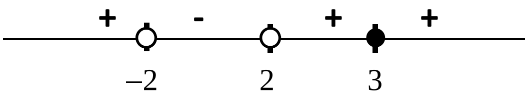
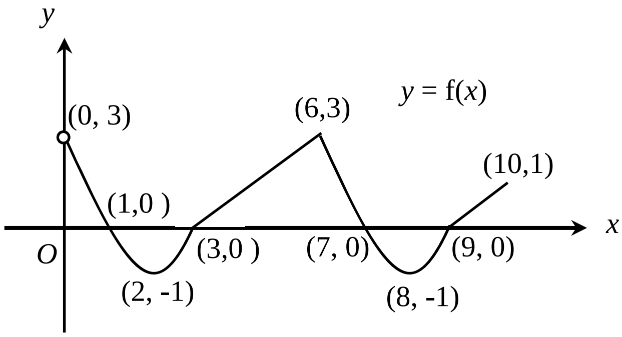
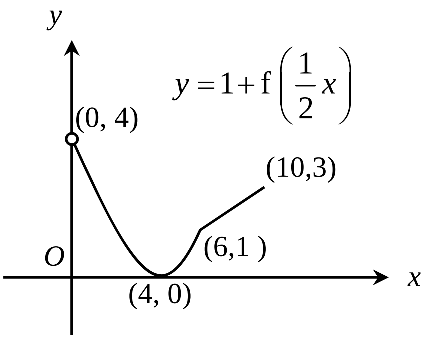
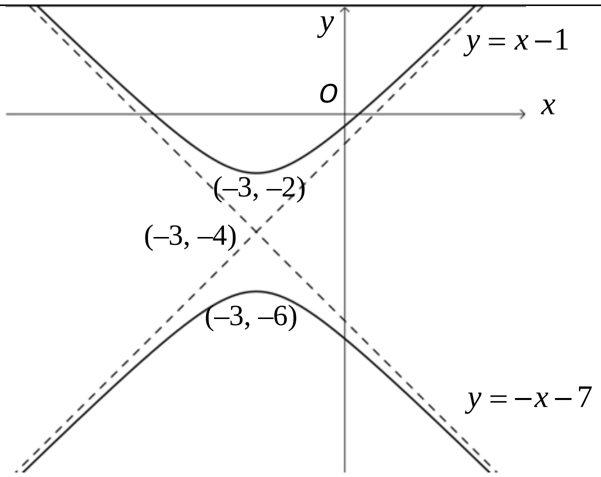
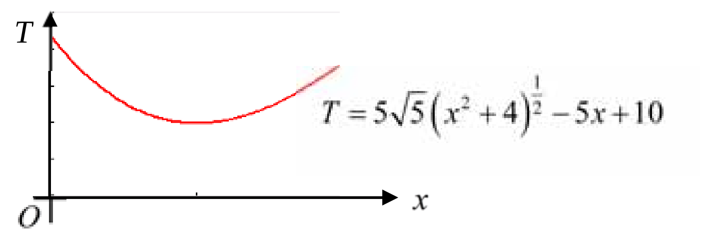
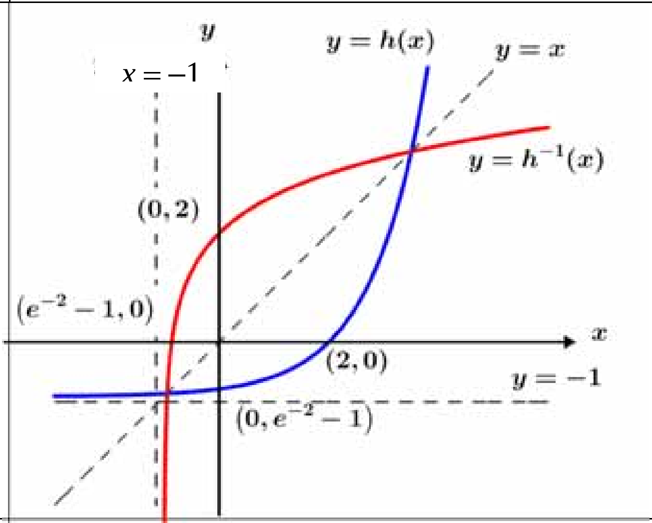
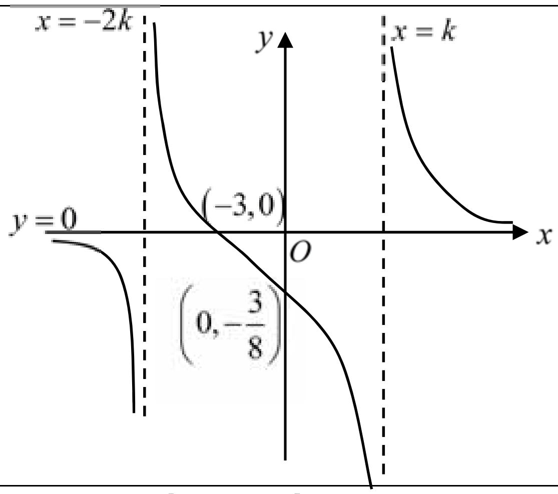
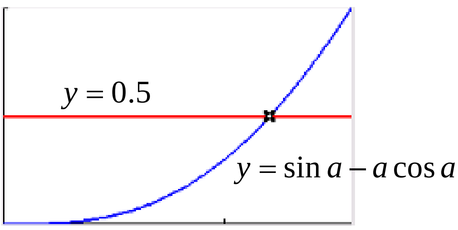
VJC H2 Maths Preliminary Examination P1 2017 Solutions

