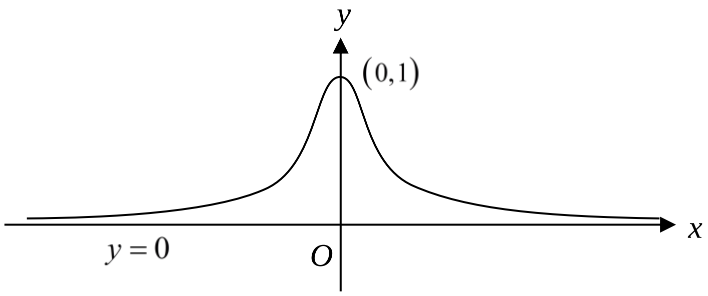
1	$\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{2}{3a}\left(\frac{3}{2}\right)(a)\left(at+\frac{3a^2}{4}\right)^{\frac{1}{2}}=\left(at+\frac{3a^2}{4}\right)^{\frac{1}{2}}$ $\frac{\mathrm{d}y}{\mathrm{d}t}=\frac{1}{2}(2t+a)=t+\frac{a}{2}$ $s=\int_0^3\sqrt{\left[\left(at+\frac{3a^2}{4}\right)^{\frac{1}{2}}\right]^2+\left(t+\frac{a}{2}\right)^2}\,\mathrm{d}t$ $=\int_0^3\sqrt{at+\frac{3a^2}{4}+t^2+at+\frac{a^2}{4}}\,\mathrm{d}t$ $=\int_0^3\sqrt{t^2+2at+a^2}\,\mathrm{d}t$ $=\int_0^3\sqrt{(t+a)^2}\,\mathrm{d}t$ $=\int_0^3\lvert t+a\rvert\,\mathrm{d}t$ $=\int_0^3(t+a)\,\mathrm{d}t$ (since $t+a>0$ for $0\le t\le 3$ as $a>0$) $=\left[\frac{t^2}{2}+at\right]_0^3$ $=\frac{9}{2}+3a$ $\frac{9}{2}+3a=\frac{21}{2}\quad\Rightarrow a=2$
2	$\cos 3\theta+\mathrm{i}\sin 3\theta=(\cos\theta+\mathrm{i}\sin\theta)^3$ $=\cos^3\theta+3\cos^2\theta(\mathrm{i}\sin\theta)+3\cos\theta(\mathrm{i}\sin\theta)^2+(\mathrm{i}\sin\theta)^3$ Real part: $\cos 3\theta=\cos^3\theta-3\cos\theta\sin^2\theta$ $=\cos^3\theta-3\cos\theta(1-\cos^2\theta)$ $=4\cos^3\theta-3\cos\theta$ $8x^3-6x+\sqrt{3}=0\quad\Rightarrow 2(4x^3-3x)=-\sqrt{3}$ Let $x=\cos\theta$: $2\cos 3\theta=-\sqrt{3}\Rightarrow\cos 3\theta=-\frac{\sqrt{3}}{2}$ $3\theta=\frac{5\pi}{6},\frac{7\pi}{6},\frac{17\pi}{6}$ $\cos\theta=\cos\left(\frac{5\pi}{18}\right),\cos\left(\frac{7\pi}{18}\right),\cos\left(\frac{17\pi}{18}\right)$ The roots of the equation are $\cos\left(\frac{5\pi}{18}\right),\cos\left(\frac{7\pi}{18}\right),\cos\left(\frac{17\pi}{18}\right)$.

3

(a)



(b) $P\left(Z \leq 1.6\right) = 0.5 + \int_0^{1.6} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{x^2}{2}} \mathrm{d}x$

Let $y = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{x^2}{2}}$

x	0	0.4	0.8	1.2	1.6
y	0.3989423	0.3682701	0.2896916	0.1941861	0.1109208

$$P\left(Z \leq 1.6\right) \approx 0.5 + \frac{0.4}{3}\left[0.3989423 + 0.3682701 + 2\left(0.2896816\right)\right.$$
$$\left. + 4\left(0.3682701 + 0.1109208\right)\right]$$
$$= 0.94521$$

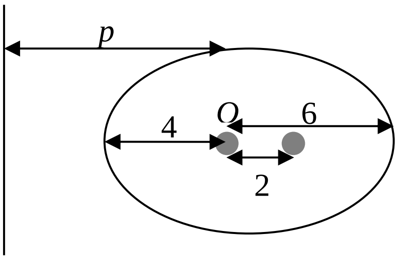
4

$\frac{a}{1-e} = 6$ and $\frac{a}{1+e} = 4$

$\therefore 6 - 6e = 4 + 4e$

$\Rightarrow e = 0.2, a = 4.8$

$\frac{1}{2}\int_0^{2\pi} \left(\frac{4.8}{1-0.2\cos\theta}\right)^2 \mathrm{d}\theta = 76.953$



is the area swept out by one complete orbit.

