1

Let $P(n)$ be the statement

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} \cos\frac{n\pi}{2} & \cos\frac{(n+1)\pi}{2} \\ \cos\frac{(n-1)\pi}{2} & \cos\frac{n\pi}{2} \end{pmatrix},\ n \in \mathbb{Z}^+$$

When $n=1$,

$$\textbf{LHS} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\textbf{RHS} = \begin{pmatrix} \cos\frac{\pi}{2} & \cos\pi \\ \cos 0 & \cos\frac{\pi}{2} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$P(1)$ is true

Assume $P(k)$ is true for some $k \in \mathbb{Z}^+$

i.e. $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^k = \begin{pmatrix} \cos\frac{k\pi}{2} & \cos\frac{(k+1)\pi}{2} \\ \cos\frac{(k-1)\pi}{2} & \cos\frac{k\pi}{2} \end{pmatrix}$

To prove $P(k+1)$ true.

i.e. $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{k+1} = \begin{pmatrix} \cos\frac{(k+1)\pi}{2} & \cos\frac{(k+2)\pi}{2} \\ \cos\frac{k\pi}{2} & \cos\frac{(k+1)\pi}{2} \end{pmatrix}$

$$\textbf{LHS} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^k \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\frac{k\pi}{2} & \cos\frac{(k+1)\pi}{2} \\ \cos\frac{(k-1)\pi}{2} & \cos\frac{k\pi}{2} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\frac{(k+1)\pi}{2} & -\cos\frac{k\pi}{2} \\ \cos\frac{k\pi}{2} & -\cos\frac{(k-1)\pi}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \cos\frac{(k+1)\pi}{2} & \cos\left(\frac{k\pi}{2}+\pi\right) \\ \cos\frac{k\pi}{2} & \cos\left(\frac{(k-1)\pi}{2}+\pi\right) \end{pmatrix} \text{ since } \cos(x+\pi) = -\cos x$$

$$= \begin{pmatrix} \cos\frac{(k+1)\pi}{2} & \cos\frac{(k+2)\pi}{2} \\ \cos\frac{k\pi}{2} & \cos\frac{(k+1)\pi}{2} \end{pmatrix}$$

$= \textbf{RHS}$

Therefore, $P(k)$ true implies $P(k+1)$ true.

And, since $P(1)$ is also true, $P(n)$ is true for all $n \in \mathbb{Z}^+$

2 **(i)** Since the daily removal of concentration level is 7% ie so the concentration remaining is 93%.

The recurrence relation is $u_n = (0.93)^7 u_{n-1} + 2.5, \quad u_0 = 0$

ie $u_n = 0.6017 u_{n-1} + 2.5, \quad u_0 = 0$

$$\begin{aligned} u_n &= 0.6017 u_{n-1} + 2.5 \\ &= 0.6017\left(0.6017 u_{n-2} + 2.5\right) + 2.5 \\ &= 0.6017^2 u_{n-2} + (0.6017)2.5 + 2.5 \\ &= 0.6017^2\left(0.6017 u_{n-3} + 2.5\right) + (0.6017)2.5 + 2.5 \\ &= 0.6017^3 u_{n-3} + \left(0.6017\right)^2 2.5 + (0.6017)2.5 + 2.5 \end{aligned}$$

Therefore,

$$\begin{aligned} u_n &= 0.6017^n u_0 + \left(0.6017\right)^{n-1} 2.5 + \left(0.6017\right)^{n-2} 2.5 + \dots + (0.6017)2.5 + 2.5 \\ &= 2.5\left(\frac{1-\left(0.6017\right)^n}{1-0.6017}\right) \quad \text{since } u_0 = 0 \\ &= 6.277\left(1-\left(0.6017\right)^n\right), \quad n \ge 0, \; n \in □ \end{aligned}$$

(ii) Since $u_n = 6.277\left(1-\left(0.6017\right)^n\right) \le 6.277$ for all $n \ge 0$, the concentration level will always be less than 7 *mg/l*. So the Local Authority should allowed the factory to go ahead with the discharge

