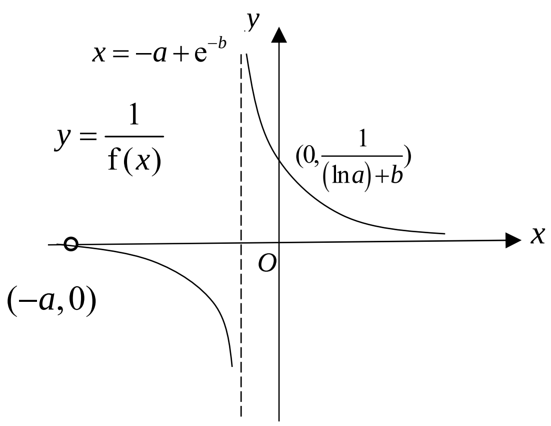
# Solutions to 2017 Y6 H2 Maths Preliminary Exam II (Paper 1)

| Question 1 [5 Marks] | | |
|---|---|---|
| (i) | **Step 1**: Translation of $a$ units in the negative $x$-axis direction;<br>**Step 2**: Translation of $b$ units in the positive $y$-axis direction.<br>OR<br><br>**Step 1**: Translation of $b$ units in the positive $y$-axis direction;<br>**Step 2**: Translation of $a$ units in the negative $x$-axis direction. | |
| (ii) | <br><br><br>$y$-intercept $\left(0,\dfrac{1}{(\ln a)+b}\right)$<br>vertical asymptote: $x=-a+e^{-b}$<br>horizontal asymptote: $y = 0$ | |

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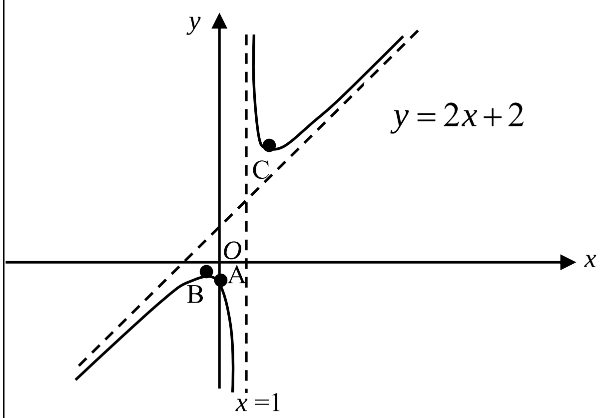
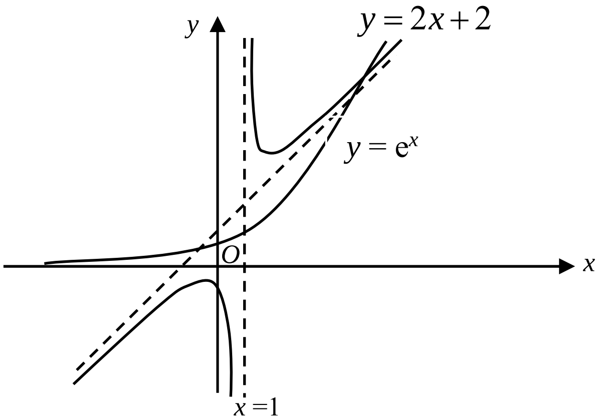
<table>
<tr><td colspan="3">Question 2 [7 Marks]</td></tr>
<tr><td>(i)</td><td>By long division,<br>$$y=\frac{2x^2+3}{x-1}$$ $$=2x+2+\frac{5}{x-1}$$<br><br>$y$-intercept A (0, −3)<br>Max point B (−0.581, −2.32)<br>Min point C (2.58, 10.3)</td><td></td></tr>
<tr><td>(ii)</td><td>$$2x+2\le \mathrm{e}^x-\frac{5}{x-1}$$ $$2x+2+\frac{5}{x-1}\le \mathrm{e}^x$$<br><br>Intersection of both curves: (2.34, 10.4)<br>$x<1$ or $x\ge 2.34$</td><td></td></tr>
<tr><td>(iii)</td><td>Replacing $x$ by $x+1$<br>$x+1<1$ or $x+1\ge 2.34$<br>$x<0$ or $x\ge 1.34$</td><td></td></tr>
</table>

