Millennia Institute H2 Mathematics 2017 Prelim II Solution

1 The curve C has the equation $4(x-1)^2+9y^2=36$.

(i) Sketch, for $y \geq 0$, the curve C, stating the coordinates of the end points and the turning point. [3]

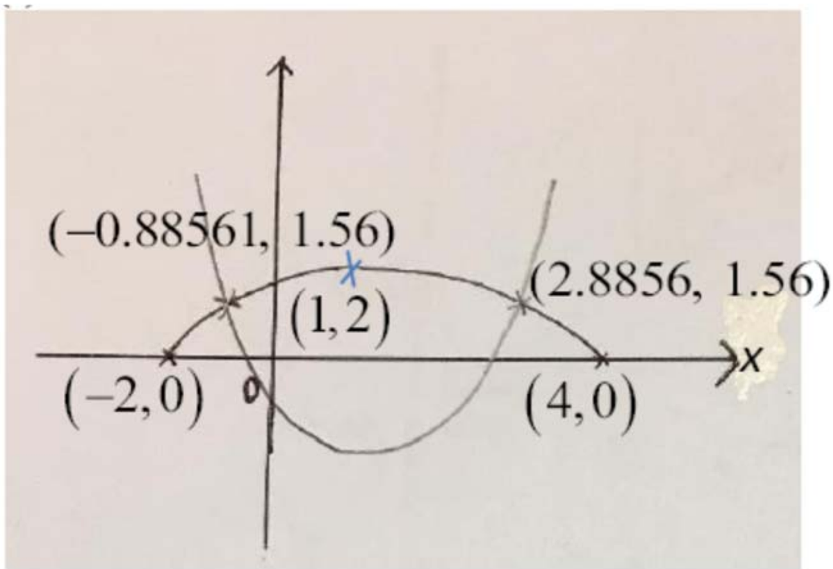
(ii) By adding a suitable graph to your sketch in part **(i)**, solve the inequality

$$2\sqrt{\left[1-\frac{(x-1)^2}{9}\right]}+2-(x-1)^2 \geq 0. \quad [2]$$

(iii) Hence, solve the inequality $2\sqrt{\left[1-\frac{(e^x-1)^2}{9}\right]} \geq \left(e^x-1\right)^2-2$. [2]

Solution:

(i)



$4(x-1)^2+9y^2=36$

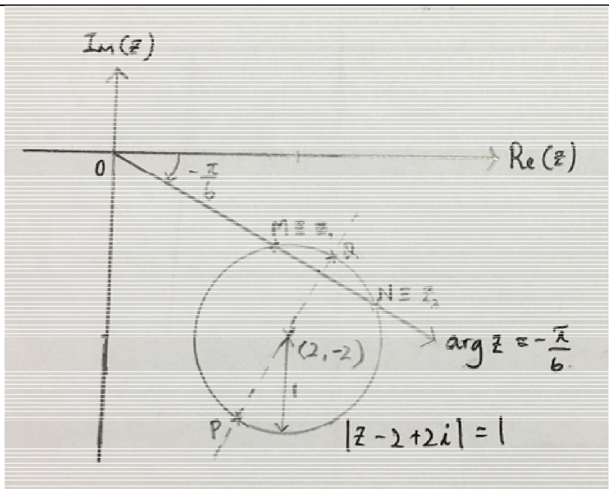
$y^2=\dfrac{36-4(x-1)^2}{9}$

$y^2=4\left[1-\dfrac{(x-1)^2}{9}\right]$

$y=2\sqrt{1-\dfrac{(x-1)^2}{9}}$ (for $y \geq 0$)

(ii)

	$2\sqrt{1-\frac{(x-1)^2}{9}}+2-(x-1)^2 \geq 0$ $2\sqrt{1-\frac{(x-1)^2}{9}} \geq (x-1)^2-2$ The suitable graph to be added is $y=(x-1)^2-2$. From the graph, $-0.88561 \leq x \leq 2.8856$ $-0.886 \leq x \leq 2.89$ (3 s.f) (iii) By comparison, $x \to \mathrm{e}^x$ $0 \leq \mathrm{e}^x \leq 2.8856$ $\ln \mathrm{e}^x \leq \ln 2.8856$ $x \leq 1.06$ (3 s.f)
2	Two loci in the Argand diagram are given by the equations $$\lvert z-2+2\mathrm{i}\rvert=1 \quad \text{and} \quad \arg z=-\frac{\pi}{6}.$$ The complex numbers z_1 and z_2, where $\lvert z_1\rvert<\lvert z_2\rvert$, correspond to the points of intersection of these loci. **(i)** Draw an Argand diagram to show both loci, and mark the points represented by z_1 and z_2. [3] **(ii)** Find the two values of z which represent points on $\lvert z-2+2\mathrm{i}\rvert=1$ such that $\lvert z-z_1\rvert=\lvert z-z_2\rvert$. [4] **(iii)** Given that the complex number w satisfies $\lvert w-2+2\mathrm{i}\rvert\leq 1$ and $\arg w\leq -\frac{\pi}{6}$, find the range of values of $\arg(w+3\mathrm{i})$. [3]
	Solution: (i) $\lvert z-2+2\mathrm{i}\rvert=1 \Rightarrow \lvert z-(2-2\mathrm{i})\rvert=1$ $\arg z=-\frac{\pi}{6}$