3 **(i)** $y = \ln(2\cos x)$

$$\frac{dy}{dx} = \frac{1}{2\cos x}(2)(-\sin x) = -\tan x$$

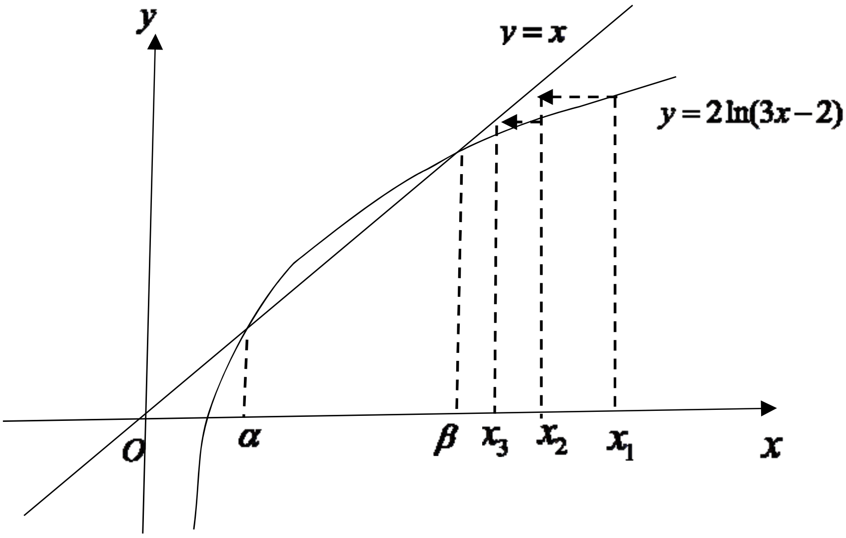
(ii)

$$\begin{aligned} L &= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx \\ &= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \sqrt{1+\tan^2 x}\, dx \\ &= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \sec x\, dx \\ &= \left[\ln(\sec x + \tan x)\right]_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \\ &= \ln\left(2+\sqrt{3}\right) - \ln\left(2-\sqrt{3}\right) \end{aligned}$$

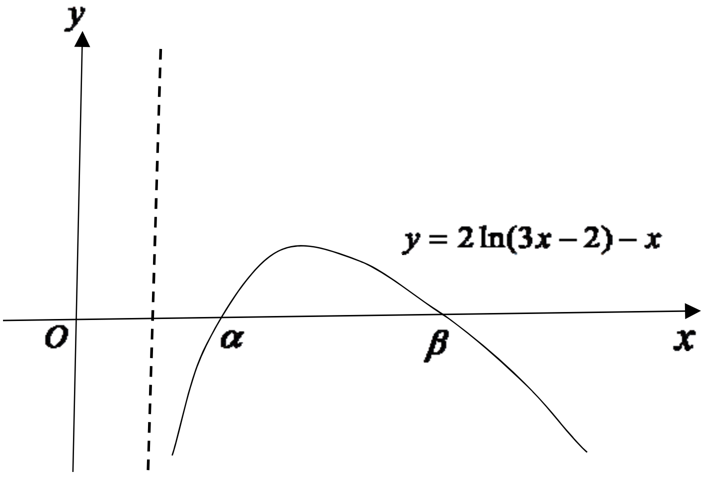
(iii)

$$S = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} 2\pi y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx = 2\pi \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \sec x \ln(2\cos x)\, dx = 7.320 \text{ (3d.p.)}$$

4	(i)	$z^9 = 1 \Rightarrow z^9 = e^{2k\pi i}$ $\Rightarrow z = e^{\frac{2k\pi}{9}i} \quad , \quad k = 0, \pm1, \pm2, \pm3, \pm4$
	(ii)	$w^9 = (w+i)^9$ $\left(\frac{w}{w+i}\right)^9 = 1$ $\frac{w}{w+i} = e^{\frac{2k\pi}{9}i}$ $\Rightarrow w = (w+i)e^{\frac{2k\pi}{9}i}$ $\Rightarrow w = \frac{ie^{\frac{2k\pi}{9}i}}{1-e^{\frac{2k\pi}{9}i}}$ $\Rightarrow w = \frac{ie^{\frac{2k\pi}{9}i}}{e^{\frac{k\pi}{9}i}\left(e^{\frac{-k\pi}{9}i} - e^{\frac{k\pi}{9}i}\right)}$ $\Rightarrow w = \frac{ie^{\frac{k\pi}{9}i}}{e^{-\frac{k\pi}{9}i} - e^{\frac{k\pi}{9}i}}$ $\Rightarrow w = \frac{i\left(\cos\left(\frac{k\pi}{9}\right) + i\sin\left(\frac{k\pi}{9}\right)\right)}{-2i\sin\left(\frac{k\pi}{9}\right)}$ $\Rightarrow w = -\frac{1}{2}\left[\cot\left(\frac{k\pi}{9}\right) + i\right] \quad , \quad k = \pm1, \pm2, \pm3, \pm4$
5	(i)	Let $f(x) = 2\ln(3x-2) - x$ $f(5) = 0.130 > 0$ $f(5.5) = -0.152 < 0$ Since there is a change in sign and the curve is continuous over the interval (5,5.5), therefore there is a root in the interval (5,5.5)
	(ii)	Take $x_1 = 5.25$ Using GC, $x_2 = 5.2421$ $x_3 = 5.2386$ $x_4 = 5.2371$ $x_5 = 5.2364$ $x_6 = 5.2362$ Since the third decimal place did not change, therefore the root β =5.24 (correct to 2 decimal places)
	(iii)	See diagram



