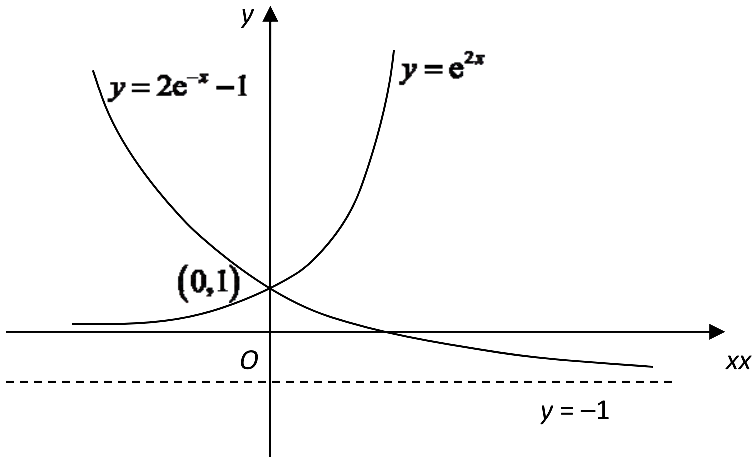
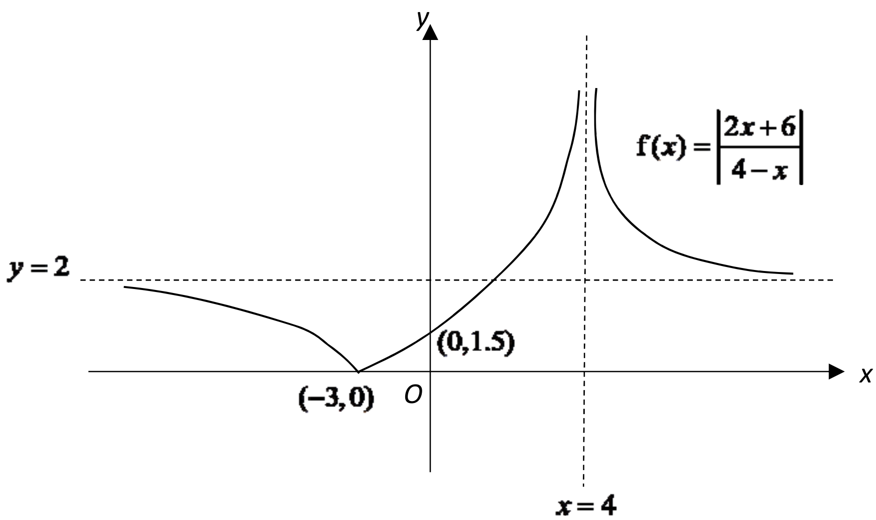
H2 Mathematics 2017 Prelim Exam Paper 1 Solution

1	$u_n = an^2 + bn + c$ $u_1 = a(1)^2 + b(1) + c = 70 \quad \Rightarrow \quad a+b+c=70 \quad (1)$ $u_2 = a(2)^2 + b(2) + c = 136 \quad \Rightarrow \quad 4a+2b+c=136 \quad (2)$ $u_3 = a(3)^2 + b(3) + c = 198 \quad \Rightarrow \quad 9a+3b+c=198 \quad (3)$ Using GC $a=-2, \quad b=72, \quad c=0$ $u_n = -2n^2 + 72n$
2	**(i)** $u_1 = u_0 + 2 - 1 = \frac{3}{2} + 2 - 1 = \frac{5}{2}$ $u_2 = u_1 + 2^2 - 2 = \frac{5}{2} + 4 - 2 = \frac{9}{2}$ $u_3 = u_2 + 2^3 - 3 = \frac{9}{2} + 8 - 3 = \frac{19}{2}$ **(ii)** $u_n - u_{n-1} = 2^n - n$ $\sum_{r=1}^{n}(u_r - u_{r-1}) = \sum_{r=1}^{n} 2^r - r$ $= \sum_{r=1}^{n} 2^r - \sum_{r=1}^{n} r$ $u_1 - u_0$ $+u_2 - u_1$ $+u_3 - u_2$ $+$ $\vdots$ $+u_{n-2} - u_{n-3} = \frac{2(1-2^n)}{1-2} - \frac{n(n+1)}{2}$ $+u_{n-1} - u_{n-2}$ $+u_n - u_{n-1}$ $u_n - u_0 = \frac{2(1-2^n)}{1-2} - \frac{n(n+1)}{2}$ $u_n = -2(1-2^n) - \frac{n(n+1)}{2} + \frac{3}{2}$ $= 2^{n+1} - \frac{1}{2} - \frac{n(n+1)}{2}$

