MERIDIAN JUNIOR COLLEGE
JC2 Preliminary Examination
Higher 2

H2 Mathematics 9758/01

Paper 1 **14 September 2017**

3 Hours

Additional Materials: Writing paper
Graph Paper
List of Formulae (MF 26)

READ THESE INSTRUCTIONS FIRST

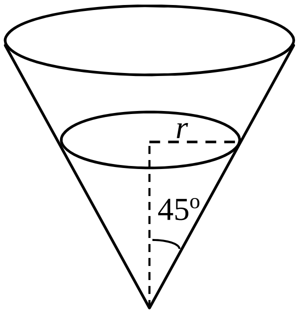
Write your name and civics group on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

1 Water is leaking at a rate of 2 cm^3 per minute from a container in the form of a cone, with its axis vertical and vertex downwards. The semi-vertical angle of the cone is 45° (see diagram). At time t minutes, the radius of the water surface is r cm. Find the rate of change of the depth of water when the depth of water in the container is 0.3 cm. [4]

[The volume of a cone of base radius r and height h is given by $V = \frac{1}{3}\pi r^2 h$.]



2 Without using a calculator, solve the inequality

$$\frac{x}{x-1} \leq \frac{4}{x+2} .$$ [5]

3 **Do not use a calculator in answering this question.**

Showing your working, find the complex numbers z and w which satisfy the simultaneous equations

$$4\mathrm{i}z - 3w = 1 + 5\mathrm{i} \quad \text{and}$$

$$2z + (1+\mathrm{i})w = 2 + 6\mathrm{i}.$$ [5]

4 **(a)** The points A and B relative to the origin O have position vectors $3\mathbf{i}-\mathbf{j}+3\mathbf{k}$ and $-3\mathbf{i}+2\mathbf{j}$ respectively.

(i) Find the angle between $\overrightarrow{OA}$ and $\overrightarrow{OB}$. [2]

(ii) Hence or otherwise, find the shortest distance from B to line OA. [2]

(b) The points C, D and E relative to the origin O have non-zero and non-parallel position vectors $\mathbf{c}$, $\mathbf{d}$ and $\mathbf{e}$ respectively. Given that $(\mathbf{c}\times\mathbf{d})\cdot\mathbf{e}=0$, state with reason(s) the relationship between O, C, D and E. [2]

5 **(i)** Prove by the method of differences that

$$\sum_{r=1}^{n}\frac{1}{r(r+1)(r+2)}=\frac{1}{4}-\frac{k}{2(n+1)(n+2)},$$

where k is a constant to be determined. [5]

(ii) Explain why $\displaystyle\sum_{r=1}^{\infty}\frac{1}{r(r+1)(r+2)}$ is a convergent series, and state its value. [2]

(iii) Using your answer in part **(i)**, show that $\displaystyle\sum_{r=1}^{n}\frac{1}{(r+2)^3}<\frac{1}{4}$. [2]

6 A curve C has equation $y=\dfrac{ax+b}{cx+1}$, where a, b and c are positive real constants and $b>\dfrac{a}{c}$.

(i) Sketch C, stating the equations of any asymptotes and the coordinates of the points where the curve crosses the axes. [3]

The curve C is transformed by a scaling parallel to y-axis by factor $\dfrac{1}{2}$ and followed by a translation of 2 units in the positive x-direction.

(ii) Find the equation of the new curve in the form of $y=\mathrm{f}(x)$. [2]

It is given that the new curve $y=\mathrm{f}(x)$ passes through the points with coordinates $\left(3,\dfrac{3}{2}\right)$ and $(6,1)$, and that $y=\dfrac{3}{4}$ is one of the asymptotes of the new curve $y=\mathrm{f}(x)$.

(iii) Find the values of a, b and c. [5]

7 **(i)** Given that $\mathrm{f}(x)=\tan\left(\frac{1}{2}x+\frac{1}{4}\pi\right)$, show that $\mathrm{f}'(x)=\frac{1}{2}\left[1+\left(\mathrm{f}(x)\right)^2\right]$, and find $\mathrm{f}(0)$, $\mathrm{f}'(0)$, $\mathrm{f}''(0)$ and $\mathrm{f}'''(0)$. Hence write down the first four non-zero terms in the Maclaurin series for $\mathrm{f}(x)$. [7]

(ii) The first three non-zero terms in the Maclaurin series for $\mathrm{f}(x)$ are equal to the first three non-zero terms in the series expansion of $\frac{\cos(ax)}{1+bx}$. By using appropriate expansions from the List of Formulae (MF26), find the possible value(s) for the constants a and b. [5]

8 10 pirates live on a pirate ship and they are ranked based on their seniority.

(a) One day, the pirates found a treasure chest that consists of some gold coins. The rule which the pirates adhered by to divide all the gold coins are based on their seniority and is as follows: The most senior pirate will get 3 gold coins more than the 2nd most senior pirate. The 2nd most senior pirate will also get 3 gold coins more than the 3rd most senior pirate and so on. Thus, the most junior pirate will get the least number of gold coins.