Question 3 [8 Marks]

(i)

Given $t = 3\sec\theta \Rightarrow \dfrac{\mathrm{d}t}{\mathrm{d}\theta} = 3\sec\theta\tan\theta$

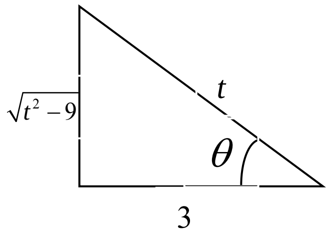
$$\int \frac{\sqrt{t^2-9}}{t}\,\mathrm{d}t$$

$$= \int \sqrt{9\sec^2\theta - 9}\left(\frac{1}{3\sec\theta}\right)\left(3\sec\theta\tan\theta\right)\mathrm{d}\theta$$

$$= 3\int \tan^2\theta\,\mathrm{d}\theta$$

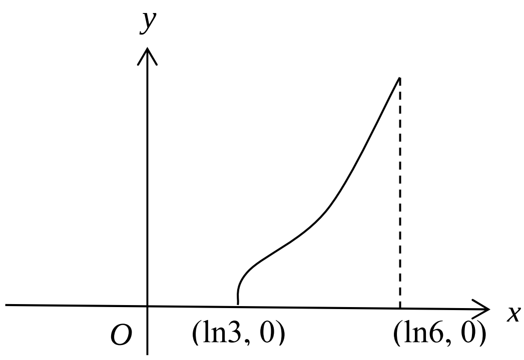
$$= 3\int \sec^2\theta - 1\,\mathrm{d}\theta$$

$$= 3\left(\tan\theta - \theta\right) + c$$




$$= 3\left(\frac{\sqrt{t^2-9}}{3} - \cos^{-1}\left(\frac{3}{t}\right)\right) + c$$

(ii)




$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{t}$$

Area of $S = \int_{\ln 3}^{\ln 6} y\,\mathrm{d}x$

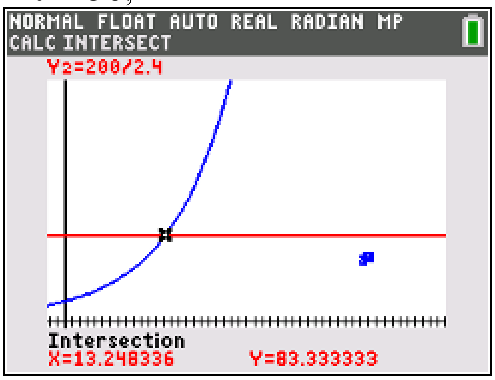
$$= \int_3^6 \sqrt{t^2-9}\left(\frac{1}{t}\right)\mathrm{d}t$$

$$= \int_3^6 \frac{\sqrt{t^2-9}}{t}\,\mathrm{d}t$$

$$= 3\left[\frac{\sqrt{t^2-9}}{3} - \cos^{-1}\left(\frac{3}{t}\right)\right]_3^6$$

$$= 3\left(\frac{\sqrt{27}}{3} - \frac{\pi}{3}\right)$$

$$= 3\sqrt{3} - \pi$$

| Question 4 [10 Marks] | | |
|---|---|---|
| (ai) | AP: $a = 2.4$, $d = 0.4$<br>Distance he runs in the 20th session<br>$= 2.4 + (20-1)(0.4)$<br>$= 10$ km | |
| (aii) | $S_n \geq 99$<br>$\Rightarrow \quad \frac{n}{2}\left[2(2.4)+(n-1)(0.4)\right] \geq 99$<br>$\Rightarrow \quad n\left[4.8+0.4n-0.4\right] \geq 198$<br>$\Rightarrow \quad 0.4n^2 + 4.4n - 198 \geq 0$<br>$\Rightarrow \quad n \leq -28.4 \quad$ or $\quad n \geq 17.4$<br>(rejected as $n > 0$)<br>Least value of $n = 18$<br>He needs a minimum of 18 sessions. | |
| (bi) | $S_{20} = \dfrac{2.4\left(\left(1+\frac{x}{100}\right)^{20}-1\right)}{\left(1+\frac{x}{100}\right)-1} = 200$<br>$\dfrac{\left(1+\frac{x}{100}\right)^{20}-1}{\frac{x}{100}} = \dfrac{200}{2.4}$<br>From GC,<br><br>$x = 13.2\%$ | |
| (bii) | Sum to infinity $= \dfrac{2.4}{1-0.95}$<br>$= 48$<br>Hence, total distance can never be greater than 200 km. | |