3	 $e^{2x} \geq 2e^{-x} - 1$ $x \geq 0$ For $x \geq 0$, $e^{2x} \geq 2e^{-x} - 1 \Rightarrow e^{2x} - 2e^{-x} + 1 \geq 0$ For $x < 0$, $e^{2x} - 2e^{-x} + 1 < 0$. $\int_{-1}^{2} \left\lvert e^{2x} - 2e^{-x} + 1 \right\rvert \, dx = \int_{-1}^{0} -(e^{2x} - 2e^{-x} + 1)\, dx + \int_{0}^{2} (e^{2x} - 2e^{-x} + 1)\, dx$ $= -\left[\frac{1}{2}e^{2x} + 2e^{-x} + x\right]_{-1}^{0} + \left[\frac{1}{2}e^{2x} + 2e^{-x} + x\right]_{0}^{2}$ $= -\left[\left(\frac{1}{2} + 2\right) - \left(\frac{1}{2}e^{-2} + 2e - 1\right)\right] + \left[\left(\frac{1}{2}e^{4} + 2e^{-2} + 2\right) - \left(\frac{1}{2} + 2\right)\right]$ $= \frac{1}{2}e^{4} + 2e + \frac{5}{2}e^{-2} - 4$
4	**(i)** $R_f = [0, \infty)$  **(ii)**

$R_f = [0, \infty)$

$D_f = (-\infty, 4) \cup (4, \infty)$ or $D_f = \mathbb{R} \setminus \{4\}$

$R_f \not\subset D_f$

f^2 does not exists.

(iii)

$k = -3$

(iv)

For $D_f = (-\infty, -3]$

$$y = -\left(\frac{2x+6}{4-x}\right)$$

$$y = \frac{2x+6}{x-4}$$

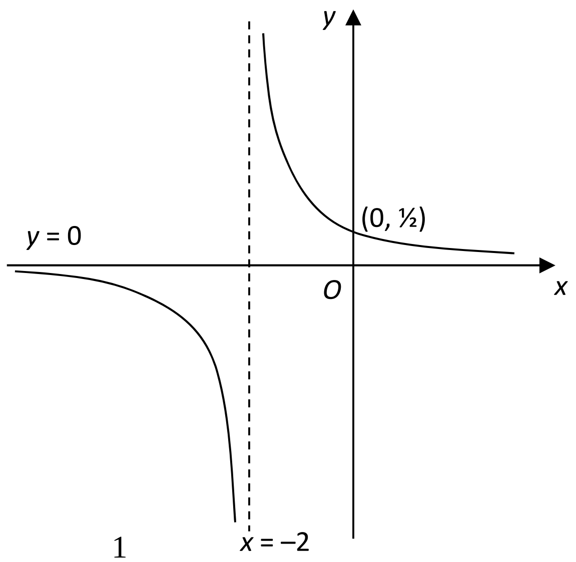
$$yx - 2x = 6 + 4y$$

$$x = \frac{6+4y}{y-2}$$

$$f^{-1} : x \mapsto \frac{6+4x}{x-2}, \qquad x \in \mathbb{R},\ \ 0 \le x < 2$$

5

(i)



(ii)

