YISHUN JUNIOR COLLEG5E
Mathematics Department

PRELIM SOLUTION

Subject : JC2 H2 MATHEMATICS 9740 P2 **Date** :

Qn	Solution
1	Let P_n be the statement $\sum_{r=1}^{n} \frac{r2^{r-1}}{(r+1)(r+2)} = \frac{2^n}{n+2} - \frac{1}{2}$, $n \in \mathbf{Z}^+$ When $n = 1$, LHS = $\frac{1}{2\times 3} = \frac{1}{6}$ RHS = $\frac{2}{1+2} - \frac{1}{2} = \frac{1}{6} =$ LHS $\therefore P_1$ is true. Assume that P_k is true for some $k \in \mathbf{Z}^+$. i.e. $\sum_{r=1}^{k} \frac{r2^{r-1}}{(r+1)(r+2)} = \frac{2^k}{k+2} - \frac{1}{2}$ Want to show that P_{k+1} is true: $\sum_{r=1}^{k+1} \frac{r2^{r-1}}{(r+1)(r+2)} = \frac{2^{k+1}}{k+3} - \frac{1}{2}$ $\sum_{r=1}^{k+1} \frac{r2^{r-1}}{(r+1)(r+2)}$ $= \sum_{r=1}^{k} \frac{r2^{r-1}}{(r+1)(r+2)} + \frac{(k+1)2^k}{(k+2)(k+3)}$ $= \frac{2^k}{k+2} - \frac{1}{2} + \frac{(k+1)2^k}{(k+2)(k+3)}$ $= \frac{2^k}{k+2}\left[1 + \frac{k+1}{k+3}\right] - \frac{1}{2}$ $= \frac{2^k}{k+2}\left[\frac{(k+3)+(k+1)}{k+3}\right] - \frac{1}{2}$ $= \frac{2^k}{k+2}\left[\frac{2(k+2)}{(k+3)}\right] - \frac{1}{2}$ $= \frac{2^{k+1}}{k+3} - \frac{1}{2}$ $\therefore P_{k+1}$ is true By induction, P_n is true for all $n \in \mathbf{Z}^+$

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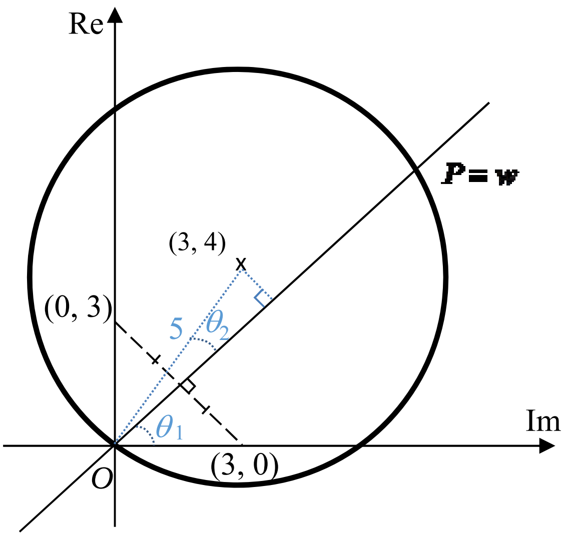
	$\sum_{r=N+1}^{2N} \frac{r2^{r-1}}{(r+1)(r+2)}$ $= \sum_{r=1}^{2N} \frac{r2^{r-1}}{(r+1)(r+2)} - \sum_{r=1}^{N} \frac{r2^{r-1}}{(r+1)(r+2)}$ $= \left(\frac{2^{2N}}{2N+2} - \frac{1}{2} \right) - \left(\frac{2^{N}}{N+2} - \frac{1}{2} \right)$ $= \frac{2^{2N-1}}{N+1} - \frac{2^{N}}{N+2}$
2(i) (ii)	Largest $\alpha = -3$ Let $y = \mathrm{g}(x) = x^2 + 6x + 8$ $= (x+3)^2 - 1$ $(x+3)^2 = y + 1$ $x + 3 = \pm\sqrt{y+1}$ $x = -3 \pm\sqrt{y+1}$ Since $x \le -3$, $x = -3 - \sqrt{y+1}$ $\mathrm{g}^{-1} : x \mapsto -3 - \sqrt{x+1}, \quad x \in [-1, \infty)$ A reflection about the line $y = x$ will transform the curve $y = \mathrm{g}(x)$ onto the curve $y = \mathrm{g}^{-1}(x)$. Since $\mathrm{R_g} = [-1, \infty) \subseteq (-2, \infty) = \mathrm{D_h}$, the composite function hg exists. $\mathrm{R_{hg}} = \left(-\infty, -\frac{1}{\mathrm{e}} \right]$
3(i)	
(ii)	$\theta_1 = \arg(w) = \frac{\pi}{4}$