| Question 5 [9 Marks] | | |
|---|---|---|
| (i) | $\mathbf{a} \cdot \mathbf{c} = 4(1)\cos\frac{\pi}{3} = 2$<br>$\lvert \mathbf{a} \cdot \mathbf{c} \rvert$ is the length of projection of $\mathbf{a}$ onto $\mathbf{c}$ | |
| (ii) | $(\mathbf{a}-\mathbf{c})\cdot(\mathbf{a}-\mathbf{c}) = k\mathbf{b}\cdot k\mathbf{b}$<br>$\mathbf{a}\cdot\mathbf{a} - \mathbf{a}\cdot\mathbf{c} - \mathbf{c}\cdot\mathbf{a} + \mathbf{c}\cdot\mathbf{c} = k^2\mathbf{b}\cdot\mathbf{b}$<br>$\lvert\mathbf{a}\rvert^2 - 2\mathbf{a}\cdot\mathbf{c} + \lvert\mathbf{c}\rvert^2 = k^2\lvert\mathbf{b}\rvert^2$<br>$16 - 2(2) + 1 = 9k^2$<br>$k^2 = \frac{13}{9}$<br>$k = \pm\frac{\sqrt{13}}{3}$ | |
| (iii) | $\overrightarrow{MC} = \frac{1}{3}\mathbf{c}$<br><br>Area of triangle $AMC$<br>$= \frac{1}{2}\left\lvert \overrightarrow{AC} \times \overrightarrow{MC} \right\rvert$<br>$= \frac{1}{2}\left\lvert (\mathbf{c}-\mathbf{a}) \times \frac{1}{3}\mathbf{c} \right\rvert$<br>$= \frac{1}{6}\lvert \mathbf{c}\times\mathbf{c} - \mathbf{a}\times\mathbf{c} \rvert$<br>$= \frac{1}{6}\lvert \mathbf{a}\times\mathbf{c} \rvert$<br>$= \frac{1}{6}\lvert\mathbf{a}\rvert\lvert\mathbf{c}\rvert\sin\left(\frac{\pi}{3}\right)$<br>$= \frac{1}{6}(4)(1)(\frac{\sqrt{3}}{2})$<br>$= \frac{\sqrt{3}}{3}$ | |

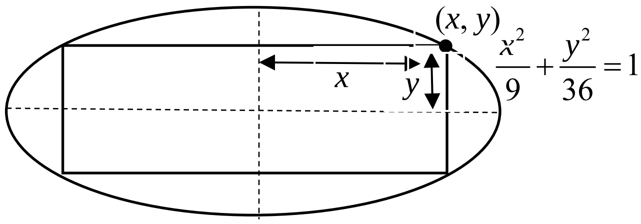
| Question 6 [10 Marks] | | |
|---|---|---|
| (a) | $z - 4w = 11 + 6\mathrm{i}$<br>$z = 4w + 11 + 6\mathrm{i}$<br><br>Sub above equation into $3z + 6iw = 27$,<br>$3(4w + 11 + 6\mathrm{i}) + 6\mathrm{i}w = 27$<br>$12w + 33 + 18\mathrm{i} + 6\mathrm{i}w = 27$<br>$w(12 + 6\mathrm{i}) = -6 - 18\mathrm{i}$<br>$w = \frac{-6-18\mathrm{i}}{12+6\mathrm{i}}$<br>$= -1 - \mathrm{i}$ | |