(ii)
The 2 values of z are as indicated as P and Q on the diagram.

$b=(1)\cos\dfrac{\pi}{6}$; $a=(1)\sin\dfrac{\pi}{6}$

$b=\dfrac{\sqrt{3}}{2}$ $\qquad$ $b=\dfrac{1}{2}$

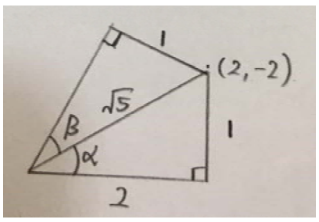
At Q: $z=\left(2+\dfrac{1}{2}\right)-\left(2-\dfrac{\sqrt{3}}{2}\right)\mathrm{i}$

At P: $z=\left(2-\dfrac{1}{2}\right)-\left(2+\dfrac{\sqrt{3}}{2}\right)\mathrm{i}$

The 2 values of z are

$\dfrac{5}{2}-\left(2-\dfrac{\sqrt{3}}{2}\right)\mathrm{i}$ and $z=\dfrac{3}{2}-\left(2+\dfrac{\sqrt{3}}{2}\right)\mathrm{i}$.

(iii)



Smallest value of $\arg\left[z-(-3\mathrm{i})\right]=0$

Since $\alpha=\beta$,

Largest value of $\arg\left[z-(-3\mathrm{i})\right]=2\tan^{-1}\dfrac{1}{2}=0.927$ (3 s.f)

$\therefore 0\le\arg\left[z+3\mathrm{i}\right]\le 0.927$ (3 s.f)

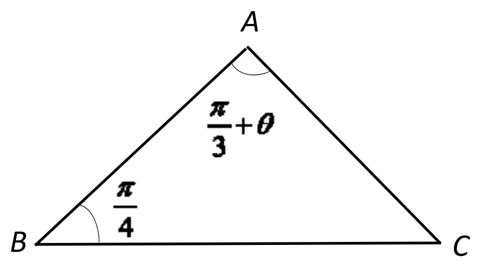
3 **(a)** It is given that $\tan^{-1}y=\ln(1+x)$.

(i) Show that $(1+x)\dfrac{\mathrm{d}y}{\mathrm{d}x}=1+y^2$. [1]

(ii) By successively differentiating this result, find the Maclaurin series for $\tan\left[\ln(1+x)\right]$, up to and including the term in x^3. [3]

(iii) It is given that $\mathrm{f}(x)=\mathrm{e}^x\tan\left[\ln(1+x)\right]$. Using your answer to part **(a)(ii)**, estimate the value of $\mathrm{f}'\left(\frac{1}{2}\right)$. [3]

(b) The diagram shows triangle *ABC*, where $AC=k$ cm, $BC=h$ cm, $\angle BAC=\frac{\pi}{3}+\theta$ and $\angle ABC=\frac{\pi}{4}$.



Given that θ is a sufficiently small angle, show that

$$\frac{h}{k}\approx\frac{\sqrt{2}}{4}\left[2\sqrt{3}+2\theta-\left(\sqrt{3}\right)\theta^2\right].$$ [3]

Solution:

(i)

$\tan^{-1}y=\ln(1+x)$

Differentiating both sides with respect to x:

$$\frac{1}{1+y^2}\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{1+x}$$

$(1+x)\frac{\mathrm{d}y}{\mathrm{d}x}=1+y^2$ (shown)

(ii)