Qn	Solution
(iii)	$\theta_2 = \tan^{-1}\frac{4}{3} - \frac{\pi}{4}$ $= 0.14190\ (5\text{ sf})$ $\cos\theta_2 = \dfrac{\frac{1}{2}\lvert w\rvert}{5}$ $\lvert w\rvert = 10\cos\theta_2$ $= 9.90\ (3\text{ sf})$
4(i)	$x = t^2, \qquad y = t - t^3.$ $\frac{dx}{dt} = 2t, \quad \frac{dy}{dt} = 1 - 3t^2$ $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{1-3t^2}{2t}$ At P, $x = p^2, \quad y = p - p^3, \quad \frac{dy}{dt} = \frac{1-3p^2}{2p}$ Equation of tangent at P, $\frac{y-(p-p^3)}{x-p^2} = \frac{1-3p^2}{2p}$ $\Rightarrow 2py - 2p(p-p^3) = \left(x-p^2\right)\left(1-3p^2\right)$ $\Rightarrow 2py - 2p^2 + 2p^4 = x\left(1-3p^2\right) - p^2 + 3p^4$ $\Rightarrow 2py = x\left(1-3p^2\right) + p^2 + p^4$ (shown) -------(1)
(ii)	At A, substitute $x = 6,\ y = 5$ into eqn (1) $2p(5) = (6)\left(1-3p^2\right) + p^2 + p^4$ $10p = 6 - 18p^2 + p^2 + p^4$ $p^4 - 17p^2 - 10p + 6 = 0$ From GC, $p = 4.35$ (rejected) or $p = -3.7261$ or $p = -1$ or $p = 0.370$ (rejected) Hence coordinates of P: (1,0) and (13.9, 48.0)
(iii)	Required area = $-\int_0^1 y \ dx$ $= -\int_0^{-1}\left(t-t^3\right)(2t)\ dt$ $= 0.267\text{ unit}^2$
5(i)	If p_1 and p_2 meet at l, then $\mathbf{m}$ is perpendicular to $\mathbf{n_2}$. $\mathbf{m} \cdot \mathbf{n_2} = 0 \Rightarrow \begin{pmatrix}-4\\1\\1\end{pmatrix} \cdot \begin{pmatrix}m\\n\\2\end{pmatrix} = 0$ $-4m + n = -2$

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	Since $(2, -0.5, 0)$ lies on p_2, $2m - 0.5n = 1$ $m = 0$ $n = -2$
(ii)	Let θ be the acute angle between p_1 and p_2. $\cos\theta = \dfrac{\left\lvert \begin{pmatrix} 1 \\ -4 \\ 8 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \right\rvert}{\sqrt{1+16+64}\sqrt{1+4+4}}$ $= \dfrac{1}{3}$ Therefore $\theta = 70.5°$
(iii)	Let $B \equiv (1, b, 5)$. Observe $A_1(4, 0, 0)$ lies on p_1 $\overrightarrow{A_1B} = \begin{pmatrix} 1 \\ b \\ 5 \end{pmatrix} - \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ b \\ 5 \end{pmatrix}$ Shortest distance of B from p_1 $= \dfrac{\left\lvert \begin{pmatrix} -3 \\ b \\ 5 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -4 \\ 8 \end{pmatrix} \right\rvert}{\sqrt{1+16+64}} = \dfrac{\lvert 37-4b \rvert}{9}$ Observe $A_2(1, 0, 0)$ lies on p_2 $\overrightarrow{A_2B} = \begin{pmatrix} 1 \\ b \\ 5 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ b \\ 5 \end{pmatrix}$ Shortest distance of B from p_1 $= \dfrac{\left\lvert \begin{pmatrix} 0 \\ b \\ 5 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \right\rvert}{\sqrt{1+4+4}} = \dfrac{\lvert 10+2b \rvert}{3}$ $\dfrac{37-4b}{9} = \dfrac{2b+10}{3}$ or $\dfrac{37-4b}{9} = -\dfrac{2b+10}{3}$ $b = \dfrac{7}{10}$ $b = -\dfrac{67}{2}$

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6(i)	Randomly select the required number of customers from each branch as follows:

Banking branch	Tua Payoh	Seety Hall	Doby Got
Sample size	18	6	16

(ii) Advantage: Stratified sampling provides a representative sample of the customers from each branch.