$x = 2t - 1 \qquad y = \dfrac{1}{2t+1}$

$\dfrac{dx}{dt} = 2 \qquad \dfrac{dy}{dt} = -\dfrac{2}{(2t+1)^2}$

$$\frac{dy}{dx} = -\frac{1}{(2t+1)^2}$$

At the point $P(-1, 1)$, $t = 0$

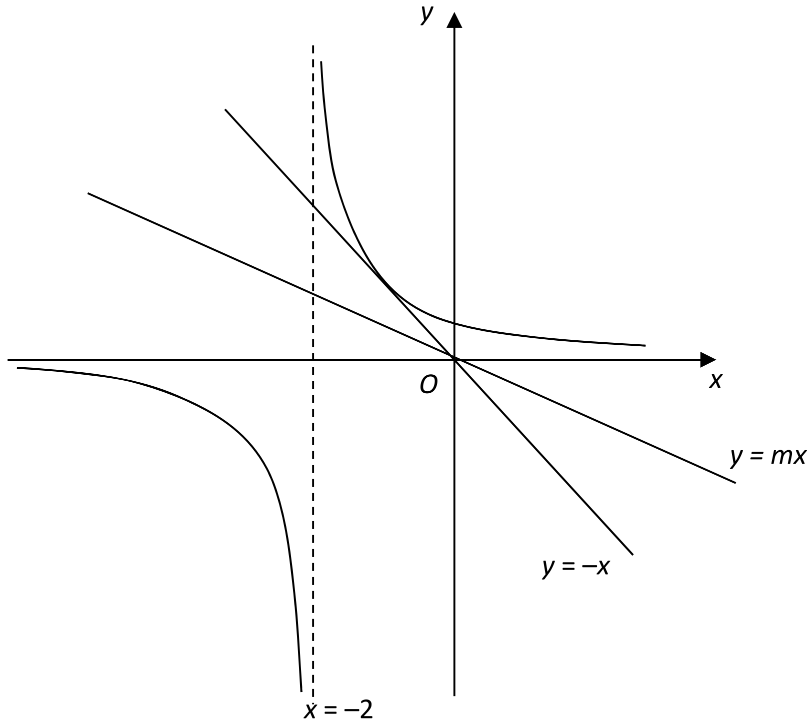
$$\frac{dy}{dx} = -1$$

Equation of tangent at P is

$y - 1 = -1(x+1)$

$y = -x$

(iii)



The line $y = mx$ does not cut the curve $\Rightarrow -1 < m \leq 0$

(iv)

Gradient of normal at $P = 1$

Equation of normal at P is

$y - 1 = x - (-1)$

$y = x + 2$

Subst $x = 2t - 1,\ y = \dfrac{1}{2t+1}$ into $y = x + 2$

$$\frac{1}{2t+1} = 2t - 1 + 2 = 2t + 1$$

$(2t+1)^2 = 1$

$2t + 1 = \pm 1$

$t = 0 \quad \text{or} \quad t = -1$

At the point Q, $t = -1$

$$x = 2(-1) - 1 = -3, \quad y = \frac{1}{2(-1)+1} = -1$$

Coordinates of Q are $(-3, -1)$

6 **(i)**

AP with $a=400$, $d=-5$

$S_n=8500$

$\frac{n}{2}\left[2(400)+(n-1)(-5)\right]=8500$

$5n^2-805n+17000=0$

$n=25$ or $n=136$ (rejected as already reached peak when $n=25$)

(ii)

GP with $a=800$, $r=90$

$S_{n(AP)}>S_{n(GP)}$

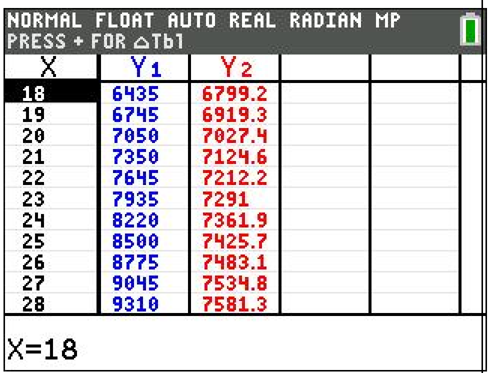
$\frac{n}{2}\left[2(400)+(n-1)(-5)\right]>\frac{800(1-0.9^n)}{1-0.9}$

$805n-5n^2>16000\left(1-0.9^n\right)$

Using GC,

$n\geq 20$

A will overtake B on the 20th day.