	$(1+x)\dfrac{\mathrm{d}y}{\mathrm{d}x}=1+y^2$ Differentiating both sides with respect to x: $(1+x)\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+\dfrac{\mathrm{d}y}{\mathrm{d}x}=2y\dfrac{\mathrm{d}y}{\mathrm{d}x}\Rightarrow(1+x)\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+(1-2y)\dfrac{\mathrm{d}y}{\mathrm{d}x}=0$ Differentiating both sides with respect to x: $(1+x)\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}+\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+(1-2y)\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+(-2)\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2=0$ $(1+x)\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}+2(1-y)\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}-2\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2=0$ When $x=0, y=0, \dfrac{\mathrm{d}y}{\mathrm{d}x}=1, \dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=-1, \dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}=4$ $y=0+(1)x+(-1)\dfrac{x^2}{2!}+(4)\dfrac{x^3}{3!}+\ldots$ $y=x-\dfrac{1}{2}x^2+\dfrac{2}{3}x^3+\ldots$ (iii) $\mathrm{f}(x)=\mathrm{e}^x\tan\left[\ln(1+x)\right]$ $=\mathrm{e}^x\left(x-\dfrac{1}{2}x^2+\dfrac{2}{3}x^3+\ldots\right)$ $=\left(1+x+\dfrac{1}{2}x^2+\ldots\right)\left(x-\dfrac{1}{2}x^2+\dfrac{2}{3}x^3+\ldots\right)$ $=x-\dfrac{1}{2}x^2+\dfrac{2}{3}x^3+x^2-\dfrac{1}{2}x^3+-\dfrac{1}{2}x^3+\ldots$ $=x+\dfrac{1}{2}x^2+\dfrac{2}{3}x^3+\ldots$ $\mathrm{f}(x)=x+\dfrac{1}{2}x^2+\dfrac{2}{3}x^3+\ldots$ $\mathrm{f}'(x)=1+x+2x^2+\ldots$ $\mathrm{f}'\left(\dfrac{1}{2}\right)=1+\dfrac{1}{2}+2\left(\dfrac{1}{2}\right)^2+\ldots\approx 2$ (b)

	$\dfrac{\sin\left(\dfrac{\pi}{3}+\theta\right)}{h}=\dfrac{\sin\left(\dfrac{\pi}{4}\right)}{k}$ $\sin\left(\dfrac{\pi}{3}+\theta\right)=\dfrac{h}{k}\left(\dfrac{1}{\sqrt{2}}\right)$ $\dfrac{h}{k}\left(\dfrac{1}{\sqrt{2}}\right)=\sin\left(\dfrac{\pi}{3}\right)\cos(\theta)+\cos\left(\dfrac{\pi}{3}\right)\sin(\theta)$ from MF15 $\dfrac{h}{k}\left(\dfrac{1}{\sqrt{2}}\right)\approx\left(\dfrac{\sqrt{3}}{2}\right)\left(1-\dfrac{\theta^2}{2}\right)+\dfrac{\theta}{2}$ $\dfrac{h}{k}\left(\dfrac{1}{\sqrt{2}}\right)\approx\dfrac{\sqrt{3}}{2}-\dfrac{\sqrt{3}\theta^2}{4}+\dfrac{\theta}{2}$ $\dfrac{h}{k}\left(\dfrac{1}{\sqrt{2}}\right)\approx\dfrac{1}{4}\left(2\sqrt{3}+2\theta-\sqrt{3}\theta^2\right)$ $\dfrac{h}{k}\approx\dfrac{\sqrt{2}}{2}\left(\sqrt{3}+\theta-\dfrac{\sqrt{3}}{2}\theta^2\right)$ (shown)

4	The plane π_1 contains the line $l_1 : \mathbf{r} = \begin{pmatrix}1\\2\\3\end{pmatrix} + \lambda\begin{pmatrix}1\\1\\1\end{pmatrix}$, where $\lambda \in \mathbb{R}$, and is parallel to the line $l_2 : \mathbf{r} = \begin{pmatrix}1\\0\\1\end{pmatrix} + \mu\begin{pmatrix}2\\0\\1\end{pmatrix}$, where $\mu \in \mathbb{R}$. **(i)** Find the vector equation of π_1 in scalar product form. [2] **(ii)** Find the position vector of the foot of the perpendicular from the point $A(1, 0, 1)$ to the plane π_1. [3] **(iii)** Find the position vector of the point A', which is the reflection of A about π_1. [2] **(iv)** Given that the angle between $l_3 : \mathbf{r} = \begin{pmatrix}1\\1\\1\end{pmatrix} + \alpha\begin{pmatrix}1\\-1\\0\end{pmatrix}$, where $\alpha \in \mathbb{R}$, and the plane $\pi_2 : ax + 2y - z = 3$, where $a \in \mathbb{R}$, is $\frac{\pi}{4}$, find the value of *a*. [2] **(v)** Find the line of intersection between the planes π_1 and π_2. [1] **(vi)** π_3 has equation $bx + y + z = c$, where $b, c \in \mathbb{R}$. Given that π_1, π_2 and π_3 have no points in common, describe the geometrical relationship between the three planes. What can be said about the values of *b* and *c*? [3]
	Solution: (i) $\underset{\sim}{n} = \begin{pmatrix}1\\1\\1\end{pmatrix} \times \begin{pmatrix}2\\0\\1\end{pmatrix} = \begin{pmatrix}1\\1\\-2\end{pmatrix}$ $\underset{\sim}{r} \cdot \begin{pmatrix}1\\1\\-2\end{pmatrix} = \begin{pmatrix}1\\2\\3\end{pmatrix} \cdot \begin{pmatrix}1\\1\\-2\end{pmatrix} \Rightarrow \underset{\sim}{r} \cdot \begin{pmatrix}1\\1\\-2\end{pmatrix} = -3$ (ii) <u>Method 1:</u>