$\frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} \left(\frac{4.8}{1-0.2\cos\theta}\right)^2 \mathrm{d}\theta = 14.372$

$\therefore$ it takes $\frac{14.372}{76.953} \times 92.3 = 17.2$ Earth days.

$0.2p = 4.8 \quad \Rightarrow p = 24$

One directrix has equation $x = -24$

$p + 2 = 26 \Rightarrow$ the other directrix has equation $x = 26$

5

If a directrix has equation $x = k$ $(k > 0)$, then by symmetry, $x = -k$ is another directrix.
Take point $(a, 0)$ on the hyperbola.
Then from the definition of eccentricity, and
considering $x = k$ & $(c, 0)$: $c - a = e(a - k)$,
considering $x = -k$ & $(-c, 0)$: $a - (-c) = e(a - (-k))$

Adding the two equations: $2c = e(2a) \quad \Rightarrow c = ae$

Translate the graph by h units in the positive x direction so that equation of asymptote is $y = \frac{x}{2}$ and the vertex is $(h, 0)$.

$\therefore$ hyperbola has equation: $\frac{x^2}{h^2} - \frac{y^2}{(\frac{1}{2}h)^2} = 1$

Eccentricity: $\left(\frac{1}{2}h\right)^2 = h^2\left(e^2 - 1\right) \quad \Rightarrow e = \frac{\sqrt{5}}{2}$

Focus then has coordinates $(3\sqrt{5} - 6 + h,\ 0)$

$"c = ae":\ \ 3\sqrt{5} - 6 + h = h\frac{\sqrt{5}}{2} \Rightarrow 3\sqrt{5} - 6 = h\left(\frac{\sqrt{5}}{2} - 1\right) \ \Rightarrow h = 6$

$\therefore$ equation of translated hyperbola is $\frac{x^2}{6^2} - \frac{y^2}{3^2} = 1$

Equation of original hyperbolic path is $\frac{\left(x+6\right)^2}{36} - \frac{y^2}{9} = 1$

6

(i) The auxiliary equation is $\lambda^2 - a\lambda - b = 0$, with solutions 4 & 1

$\therefore \left(\lambda - 4\right)\left(\lambda - 1\right) = 0 \Rightarrow \lambda^2 - 5\lambda + 4 = 0$

Comparing coefficients, $a = 5$ and $b = -4$.

(ii) Let P_n be the proposition $4^{n-1} \ldots n^2$, for all positive integers n.

Check P_1: RHS = 1^2 = 1

LHS = 4^0 = 1 …RHS

$\therefore$ P_1 is true.

Assume that P_k is true for some positive integer k, i.e. $4^{k-1} \ldots k^2$

LHS of P_{k+1} = 4^k

$\ldots 4k^2$

$= \left(2k\right)^2$

$\ldots \left(k+1\right)^2 = \text{RHS}$

Since P_1 is true and P_k is true $\Rightarrow P_{k+1}$ is true, by mathematical induction, $4^{n-1} \ldots n^2$, for all positive integers n.

$$9 + 27 + 45 + \ldots + \left(18n + 9\right) = \frac{n+1}{2}\left[9 + \left(18n + 9\right)\right]$$

$$= 9\left(n+1\right)^2$$

$$,, \ 9\left(4^n\right)$$

$$< 9\left(4^n\right) + 2$$

$$= u_n$$

7

$$\begin{pmatrix} 6 & k & 1 \\ -6 & -1 & 3 \\ 8 & 8 & k \end{pmatrix}\begin{pmatrix} k \\ -9 \\ 2k \end{pmatrix} = \begin{pmatrix} -k \\ 9 \\ 8k-72+2k^2 \end{pmatrix} = -1\begin{pmatrix} k \\ -9 \\ 2k \end{pmatrix}$$

$\therefore$ eigenvalue = -1,

$$\begin{pmatrix} 6 & k & 1 \\ -6 & -1 & 3 \\ 8 & 8 & k \end{pmatrix}\begin{pmatrix} 5 \\ -6 \\ k \end{pmatrix} = \begin{pmatrix} 30-5k \\ -24+3k \\ -8+k^2 \end{pmatrix}$$