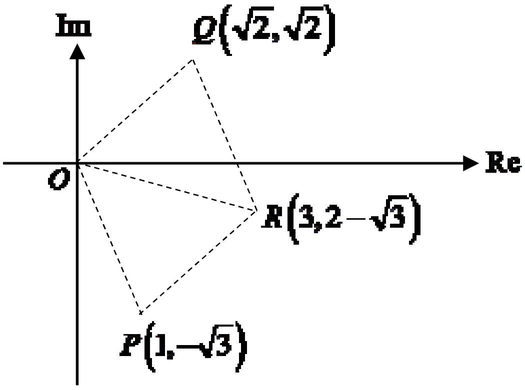
Q	Solution	Comments
1	$\dfrac{6x-13}{x^2-4} \leq 1$ $\dfrac{6x-13-x^2+4}{x^2-4} \leq 0$ $\dfrac{(x-3)^2}{(x+2)(x-2)} \leq 0$  $\therefore -2 < x < 2 \quad \text{or} \quad x = 3$	
2i	Let \$$E$, \$$W$ and \$$G$ be the unit cost of electricity, water and gas, respectively. $514E + 18.8W + 134G = 155.54$ $309E + 11.3W + 89G = 94.99$ $639(1.2)E + 21.7W + 108G = 208.40$ Using G.C, $E = 0.2137, \quad W = 1.1749, \quad G = 0.1761.$	
2ii	Let w be the water usage for August 2017 $(0.2137)(1.2)(555) + 1.1749w + 0.1761(128) = 184.84$ $w = 17$	
3i		
3ii		

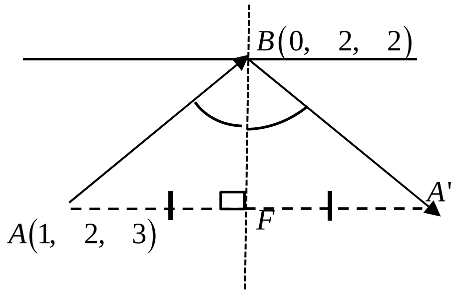
Q	Solution	Comments
4	 Intersect at least once: $\{m \in \mathbb{R} : \quad m < -1 \text{ or } \quad m > 1\}$	
5i	$\frac{\mathrm{d}T}{\mathrm{d}x} = \frac{5\sqrt{5}x}{\sqrt{x^2+4}} - 5$ $T = \int \left(\frac{5\sqrt{5}x}{\sqrt{x^2+4}} - 5 \right) \mathrm{d}x$ $= \frac{5\sqrt{5}}{2} \int 2x\left(x^2+4\right)^{-\frac{1}{2}} \mathrm{d}x - \int 5 \,\mathrm{d}x$ $= \frac{5\sqrt{5}}{2} \frac{\left(x^2+4\right)^{\frac{1}{2}}}{\frac{1}{2}} - 5x + C$ $= 5\sqrt{5}\left(x^2+4\right)^{\frac{1}{2}} - 5x + C \qquad ---(1)$	
5ii	When t = 30, $\frac{\mathrm{d}T}{\mathrm{d}x} = 0$: $\frac{5\sqrt{5}x}{\sqrt{x^2+4}} = 5 \Rightarrow \sqrt{5}x = \sqrt{x^2+4}$ $5x^2 = x^2 + 4 \Rightarrow x = \pm 1$ Since $x > 0,\ x = 1$ Substitute $x = 1$ and $T = 30$ into equation (1) $30 = 5\sqrt{5}(1+4)^{\frac{1}{2}} - 5 + C \Rightarrow C = 10$ $T = 5\sqrt{5}\left(x^2+4\right)^{\frac{1}{2}} - 5x + 10$	
5iii	 When x =0, $T = 32.361$. When x =2, $T = 31.623$ Longest time taken by Alvin is 32.4 mins.	