(iv) See diagram



(1) When the starting value is to the left and close to the turning point, the next approximation will be on the other side of the asymptote.

(2) When the starting value is to the right side of the turning point, the iterations will converge to the root β and not α .

(v) Newton Raphson Method:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$ie \quad x_{n+1} = x_n - \frac{2\ln(3x_n - 2) - x_n}{\frac{6}{3x_n - 2} - 1}$$

Using GC,

$x_1 = 1.5$

$x_2 = 1.2624$

$x_3 = 1.3053$

$x_4 = 1.3076$

$x_5 = 1.3076$

Since the third decimal place did not change, therefore the root α =1.31 (correct to 2 decimal places)

6	Consider DE: $x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+2(x+1)\frac{\mathrm{d}y}{\mathrm{d}x}+(2+x)y=e^{2x}$ $\left(x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+2\frac{\mathrm{d}y}{\mathrm{d}x}\right)+2\left(x\frac{\mathrm{d}y}{\mathrm{d}x}+y\right)+xy=e^{2x}$ ---(1) Let $u=xy$ $\frac{du}{dx}=y+x\frac{dy}{dx}$ $\frac{d^2u}{dx^2}=\frac{dy}{dx}+\frac{dy}{dx}+x\frac{d^2y}{dx^2}=2\frac{dy}{dx}+x\frac{d^2y}{dx^2}$ Substitute into (1): $\frac{\mathrm{d}^2u}{\mathrm{d}x^2}+2\frac{\mathrm{d}u}{\mathrm{d}x}+u=e^{2x}$ ---(2) Characteristic equation: $r^2+2r+1=0$ $r=-1$ So, $u_c=c_1e^{-x}+c_2xe^{-x}$ Let $u_p=Ae^{2x}$ $\frac{\mathrm{d}u_p}{\mathrm{d}x}=2Ae^{2x}$ $\frac{\mathrm{d}^2u_p}{\mathrm{d}x^2}=4Ae^{2x}$ Substitute into (2): $4Ae^{2x}+2(2Ae^{2x})+Ae^{2x}=e^{2x}$ $9Ae^{2x}=e^{2x}$ $A=\frac{1}{9}$ So, $u_p=\frac{1}{9}e^{2x}$ Combining, $u=u_p+u_c=c_1e^{-x}+c_2xe^{-x}+\frac{1}{9}e^{2x}$ $xy=c_1e^{-x}+c_2xe^{-x}+\frac{1}{9}e^{2x}$ $y=\frac{c_1e^{-x}}{x}+c_2e^{-x}+\frac{e^{2x}}{9x}$ (where c_1 and c_2 are arbitrary constants)
7	$z+z^{-1}=(\cos\theta+i\sin\theta)+(\cos(-\theta)+i\sin(-\theta))$ $=(\cos\theta+i\sin\theta)+(\cos\theta-i\sin\theta)$ $=2\cos\theta$ Similarly, $z^m+z^{-m}=2\cos(m\theta)$