| | | |
|---|---|---|
| | $z=4w+11+6\mathrm{i}$<br>$=4(-1-\mathrm{i})+11+6\mathrm{i}$<br>$=7+2\mathrm{i}$<br><u>ALT</u><br>$z-4w=11+6\mathrm{i}$<br>$\times 3,\quad 3z-12w=33+18\mathrm{i}\ \ldots(1)$<br>$3z+6\mathrm{i}w=27\ \ldots(2)$<br>(2) − (1),<br>$6\mathrm{i}w+12w=-6-18\mathrm{i}$<br>$w=\dfrac{-6-18\mathrm{i}}{12+6\mathrm{i}}$<br>$=-1-\mathrm{i}$<br>$z=4w+11+6\mathrm{i}$<br>$=4(-1-\mathrm{i})+11+6\mathrm{i}$<br>$=7+2\mathrm{i}$ | |
| (bi) | $\lvert z\rvert=4$ & $\arg z=-\dfrac{\pi}{3}$<br>$\lvert w\rvert=\sqrt{1^2+\left(\sqrt{3}\right)^2}=2$ & $\arg w=\tan^{-1}\sqrt{3}=\dfrac{\pi}{3}$<br>$\left\lvert\dfrac{w^*}{z^2}\right\rvert=\dfrac{\lvert w^*\rvert}{\lvert z^2\rvert}=\dfrac{\lvert w\rvert}{\lvert z\rvert^2}=\dfrac{2}{16}=\dfrac{1}{8}$<br>$\arg\left(\dfrac{w^*}{z^2}\right)=\arg\left(w^*\right)-\arg\left(z^2\right)$<br>$=-\arg w-2\arg z$<br>$=-\left(\dfrac{\pi}{3}\right)-2\left(-\dfrac{\pi}{3}\right)$<br>$=\dfrac{\pi}{3}$ | |
| (bii) | $\dfrac{w^*}{z^2}=\dfrac{1}{8}\left[\cos\left(\dfrac{\pi}{3}\right)+i\sin\left(\dfrac{\pi}{3}\right)\right]$<br>$\left(\dfrac{w^*}{z^2}\right)^n=\left(\dfrac{1}{8}\right)\left[\cos\left(\dfrac{n\pi}{3}\right)+i\sin\left(\dfrac{n\pi}{3}\right)\right]$<br>For $\left(\dfrac{w^*}{z^2}\right)^n$ to be purely imaginary,<br>$\cos\left(\dfrac{n\pi}{3}\right)=0$<br>$\dfrac{n\pi}{3}=\pm\dfrac{\pi}{2},\pm\dfrac{3\pi}{2},\pm\dfrac{5\pi}{2},\pm\dfrac{7\pi}{2},\ldots$<br>$=\dfrac{(2m+1)\pi}{2},\ m\in\mathbb{Z}$<br>$n=\dfrac{3(2m+1)}{2},\ m\in\mathbb{Z}$<br>$\left\{n:n=\dfrac{3(2m+1)}{2},\ m\in\mathbb{Z}\right\}$ | |

| Question 7 [11 Marks] | | |
|---|---|---|
| (a) | $\int \sqrt{5-x^2}\ \mathrm{d}x = x\sqrt{5-x^2} - \int \frac{-x^2}{\sqrt{5-x^2}}\ \mathrm{d}x$<br>$= x\sqrt{5-x^2} - \int \frac{\left(5-x^2\right)-5}{\sqrt{5-x^2}}\ \mathrm{d}x$<br>$= x\sqrt{5-x^2} - \int \sqrt{5-x^2}\ \mathrm{d}x + 5\int \frac{1}{\sqrt{5-x^2}}\ \mathrm{d}x$<br>$= x\sqrt{5-x^2} - \int \sqrt{5-x^2}\ \mathrm{d}x + 5\sin^{-1}\left(\frac{x}{\sqrt{5}}\right)$<br>$\Rightarrow 2\int \sqrt{5-x^2}\ \mathrm{d}x = x\sqrt{5-x^2} + 5\sin^{-1}\left(\frac{x}{\sqrt{5}}\right) + c'$<br>$\Rightarrow \int \sqrt{5-x^2}\ \mathrm{d}x = \frac{x}{2}\sqrt{5-x^2} + \frac{5}{2}\sin^{-1}\left(\frac{x}{\sqrt{5}}\right) + c$ | |
| (bi) | $y^4 + x^2 = 5$<br>Differentiating wrt $x$,<br>$4y^3 \frac{\mathrm{d}y}{\mathrm{d}x} = -2x$<br>When $x=1$, $y^4 = 4$<br>$y = \pm\sqrt{2}$<br>At $\left(1,\sqrt{2}\right)$, $4\left(\sqrt{2}\right)^3 \frac{\mathrm{d}y}{\mathrm{d}x} = -2$<br>$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{4\sqrt{2}}$<br>Gradient of normal at $\left(1,\sqrt{2}\right)$<br>$= -\frac{1}{-\frac{1}{4\sqrt{2}}}$<br>$= 4\sqrt{2}$ (shown)<br>Equation of normal: $y-\sqrt{2} = 4\sqrt{2}\left(x-1\right)$<br>$y = 4\sqrt{2}x - 3\sqrt{2}$ | |
| (iii) | Volume of $S$= $\pi\int_{-\sqrt{5}}^{\sqrt{5}} y^2\ \mathrm{d}x = 2\pi\int_{0}^{\sqrt{5}} \sqrt{5-x^2}\ \mathrm{d}x$<br>$= 2\pi\left[\frac{x}{2}\sqrt{5-x^2} + \frac{5}{2}\sin^{-1}\left(\frac{x}{\sqrt{5}}\right)\right]_0^{\sqrt{5}}$<br>$= 2\pi\left[\frac{5}{2}\left(\frac{\pi}{2}\right) - 0\right] = \frac{5}{2}\pi^2$ | |