$$l_{AN} : \underset{\sim}{r} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \alpha \in \mathbb{R}$$

$$\overrightarrow{ON} = \begin{pmatrix} 1+\alpha \\ \alpha \\ 1-2\alpha \end{pmatrix}, \text{ for some } \alpha \in \mathbb{R}$$

Since N is the intersection point of line AN and plane,

$$\begin{pmatrix} 1+\alpha \\ \alpha \\ 1-2\alpha \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = -3$$

$$1+\alpha+\alpha-2+4\alpha = -3$$

$$\alpha = -\frac{1}{3}$$

$$\overrightarrow{ON} = \begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ \frac{5}{3} \end{pmatrix}$$

<u>Method 2:</u>

$$\overrightarrow{AN} = \left(\overrightarrow{AB} \cdot \frac{\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}}{\sqrt{6}} \right) \frac{\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}}{\sqrt{6}}, \text{ where } \overrightarrow{OB} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$\overrightarrow{ON} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \left(\begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} \cdot \frac{\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}}{\sqrt{6}} \right) \frac{\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}}{\sqrt{6}}$$

$$\overrightarrow{ON} = \begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ \frac{5}{3} \end{pmatrix}$$

(iii)

$$\overrightarrow{ON} = \frac{\overrightarrow{OA} + \overrightarrow{OA'}}{2}$$

	$\overrightarrow{OA'} = 2\overrightarrow{ON} - \overrightarrow{OA} = \begin{pmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ \frac{7}{3} \end{pmatrix}$ (iv) $\pi_2 : \underset{\sim}{r} \cdot \begin{pmatrix} a \\ 2 \\ -1 \end{pmatrix} = 3$ $\sin\theta = \dfrac{\left\lvert \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} a \\ 2 \\ -1 \end{pmatrix} \right\rvert}{\left(\sqrt{2}\right)\left(\sqrt{\left(a^2+4+1\right)}\right)}$ Since $\theta = \dfrac{\pi}{4}$, $\dfrac{\sqrt{2}}{2} = \dfrac{\lvert a-2 \rvert}{\left(\sqrt{2}\right)\sqrt{\left(a^2+5\right)}}$ $\sqrt{\left(a^2+5\right)} = \lvert a-2 \rvert$ $\left(a^2+5\right) = a^2 - 4a + 4$ $a = -\dfrac{1}{4}$ (v) Using GC: Equation of line of intersection: $\underset{\sim}{r} = \begin{pmatrix} -4 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix}, \beta \in \mathbb{R}$ (vi) Geometrical interpretation: Either: the three planes are the sides of a triangular prism OR: π_3 is parallel to the line of intersection of π_1 and π_2, but does not contain it. $\pi_3 : \underset{\sim}{r} \cdot \begin{pmatrix} b \\ 1 \\ 1 \end{pmatrix} = c$, $\begin{pmatrix} b \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix} = 0 \Rightarrow b = -\dfrac{5}{4}$ $\begin{pmatrix} -4 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} b \\ 1 \\ 1 \end{pmatrix} \neq c \Rightarrow c \neq 6$
5	Resilience Primary School has 500 students who are either Chinese, Indian or Malay, as

seen in the table below.

	Chinese	Indian	Malay
Boys	114	8	93
Girls	122	77	86

The National Eye Centre wishes to conduct a survey at Resilience Primary School to find out the number of hours students spend on electronic devices each week, using a sample of 50 students.