If eigenvalue = λ, $\quad 30-5k=5\lambda$

and $\quad -24+3k=-6\lambda$

Solving: $\lambda = 2, k = 4$

Consider $|\mathbf{M}-\lambda\mathbf{I}|=0$

$$\Rightarrow \begin{vmatrix} 6-\lambda & 4 & 1 \\ -6 & -1-\lambda & 3 \\ 8 & 8 & 4-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (6-\lambda)\left[\lambda^2-3\lambda-28\right]-4[6\lambda-48]+1[-40+8\lambda]=0$$

$$\Rightarrow \lambda = -1, 2, 8$$

When $\lambda = 8, (\mathbf{M}-8\mathbf{I})\mathbf{x}=\mathbf{0}$ $\quad \Rightarrow \begin{pmatrix} -2 & 4 & 1 \\ -6 & -9 & 3 \\ 8 & 8 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

From the GC, $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0.5z \\ 0 \\ z \end{pmatrix}$

$\therefore$ an eigenvector corresponing to eigenvalue 8 = $\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$.

$$\mathbf{P} = \begin{pmatrix} 4 & 5 & 1 \\ -9 & -6 & 0 \\ 8 & 4 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} -\frac{2}{9} & -\frac{1}{9} & \frac{1}{9} \\ \frac{1}{3} & 0 & -\frac{1}{6} \\ \frac{2}{9} & \frac{4}{9} & \frac{7}{18} \end{pmatrix},$$

$$\mathbf{D} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{pmatrix}^{3} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 512 \end{pmatrix}$$

8

When z = 2500,

$$14 = -0.5 + 0.8\ln 2500 - 0.01x + 2\ln x$$

$$2\ln x - 0.01x - 8.2408 = 0$$

Let $\mathrm{f}\left(x\right) = 2\ln x - 0.01x - 8.2408$

f(100) = –0.0305 < 0, f(110) = 0.0602 > 0

and f is a continuous function in the interval $\left[100, 110\right]$, there is at a solution α in the interval.

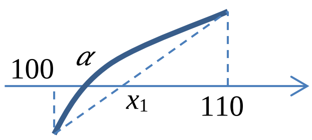
$$x_1 = \frac{100\left(2\ln 110 - 1.1 - 8.2408\right) - 110\left(2\ln 100 - 1 - 8.2408\right)}{\left(2\ln 110 - 1.1 - 8.2408\right) - \left(2\ln 100 - 1 - 8.2408\right)}$$

$$= 103.36$$

<u>Method 1</u> Since f(103.36) > 0, $\alpha \in \left[100, 103.36\right]$, hence 103.36 is an overestimate.

<u>Method 2</u>

$\mathrm{f}'\left(x\right) = \dfrac{2}{x} - 0.01$, $\mathrm{f}''\left(x\right) = -\dfrac{2}{x^2} < 0$,



103.36 is an overestimate

$$x_{n+1} = x_n - \frac{2\ln x_n - 0.01x_n - 8.2408}{\dfrac{2}{x_n} - 0.01}$$

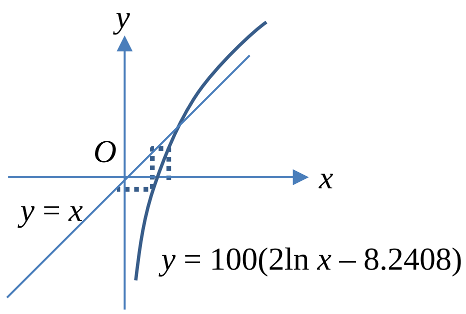
$$x_2 = 103.1422$$

$$x_3 = 103.1427$$

f(103.135) < 0, f(103.145) > 0 $\Rightarrow 103.135 < \alpha < 103.145$

$\therefore\ \alpha \approx 103.14$

By considering the graphs of $y = 100\left(2\ln x - 8.2408\right)$ and $y = x$,



From the graph above, we will eventually obtain a negative value of x, which is undefined when we substitute it into $100\left(2\ln x - 8.2408\right)$. Hence we are not able to obtain the value of α.

