NANYANG JUNIOR COLLEGE

JC2 PRELIMINARY EXAMINATION

Higher 2

MATHEMATICS **9758/01**

Paper 1 **13th September 2017**

3 Hours

Additional Materials: Answer Paper

List of Formulae (MF26)

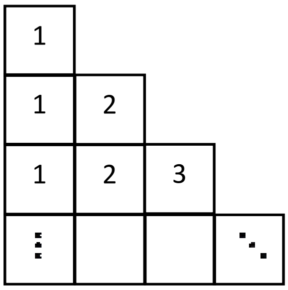
READ THESE INSTRUCTIONS FIRST

Write your name and class on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphic calculator.
Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

1 A board is such that the n^{th} row from the top has n tiles, and each row is labelled from left to right in ascending order such that the i^{th} tile is labelled i, where n and i are positive integers.



Given that $\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$, by finding the sum of the numbers in the r^{th} row, show that the sum of all the numbers in n rows of tiles is $\frac{1}{6}(n)(n+1)(n+2)$. [4]

2 The curve C has equation $2x - y^2 = (x+y)^2$.

(i) Find the equations of the tangents to C which are parallel to the x-axis. [4]

(ii) The line l is tangent to C at $A(2,-2)$. If the normal to C at the origin O meets l at the point B, find the area of triangle OAB. [4]

3 **Do not use a calculator in answering this question.**

(i) Explain why the equation $z^3 + az^2 + az + 7 = 0$ cannot have more than two non-real roots, where a is a real constant. [1]

(ii) Given that $z = -7$ is a root of the equation in **(i)**, find the other roots, leaving your answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \le \pi$. [4]

(iii) Hence, solve the equation $\mathrm{i}z^3 + 8z^2 - 8\mathrm{i}z - 7 = 0$, leaving your answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \le \pi$. [2]

4 **(i)** By using the substitution $x-1=3\tan\theta$, find $\displaystyle\int \frac{1}{\sqrt{x^2-2x+10}}\,\mathrm{d}x$. [5]

(ii) By expressing $x+3=A\left(2x-2\right)+B$, find $\displaystyle\int \frac{x+3}{\sqrt{x^2-2x+10}}\,\mathrm{d}x$. [3]

5 **(i)** By considering $\mathrm{f}\left(r\right)-\mathrm{f}\left(r+1\right)$, where $\mathrm{f}\left(r\right)=\dfrac{\sqrt{r}}{2\sqrt{r}+1}$, find

$$\sum_{r=1}^{n}\frac{\sqrt{r}-\sqrt{r+1}}{\left(2\sqrt{r}+1\right)\left(2\sqrt{r+1}+1\right)}$$

in terms of n. [3]

(ii) Hence, find $\displaystyle\sum_{r=1}^{\infty}\frac{\sqrt{r}-\sqrt{r+1}}{\left(2\sqrt{r}+1\right)\left(2\sqrt{r+1}+1\right)}$. [2]

(iii) Find the smallest integer n such that

$$\sum_{r=1}^{n}\frac{\sqrt{r+1}-\sqrt{r+2}}{\left(2\sqrt{r+1}+1\right)\left(2\sqrt{r+2}+1\right)}<-0.1.$$ [3]

6 The curve C has equation

$$y=1+\frac{2x+p}{(x-2)(x+3)},$$

where p is a constant.

(i) Find the range of values of p for which C has more than one stationary point. [4]

(ii) Sketch C for p = 7, stating the coordinates of the turning point(s) and the points of intersection with the axes and the equations of any asymptotes. [3]

(iii) By sketching a suitable graph on the same diagram, solve the inequality

$$1+\sqrt{12-x^2}\geq\frac{2x+7}{(2-x)(x+3)}.$$ [3]

7 The functions f and g are defined by

$$\mathrm{f}: x \mapsto \mathrm{e}^{-x^2}, \quad x \in \mathbf{R},\ x < 0,$$

$$\mathrm{g}: x \mapsto \frac{1}{x+3}, \quad x \in \mathbf{R},\ x \neq -3.$$

(i) Show that g^{-1} exists, and define g^{-1} in a similar form. [2]

(ii) State the solution set for $\mathrm{g}\,\mathrm{g}^{-1}(x) = x$. [1]

(iii) Explain why fg^{-1} does not exist. [1]

Let the function h be defined by

$$\mathrm{h}: x \mapsto \mathrm{g}(x), \quad x \in \mathbf{R},\ x < k,$$

where k is a real constant.