(i) Explain how stratified sampling can be carried out in this context. [2]

(ii) Give two reasons why systematic sampling may not be appropriate. [2]

Solution:

(i)

	Chinese	Indian	Malay
Boys	$\frac{114}{500}\times 50\approx 11$	$\frac{8}{500}\times 50\approx 1$	$\frac{93}{500}\times 50\approx 9$
Girls	$\frac{122}{500}\times 50\approx 12$	$\frac{77}{500}\times 50\approx 8$	$\frac{86}{500}\times 50\approx 9$

Split the students into the stratas for Chinese, Indian, Malay boys or girls as shown in the table above. Arrange the students within each strata in alphabetical order (for example). Using simple random sampling, obtain the required number in each strata.

(ii)

$$k=\frac{500}{50}=10$$

Since $k=10>8=$ number of Indian boys available, there is a possibility the Indian boys may not be represented.

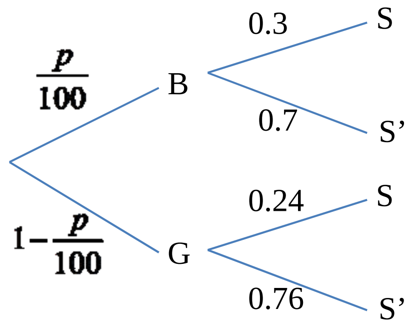
Systematic sampling does not ensure equal proportions of students being taken from each strata.

6 In another survey conducted by the National Eye Centre, it was found that p% are boys and the remaining are girls. The probability that a randomly chosen boy wears spectacles is 0.3 and the probability that a randomly chosen girl wears spectacles is 0.24.

(i) Find the value of p, given that the probability that a randomly chosen child wears spectacles is 0.267. [2]

(ii) For a general value of p, the probability that a randomly chosen child that wears spectacles is a girl is denoted by $\mathrm{f}(p)$. Show that $\mathrm{f}(p)=\dfrac{4(100-p)}{(400+p)}$. Prove by differentiation that f is a decreasing function for $0 \le p \le 100$, and explain what this statement means in the context of the question. [5]

Solution:



B = Boys

G = Girls

S = wear spectacles

S' = don't wear spectacles

(i)

$$\frac{p}{100}(0.3)+\left(1-\frac{p}{100}\right)(0.24)=0.267$$

$$0.0006p=0.027$$

$$p=45$$

(ii)

$$\mathrm{P}(\text{Girl}\mid\text{spectacles})=\frac{0.24\left(1-\frac{p}{100}\right)}{0.3\left(\frac{p}{100}\right)+0.24\left(1-\frac{p}{100}\right)}$$

$$=\frac{0.24-0.0024p}{0.003p+0.24-0.0024p}$$

$$=\frac{0.0024(100-p)}{0.0006(400+p)}$$

$$\mathrm{f}(p)=\frac{4(100-p)}{(400+p)} \quad \text{(shown)}$$

$$\mathrm{f}'(p)=\frac{(400+p)(-4)-(400-4p)}{(400+p)^2}$$

$$=\frac{-2000}{(400+p)^2}$$

Since $(400+p)^2>0$,

$$\mathrm{f}'(p)=\frac{-2000}{(400+p)^2}<0,\ \forall p\in\mathbb{R}$$

$\therefore$ f is a decreasing function.

	Context: As the percentage of boys in the survey increases, the percentage that a girl wears spectacles decreases.
7	In this question you should state clearly all distributions that you use, together with the values of the appropriate parameters. The mass, in grams, of broccoli and carrots are normally distributed with means and standard deviations as shown in the table below. (table below) **(i)** Given that the probability that the mass of a randomly chosen broccoli does not exceed 250g is 0.788 and the probability that the mass of a randomly chosen broccoli exceeds 236g is 0.625, find the values of μ and σ. [3] **(ii)** Find the probability that the mass of a randomly chosen broccoli lies within 5 grams of a randomly chosen carrot. [2] **(iii)** 120 broccoli are randomly chosen. Using a suitable approximation, find the probability that there are fewer than 90 broccoli with a mass not exceeding 250g. [3] **(iv)** Determine, with explanation, whether the mass of a vegetable chosen randomly from a basket containing an equal number of broccoli and carrots follows a normal distribution. [1]

	Mean (g)	**Standard deviation (g)**
Broccoli	μ	σ
Carrot	180	15

Solution:

Let X and Y be the random variable, the mass of a broccoli and the mass of a carrot respectively

$X \sim \mathrm{N}\left(\mu, \sigma^2\right)$, $Y \sim \mathrm{N}\left(180, 15^2\right)$