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X	Y1	Y2
18	6435	6799.2
19	6745	6919.3
20	7050	7027.4
21	7350	7124.6
22	7645	7212.2
23	7935	7291
24	8220	7361.9
25	8500	7425.7
26	8775	7483.1
27	9045	7534.8
28	9310	7581.3

X=18

(iii)

$S_\infty=\frac{800}{1-0.9}=8000(<8500)$

Hence, Team B will never be able to reach the peak.

(iv)

$T_{15}=800\left(0.9^{15-1}\right)=183.014$

$S_{15}=\frac{800\left(1-0.9^{15}\right)}{1-0.9}=6352.871$

Remaining distance $=8500-6352.871=2147.129$

First term of new GP $=183.014\times 0.95=173.864$

$S_{n(New\,GP)}=2147.129$

$\frac{173.864\left(1-0.95^n\right)}{1-0.95}=2147.129$

$0.95^n=0.38253$

$n=18.7$

Team B will take $15+19=34$ days

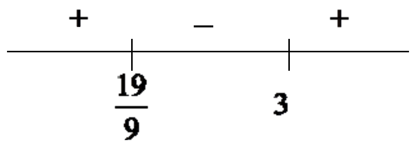
Hence, Team A will reach the peak first.

7 **(i)**

Consider the graph of $y=2+\frac{x-3}{(x-2)(x+1)}$ and $y=p$ intersecting.

$p=2+\frac{x-3}{(x-2)(x+1)}$

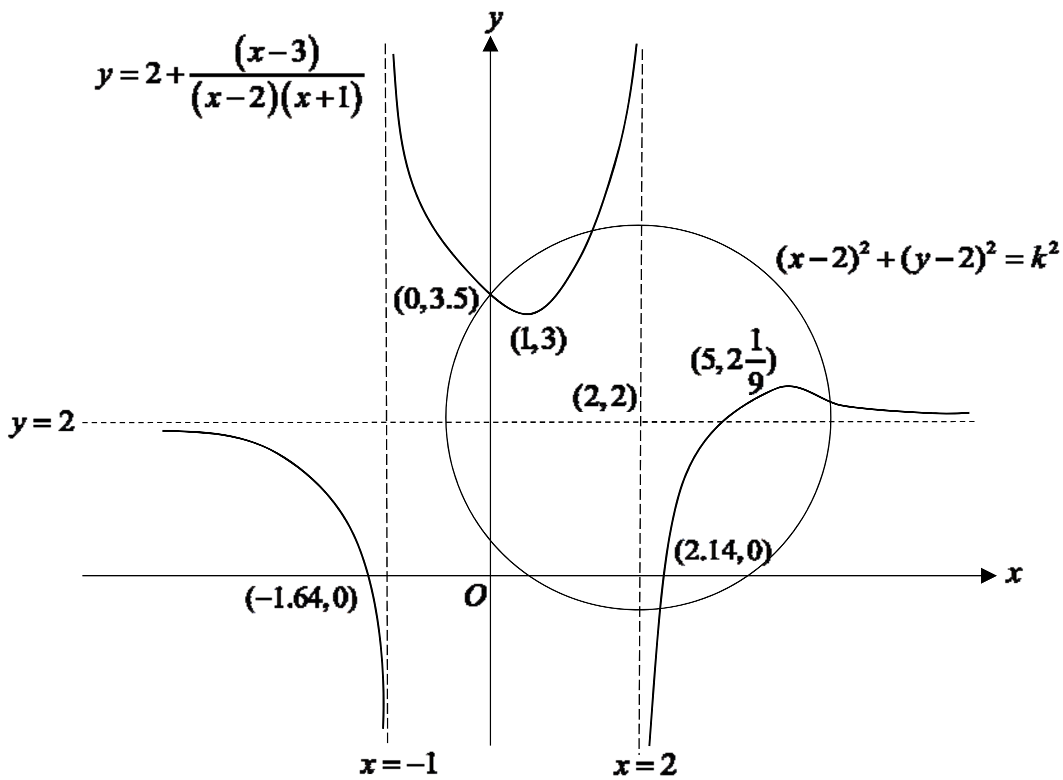
$$p-2=\frac{x-3}{x^2-x-2}$$
$$px^2-px-2p-2x^2+2x+4=x-3$$
$$(p-2)x^2+(1-p)x+(7-2p)=0$$
Discriminant ≥ 0
$$(1-p)^2-4\left(p-2\right)\left(7-2p\right)\geq 0$$
$$1-2p+p^2-28p+8p^2+56-16p\geq 0$$
$$9p^2-46p+57\geq 0$$
$$\left(9p-19\right)\left(p-3\right)\geq 0$$