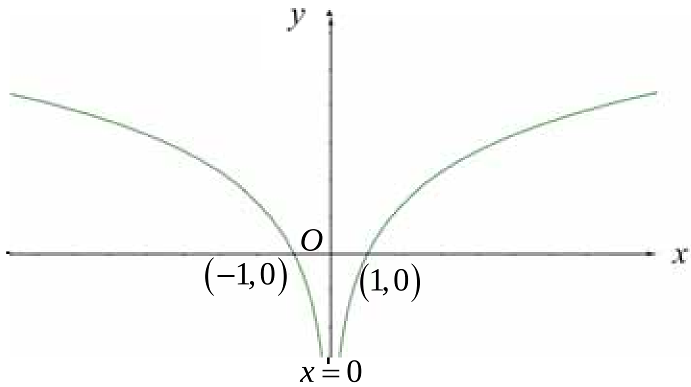
Q	Solution	Comments
6i	$y = \mathrm{e}^{x-2} - 1$ $x = \ln(y+1) + 2$ $\mathrm{h}^{-1}(x) = \ln(x+1) + 2$ Domain of h^{-1} = range of $\mathrm{h} = (-1, \infty)$	
6ii		
6iii	Using G.C, $y = \mathrm{h}^{-1}(x)$ and $y = \mathrm{h}(x)$ intersects at $x = -0.94753$ and $x = 3.50524$ Set of values of $x = \{ x \in \mathbb{R} : -0.948 < x < 3.51 \}$.	
7i		
ii	Since $x = -3$ is the vertical asymptote, $c = 3$ Given that $y = x - 1$ is an oblique asymptote, $\mathrm{f}(x) = x - 1 + \dfrac{A}{x+3}$ $= \dfrac{(x-1)(x+3) + A}{x+3} = \dfrac{x^2 + 2x - 3 + A}{x+3}$ By comparing coefficient of x with $\dfrac{x^2 + ax + b}{x+3}$: $a = 2$	

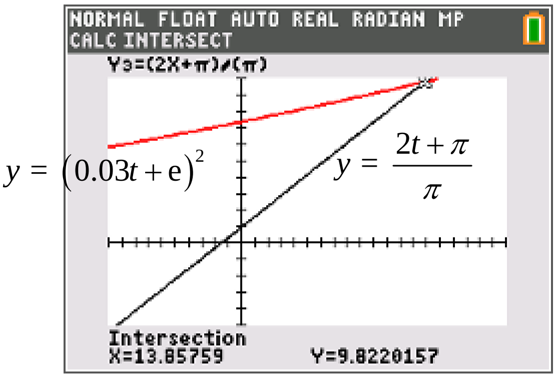
Q	Solution	Comments
	Since $\left(0,-\frac{8}{3}\right)$ is on the curve, $\frac{(0)^2+2(0)+b}{(0)+3}=-\frac{8}{3}$ $b=-8$	
8i	$\sum_{r=1}^{\infty}\frac{r^2+(-1)^r}{3^r}=\sum_{r=1}^{\infty}\frac{r^2}{3^r}+\sum_{r=1}^{\infty}\left(-\frac{1}{3}\right)^r$ $=\frac{3}{2}+\frac{\left(-\frac{1}{3}\right)}{1-\left(-\frac{1}{3}\right)}$ $=\frac{5}{4}$	
8ii	$\sum_{r=4}^{n}\frac{(r-2)^2}{3^{r-2}}=\sum_{r+2=4}^{r+2=n}\frac{r^2}{3^r}$ (replace r with $r+2$) $=\sum_{r=2}^{n-2}\frac{r^2}{3^r}$ $=\sum_{r=1}^{n-2}\frac{r^2}{3^r}-\frac{(1)^2}{3^1}$ $=\frac{3}{2}-\frac{(n-2)^2+3(n-2)+3}{2\left(3^{n-2}\right)}-\frac{1}{3}$ $=\frac{7}{6}-\frac{n^2-4n+4+3n-6+3}{2\left(3^{n-2}\right)}$ $=\frac{7}{6}-\frac{n^2-n+1}{2\left(3^{n-2}\right)}$ $\therefore p=7,\quad q=6,\quad a=1$	
9a	$\int_0^a x\sin x\mathrm{d}x = 0.5$ $\left[-x\cos x\right]_0^a+\int_0^a\cos x\mathrm{d}x=0.5$ $\left[-a\cos a+0\right]+\left[\sin x\right]_0^a=0.5$ $-a\cos a+\sin a=0.5$ --- (1)  Using GC, a = 1.20249 =1.20 (3 s.f.)	