Question 8 [13 marks]

(a)




Let $(x, y)$ be a point on the ellipse.

Area of rectangle, $A$

$$\begin{aligned}
&= (2x)(2y) \\
&= 4xy \\
&= 4x\sqrt{36-4x^2} \\
&= 8\sqrt{9x^2-x^4} \\
&= 8\left(9x^2-x^4\right)^{\frac{1}{2}}
\end{aligned}$$

$$\begin{aligned}
\frac{\mathrm{d}A}{\mathrm{d}x} &= 8\left(\frac{1}{2}\right)\left(9x^2-x^4\right)^{-\frac{1}{2}}\left(18x-4x^3\right) \\
&= \frac{4\left(18x-4x^3\right)}{\sqrt{9x^2-x^4}}
\end{aligned}$$

When the area is the largest,

$$\frac{\mathrm{d}A}{\mathrm{d}x} = 0$$

$$\frac{4\left(18x-4x^3\right)}{\sqrt{9x^2-x^4}} = 0$$

$$18x-4x^3 = 0$$

$$2x\left(3-\sqrt{2}x\right)\left(3+\sqrt{2}x\right) = 0$$

$x = 0$ (rejected since $x \neq 0$)

or $x = \frac{3}{\sqrt{2}}$

or $x = -\frac{3}{\sqrt{2}}$ (rejected since $x > 0$)

When $x = \frac{3}{\sqrt{2}}$, $y = 3\sqrt{2}$

| $x$ | 2.115 | $\frac{3}{\sqrt{2}} \approx 2.12$ | 2.125 |
|---|---|---|---|
| $\frac{dA}{dx}$ | 0.2013 | 0 | -0.118 |
| Slope | | | |

Area of the rectangle is a maximum
Maximum area

$$= 8\left(9x^2 - x^4\right)^{\frac{1}{2}}$$

$$= 8\sqrt{9\left(\frac{3}{\sqrt{2}}\right)^2 - \left(\frac{3}{\sqrt{2}}\right)^4}$$

$$= 36 \text{ units}^2$$

ALT

Note: $A = 8\left(9x^2 - x^4\right)^{\frac{1}{2}}$

Since $x$, $y > 0$, value of $x$ that maximises $A$ also maximises $A^2$

$$A^2 = 64\left(9x^2 - x^4\right)$$

$$\frac{dA^2}{dx} = 64\left(18x - 4x^3\right) = 0$$

$$\Rightarrow x = \frac{3}{\sqrt{2}}$$

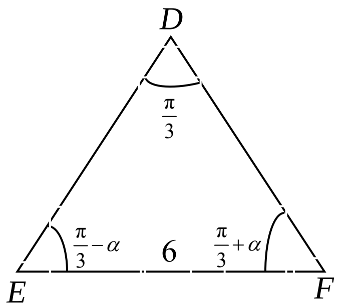
(b) Using Sine rule,

$$\frac{DF}{\sin\left(\frac{\pi}{3} - \alpha\right)} = \frac{6}{\sin\left(\frac{\pi}{3}\right)}$$

$$DF = 4\sqrt{3}\sin\left(\frac{\pi}{3} - \alpha\right)$$

$$\frac{DE}{\sin\left(\frac{\pi}{3} + \alpha\right)} = \frac{6}{\sin\left(\frac{\pi}{3}\right)}$$