$p\leq\frac{19}{9}$ or $p\geq 3$

$y\leq 2\frac{1}{9}$ or $y\geq 3$

(ii)



(iii)

$$\left(x-2\right)^2+\frac{\left(x-3\right)^2}{\left(x-2\right)^2\left(x+1\right)^2}=k^2$$
$$\left(x-2\right)^2+\left(y-2\right)^2=k^2$$

	Distance from centre of circle to the y – intercept of $y=2+\frac{(x-3)}{(x-2)(x+1)}$ $=\sqrt{2^2+\left(\frac{7}{2}-2\right)^2}=\frac{5}{2}$ $k<-2.5$ or $k>2.5$
8	**(a)**

$\int\sqrt{\frac{1-x}{x}}\,\mathrm{d}x$

$=\int\sqrt{\frac{1-\sin^2\theta}{\sin^2\theta}}2\sin\theta\cos\theta\,\mathrm{d}\theta$

$=\int 2\cos^2\theta\,\mathrm{d}\theta$

$=\int(1+\cos 2\theta)\,\mathrm{d}\theta$

$=\theta+\frac{1}{2}\sin 2\theta+C$

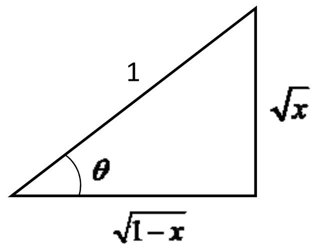
$=\theta+\sin\theta\cos\theta+C$

$=\sin^{-1}\left(\sqrt{x}\right)+\sqrt{x(1-x)}+C$

$x=\sin^2\theta$

$\frac{\mathrm{d}x}{\mathrm{d}\theta}=2\sin\theta\cos\theta$

Since $\sqrt{x}=\sin\theta$
Consider a right angle triangle or use trigo identity
$\cos^2\theta+\sin^2\theta=1$



(b)

$y=\cos\left(x^2\right)$

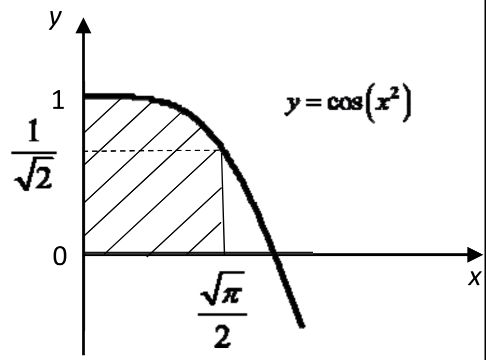
$x=0\Rightarrow y=1$

$x=\frac{\sqrt{\pi}}{2}\Rightarrow y=\cos\frac{\pi}{4}=\frac{1}{\sqrt{2}}$

Required volume

$=\pi\left(\frac{\sqrt{\pi}}{2}\right)^2\left(\frac{1}{\sqrt{2}}\right)+\pi\int_{\frac{1}{\sqrt{2}}}^{1}\cos^{-1}y\,\mathrm{d}y$

