SAINT ANDREW'S JUNIOR COLLEGE

Preliminary Examination

MATHEMATICS

Higher 2 **9758/01**

Thursday **24 August 2017** **3 hours**

Additional materials : Answer paper
List of Formulae (MF26)
Cover Sheet

READ THESE INSTRUCTIONS FIRST

Write your name, civics group and index number on all the work you hand in.

Write in dark blue or black pen on both sides of the paper.

You may use a soft pencil for any diagrams or graphs.

Answer **all** the questions. Total marks : **100**

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of agles in degrees, unless a different level of accuracy is specified in the question.

You are expected to use a graphic calculator.

Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.

Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.

You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

At the end of the examination, fasten all your work securely together.

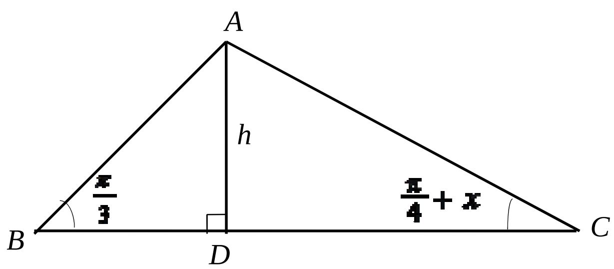
This document consists of 6 printed pages including this page.

[Turn over

1 The volume of a spherical bubble is increasing at a constant rate of λ cm^3 per second. Assuming that the initial volume of the bubble is negligible, find the exact rate in terms of λ at which the surface area of the bubble is increasing when the volume of the bubble is 20 cm^3. **[5]**

[The volume of a sphere, $V = \frac{4}{3}\pi r^3$ and the surface area of a sphere, $A = 4\pi r^2$ where r is the radius of the sphere.]

2 The diagram shows the triangle ABC. It is given that the height AD is h units, $\angle ABD = \frac{\pi}{3}$ and $\angle ACD = \frac{\pi}{4} + x$.



Show that if x is sufficiently small for x^3 and higher powers of x to be neglected, then

$$BC = \frac{h}{\sqrt{3}} + \frac{h}{\tan\left(\frac{\pi}{4} + x\right)} \approx h\left(p + qx + rx^2\right),$$

for constants p, q, r to be determined in exact form. **[5]**

3 It is given that

$$f(x) = \begin{cases} b\sqrt{1 - \dfrac{x^2}{a^2}} & \text{for } -a < x \le a \\ -a\sqrt{1 - \dfrac{(x-2a)^2}{a^2}} & \text{for } a < x \le 3a \end{cases}$$

and that $f(x + 4a) = f(x)$ for all real values of x, where a and b are real constants and $0 < a < b$.

(i) Sketch the graph of $y = f(x)$ for $-a \le x \le 8a$. **[3]**

(ii) Use the substitution $x = a\cos\theta$ to find the exact value of $\int_{3a}^{4a} f(x)\,dx$ in terms of a, b and π. **[5]**

4 **(i)** State a sequence of transformations that would transform the curve with equation $y = \mathrm{e}^{x^2}$ onto the curve with equation $y = \mathrm{f}(x)$, where $\mathrm{f}(x) = \mathrm{e}^{ax^2} - b$, $a > 0$ and $b > 1$. **[2]**

(ii) Sketch the curve $y = \mathrm{f}(x)$ and the curve $y = \dfrac{1}{\mathrm{f}(x)}$.

You should state clearly the equations of any asymptotes, coordinates of turning points and axial intercepts. **[5]**

5 It is given that $\mathbf{u} + \mathbf{v} - \mathbf{w}$ is perpendicular to $\mathbf{u} - \mathbf{v} + \mathbf{w}$, where $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are unit vectors.

(i) Show that the angle between $\mathbf{v}$ and $\mathbf{w}$ is 60°. **[4]**

Referred to the origin O, the points U, V and W have position vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ respectively.

(ii) Find the exact area of triangle OVW. **[2]**

(iii) Given that $\mathbf{u}$ and $\mathbf{v} \times \mathbf{w}$ are parallel, find the exact volume of the solid $OUVW$. **[2]**

[The volume of a pyramid is $\dfrac{1}{3}bh$, where b is the base area and h is the height of the pyramid.]