7(i)

P(score of 4)

$=$P(obtain 1 and 3 for the first 2 cards, and obtain 2 for the third card)

$$=\left(\frac{1}{4}\times\frac{1}{3}\times 2\right)\times\frac{1}{2}=\frac{1}{12}$$

(ii)

P(score < 5 | draws three cards)

$$=\frac{\text{P(score <5 and draws three cards)}}{\text{P(draws three cards)}}$$

$$=\frac{\text{P(score 4 and 3 cards)+P(score 1 and 3 cards)}}{\text{P(obtain 1,3 or 2,4 for first two cards)}}$$

$$=\frac{\frac{1}{12}+\frac{1}{12}}{\left(\frac{1}{4}\times\frac{1}{3}\times 2\right)\times 2}=\frac{1}{2}$$

8

Let X be the random variable 'number of rocks that contain fossils out of 25 rocks'

$X \sim \mathrm{B}(25, 0.1)$

$\mathrm{P}(4 < X \leq 10) = \mathrm{P}\left(X \leq 10\right) - \mathrm{P}\left(X \leq 4\right)$

≈ 0.0979819403

≈ 0.0980 (3 sig fig)

Let Y be the 'number of rocks that contain fossils out of 20 rocks in the new area'

$Y \sim \mathrm{B}(20, \frac{p}{100})$

$\mathrm{P}(Y = 2) = 0.17$

Using g.c.

$\frac{p}{100} = 0.045473$ or $\frac{p}{100} = 0.1815827$

Since p >10, p = 18.16 (2 d.p)

9(a)(i)

No. of ways $= \frac{3!\times {}^4C_3}{2!} = 12$

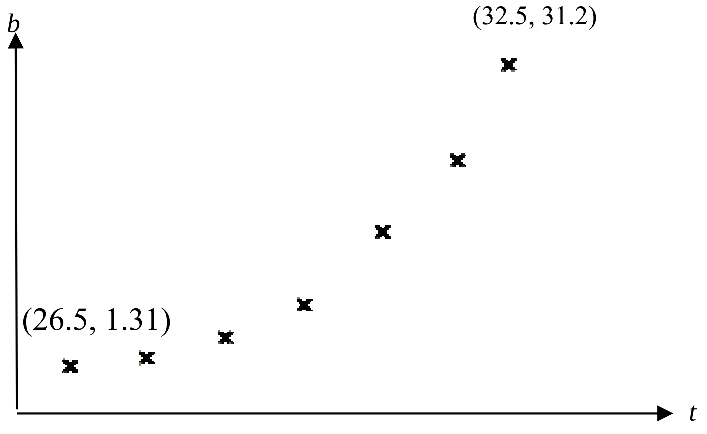
(ii)

Case 1: Ending with "T"

No. of ways $= \frac{5!}{3!} = 20$

Case 2: Ending with "R"

Qn	Solution
(b)	No. of ways $=\dfrac{5!}{3!2!}=10$ Total no. of ways $=20+10=30$ Choose the two sizes that have the same colour: ${}^{3}C_{2}=3$ Choose colour that is same for two sizes: ${}^{4}C_{1}=4$ Choose colour of remaining size: ${}^{3}C_{1}=3$ No. of ways = ${}^{3}C_{2}\times{}^{4}C_{1}\times{}^{3}C_{1}=36$
10(a) (i) (ii)	$X \sim \text{Po}(1.2)$ $Y=3X$ $\text{E}(Y) = 3(1.2)=3.6$ $\text{Var}(Y) = 9(1.2) =10.8$ Since $\text{E}(Y) \neq \text{Var}(Y)$, Y does not have a Poisson distribution. Or Since Y=3X does not take all integral values, Y does not have a Poisson distribution
(b) (i) (ii) (iii)	Let X be the random variable 'number of flaws in a dress' $X \sim \text{P}_\text{O}(0.812)$ $\text{P}(2 \leq X < 5) = \text{P}(X \leq 4) - \text{P}(X \leq 1)$ $\approx 0.9984944372 - 0.8044722615$ $= 0.194$ (3 s.f.) Required Prob = $\text{P}(X \geq 1)(0.04)$ $= (1 - \text{P}(X = 0))(0.04)$ $\approx (0.5560307606)(0.04)$ ≈ 0.0222412304 $= 0.0222$ (3 s. f.) Let Y be the random variable 'number of outfits rejected out of 200' $Y \sim B(200,\ 0.022241)$ Since n = 200 is large, p = 0.022241 is small, np = 4.4482 < 5, $Y \sim \text{P}_\text{O}(4.4482)$ approximately, $\text{P}(Y \leq 5) =$ 0.712 (3 s.f.)
11(i)	$b = -\ 129.368 + 4.75214t$ From GC, $\bar{t} = 29.5$ $\bar{b} = -\ 129.368 + 4.75214\bar{t}$ $\bar{b} = -\ 129.368 + 4.75214(29.5)$ $= 10.82013$ $\dfrac{1.31+2.1+3.65+5.8+\alpha+19.56+31.2}{7} = 10.82013$ $\alpha = 12.121 = 12.12$ (2 dp)