$=\frac{\pi^2}{4\sqrt{2}}+\pi\int_{\frac{1}{\sqrt{2}}}^{1}\cos^{-1}y\,\mathrm{d}y$

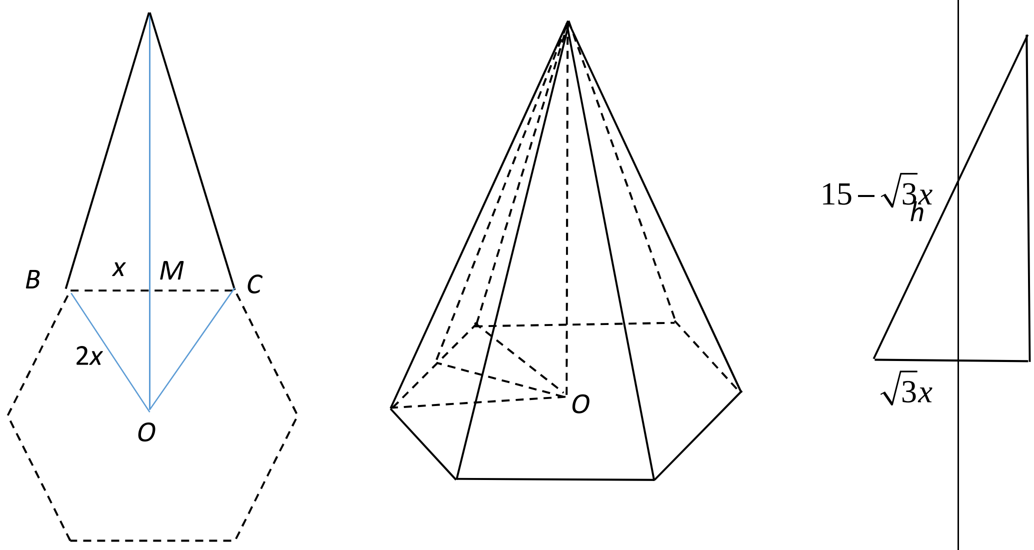


$\int\cos^{-1}y\,\mathrm{d}y$

$=y\cos^{-1}y-\int-\frac{y}{\sqrt{1-y^2}}\,\mathrm{d}y$

Let $u=\cos^{-1}y$ $\qquad$ $\frac{\mathrm{d}v}{\mathrm{d}y}=1$

$\frac{\mathrm{d}u}{\mathrm{d}y}=-\frac{1}{\sqrt{1-y^2}}$ $\qquad$ $v=y$

	$= y\cos^{-1} y - \frac{1}{2}\int \frac{-2y}{\sqrt{1-y^2}}\,\mathrm{d}y$ $= y\cos^{-1} y - \frac{1}{2}[2(1-y^2)^{\frac{1}{2}}] + c$ $= y\cos^{-1} y - \sqrt{1-y^2} + c$ Required volume = $\frac{\pi^2}{4\sqrt{2}} + \pi\left[y\cos^{-1} y - \sqrt{1-y^2}\right]_{\frac{1}{\sqrt{2}}}^{1}$ $= \frac{\pi^2}{4\sqrt{2}} + \pi\left[0 - \left(\frac{1}{\sqrt{2}}\frac{\pi}{4} - \frac{1}{\sqrt{2}}\right)\right]$ $= \frac{\pi}{\sqrt{2}}$
9	 $OM = \sqrt{(2x)^2 - x^2} = \sqrt{3}x$ $\therefore \quad AM = 15 - \sqrt{3}x$ Let h cm be the height of the pyramid. $h^2 = (15 - \sqrt{3}x)^2 - (\sqrt{3}x)^2$ $= 225 - 30\sqrt{3}x + 3x^2 - 3x^2$ $= 225 - 30\sqrt{3}x$ (shown)