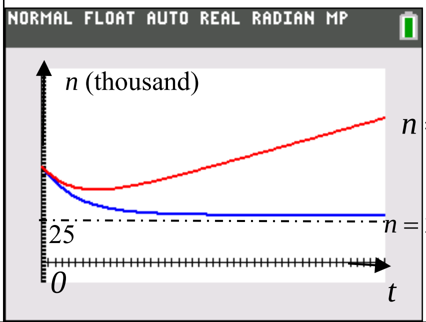
$$DE = 4\sqrt{3}\sin\left(\frac{\pi}{3} + \alpha\right)$$



| | | |
|---|---|---|
| | $DF-DE$<br>$=4\sqrt{3}\sin\left(\frac{\pi}{3}-\alpha\right)-4\sqrt{3}\sin\left(\frac{\pi}{3}+\alpha\right)$<br>$=4\sqrt{3}\left(\frac{\sqrt{3}}{2}\cos\alpha-\frac{1}{2}\sin\alpha\right)-4\sqrt{3}\left(\frac{\sqrt{3}}{2}\cos\alpha+\frac{1}{2}\sin\alpha\right)$<br>$\approx 4\sqrt{3}\left[\frac{\sqrt{3}}{2}\left(1-\frac{\alpha^2}{2}\right)-\frac{1}{2}\alpha-\frac{\sqrt{3}}{2}\left(1-\frac{\alpha^2}{2}\right)-\frac{1}{2}\alpha\right]$<br>$=-4\sqrt{3}\alpha$ | |

| Question 9 [13 Marks] | | |
|---|---|---|
| (i) | $l:\mathbf{r}=\begin{pmatrix}2\\2\\-3\end{pmatrix}+\lambda\begin{pmatrix}4\\0\\1\end{pmatrix},\ \lambda\in\mathbb{R}$<br><br>Let $\theta$ be the angle between the line $l$ and the plane $p_1$.<br><br>$\sin\theta=\left\|\frac{\begin{pmatrix}4\\0\\1\end{pmatrix}.\begin{pmatrix}1\\1\\1\end{pmatrix}}{\sqrt{17}\sqrt{3}}\right\|$<br>$=\left\|\frac{5}{\sqrt{17}\sqrt{3}}\right\|$<br>$\theta=44.4^\circ$ | |
| (ii) | $l_{AN}:\mathbf{r}=\begin{pmatrix}2\\2\\-3\end{pmatrix}+\mu\begin{pmatrix}1\\1\\1\end{pmatrix},\ \mu\in\mathbb{R}$<br>$\begin{pmatrix}2+\mu\\2+\mu\\-3+\mu\end{pmatrix}\bullet\begin{pmatrix}1\\1\\1\end{pmatrix}=16$<br>$2+\mu+2+\mu-3+\mu=16$<br>$3\mu=15$<br>$\mu=5$<br><br>Coordinates of $N=(7,7,2)$ | |