Qn	Solution
(ii)	
(iii)	From (ii), the scatter diagram shows that as t increases, b increases at an increasing rate which would not be the case if the data follows a linear model. Hence the model $b = kt^3 + l$ is a better model. $b = -37.370 + 0.0018516t^3$ $= -37.4 + 0.00185t^3 \ (3\text{s.f.})$
(iv)	When $t = 33$, $b = -37.370 + 0.0018516(33)^3$ $= 29.171$ $= 29.2 (3\text{s.f.})$ The population of the bacteria is 29.2 million. Since the estimate is obtained via extrapolation, the estimate is not reliable.
(v)	$b = -37.370 + 0.0018516\left(\frac{T-32}{1.8}\right)^3$ $= -37.370 + (3.1749 \times 10^{-4})(T-32)^3$ $= -37.4 + (3.17 \times 10^{-4})(T-32)^3 \ (3 \text{ s.f.})$
12(i)	The time taken by a machine to assembly a smartphone is assumed to be normally distributed. $H_0 : \mu = 53$ $H_1 : \mu < 53$ Under H_0, the test statistic $T = \frac{\overline{X} - \mu}{s / \sqrt{n}} \sim t(5)$ where $\mu = 53, s = 4.3213, \overline{x} = 52.15, \ n = 6$. By GC, p-value $= 0.325 (3 \text{ s.f.})$.

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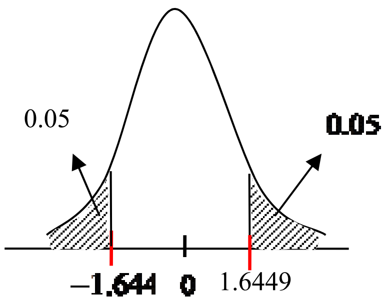
	Since p-value > 0.1, we do not reject H_0 and conclude at 10% level that there is no sufficient evidence that average time taken by a machine to assembly a smartphone has decreased.
(ii)	$\mathrm{H}_0 : \mu = 45$ $\mathrm{H}_1 : \mu \neq 45$ Under H_0, the test statistic $Z = \dfrac{\overline{X}-\mu}{\sigma/\sqrt{n}} \sim \mathrm{N}(0,1)$ approx. by CLT, where $\mu = 45, \sigma = \sqrt{9},\ n = 50$.  Since H_0 is rejected, $\dfrac{\overline{x}-45}{\sqrt{9}/\sqrt{50}} < -1.6449$ or $\dfrac{\overline{x}-45}{\sqrt{9}/\sqrt{50}} > 1.6449$ $\overline{x} < 44.3021$ $\quad$ $\overline{x} > 45.698$ $\overline{x} < 44.3\,(3\text{ s.f.})$ $\quad$ $\overline{x} > 45.7\,(3\text{ s.f.})$ Range of values of $\overline{x}$: $\overline{x} < 44.3\,(3\text{ s.f.})$ or $\overline{x} > 45.7\,(3\text{ s.f.})$
13(i)	Let X, Y and W be the random variable ‘amount (in kg) of impact modifier, Polymer A and Polymer B in a batch of resin’ respectively. $X \sim N(1400, 26.6^2)$ $Y \sim N(2030, 44.8^2)$, $W \sim N(1563, 22.7^2)$ Total cost of a batch, $T = 2.20Y + 2.80W + 1.50X \sim N(10942.4, 15345.9572)$ Total cost of 2 batches, $T_1 + T_2 \sim N(21884.8, 30715.9144)$ $P(T_1 + T_2 > 22000) = 0.255$ (3.s.f.)

Qn	Solution
(ii)	Let H be the r.v.' number of batches of resin with more than 1414 kg of impact modifier out of n batches.' $H \sim B(n,\ \mathrm{P}(X > 1414))$ $H \sim B(n,\ 0.29933)$ Since $n > 50$, $0.29933n > 5$, $0.70067n > 5$, $H \sim N(0.29933n, 0.20973n)$ approx $\mathrm{P}(H \le 6) < 0.001$ $\mathrm{P}(H \le 6.5) < 0.001$ (using c.c.) Using GC. When $n = 57$, $\mathrm{P}(H \le 6.5) = 0.00113 > 0.001$ When $n = 58$, $\mathrm{P}(H \le 6.5) = 9.23 \times 10^{-4} < 0.001$ Therefore, least $n = 58$
(iii)	Let S be the r.v.' tensile strength (in N/m^2) of resin in a car seat' E(S) = 125, Var(S) = 17^2 $\overline{S} = \dfrac{S_1 + S_2 + \ldots + S_{50}}{50}$ $\overline{S} \sim N\left(125, \dfrac{17^2}{50}\right)$ approx by Central Limit Thm $\mathrm{P}(\overline{S} < 130) = 0.981$ (3 s.f.)