Q	Solution	Comments
9b	Area = $\displaystyle\int_0^4 \frac{\sqrt{x}}{3-\sqrt{x}}\ \mathrm{d}x$ Let $u = 3-\sqrt{x}$ $\dfrac{\mathrm{d}u}{\mathrm{d}x} = -\dfrac{1}{2\sqrt{x}} \Rightarrow \dfrac{\mathrm{d}x}{\mathrm{d}u} = -2(3-u)$ When $x=0,\ u=3$ When $x=4,\ u=1$ $\displaystyle\int_0^4 \frac{\sqrt{x}}{3-\sqrt{x}}\ \mathrm{d}x = \int_3^1 \left(\frac{3-u}{u}\right)\left[(-2)(3-u)\right]\ \mathrm{d}u$ $\displaystyle = \int_1^3 \frac{2(3-u)^2}{u}\ \mathrm{d}u$ $\displaystyle = 2\int_1^3 \frac{9-6u+u^2}{u}\ \mathrm{d}u$ $\displaystyle = 2\int_1^3 \left(\frac{9}{u}-6+u\right)\ \mathrm{d}u$ $\displaystyle = 2\left[9\ln u - 6u + \frac{u^2}{2}\right]_1^3$ $\displaystyle = 2\left(9\ln 3 - 18 + \frac{9}{2}\right) - 2\left(-6+\frac{1}{2}\right)$ $= 18\ln 3 - 16$	
10	Since $1-\sqrt{3}\mathrm{i}$ is a root, $2\left(1-\mathrm{i}\sqrt{3}\right)^3 + p\left(1-\mathrm{i}\sqrt{3}\right)^2 + q\left(1-\mathrm{i}\sqrt{3}\right) - 4 = 0$ $2(-8) + p\left(-2-2\sqrt{3}\mathrm{i}\right) + q\left(1-\mathrm{i}\sqrt{3}\right) - 4 = 0$ $\left(-20-2p+q\right) + \left(-2\sqrt{3}p - \sqrt{3}q\right)\mathrm{i} = 0$ Compare real and imaginary parts: $-2p+q = 20 \qquad ---(1)$ $-2\sqrt{3}p - \sqrt{3}q = 0 \qquad ---(2)$ $\therefore p = -5, \qquad q = 10$	
10i	Since $1-\sqrt{3}\mathrm{i}$ is a root, and all coefficients are real $\Rightarrow 1+\sqrt{3}\mathrm{i}$ is also a root. $2z^3 - 5z^2 + 10z - 4 = \left(z-\left(1-\sqrt{3}\mathrm{i}\right)\right)\left(z-\left(1+\sqrt{3}\mathrm{i}\right)\right)(2z+a)$ $= \left(z^2 - 2z + 4\right)(2z+a)$	

Q	Solution	Comments
	By observation: $a=-1$ $\therefore z_2 = \frac{1}{2}, \qquad z_3 = 1+\sqrt{3}\mathrm{i}$	
10ii	$\lvert z_1 \rvert = \sqrt{1+3} = 2 \qquad \lvert w \rvert = \sqrt{2+2} = 2$ $\arg z_1 = \arg\left(1-\sqrt{3}\mathrm{i}\right)$ $= -\tan^{-1}\left(\frac{\sqrt{3}}{1}\right)$ $= -\frac{\pi}{3}$ $\arg w = \arg\left(\sqrt{2}+\mathrm{i}\sqrt{2}\right)$ $= \frac{\pi}{4}$ $\therefore z_1 = 2\mathrm{e}^{-\frac{\pi}{3}\mathrm{i}}, \quad w = 2\mathrm{e}^{\frac{\pi}{4}\mathrm{i}}$	
10iii	 Quadrilateral $OPRQ$ is a rhombus	
10iv	$4\arg\left(z_1\right)+\arg\left(w^*\right) = 4\arg\left(z_1\right)-\arg\left(w\right)$ $= -\frac{4\pi}{3}-\frac{\pi}{4}$ $= -\frac{19\pi}{12}$ $\arg\left(z_1{}^4 w^*\right) = -\frac{19\pi}{12}+2\pi$ $= \frac{5\pi}{12}$	
11	$\underset{\sim}{r} = \begin{pmatrix}1\\2\\3\end{pmatrix}+\lambda\begin{pmatrix}1\\0\\1\end{pmatrix}, \qquad \lambda \in \mathbb{R}$	