(i) $\mathrm{P}\left(X \le 250\right) = 0.788$

$$\mathrm{P}\left(Z \le \frac{250-\mu}{\sigma}\right) = 0.788$$

$$\frac{250-\mu}{\sigma} = 0.79950$$

$\mu + 0.79950\sigma = 250 \quad ---(1)$

$\mathrm{P}\left(X > 236\right) = 0.625$

$$\mathrm{P}\left(Z \le \frac{236-\mu}{\sigma}\right) = 0.375$$

$$\frac{236-\mu}{\sigma}=-0.31864$$

$\mu-0.31864\sigma=236 \quad ---(2)$

Using GC:

$\mu \approx 239.99 \approx 240$ (3 s.f.) and $\sigma \approx 12.521 \approx 12.5$ (3 s.f.)

(ii)

$X-Y \sim \mathrm{N}(59.99, 381.78)$

$\mathrm{P}(|X-Y| \le 5) = \mathrm{P}(-5 \le X-Y \le 5)$

$= 0.00200$ (3 s.f.)

(iii)

Let W be the random variable, the number of broccoli with mass not exceeding 250g

$W \sim \mathrm{B}(120, 0.788)$

Since $n=120>50$, $np=94.56>5$, $nq=25.44>5$

$W \sim \mathrm{N}(94.56, 20.047)$ approx.

$\mathrm{P}(W<90)=\mathrm{P}(W \le 89)$

$=\mathrm{P}(W<89.5)$ (using Continuity Correction)

$= 0.129$ (3 s.f.)

(iv) It will not follow normal distribution as the mass of a randomly chosen vegetable from a basket containing an equal number of broccoli and carrot follows a bimodal distribution.

8 The table gives the values of eight observations of bivariate data, x and y.

x	1	2	3	4	5	6	7	8
y	5	1	18	23	28	31	33	34

(i) Draw a scatter diagram for these values, labelling the axes clearly. Determine the outlier by labelling it as P in your scatter diagram. [2]

(ii) By omitting P, explain if $y=ax^2+b$ or $y=a\ln x+b$ is the better model for the data. [2]

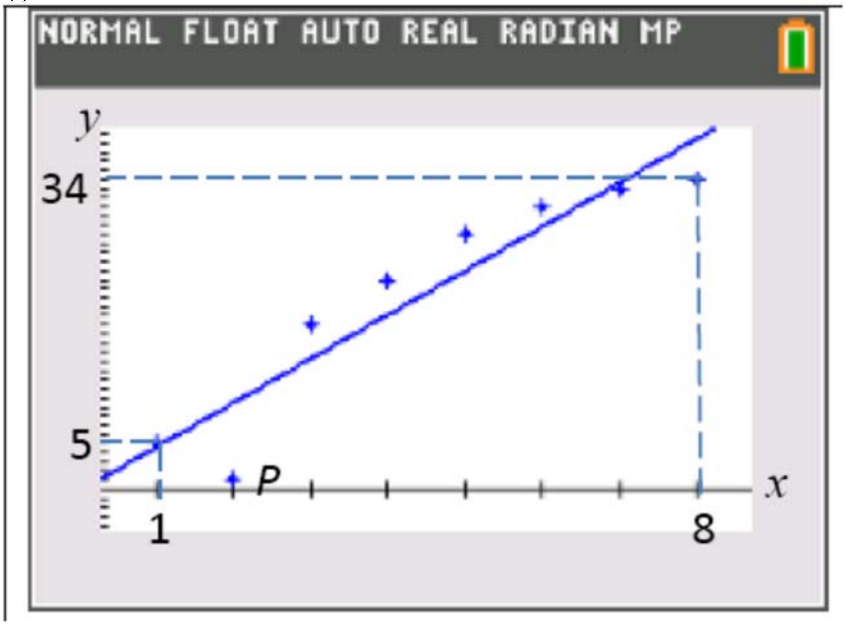
(iii) Using the more appropriate model found in part **(ii)**, calculate the equation of the least-squares regression line. [1]

(iv) Interpret, in the context of the question, the least squares estimates of a and b. [2]

(v) Use the regression line found in part **(iii)** to predict the value of y when x = 4.5. Comment on the reliability of your answer. [2]

Solution:

(i)



(ii) $y = ax^2 + b$: $r = 0.880$ (3 s.f.)

$y = a\ln x + b$: $r = 0.994$ (3 s.f.)

Since $y = a\ln x + b$ has $|r|$ closer to 1, $y = a\ln x + b$ is the better model.