Now,

$$\left(2\cos\theta\right)^{2n+1}=\left(z+z^{-1}\right)^{2n+1}$$

$$=\binom{2n+1}{0}z^{2n+1}+\binom{2n+1}{1}z^{2n-1}+\cdots+\binom{2n+1}{n}z+\binom{2n+1}{n+1}z^{-1}+\cdots+\binom{2n+1}{2n}z^{-(2n-1)}+\binom{2n+1}{2n+1}z^{-(2n+1)}$$

$$=\binom{2n+1}{0}\left(z^{2n+1}+z^{-(2n+1)}\right)+\binom{2n+1}{1}\left(z^{2n-1}+z^{-(2n-1)}\right)+\cdots+\binom{2n+1}{n}\left(z+z^{-1}\right)$$

$$\left[\because\binom{2n+1}{r}=\binom{2n+1}{2n+1-r}\right]$$

$$=\sum_{r=0}^{n}\left(\binom{2n+1}{r}\left(z^{2n+1-2r}+z^{-(2n+1-2r)}\right)\right)$$

$$=\sum_{r=0}^{n}\left(\binom{2n+1}{r}2\cos\left((2n+1-2r)\theta\right)\right)\qquad\because z^{m}+z^{-m}=2\cos\left(m\theta\right)$$

$$\Rightarrow 2^{2n+1}\cos^{2n+1}\theta=2\sum_{r=0}^{n}\left(\binom{2n+1}{r}\cos\left((2n+1-2r)\theta\right)\right)$$

$$\Rightarrow \cos^{2n+1}\theta=2^{-2n}\sum_{r=0}^{n}\left(\binom{2n+1}{r}\cos\left((2n+1-2r)\theta\right)\right)\ \text{(shown)}$$

$$V=\int_{\frac{7}{4}\pi}^{\frac{13}{6}\pi}2\pi xy\,\mathrm{d}x=\int_{\frac{7}{4}\pi}^{\frac{13}{6}\pi}2\pi\cos^{5}x\,\mathrm{d}x$$

To find $\cos^{5}x$ (let $n=2$):

$$\cos^{5}\theta=2^{-4}\sum_{r=0}^{2}\left(\binom{5}{r}\cos\left((5-2r)\theta\right)\right)$$

$$=2^{-4}\left(\cos5\theta+5\cos3\theta+10\cos\theta\right)$$

So, $$\int_{\frac{7}{4}\pi}^{\frac{13}{6}\pi}2\pi\cos^{5}x\,\mathrm{d}x=\frac{\pi}{8}\int_{\frac{7}{4}\pi}^{\frac{13}{6}\pi}\cos5x+5\cos3x+10\cos x\,\mathrm{d}x$$

$$=\frac{\pi}{8}\left[\frac{\sin5x}{5}+\frac{5\sin3x}{3}+10\sin x\right]_{\frac{7}{4}\pi}^{\frac{13}{6}\pi}$$

$$=\frac{\pi}{8}\left[\frac{1}{10}+\frac{5}{3}+5\right]-\frac{\pi}{8}\left[\frac{\sqrt{2}}{10}-\frac{5\sqrt{2}}{6}-5\sqrt{2}\right]$$

$$=\frac{\pi}{240}\left[172\sqrt{2}+203\right]$$

8 (i)

$$r=\frac{d}{1+\cos\left(\theta-\theta_0\right)}$$

Using P,

$$2\sqrt{2}=\frac{d}{1+\cos\left(\frac{\pi}{2}-\theta_0\right)}=\frac{d}{1+\sin\theta_0}$$

Using Q,

$$3=\frac{d}{1+\cos\left(\frac{\pi}{4}-\theta_0\right)}$$

$$=\frac{d}{1+\cos\frac{\pi}{4}\cos\theta_0+\sin\frac{\pi}{4}\sin\theta_0}$$

$$=\frac{d}{1+\frac{1}{\sqrt{2}}\cos\theta_0+\frac{1}{\sqrt{2}}\sin\theta_0}$$

$$\therefore\quad 2\sqrt{2}\left(1+\sin\theta_0\right)=3\left(1+\frac{1}{\sqrt{2}}\cos\theta_0+\frac{1}{\sqrt{2}}\sin\theta_0\right)$$