(iv) Given that $\mathrm{f}\,\mathrm{h}^{-1}$ exists, state the maximum value of k. [1]

(v) For the value of k found in **(iv)**,

(a) find the exact range of $\mathrm{f}\,\mathrm{h}^{-1}$, [2]

(b) solve $\mathrm{h}(x) = \mathrm{h}^{-1}(x)$. [2]

8 A curve *C* has parametric equations

$$x = 1 + \mathrm{e}^{t} + \mathrm{e}^{-t}, \quad 2y = \mathrm{e}^{t} - \mathrm{e}^{-t}, \quad t \in \mathbf{R}.$$

(i) Show that the Cartesian equation of *C* is $\dfrac{(x-1)^2}{2^2} - y^2 = 1$. [2]

(ii) Sketch *C*, showing clearly the equations of any asymptotes and coordinates of the centre and the point(s) where the curve cuts the *x*-axis. [3]

(iii) Find the exact area of the region bounded by *C* and the line $x = 1 + \mathrm{e} + \mathrm{e}^{-1}$. [4]

(iv) Find the volume of the solid of revolution when the region bounded by *C* and the lines $x = 3$ and $y = 4$ is rotated completely about the *y*-axis. [2]

9 With reference to an origin O, a particle P moves in space with position vector $(\lambda-\mu)\mathbf{i}+(1+2\mu)\mathbf{j}+(2-3\lambda)\mathbf{k}$. Another particle Q moves along the line l with equation $\mathbf{r}=\begin{pmatrix}1\\1\\0\end{pmatrix}+t\begin{pmatrix}2\\-2\\-3\end{pmatrix},\ t\in\mathbb{R}$.

(i) State the locus of P. [1]

(ii) Determine if the particles P and Q can meet. [3]

(iii) Find the shortest possible distance between P and Q. [2]

Another particle R moves along the line m with equation $\mathbf{r}=\begin{pmatrix}1\\k\\0\end{pmatrix}+s\begin{pmatrix}2\\-2\\-3k\end{pmatrix},\ s\in\mathbb{R}$, where k is a constant.

(iv) Find condition(s) satisfied by k if lines l and m are skew lines. [3]

(v) A particle is shot from $X(0,-1,-5)$ perpendicularly toward the path of Q. Find the coordinates of the point where it crosses the path of Q. [2]

10 A car is travelling at a speed of 30 m/s on a road heading towards a perpendicular train track, which is elevated 30 m above the ground. The front of the car is 40 m away from the track when the front of the train first crossed the road.

If the train is travelling at 20 m/s, show that the distance between the front of the train and the car is $\sqrt{1300t^2-2400t+2500}$ m. [2]

(i) How fast is the front of both the train and the car separating 1 second later? [2]

(ii) Find the distance when the front of the train and the front of the car are closest. [4]

(iii) Find the rate of change of the angle of elevation of the front of the train from the car 1 second later. [4]

11 Suppose a point P on the rim of a wheel of radius r is initially at the point O. As the wheel roll along the x-axis without slippage, the locus of P, known as a *cycloid*, has parametric equations given by

$$x=r(\theta-\sin\theta),\ y=r(1-\cos\theta),\ \theta\geq 0.$$

(i) Sketch the locus of P for $0\leq\theta\leq 4\pi$. [2]

(ii) Show that $\frac{\mathrm{d}y}{\mathrm{d}x}=\cot\frac{\theta}{2}$. [3]

(iii) Show that the curve is a solution to the differential equation $\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2=\frac{2r}{y}-1.$ [3]

(iv) Find the exact area bounded by the locus of P and the x-axis for $0\leq x\leq 2\pi r$. [4]

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