Area of hexagon = $6\times$ area of triangle OBC

$$= 6(\frac{1}{2})(2x)(\sqrt{3}x)$$
$$= 6\sqrt{3}x^2$$

$$\therefore \quad V = \frac{1}{3}(6\sqrt{3}x^2)\sqrt{225-30\sqrt{3}x}$$

$V^2 = 180x^4(15-2\sqrt{3}x)$ (shown)

$V^2 = 180(15x^4 - 2\sqrt{3}x^5)$

Differentiating wrt x,

$$2V\frac{dV}{dx} = 180\left(60x^3 - 10\sqrt{3}x^4\right)$$
$$= 1800x^3\left(6-\sqrt{3}x\right)$$

$$\frac{\mathrm{d}V}{\mathrm{d}x} = 0 \Rightarrow x = 0 \quad \text{or} \quad x = \frac{6}{\sqrt{3}} = 2\sqrt{3}$$

(NA as $x > 0$)

Alternatively,

$$V = 6\sqrt{5}x^2)\sqrt{15-2\sqrt{3}x}$$

$$\frac{\mathrm{d}V}{\mathrm{d}x} = (6\sqrt{5}x^2)\frac{1}{2}(15-2\sqrt{3}x)^{-\frac{1}{2}}(-2\sqrt{3}) + (15-2\sqrt{3}x)^{\frac{1}{2}}(12\sqrt{5}x)$$

$$= 12\sqrt{5}x(15-2\sqrt{3}x)^{\frac{1}{2}} - 6\sqrt{15}x^2(15-2\sqrt{3}x)^{-\frac{1}{2}}$$

$$= 6\sqrt{5}x(15-2\sqrt{3}x)^{-\frac{1}{2}}[2(15-2\sqrt{3}x)-\sqrt{3}x]$$

$$= \frac{30\sqrt{5}x(6-\sqrt{3}x)}{\sqrt{(15-2\sqrt{3}x)}}$$

$$\frac{\mathrm{d}V}{\mathrm{d}x} = 0 \Rightarrow x = 0 \quad \text{or} \quad x = \frac{6}{\sqrt{3}} = 2\sqrt{3}$$

(NA as $x > 0$)

To Prove Maximum

Method 1

$$2V\frac{d^2V}{dx^2} + 2\left(\frac{dV}{dx}\right)^2 = 180\left[180x^2 - 40\sqrt{3}x^3\right]$$

$$x = 2\sqrt{3},\ \frac{d^2V}{dx^2} = \frac{180}{2V}\left[180\left(2\sqrt{3}\right)^2 - 40\sqrt{3}\left(2\sqrt{3}\right)^3\right] = -\frac{64800}{V} < 0 \text{ since } V > 0$$

Method 2

x	3.4	$2\sqrt{3} = 3.46$	3.5
$\frac{\mathrm{d}V}{\mathrm{d}x}$	$\approx \frac{7855}{2V} > 0$	0	$\approx -\frac{4799}{2V}$

V is maximum when $x = 2\sqrt{3}$ cm.

Max $V = 72\sqrt{15}$ cm^3.

(iv)

When $x = 2\sqrt{3}$, $h^2 = 225 - 30\sqrt{3}(2\sqrt{3}) = 45$

$h = 3\sqrt{5}$ cm (reject $h = -3\sqrt{5}$ as $h > 0$)