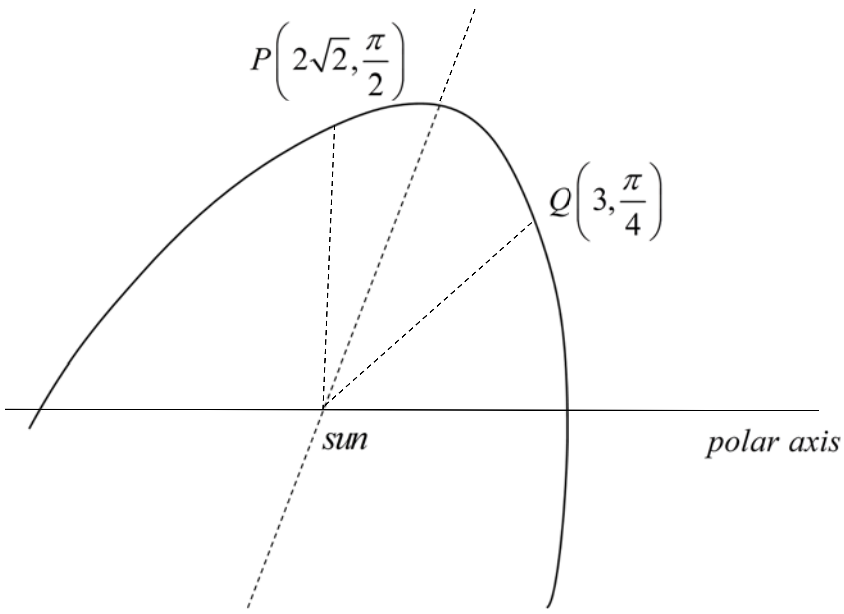
$$2\sqrt{2}+2\sqrt{2}\sin\theta_0=3+\frac{3}{\sqrt{2}}\cos\theta_0+\frac{3}{\sqrt{2}}\sin\theta_0$$

$$4+4\sin\theta_0=3\sqrt{2}+3\cos\theta_0+3\sin\theta_0$$

$$\therefore\quad 3\cos\theta_0-\sin\theta_0+\left(3\sqrt{2}-4\right)=0$$

(ii) From GC, $\theta_0=1.32585$ radians (5 dec pl)

(iii)



(iv) $$2\sqrt{2}=\frac{d}{1+\sin\left(1.32585\right)}$$

$$\therefore\quad d=5.57243$$

The equation of the trajectory is $r=\dfrac{5.57243}{1+\cos\left(\theta-1.32585\right)}$

When $\theta=1.32585$, $r=2.78621=2.79\ (3sf)$

Therefore the closest the comet can get to the sun is 2.79 AU

(v) $$r=\frac{5.57243}{1+\cos\left(\theta-1.32585\right)}$$

$$\frac{dr}{d\theta}=\frac{5.57243\sin(\theta-1.32585)}{\left[1+\cos(\theta-1.32585)\right]^2}$$

Length of the arc from P to Q

$$=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

$$=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\sqrt{\left(\frac{5.57243}{1+\cos(\theta-1.32585)}\right)^2+\left(\frac{5.57243\sin(\theta-1.32585)}{\left[1+\cos(\theta-1.32585)\right]^2}\right)^2}\,d\theta$$

$$=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\sqrt{\left(\frac{5.57243}{1+\cos(\theta-1.32585)}\right)^2+\left(\frac{5.57243\sin(\theta-1.32585)}{\left[1+\cos(\theta-1.32585)\right]^2}\right)^2}\,d\theta$$

$= 2.25072$ AU

Average speed of the comet along arc PC

$$=\frac{2.25072\times 150,000,000}{30}\quad km/day$$

$$=\frac{2.25072\times 150,000,000}{30\times 24\times 60\times 60}\quad km/s$$

$$=130.25\quad km/s$$

$$=130.3\quad km/s\ (3sf)$$

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9 (i) For any $\lambda\in\mathbb{R}$,

Take $\mathbf{x}_1,\mathbf{x}_2\in\mathbb{R}^3$,

$$\mathbf{A}(\mathbf{x}_1+\mathbf{x}_2)=\mathbf{A}\mathbf{x}_1+\mathbf{A}\mathbf{x}_2=\lambda\mathbf{x}_1+\lambda\mathbf{x}_2=\lambda(\mathbf{x}_1+\mathbf{x}_2)$$

$$\therefore \mathbf{x}_1+\mathbf{x}_2\in S_\lambda$$

S_λ is closed under vector addition.

Take $\phi\in\mathbb{R}$ and $\mathbf{x}\in\mathbb{R}^3$,

$$\mathbf{A}(\phi\mathbf{x})=\phi\mathbf{A}\mathbf{x}=\phi\lambda\mathbf{x}=\lambda(\phi\mathbf{x})$$

$$\therefore \phi\mathbf{x}\in S_\lambda$$

S_λ is closed under scalar multiplication.