9

Consider auxiliary equation $r^2 + 2r = 0 \Rightarrow r = 0$ or -2

Complementary function is $x = Ae^0 + Be^{-2t} = A + Be^{-2t}$

Try particular integral $x_p = \lambda t e^{-2t}$

$x_p' = \lambda e^{-2t} - 2\lambda t e^{-2t}$

$x_p'' = -2\lambda e^{-2t} - 2\lambda e^{-2t} + 4\lambda t e^{-2t}$

$\quad = -4\lambda e^{-2t} + 4\lambda t e^{-2t}$

$\therefore -4\lambda e^{-2t} + 4\lambda t e^{-2t} + 2\left(\lambda e^{-2t} - 2\lambda t e^{-2t}\right) = e^{-2t}$

$\Rightarrow \lambda = -\dfrac{1}{2}$

$$x = A + Be^{-2t} - \frac{1}{2}te^{-2t}$$

When $t = 0$, $x = 0$, $A + B = 0$, i.e $A = -B$

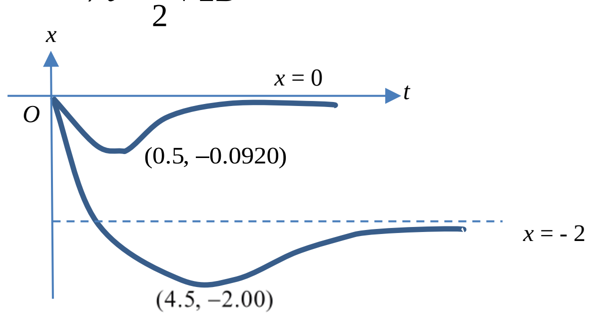
$\therefore x = B\left(e^{-2t} - 1\right) - \dfrac{1}{2}te^{-2t}$

As $t \to \infty$, $x \to -B$

$\therefore$ sketch $x = -\dfrac{1}{2}te^{-2t}$ and $x = 2\left(e^{-2t} - 1\right) - \dfrac{1}{2}te^{-2t}$

$\dfrac{dx}{dt} = 0 \Rightarrow B\left(-2e^{-2t}\right) - \dfrac{1}{2}e^{-2t} + te^{-2t} = 0$

$\qquad \Rightarrow t = \dfrac{1}{2} + 2B$



When $t = 0$, $x = 0$ and $\dfrac{dx}{dt} = 0$, $B = -\dfrac{1}{4}$.

$\therefore x = -\dfrac{1}{4}\left(e^{-2t} - 1\right) - \dfrac{1}{2}te^{-2t}$

$\therefore$ When $t \to \infty$, $x \to \dfrac{1}{4}$

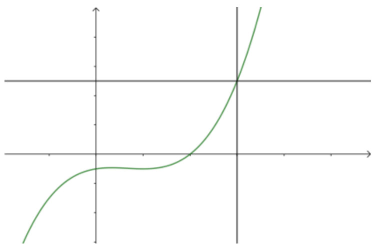
Displacement $= \dfrac{1}{4}$ cm.

10

When $y=5$, $4x^3-4x^2+x-1=5$

$\Rightarrow 4x^3-4x^2+x-6=0 \;\Rightarrow x=\dfrac{3}{2}$

When $y=0$, $x=1$



$$\text{Volum} = \pi\left(\frac{3}{2}\right)^2(5)-\int_1^{\frac{3}{2}} 2\pi xy\,\mathrm{d}x$$

$$=\frac{45\pi}{4}-2\pi\int_1^{\frac{3}{2}} x\left(4x^3-4x^2+x-1\right)\mathrm{d}x$$

$$=\frac{45\pi}{4}-2\pi\int_1^{\frac{3}{2}} \left(4x^4-4x^3+x^2-x\right)\mathrm{d}x$$

$$=\frac{45\pi}{4}-2\pi\left[\frac{4}{5}x^5-x^4+\frac{1}{3}x^3-\frac{1}{2}x^2\right]_1^{\frac{3}{2}}$$

$$=\frac{45\pi}{4}-2\pi\left(\frac{243}{40}-\frac{81}{16}+\frac{9}{8}-\frac{9}{8}-\frac{4}{5}+1-\frac{1}{3}+\frac{1}{2}\right)$$

$$=\frac{45\pi}{4}-\frac{331\pi}{120}=\underline{\underline{\frac{1019}{120}\pi}}$$

$$\int x\ln x\,\mathrm{d}x=\frac{x^2}{2}\ln x-\int\frac{x^2}{2}\times\frac{1}{x}\,\mathrm{d}x=\frac{x^2}{2}\ln x-\int\frac{x}{2}\,\mathrm{d}x$$

$$=\frac{x^2}{2}\ln x-\frac{x^2}{4}+C \quad \underline{\underline{\text{(shown)}}}$$