6 **(a)** **(i)** Find $\int \mathrm{e}^x \cos nx \, \mathrm{d}x$, where n is a positive integer. **[4]**

(ii) Hence, without the use of a calculator, find $\int_{\pi}^{2\pi} \mathrm{e}^x \cos nx \, \mathrm{d}x$ in terms of n, when n is odd. **[3]**

(b) The region bounded by the curve $y = \dfrac{\sqrt{x}}{16 - x^2}$, the y-axis and the line $y = \dfrac{\sqrt{2}}{12}$ is rotated 2π radians about the x-axis. Find the exact volume of the solid obtained. **[5]**

7 **(i)** Show that for any complex number $z = r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$, and $-\pi < \theta \leq \pi$,

$$\frac{z}{z-r} = \frac{1}{2} - \frac{1}{2}\left(\cot\frac{\theta}{2}\right)\mathrm{i}\,.$$ **[3]**

(ii) Given that $z = 2\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{3}\right)}$ is a root of the equation $z^2 - 2z + 4 = 0$. State, in similar form, the other root of the equation. **[1]**

(iii) Using parts **(i)** and **(ii)**, solve the equation $\dfrac{4w^2}{(w-1)^2} - \dfrac{4w}{w-1} + 4 = 0$. **[4]**

8 In a training session, an athlete runs from a starting point *S* towards his coach in a straight line as shown in the diagrams below. When he reaches the coach, he runs back to *S* along the same straight line. A lap is completed when he returns to *S*. At the beginning of the training session, the coach stands at A_1 which is 30 m away from *S*. After the first lap, the coach moves from A_1 to A_2 and after the second lap, he moves from A_2 to A_3 and so on. The distance between A_i to A_{i+1} is denoted by A_iA_{i+1}, $i \in \mathbb{Z}^+$.

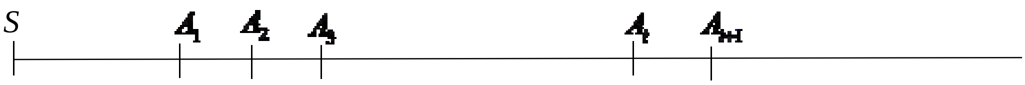


Figure 1

(i) For training regime 1 (shown in Figure 1), the coach ensures that the distance $A_iA_{i+1} = 3$ m for $i \in \mathbb{Z}^+$. Find the least number of laps that the athlete must complete so that he covers a total distance of more than 3000 m. **[3]**

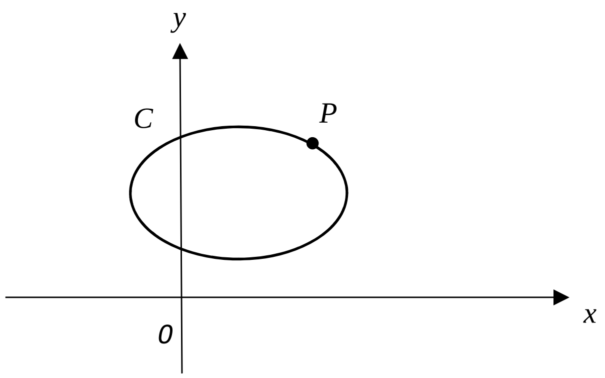
Figure 2

(ii) For training regime 2 (shown in Figure 2), after the first lap, the coach ensures that the distances $A_1A_2 = 2$ m, $A_2A_3 = 6$ m and the distance $A_{i+1}A_{i+2} = 3A_iA_{i+1}$ where $i \in \mathbb{Z}^+$. Show that the distance the coach is away from *S* **just before** the athlete completed r laps is $\left(3^{r-1} + 29\right)$m.

Hence find the distance run by the athlete after *n* complete laps. Also find how far the athlete is from the coach after he has run 8 km. **[6]**

9 The diagram below shows the curve C with parametric equations

$$x = 1 + 2\sin\theta, \quad y = 4 + \sqrt{3}\cos\theta, \ \text{ for } \ -\pi < \theta \le \pi.$$



The point P is where $\theta = \dfrac{\pi}{6}$.