And, since $\mathbf{0}\in S_\lambda\subseteq\mathbb{R}^3$, therefore S_λ is a vector space for all $\lambda\in\mathbb{R}$

(ii) Since $\det\begin{pmatrix}\alpha & -2 & 0\\ 0 & -\alpha & 0\\ -1 & \beta & 2\end{pmatrix}=-2\alpha^2\neq 0$, the special case where $\mathbf{A}\mathbf{x}=\mathbf{0}$ yields the unique solution $\mathbf{x}=\mathbf{0}$, therefore $\dim(S_0)=0$.

For $\dim(S_\lambda)\neq 0$, consider non-trivial solutions for $\mathbf{A}\mathbf{x}=\lambda\mathbf{x}$.

$$\det\begin{pmatrix}\alpha-\lambda & -2 & 0\\ 0 & -\alpha-\lambda & 0\\ -1 & \beta & 2-\lambda\end{pmatrix}=0$$

$$(\alpha-\lambda)(-\alpha-\lambda)(2-\lambda)=0$$

$\lambda=\alpha$ or $\lambda=-\alpha$ or $\lambda=2$

(iii) It is clear that $\dim(S_\lambda)=0\ \ \forall\lambda\in\mathbb{R}\setminus\{\alpha,-\alpha,2\}$.

We will now check the cases where $\forall\lambda \in \{\alpha, -\alpha, 2\}$

When $\lambda = \alpha$:

$$\begin{pmatrix} 0 & -2 & 0 \\ 0 & -2\alpha & 0 \\ -1 & \beta & 2-\alpha \end{pmatrix} \to \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ -1 & \beta & 2-\alpha \end{pmatrix} \because \alpha \neq 0$$

One row of zeros $\therefore \dim(S_\alpha) = 1$

Let $z = \phi$, $\phi \in \mathbb{R}$, we have $y = 0$ and $x = (2-\alpha)\phi$

So, $\mathbf{e}_{\boldsymbol{\alpha}} = \begin{pmatrix} 2-\alpha \\ 0 \\ 1 \end{pmatrix}$

When $\lambda = -\alpha$:

$$\begin{pmatrix} 2\alpha & -2 & 0 \\ 0 & 0 & 0 \\ -1 & \beta & 2+\alpha \end{pmatrix} \to \begin{pmatrix} \alpha & -1 & 0 \\ 0 & 0 & 0 \\ 0 & \alpha\beta - 1 & \alpha(2+\alpha) \end{pmatrix}$$

Consider the scenario where there is 2 rows of zeros,

$\alpha(2+\alpha) = 0$ and $\alpha\beta - 1 = 0$

$\alpha = -2$ and $\alpha\beta = 1$ since $\alpha \neq 0$

$\alpha = -2$ and $\beta = -\frac{1}{2}$

So, for $\dim(S_{-\alpha}) \leq 1$, we must have

$\alpha \neq -2$ or $\beta \neq -\frac{1}{2}$

To find eigenvector,

Let $z = \varphi$, $\varphi \in \mathbb{R}$,

$$y = -\frac{\alpha(2+\alpha)}{\alpha\beta - 1}\varphi$$

$$x = -\frac{2+\alpha}{\alpha\beta - 1}\varphi$$

So, $\mathbf{e}_{\boldsymbol{-\alpha}} = \begin{pmatrix} 2+\alpha \\ \alpha(2+\alpha) \\ 1-\alpha\beta \end{pmatrix}$

When $\lambda = 2$:

In the analysis below, it is fine to assume $\alpha \neq -2$ and $\alpha \neq 2$. This is because we have already considered these cases earlier.

$$\begin{pmatrix} \alpha - 2 & -2 & 0 \\ 0 & -\alpha - 2 & 0 \\ -1 & \beta & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -\beta & 0 \\ 0 & \alpha + 2 & 0 \\ 0 & \alpha\beta - 2\beta - 2 & 0 \end{pmatrix}$$

Since we do not consider $\alpha = -2$, there cannot be two rows of zeros. So the condition $\dim(S_2) \leq 1$ will always be satisfied.

To find eigenvector,

consider $\mathbf{A}-2\mathbf{I}=\begin{pmatrix}\alpha-2 & -2 & 0\\ 0 & -\alpha-2 & 0\\ -1 & \beta & 0\end{pmatrix}\to\begin{pmatrix}1 & -\beta & 0\\ 0 & \alpha+2 & 0\\ 0 & 0 & 0\end{pmatrix}$

Let $z=\gamma$, $\gamma\in\mathbb{R}$, we have $y=0$ and $x=0$.