| | | |
|---|---|---|
| (iii) | Since $N$ is the midpoint of $A$ and $B$, using ratio theorem,<br>$\overrightarrow{ON}=\dfrac{\overrightarrow{OA}+\overrightarrow{OB}}{2}$<br>$\overrightarrow{OB}=2\overrightarrow{ON}-\overrightarrow{OA}$<br>$=2\begin{pmatrix}7\\7\\2\end{pmatrix}-\begin{pmatrix}2\\2\\-3\end{pmatrix}=\begin{pmatrix}12\\12\\7\end{pmatrix}$<br>Coordinates of $B=(12,12,7)$ | |
| (iv) | Let $C$ be the point of intersection of the line $l$ and the plane $p_1$.<br>$\begin{pmatrix}2+4\lambda\\2\\-3+\lambda\end{pmatrix}\bullet\begin{pmatrix}1\\1\\1\end{pmatrix}=16$<br>$2+4\lambda+2-3+\lambda=16$<br>$5\lambda=15$<br>$\lambda=3$<br>$\overrightarrow{OC}=\begin{pmatrix}14\\2\\0\end{pmatrix}$<br>$\overrightarrow{BC}=\begin{pmatrix}14\\2\\0\end{pmatrix}-\begin{pmatrix}12\\12\\7\end{pmatrix}=\begin{pmatrix}2\\-10\\-7\end{pmatrix}$<br>$l_{BC}:\mathbf{r}=\begin{pmatrix}14\\2\\0\end{pmatrix}+s\begin{pmatrix}2\\-10\\-7\end{pmatrix},\ s\in\mathbb{R}$ | |
| (v) | Since $AN=BN$,<br>$BN=\sqrt{(2-7)^2+(2-7)^2+(-3-2)^2}$<br>$=\sqrt{(-5)^2+(-5)^2+(-5)^2}$<br>$=\sqrt{75}$units<br>$=5\sqrt{3}$units | |

| Question 10 [14 marks] | | |
|---|---|---|
| (ai) | Let $n$ denote the population of prawns in thousands at time $t$<br>$\frac{\mathrm{d}^2 n}{\mathrm{d}t^2} = \mathrm{e}^{-\frac{t}{5}}$<br>$\frac{\mathrm{d}n}{\mathrm{d}t} = -5\mathrm{e}^{-\frac{t}{5}} + C$<br>$n = 25\mathrm{e}^{-\frac{t}{5}} + Ct + D$ | |
| (aii) | Given $n = 50,\ t = 0,$<br>$50 = 25 + D \Rightarrow D = 25$<br>$n = 25\mathrm{e}^{-\frac{t}{5}} + Ct + 25$<br><br>I Requires $C > 0$ so that $n = 25\mathrm{e}^{-\frac{t}{5}} + Ct + 25 \to \infty$ as $t \to \infty$<br><br>II Requires $C = 0$ so that $n = 25\mathrm{e}^{-\frac{t}{5}} + 25$<br>Then as $t \to \infty$, $n \to 25$<br><br><br><br>$n = 25\mathrm{e}^{-\frac{t}{5}} + Ct + 25, C > 0$<br>$n = 25\mathrm{e}^{-\frac{t}{5}} + 25$ | |
| (bi) | $\frac{\mathrm{d}x}{\mathrm{d}t} = -kx$<br><br>$\int \frac{1}{x}\ \mathrm{d}x = -k\int 1\ \mathrm{d}t$<br><br>$\ln\lvert x\rvert = -kt + C$<br>$x = A\mathrm{e}^{-kt}$ where $A = \pm \mathrm{e}^{C}$<br>At $t = 0$, $x = 0.1$,<br>$\therefore A = 0.1$<br>$x = \frac{1}{10}\mathrm{e}^{-kt}$ (shown) | |

| | | |
|---|---|---|
| (bii) | At $t = 4$, $x = 0.05$,<br>$\therefore 0.05 = 0.1\mathrm{e}^{-4k}$<br>$\Rightarrow \mathrm{e}^{-4k} = \frac{1}{2}$<br>$\Rightarrow -k = \frac{\ln\frac{1}{2}}{4}$<br>$\Rightarrow k = -\frac{1}{4}\ln\frac{1}{2} = \frac{\ln 2}{4}$ | |
| (biii) | Total amount of drug present in the prawn's body at any time $t$<br>$< 0.1 + 0.1\mathrm{e}^{-\left(\frac{\ln 2}{4}\right)8} + 0.1\mathrm{e}^{-2\left(\frac{\ln 2}{4}\right)8} + 0.1\mathrm{e}^{-3\left(\frac{\ln 2}{4}\right)8} + \ldots$<br>$= \frac{0.1}{1-\mathrm{e}^{-\left(\frac{\ln 2}{4}\right)8}}$<br>$= \frac{2}{15} < 0.15$<br>$\therefore$ The total amount of drug present in the prawn's body at any time $t$ is always less than 0.15 mg. | |