$$\text{Surface area } = \int_2^4 2\pi y\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x$$

$$=2\pi\int_2^4\left(\frac{1}{2}x^2-\frac{1}{4}\ln x\right)\sqrt{1+\left(x-\frac{1}{4x}\right)^2}\,\mathrm{d}x$$

$$=2\pi\int_2^4\left(\frac{1}{2}x^2-\frac{1}{4}\ln x\right)\sqrt{1+x^2-\frac{1}{2}+\frac{1}{16x^2}}\,\mathrm{d}x$$

$$=2\pi\int_2^4\left(\frac{1}{2}x^2-\frac{1}{4}\ln x\right)\sqrt{x^2+\frac{1}{2}+\frac{1}{16x^2}}\,\mathrm{d}x$$

$$=2\pi\int_2^4\left(\frac{1}{2}x^2-\frac{1}{4}\ln x\right)\sqrt{\left(x+\frac{1}{4x}\right)^2}\,\mathrm{d}x$$

$$=2\pi\int_2^4\left(\frac{1}{2}x^2-\frac{1}{4}\ln x\right)\left|x+\frac{1}{4x}\right|\mathrm{d}x$$

$$=2\pi\int_2^4\left(\frac{1}{2}x^2-\frac{1}{4}\ln x\right)\left(x+\frac{1}{4x}\right)\mathrm{d}x \text{ (since } x+\frac{1}{4x}>0 \text{ for } 2\leq x\leq 4)$$

	$= 2\pi \int_2^4 \left(\frac{1}{2}x^3 + \frac{1}{8}x - \frac{1}{4}x\ln x - \frac{1}{16x}\ln x \right) \mathrm{d}x$ $= \pi \left[\frac{x^4}{4} + \frac{x^2}{8} - \frac{x^2 \ln x}{4} + \frac{x^2}{8} - \frac{(\ln x)^2}{16} \right]_2^4$ $= \pi \left[\frac{4^4 - 2^4}{4} + 2 \times \frac{4^2 - 2^2}{8} - \frac{16\ln 4 - 4\ln 2}{4} - \frac{(\ln 4)^2 - (\ln 2)^2}{16} \right]_2^4$ $= \pi \left(63 - 7\ln 2 - \frac{3(\ln 2)^2}{16} \right)$
11	(i) Rate of X going in = $30 \times 4 = 120$ grams/minute Amount of mixture in cooler at time t = $50 + 4t - 3t = 50 + t$ Rate of X going out = $\frac{3x}{50+t}$ Hence $\frac{\mathrm{d}x}{\mathrm{d}t} = 120 - \frac{3x}{50+t}$. (ii) $\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{3x}{50+t} = 120$ Integrating factor: $\mathrm{e}^{\int \frac{3}{50+t}\mathrm{d}t} = \mathrm{e}^{3\ln(50+t)} = (50+t)^3$ $\frac{\mathrm{d}x}{\mathrm{d}t}(50+t)^3 + 3x(50+t)^2 = 120(50+t)^3$ $\frac{\mathrm{d}}{\mathrm{d}t}\left[x(50+t)^3 \right] = 120(50+t)^3$ $x(50+t)^3 = 120\frac{(50+t)^4}{4} + C$ $x = 30(50+t) + C(50+t)^{-3}$ When $t = 0$, $x = 0$. Hence $C = -187500000 \Rightarrow x = 30(50+t) - \frac{187500000}{(50+t)^3}$ There are 51 litres in the cooler when $t = 1$. Then $x = 30(51) - \frac{187500000}{(51)^3} = 116.52$ Amount of chemical X is 117 grams (3 s.f.) Let $\mathrm{f}(\theta, t) = \frac{\mathrm{d}\theta}{\mathrm{d}t} = 60\mathrm{e}^{0.04t}\left(\cos(0.01\theta) - 1\right)$ $\theta_1 = 60, t_1 = 0$ $u_2 = 60 + 0.5\mathrm{f}(60, 0) = 54.760$ $\theta_2 = 60 + \frac{0.5}{2}\left[\mathrm{f}(60, 0) + \mathrm{f}(54.760, 0.5) \right] = 55.142$ $u_3 = 55.142 + 0.5\mathrm{f}(55.142, 0.5) = 50.606$ $\theta_3 = 55.142 + \frac{0.5}{2}\left[\mathrm{f}(55.142, 0.5) + \mathrm{f}(50.606, 1) \right] = 50.917$ Hence after 1 minute, temperature of mixture is 50.9 °C (1 d.p.).