Q	Solution	Comments
11i	$\cos\alpha = \dfrac{\begin{pmatrix}1\\0\\1\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}}{\sqrt{2}\sqrt{3}}$ $\alpha = 35.3^{o}$ $\theta = 90^\circ - 35.3^\circ$ $= 54.7^\circ$	
11ii	Intersection of light beam with reflective surface: $\left[\begin{pmatrix}1\\2\\3\end{pmatrix}+\lambda\begin{pmatrix}1\\0\\1\end{pmatrix}\right]\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}=4$ $6+2\lambda=4$ $\lambda=-1$ Coordinates of point of intersection = $(0,\quad 2,\quad 2)$.	
11iii	Let F be the foot of perpendicular from device to normal line and A be the point $(1,\quad 2,\quad 3)$:  $\overrightarrow{BF} = \left(\overrightarrow{BA}\cdot\hat{\underset{\sim}{n}}\right)\hat{\underset{\sim}{n}}$ $= \left[\dfrac{\begin{pmatrix}1-0\\2-2\\3-2\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}}{\sqrt{3}}\right]\dfrac{\begin{pmatrix}1\\1\\1\end{pmatrix}}{\sqrt{3}}$ $=\dfrac{2}{3}\begin{pmatrix}1\\1\\1\end{pmatrix}$ Using Ratio Theorem,	

Q	Solution	Comments
	$\overrightarrow{BF}=\dfrac{\overrightarrow{BA}+\overrightarrow{BA'}}{2}$ $\overrightarrow{BA'}=\dfrac{4}{3}\begin{pmatrix}1\\1\\1\end{pmatrix}-\begin{pmatrix}1\\0\\1\end{pmatrix}=\dfrac{1}{3}\begin{pmatrix}1\\4\\1\end{pmatrix}$ Equation of reflected light path: $r=\begin{pmatrix}0\\2\\2\end{pmatrix}+\alpha\begin{pmatrix}1\\4\\1\end{pmatrix},\qquad \alpha\in\mathbb{R}$	
12i		
ii	Volume of the vase = $\pi\int_{-2}^{k}x^2\,\mathrm{d}y$ $=\pi\int_{-2}^{k}\mathrm{e}^y\,\mathrm{d}y$ $=\pi\left[\mathrm{e}^y\right]_{-2}^{k}$ $=\pi\left[\mathrm{e}^k-\mathrm{e}^{-2}\right]$	
iii	Volume of water , $V=\pi\int_{-2}^{y}\mathrm{e}^y\,\mathrm{d}y$ $=\pi\left[\mathrm{e}^y-\mathrm{e}^{-2}\right]$ $=\pi\left[\mathrm{e}^{\ln x^2}-\mathrm{e}^{-2}\right]$ $=\pi\left[x^2-\mathrm{e}^{-2}\right]$ Given $\dfrac{\mathrm{d}V}{\mathrm{d}t}=2$, $\dfrac{\mathrm{d}x}{\mathrm{d}t}=\dfrac{\mathrm{d}V}{\mathrm{d}t}\times\dfrac{\mathrm{d}x}{\mathrm{d}V}$ $=2\times\dfrac{1}{2\pi x}$ $=\dfrac{1}{\pi x}$	

Q	Solution	Comments
	Hence the rate at which the radius of the water surface is increasing is $\frac{1}{\pi x}$ cm per second.	
iv	For the insect, $\frac{\mathrm{d}x}{\mathrm{d}t} = 0.03$. t seconds later, the location of the insect is at $x = 0.03t + \mathrm{e}$ For the movement of the water, $\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{\pi x}$ $\int \pi x \, \mathrm{d}x = \int 1 \, \mathrm{d}t$ $\frac{\pi x^2}{2} = t + C$ When $t = 0$, $x = 1$ $C = \frac{\pi}{2}$ $\frac{\pi x^2}{2} = t + \frac{\pi}{2}$ When the insect first comes into contact with water, $\frac{\pi(0.03t + \mathrm{e})^2}{2} - \frac{\pi}{2} = t$ $\pi(0.03t + \mathrm{e})^2 - \pi = 2t$ $(0.03t + \mathrm{e})^2 = \frac{2t + \pi}{\pi}$  Using GC, $t = 13.858$ $x = 0.03(13.858) + \mathrm{e} = 3.1340$ $y = \ln(3.1340)^2 = 2.28$ Hence coordinates of the point = $(3.13, 2.28)$	