(i) Using a non-calculator method, find the equation of the normal at P. **[4]**

(ii) The normal at the point P cuts C again at point Q, where $\theta = \alpha$. Show that $8\sin\alpha - 2\sqrt{3}\cos\alpha = 1$ and hence deduce the coordinates of Q. **[3]**

(iii) Find the area of the region bounded by the curve C, the normal at point P and the vertical line passing through the point Q. **[4]**

10 A population of 15 foxes has been introduced into a national park. A zoologist believes that the population of foxes, x, at time t years, can be modelled by the Gompertz equation given by:

$$\frac{dx}{dt} = cx\ln\left(\frac{40}{x}\right),$$

where c is a constant.

(i) Using the substitution $u = \ln\left(\dfrac{40}{x}\right)$, show that the differential equation can be written as $\dfrac{du}{dt} = -cu$. **[2]**

(ii) Hence find u in terms of t and show that $x = 40e^{-Be^{-ct}}$, where B is a constant. **[5]**

After 3 years, the population of foxes is estimated to be 20.

(iii) Find the values of B and c. **[3]**

(iv) Find the population of foxes in the long run. **[1]**

(v) Hence, sketch the graph showing the population of foxes over time. **[2]**

[Turn over

11 A computer-controlled machine can be programmed to make plane cuts by keying in the equation of the plane of the cut, and drill holes in a straight line through an object by keying in the equation of the drill line.

A 10cm × 20 cm ×30 cm cuboid is to be cut and drilled. The cuboid is positioned relative to the *x*-, *y*-and *z*-axes as shown in Figure 1.

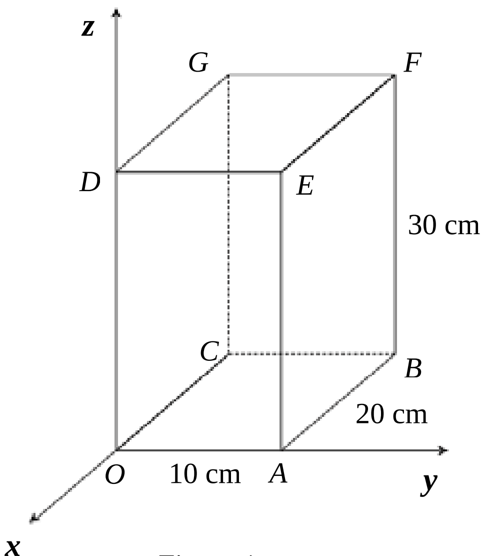


Figure 1

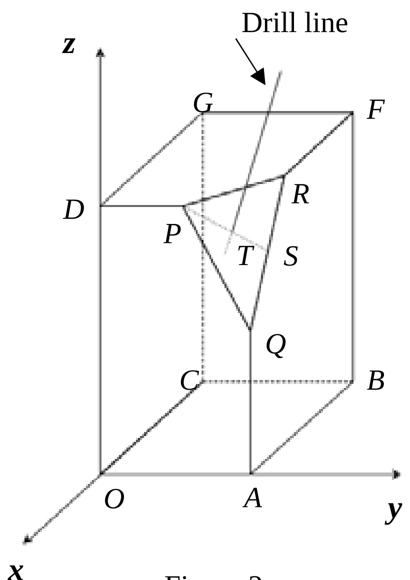


Figure 2

First, a plane cut is made to remove the corner at *E*. The cut goes through the points *P*, *Q* and *R* which are the midpoints of the sides *ED*, *EA* and *EF* respectively.

(i) Show that $\overrightarrow{PQ} = \begin{pmatrix} 0 \\ 5 \\ -15 \end{pmatrix}$ and $\overrightarrow{PR} = \begin{pmatrix} -10 \\ 5 \\ 0 \end{pmatrix}$. **[2]**

(ii) Find the cartesian equation of the plane, *p* that contains *P*, *Q* and *R*. **[2]**

(iii) Find the acute angle between *p* and the plane *DEFG*. **[2]**

A hole is then drilled perpendicular to triangle *PQR*, as shown in Figure 2. The hole passes through the triangle at the point *T* which divides the line *PS* in the ratio of 4:1, where *S* is the midpoint of *QR*.

(iv) Show that the point *T* has coordinates $(-4,\ 9,\ 24)$. **[3]**

(v) State the vector equation of the drill line. **[1]**

(vi) If the computer program continues drilling through the cuboid along the same line as in part **(v)**, determine the side of the cuboid that the drill exits from. Justify your answer. **[4]**

End of Paper