each selection.

(ii) Given $X \sim \mathrm{B}\left(10, \frac{2}{5}\right)$

$$\begin{aligned} \mathrm{P}(X \geq 4) &= 1 - \mathrm{P}(X \leq 3) \\ &= \underline{\underline{0.618}} \text{ (3 sf)} \end{aligned}$$

(iii) Given

$$\begin{aligned} \mathrm{E}(X) &= 4.8 \\ \Rightarrow \frac{2}{5}n &= 4.8 \\ n &= \underline{\underline{12}} \end{aligned}$$

9(iv) Given $X \sim \mathrm{B}\left(n, \frac{2}{5}\right)$

$$\begin{aligned} \mathrm{P}(X = 0 \text{ or } 1) &< 0.01 \\ \Rightarrow \mathrm{P}(X = 0) + \mathrm{P}(X = 1) &< 0.01 \\ \Rightarrow \left(\frac{3}{5}\right)^n + n\left(\frac{2}{5}\right)\left(\frac{3}{5}\right)^{n-1} &< 0.01 \end{aligned}$$

From GC, least $n = \underline{\underline{14}}$

(v) Without replacement,

P(Shawn wins the game)

$$= \frac{2}{5} + \frac{3}{5}\left(\frac{2}{4}\right)\left(\frac{2}{3}\right)$$

$$= \frac{3}{5} \qquad \text{(Shown)}$$

(vi) With replacement,

P(Shawn wins the game)

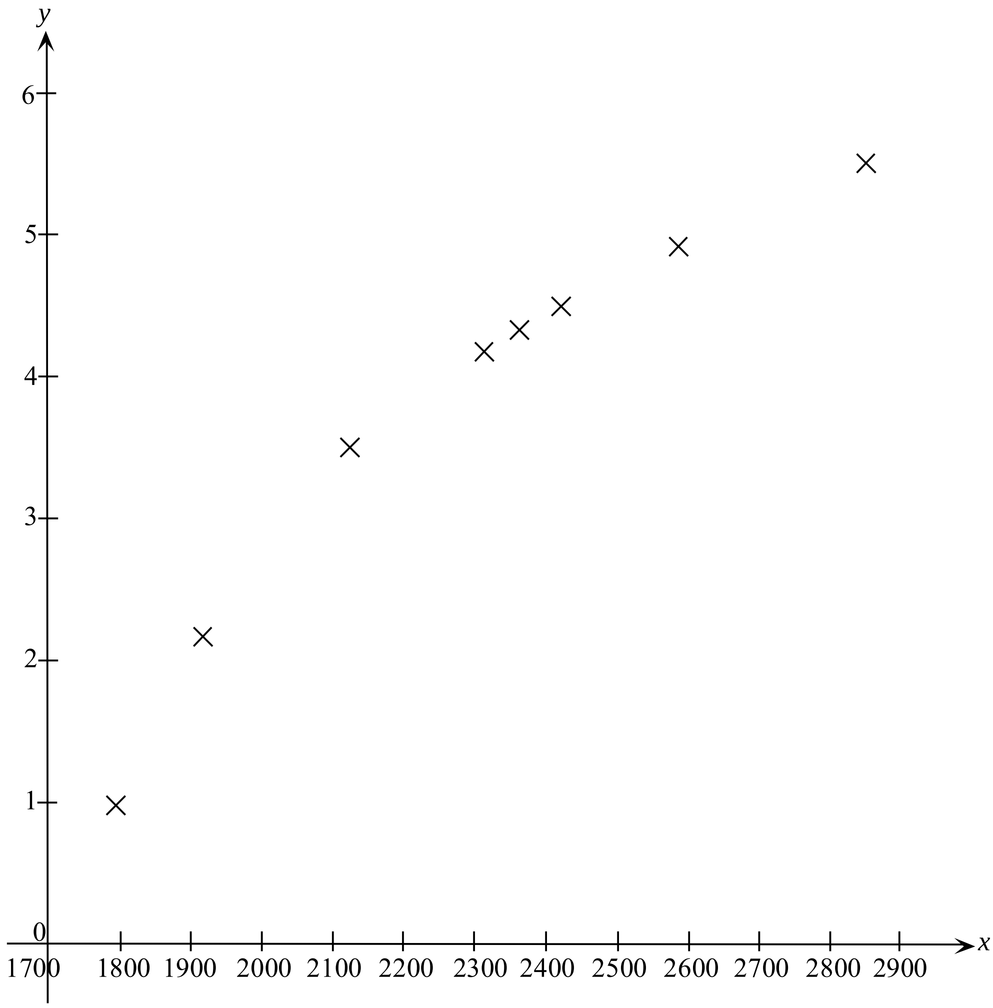
$$= \frac{2}{5} + \frac{3}{5}\left(\frac{3}{5}\right)\left(\frac{2}{5}\right) + \frac{3}{5}\left(\frac{3}{5}\right)\left(\frac{3}{5}\right)\left(\frac{3}{5}\right)\left(\frac{2}{5}\right) + \ldots$$

$$= \frac{2}{5} + \frac{2}{5}\left(\frac{3}{5}\right)^2 + \left(\frac{2}{5}\right)\left(\frac{3}{5}\right)^4 + \ldots$$

$$= \frac{\frac{2}{5}}{1 - \left(\frac{3}{5}\right)^2}$$

$$= \underline{\underline{\frac{5}{8}}} \quad \text{or} \quad \underline{\underline{0.625}}$$

10 (i)



(ii) **(a)** Between x and y: $r = \underline{\underline{0.959}}$

(b) Between x and y^2: $r = \underline{\underline{0.995}}$

(iii) From **(i)**, since as x increases, y increases at a decreasing rate, the points on the scatter diagram take the shape of the graph of $y^2 = c + dx$.

Or: From **(i)**, the points on the scatter diagram seem to lie on a concave downward curve.

From **(ii)**, the product moment correlation coefficient between x and y^2 is closer to 1, as compared to that between x and y,

$\therefore$ the model $\underline{\underline{y^2 = c + dx}}$ is the better model.

(iv) From GC, the regression line of y^2 on x is

$$y^2 = 0.027897x - 47.985$$

$$\underline{\underline{y^2 = 0.0279x - 48.0}} \text{ (3 sf)}$$

When $x = 2000$,

$$y^2 = 0.027897(2000) - 47.985$$

$$= 7.809$$

$$\therefore y = \underline{\underline{2.79}} \text{ (3 sf) or } \underline{\underline{2.8}} \text{ (1 dp, as shown in the table of values)}$$

(v) May not be valid as correlation does not necessarily imply causation.

Or: May not be valid as there could be other factors relating traffic flow and air pollution.

10 **(b)**

$$\begin{aligned} y &= 2.5x + 3.8 \\ \overline{y} &= 2.5\overline{x} + 3.8 \\ &= 2.5(4.4) + 3.8 \\ &= 14.8 \end{aligned}$$

Let

$$\begin{aligned} x &= 1.5y - k \\ \overline{x} &= 1.5\overline{y} - k \\ 4.4 &= 1.5(14.8) - k \\ k &= 22.2 - 4.4 \\ &= \underline{\underline{17.8}} \end{aligned}$$