So, $\mathbf{e_2}=\begin{pmatrix}0\\0\\1\end{pmatrix}$

Therefore, the conditions required for part (iii) are $\alpha\neq-2$ or $\beta\neq-\frac{1}{2}$.

(iv) Corresponding to $\lambda=2,\ \alpha$ and $-\alpha$, the basis for S_2, S_α and $S_{-\alpha}$ are $\left\{\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$, $\left\{\begin{pmatrix}2-\alpha\\0\\1\end{pmatrix}\right\}$

and $\left\{\begin{pmatrix}2+\alpha\\ \alpha(2+\alpha)\\ 1-\alpha\beta\end{pmatrix}\right\}$respectively.

(v) Given $\alpha=1$, we have $\mathbf{P}=\begin{pmatrix}0 & 1 & 3\\ 0 & 0 & 3\\ 1 & 1 & 1-\beta\end{pmatrix}$ and $\mathbf{D}=\begin{pmatrix}2 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix}$.

$\mathbf{P}^{-1}=\begin{pmatrix}-1 & \frac{2+\beta}{3} & 1\\ 1 & -1 & 0\\ 0 & \frac{1}{3} & 0\end{pmatrix}$

(vi) To find $\mathbf{P}^{-1}$:

$$\left(\begin{array}{ccc|ccc}0 & 1 & 3 & 1 & 0 & 0\\ 0 & 0 & 3 & 0 & 1 & 0\\ 1 & 1 & 1-\beta & 0 & 0 & 1\end{array}\right)\to\left(\begin{array}{ccc|ccc}0 & 1 & 0 & 1 & -1 & 0\\ 0 & 0 & 3 & 0 & \frac{1}{3} & 0\\ 1 & 0 & -2-\beta & -1 & 0 & 1\end{array}\right)$$

$$\to\left(\begin{array}{ccc|ccc}0 & 1 & 0 & 1 & -1 & 0\\ 0 & 0 & 1 & 0 & \frac{1}{3} & 0\\ 1 & 0 & 0 & -1 & \frac{2+\beta}{3} & 1\end{array}\right)$$

$$\to\left(\begin{array}{ccc|ccc}1 & 0 & 0 & -1 & \frac{2+\beta}{3} & 1\\ 0 & 1 & 0 & 1 & -1 & 0\\ 0 & 0 & 1 & 0 & \frac{1}{3} & 0\end{array}\right)$$

$$\mathbf{A}^9=\begin{pmatrix}0 & 1 & 3\\ 0 & 0 & 3\\ 1 & 1 & 1-\beta\end{pmatrix}\begin{pmatrix}2 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix}^9\begin{pmatrix}-1 & \frac{2+\beta}{3} & 1\\ 1 & -1 & 0\\ 0 & \frac{1}{3} & 0\end{pmatrix}$$

$$=\begin{pmatrix}0 & 1 & 3\\ 0 & 0 & 3\\ 1 & 1 & 1-\beta\end{pmatrix}\begin{pmatrix}512 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix}\begin{pmatrix}-1 & \frac{2+\beta}{3} & 1\\ 1 & -1 & 0\\ 0 & \frac{1}{3} & 0\end{pmatrix}$$

$$=\begin{pmatrix}0 & 1 & -3\\ 0 & 0 & -3\\ 512 & 1 & \beta-1\end{pmatrix}\begin{pmatrix}-1 & \frac{2+\beta}{3} & 1\\ 1 & -1 & 0\\ 0 & \frac{1}{3} & 0\end{pmatrix}$$

$$=\begin{pmatrix}1 & -2 & 0\\ 0 & -1 & 0\\ -511 & \frac{1020+513\beta}{3} & 512\end{pmatrix}$$

10

(i) Since the tank experiences constant net flow of 1 litre per minute out of the tank,

$V=100-T\ ,\ 0<T<75$

$$\frac{\mathrm{d}S}{\mathrm{d}T}=1-2\left(\frac{S}{V}\right)=1-2\left(\frac{S}{100-T}\right)=\frac{100-T-2S}{100-T}$$