(iii)

$y = 4.0144 + 14.518\ln x$

$\approx 4.01 + 14.5\ln x\ (3\text{ s.f.})$

(iv)

The expected value of y when $\ln x$ is 0 is 4.01.

For every increase in $\ln x$ by 1 unit, expected value of y increases by 14.5 units.

(v)

At x = 4.5, $y = 4.0144 + 14.518\ln(4.5) = 25.9$ (3 s.f.)

Reliable because x = 4.5 lies within the data range and $|r|$ is close to 1

9 Based on past records, the mean number of rainy days per year in Singapore was reported as 178. The authorities suspect that due to global warming, the number of rainy days has changed. A random sample of 12 years is taken and the number of rainy days per year, X, is summarised by

$$\sum(x-8) = 2017.7, \qquad \sum x^2 = 372\,500.$$

(i) Calculate the unbiased estimates of the mean and variance of X. [2]

	(ii) Test, at the 5% level of significance, whether the mean number of rainy days per year has changed. State any assumptions used in your calculations. [4] **(iii)** Explain, in the context of the question, the meaning of the p-value. [1] **(iv)** The population variance is found to be 9 and the assumption used in part **(ii)** holds true. A test at the 5% level of significance whether the mean number of rainy days per year has changed was conducted. Find the range of values of $\bar{x}$ such that the null hypothesis is not rejected. [3]
	Solution: (i) $\bar{x}=\frac{2017.7}{12}+8\approx 176.14\approx 176$ (3 s.f.) <u>Method 1</u> $s^2=\frac{1}{n-1}\left(\sum x^2-n\left(\bar{x}\right)^2\right)$ $=\frac{1}{11}\left(372500-12\left(176.14\right)^2\right)$ $\approx 17.855\approx 17.9$ (3 s.f.) <u>Method 2</u> $\sum x=2017.7+8\left(12\right)=2113.7$ $s^2=\frac{1}{n-1}\left(\sum x^2-\frac{\left(\sum x\right)^2}{n}\right)$ $=\frac{1}{11}\left(372\,500-\frac{\left(2113.7\right)^2}{12}\right)$ $=17.214\approx 17.2\ (3\,\text{sf})$ (ii) Let X be the random variable, the number of rainy days per year in Singapore $\text{H}_0:\mu=178$ $\text{H}_1:\mu\neq 178$ Assume H_0 is true. $\alpha=0.05$. Assume X follows normal distribution. Since $n=12<50$, population variance unknown, $T\sim \text{t}\left(11\right)$ approx. 2 tail t-test used. <u>Method 1:</u>

	Using GC, p-value = 0.156 (3 s.f.) > 0.05 if $s^2 = 17.855$ used [Alt: p-value = 0.149 (3 s.f.) > 0.05 if $s^2 = 17.214$ used] Do not reject H_0 <u>Method 2:</u> Test-statistic value: $t = \dfrac{176.14 - 178}{\sqrt{\dfrac{17.855}{12}}} \approx -1.52$ (3 s.f.) if $s^2 = 17.855$ used [Alt: $t = \dfrac{176.14 - 178}{\sqrt{\dfrac{17.214}{12}}} \approx -1.55$ (3 s.f.) if $s^2 = 17.214$ used] Critical region: $t \le -2.20$ (3 s.f.) or $t \ge 2.20$ (3 s.f.) Since test-statistic does not lie in the critical region, H_0 is not rejected. There is insufficient evidence at 5% level of significance to conclude that the mean number of rainy days per year has changed. (iii) <u>Either</u> p-value is the smallest level of significance for which the null hypothesis of the mean number of rainy days per year is 178 will be rejected. <u>Or</u> p-value is twice the probability of obtaining a test statistic less than or equal to -1.52, assuming the null hypothesis of the mean number of rainy days per year is 178 is true. (iv) $H_0 : \mu = 178$ $H_1 : \mu \ne 178$ Assume H_0 is true. Since X is normal, $\overline{X} \sim N\left(178, \dfrac{9}{12}\right)$ 2 tail z-test used. Since H_0 is not rejected at the 5% level of significance, $-1.9600 < \dfrac{\overline{x} - 178}{\left(\sqrt{\left(\dfrac{3}{4}\right)}\right)} < 1.9600$ $-1.9600\sqrt{\left(\dfrac{3}{4}\right)} < \overline{x} - 178 < 1.9600\sqrt{\left(\dfrac{3}{4}\right)}$ $176 < \overline{x} < 180$ (3 s.f.)
10	**(a)** Find the number of ways in which the letters of the word MILLENNIUM can be

arranged if

(i) there are no restrictions, [1]