10	**(i)** $l: x+2=\dfrac{4-y}{3}, z=0$ $l: \mathbf{r}=\begin{pmatrix}-2\\4\\0\end{pmatrix}+\lambda\begin{pmatrix}1\\-3\\0\end{pmatrix} \qquad \lambda\in\mathbb{R}$ $\begin{pmatrix}-2\\4\\0\end{pmatrix}-\begin{pmatrix}-3\\4\\-2\end{pmatrix}=\begin{pmatrix}1\\0\\2\end{pmatrix}$ $\begin{pmatrix}1\\-3\\0\end{pmatrix}\times\begin{pmatrix}1\\0\\2\end{pmatrix}=\begin{pmatrix}-6\\-2\\3\end{pmatrix}$ $r\bullet\begin{pmatrix}6\\2\\-3\end{pmatrix}=\begin{pmatrix}-3\\4\\-2\end{pmatrix}\bullet\begin{pmatrix}6\\2\\-3\end{pmatrix}=-4$ $p: 6x+2y-3z=-4$ **(ii)** To find intersection between *y*-axis and *l*, sub $x=0$ into *l* $0+2=\dfrac{4-y}{3}\Rightarrow y=-2$ Thus, point of intersection is $(0,-2,0)$. Point of reflection of $(-2,4,0)$ about *y*-axis is $(2,4,0)$ $\begin{pmatrix}2\\4\\0\end{pmatrix}-\begin{pmatrix}0\\-2\\0\end{pmatrix}=\begin{pmatrix}2\\6\\0\end{pmatrix}=2\begin{pmatrix}1\\3\\0\end{pmatrix}$ Line of reflection, $l': \mathbf{r}=\begin{pmatrix}0\\-2\\0\end{pmatrix}+s\begin{pmatrix}1\\3\\0\end{pmatrix} \qquad s\in\mathbb{R}$ **(iii)** $\begin{pmatrix}-9\\9\\-6\end{pmatrix}-\begin{pmatrix}-3\\4\\-2\end{pmatrix}=\begin{pmatrix}-6\\5\\-4\end{pmatrix}$

	$$\left\|\begin{pmatrix}-6\\5\\-4\end{pmatrix}\bullet\begin{pmatrix}1\\-3\\0\end{pmatrix}\right\| = \left\|\begin{pmatrix}-6\\5\\-4\end{pmatrix}\right\|\left\|\begin{pmatrix}1\\-3\\0\end{pmatrix}\right\|\cos\theta$$ $$\cos\theta = \frac{21}{\sqrt{(-6)^2+5^2+(-4)^2}\sqrt{1^2+3^2}} = \frac{21}{\sqrt{770}}$$ $\theta = 40.8^{\circ}$ **(iv)** Let the point that is equidistant from both planes be *C*. $$\overrightarrow{OC} = \begin{pmatrix}-3\\4\\-2\end{pmatrix} + t\begin{pmatrix}6\\-5\\4\end{pmatrix} \text{ for some } t \in \mathbb{R}$$ Distance of *C* from *p* = Distance of *C* from *x*-*y* plane $$\frac{\left\|\left[\begin{pmatrix}-3+6t\\4-5t\\-2+4t\end{pmatrix}-\begin{pmatrix}-3\\4\\-2\end{pmatrix}\right]\bullet\begin{pmatrix}6\\2\\-3\end{pmatrix}\right\|}{\sqrt{6^2+2^2+3^2}} = \frac{\left\|\left[\begin{pmatrix}-3+6t\\4-5t\\-2+4t\end{pmatrix}-\begin{pmatrix}0\\0\\0\end{pmatrix}\right]\bullet\begin{pmatrix}0\\0\\1\end{pmatrix}\right\|}{\sqrt{0^2+0^2+1^2}}$$ $$\frac{\|36t-10t-12t\|}{7} = \|-2+4t\|$$ $\|t\| = \|2t-1\|$ $t^2 = 4t^2-4t+1$ $3t^2-4t+1=0$ $t = 2t-1$ or $t = -2t+1$ $t=1$ or $t=\frac{1}{3}$ $$\overrightarrow{OC} = \begin{pmatrix}-3\\4\\-2\end{pmatrix}+(1)\begin{pmatrix}6\\-5\\4\end{pmatrix} = \begin{pmatrix}3\\-1\\2\end{pmatrix} \text{ or } \overrightarrow{OC} = \begin{pmatrix}-3\\4\\-2\end{pmatrix}+\left(\frac{1}{3}\right)\begin{pmatrix}6\\-5\\4\end{pmatrix} = \left(\frac{1}{3}\right)\begin{pmatrix}-3\\7\\-2\end{pmatrix}$$ The 2 points are $(3,-1,2)$ and $\left(-1,\frac{7}{3},-\frac{2}{3}\right)$.