(ii) $$\frac{\mathrm{d}S}{\mathrm{d}T}=\frac{100-T-2S}{100-T}$$

$$\frac{\mathrm{d}S}{\mathrm{d}T}+\left(\frac{2}{100-T}\right)S=1 \qquad \text{---(1)}$$

Integrating factor (IF) $=e^{\int\frac{2}{100-T}\mathrm{d}T}=e^{-2\ln(100-T)}=(100-T)^{-2}$

Multiplying (1) by IF:

$$\frac{S}{(100-T)^2}=\int(100-T)^{-2}\,\mathrm{d}T$$

$$=\frac{1}{100-T}+C$$

So, $S=(100-T)+C(100-T)^2$

When $T=0, S=0$,

$$0=100+C(100)^2\Rightarrow C=-\frac{1}{100}$$

So, $S=(100-T)-\frac{1}{100}(100-T)^2=\frac{T(100-T)}{100}$

Therefore, at the time 75 min later,

$$S_{75}=\frac{75(100-75)}{100}=18.75$$

(iii) $$\frac{\mathrm{d}v}{\mathrm{d}t}=1-k\sqrt{v}$$

Water level stay constant at 36 litres, i.e. $\frac{\mathrm{d}v}{\mathrm{d}t}=0, v=36$,

$$0=1-k\sqrt{36}\Rightarrow k=\frac{1}{6}$$

Therefore, $\frac{\mathrm{d}v}{\mathrm{d}t}=1-\frac{1}{6}\sqrt{v}=\frac{6-\sqrt{v}}{6}$ (shown)

(a) $\dfrac{\mathrm{d}v}{\mathrm{d}t}=\dfrac{6-\sqrt{v}}{6}\Rightarrow\dfrac{\mathrm{d}t}{\mathrm{d}v}=\dfrac{6}{6-\sqrt{v}}$

Let $t_0=0, v_0=25, v_1=25.5$

$$t_1=t_0+0.5\left(\frac{6}{6-\sqrt{v_0}}\right)=3$$

$$t_2=t_1+0.5\left(\frac{6}{6-\sqrt{v_1}}\right)=6.16$$

(b) $\dfrac{\mathrm{d}v}{\mathrm{d}t}=\dfrac{6-\sqrt{v}}{6}$

$$\int_0^{t_{26}}1\,\mathrm{d}t=\int_{25}^{26}\frac{6}{6-\sqrt{v}}\,\mathrm{d}v$$

$$t_{26}\approx\frac{1}{6}(26-25)\left(\frac{6}{6-\sqrt{25}}+4\left(\frac{6}{6-\sqrt{25.5}}\right)+\frac{6}{6-\sqrt{26}}\right)$$

$$=6.32$$

(iv) $\dfrac{\mathrm{d}s}{\mathrm{d}t}=$(Rate of change of salt into the tank) – (rate of change of salt out of the tank)

where rate of change of salt into the tank = 1

and rate of change of salt out of the tank = concentration $\times$ rate of mixture outflow

$$=\left(\frac{s}{v}\right)\times\left(\frac{\sqrt{v}}{6}\right)$$

Therefore,

$$\frac{\mathrm{d}s}{\mathrm{d}t}=1-\left(\frac{s}{v}\right)\frac{\sqrt{v}}{6}$$

$$\frac{\mathrm{d}s}{\mathrm{d}t}=\frac{6\sqrt{v}-s}{6\sqrt{v}}$$

By chain rule,

$$\frac{\mathrm{d}s}{\mathrm{d}v}\frac{\mathrm{d}v}{\mathrm{d}t}=\frac{\mathrm{d}s}{\mathrm{d}t}$$

$$\frac{\mathrm{d}s}{\mathrm{d}v}\frac{6-\sqrt{v}}{6}=\frac{6\sqrt{v}-s}{6\sqrt{v}}$$

$$\frac{\mathrm{d}s}{\mathrm{d}v}=\frac{6\sqrt{v}-s}{6\sqrt{v}-v}$$

Let $s_1=18.75, v_1=25, v_2=26, h=1$

$$u_2=18.75+1\left(\frac{6\sqrt{25}-18.75}{6\sqrt{25}-25}\right)=21$$

$$s_2=18.75+\frac{1}{2}\left(\frac{6\sqrt{25}-18.75}{6\sqrt{25}-25}+\frac{6\sqrt{26}-21}{6\sqrt{26}-26}\right)$$

$$=20.9192 \quad \text{(to 4 d.p.)}$$