(ii) the first and last letters are the same, and the letters E and U must be separated. [2]

Four letters are randomly selected from the letters of the word MILLENNIUM to form a code word. Find the number of possible code words that can be formed. [2]

(b) Mr See (together with his wife and daughter) and Mrs Saw (together with her husband and two sons) came to visit their former teacher Mdm Rain during Millennium Institute's Homecoming Day. Find the number of ways Mr See and his family, Mrs Saw and her family, and Mdm Rain can be arranged if

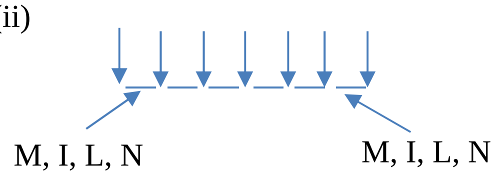
(i) they are around a table with ten indistinguishable chairs, such that the children are seated together. [2]

(ii) the two empty chairs are removed and Mr See's daughter is seated between her parents and the See family are to be seated directly opposite Mdm Rain. [3]

Solution:

(a)(i) No. of ways = $\dfrac{10!}{2!2!2!2!} = 226\,800$

(ii)



No. of ways $= {}^4C_1 \times \dfrac{6!}{2!2!2!} \times {}^7C_2 \times 2!$

$= 15\,120$

(a)(last part)

<u>Case 1: 2 Repeats</u>

No. of ways $= {}^4C_2 \times \dfrac{4!}{2!2!} = 36$

<u>Case 2: 1 Repeat</u>

No. of ways $= {}^4C_1 \times {}^5C_2 \times \dfrac{4!}{2!} = 480$

<u>Case 3: No Repeat</u>

No. of ways $= {}^6C_4 \times 4! = 360$

Total ways = 876

	(b)(i) No. of ways $=\dfrac{8!}{8(2!)}\times 3!$ = 15 120 (b)(ii) No. of ways $=\dfrac{2!}{2}\times 2!\times 4!$ = 48
11	In this question you should state clearly all distributions that you use, together with the values of the appropriate parameters. The number of people queuing to buy coffee at CoffeeVille in a period of 1 minute during lunch hour (12pm to 2pm) is a random variable with an average number of 2.9. State, in context, a condition under which a Poisson distribution would be a suitable probability model. [1] Assume that the number of people queuing to buy coffee at CoffeeVille in a period of 1 minute during the lunch hour follows the distribution $\text{Po}(2.9)$. **(i)** State the most probable number of people queuing in 1 minute. [1] **(ii)** Find the probability that in a period of 3 minutes, there are at most 5 people queuing to buy coffee. [2] **(iii)** *N* periods of 3 minutes are taken. Given that the probability that at least 7 periods of 3 minutes have at most 5 people queuing to buy coffee is more than 0.99, find the least value of *N*. [3] **(iv)** A random sample of 120 periods of 3 minutes is taken. Using a suitable approximation, find the probability that more than 12 periods of 3 minutes have exactly 4 people queuing. [3] **(v)** Explain why the Poisson model would probably not be valid if applied to the operating hours of CoffeeVille from 11am to 10pm. [1]
	Solution: Average number of people queuing to buy coffee is a constant (i) Let X be the random variable, for the number of people queuing to buy coffee in 1 min. $X \sim \text{Po}(2.9)$ Using GC: Mode = 2

(ii)
Let Y be the random variable, for the number of people queuing to buy coffee in 3 min.

$Y \sim \text{Po}(8.7)$

$\text{P}(Y \le 5) = 0.13516 \approx 0.135$ (3 s.f.)

(iii)
Let W be the random variable, for the number of periods of 3 min with $Y \le 5$

$W \sim \text{B}(n, 0.13516)$

$\text{P}(W \ge 7) > 0.99$

$1 - \text{P}(W \le 6) > 0.99$

$\text{P}(W \le 6) < 0.01$

Using GC:

N	$\text{P}(W \le 6)$
103	0.0104 > 0.01
104	0.00947 < 0.01
105	0.00864 < 0.01

Least value of N is 104

(iv)
Let V be the random variable, for the number of periods of 3 min with $Y = 4$

$V \sim \text{B}(120, 0.039765)$

Since $n = 120 > 50$, $np = 4.7718 < 5$
$V \sim \text{Po}(4.7718)$ approx.

$\text{P}(V > 12) = 1 - \text{P}(V \le 12) \approx 0.00135$ (3 s.f.)

(v)
Mean number of people queuing varies throughout the day.