(i) If the treasure chest contains 305 gold coins, find the number of gold coins the most senior pirate will get. [3]

(ii) Find the least number of gold coins the treasure chest must contain if all pirates get some (at least one) gold coins each. [2]

(b) The pirates need to take turns, one at a time, to be on the lookout for their ship. Each day (24 hours) is divided into 10 shifts rotated among the 10 pirates. The 1st lookout shift starts from 10pm daily and it starts with the most junior pirate to the most senior pirate. The length of their shift is also based on their seniority. The length of shift for the most senior pirate is 10% less than that of the 2nd most senior pirate. The length of shift for the 2nd most senior pirate is 10% less than that of the 3rd most senior pirate and so on. Thus, the most junior pirate has the longest shift.

(i) Show that the length of shift for the most junior pirate is 3.6848 hours, correct to 4 decimal places. [2]

(ii) Calculate the length of shift for the 6th most junior pirate. Find the start time of his shift, giving your answer to the nearest minute. [4]

9 A curve C has parametric equations

$$x = \sqrt{2}\cos\frac{t}{2}, \quad y = \sqrt{2}\sin t, \quad \text{for } -2\pi \le t \le 2\pi.$$

(i) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ and verify that curve C has a stationary point at P with parameter $\frac{\pi}{2}$. Hence find the equation of the normal to the curve at point P. [3]

(ii) Sketch C, indicating clearly all turning points and axial intercepts in exact form. [4]

(iii) Find the exact area bounded by the curve C. (You may first consider the area bounded by the curve C and the positive x-axis in the first quadrant.) [6]

10 The plane p_1 has equation $\mathbf{r} = \begin{pmatrix} -1 \\ 1 \\ 16 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$, where λ and μ are real parameters. The point A has position vector $5\mathbf{i} - 6\mathbf{j} + 7\mathbf{k}$.

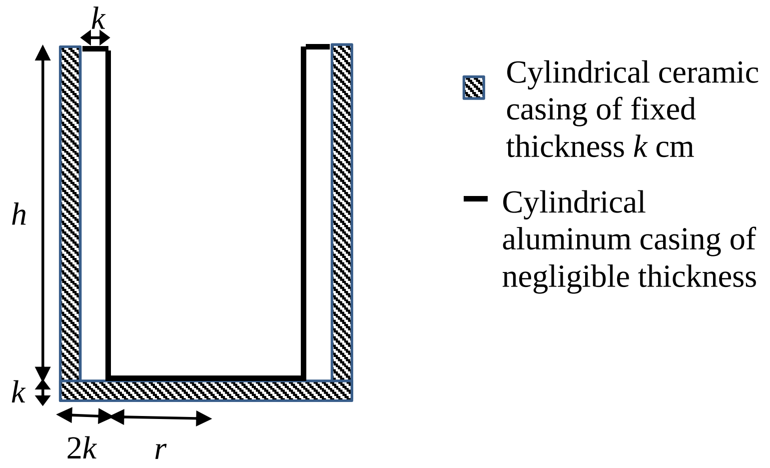
(i) Find a cartesian equation of p_1. [3]

(ii) Find the position vector of the foot of perpendicular from A to p_1. [4]

The plane p_2 has equation $\mathbf{r} \cdot \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix} = 52$. The plane p_3 is obtained by reflecting p_2 about p_1. By considering the relationship between A and p_2, or otherwise, find a cartesian equation of p_3. [6]

[Question 11 is printed on the next page.]

11 A company intends to manufacture a cylindrical double-walled ceramic vacuum flask which can hold a fixed V cm^3 of liquid when filled to the brim. The cylindrical vacuum flask is made up of an inner cylindrical aluminum casing (of negligible thickness) with height h cm and radius r cm and an outer cylindrical ceramic casing of fixed thickness k cm. There is a fixed k cm gap between the sides of the inner casing and outer casing where air has been removed to form a vacuum. The diagram below shows the view of the vacuum flask if it is dissected vertically through the centre.



Let the volume of the outer ceramic casing be C cm^3.

(i) Show that the volume of the ceramic casing can be expressed as $C = k\left(\frac{2V}{r} + \frac{3kV}{r^2} + \pi\left(r+2k\right)^2\right)$. [4]

(ii) Let r_1 be the value of r which gives the minimum value of C. Show that r_1 satisfies the equation $\pi r^4 + 2\pi k r^3 - rV - 3kV = 0$. [3]

For the rest of the question, it is given that $k = \frac{1}{4}$ and $V = 250$.

(iii) Find the minimum volume of the ceramic casing, proving that it is a minimum. [3]

(iv) Sketch the graph showing the volume of the ceramic casing as the radius of the aluminum casing varies. [2]

End of Paper