1

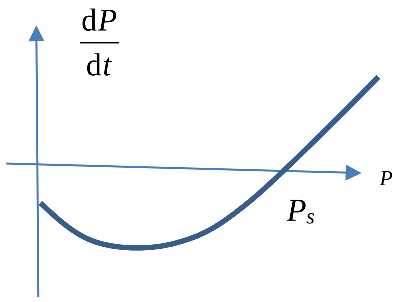
(i) $\frac{1}{15}P^2-\frac{1}{5}P-h=0$

$P^2-3P-15h=0$

$P=\frac{3\pm\sqrt{9-4(-15h)}}{2}=\frac{3\pm\sqrt{9+60h}}{2}$

Since $P>0$, $P_s=\frac{3+\sqrt{9+60h}}{2}$

(ii)



When $0<P_0<P_s$, $\frac{dP}{dt}<0$. The population of the giant otters will decrease and become extinct after a finite number of years.

When $P_0>P_s$, $\frac{dP}{dt}>0$. The population of the giant otters will increase indefinitely.

(iii) $P_0\geq P_s$

$\Rightarrow 6\geq\frac{3+\sqrt{9+60h}}{2}$

$\Rightarrow \sqrt{9+60h}\leq 9$

$\Rightarrow h\leq 1.2$

Maximum number of otters that can be hunted is 1200.

2

$\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3$ are linearly dependent

$\Leftrightarrow \exists\lambda_1,\lambda_2,\lambda_3$ <u>not all zero</u> such that $\lambda_1\mathbf{x}_1+\lambda_2\mathbf{x}_2+\lambda_3\mathbf{x}_3=0$

Consider $$\begin{aligned}\lambda_1\mathbf{Mx}_1+\lambda_2\mathbf{Mx}_2+\lambda_3\mathbf{Mx}_3&=\mathbf{M}\left(\lambda_1\mathbf{x}_1+\lambda_2\mathbf{x}_2+\lambda_3\mathbf{x}_3\right)\\&=\mathbf{M}(\mathbf{0})=\mathbf{0}\end{aligned}$$

ie. we have also found $\lambda_1,\lambda_2,\lambda_3$ <u>not all zero</u> such that $\lambda_1\mathbf{Mx}_1+\lambda_2\mathbf{Mx}_2+\lambda_3\mathbf{Mx}_3=\mathbf{0}$

$\therefore$ $\mathbf{Mx}_1,\mathbf{Mx}_2,\mathbf{Mx}_3$ are linearly dependent.

(i) $11\begin{pmatrix}1\\3\\4\end{pmatrix}-3\begin{pmatrix}-4\\2\\1\end{pmatrix}-\begin{pmatrix}23\\27\\41\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$,

$\therefore \mathbf{y}_1,\mathbf{y}_2,\mathbf{y}_3$ are linearly dependent.

(ii) Since $\mathbf{Ay}_1,\mathbf{Ay}_2,\mathbf{Ay}_3$ are linearly dependent (from previous part), but

$\mathbf{Ay}_1=2\begin{pmatrix}1\\10\\17\end{pmatrix}$, $\mathbf{Ay}_2=2\begin{pmatrix}3\\5\\1\end{pmatrix}$ are linearly independent, $\therefore$ a basis is $\left\{\begin{pmatrix}1\\10\\17\end{pmatrix},\begin{pmatrix}3\\5\\1\end{pmatrix}\right\}$

(iii) Solving $\mathbf{Ay} = \mathbf{0}$ by GC, a basis for the null space is $\left\{ \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \right\}$.

(iv) $\mathrm{T}(\mathbf{0}) = \mathbf{0} \neq \begin{pmatrix} 1 \\ 6 \\ 9 \end{pmatrix}$, $\therefore$ the set of solutions is not a vector space.

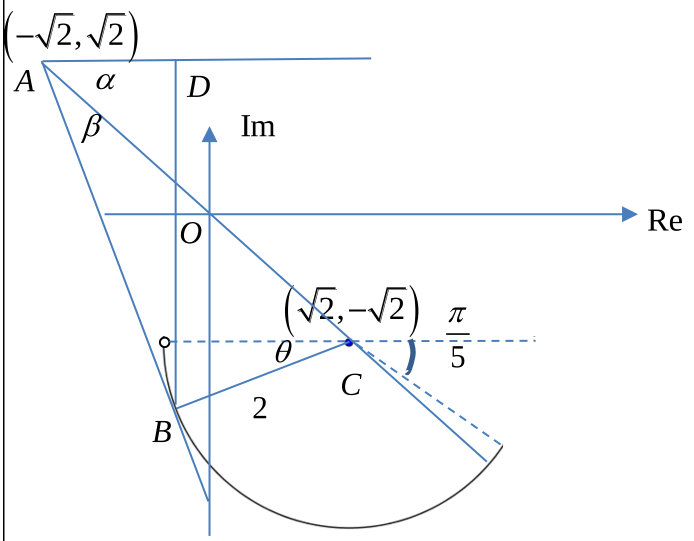
General solution is $\mathbf{x} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$, $\lambda \in \mathbb{R}$

$$|\mathbf{x}| = \sqrt{\lambda^2 + (1+\lambda)^2 + (1-2\lambda)^2} = \sqrt{6\lambda^2 - 2\lambda + 2}$$

$$= \sqrt{6\left(\lambda - \frac{1}{6}\right)^2 + \frac{11}{6}}$$

$\therefore$ least $|\mathbf{x}|$ occurs when $\lambda = \frac{1}{6}$ corresponding to $\mathbf{x} = \frac{1}{6}\begin{pmatrix} 1 \\ 7 \\ 4 \end{pmatrix}$

3



(ii) Max $|z| = 4$

(iii) $\alpha = \tan^{-1} 1 = \frac{\pi}{4}$, $\beta = \sin^{-1} \frac{1}{2} = \frac{\pi}{6}$

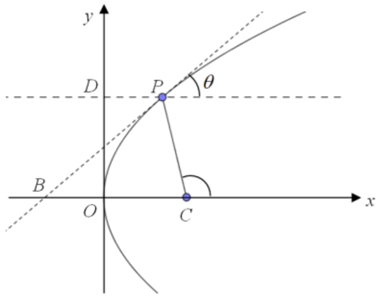
$$\alpha + \beta = \frac{5\pi}{12}$$

$$AB = \sqrt{16 - 4} = 2\sqrt{3}$$

$$AD = AB \cos\left(\frac{5\pi}{12}\right)$$

$$\mathrm{Re}(z) = -\sqrt{2} + 2\sqrt{3}\cos\left(\frac{5\pi}{12}\right) = -\sqrt{2} + 2\sqrt{3}\sin\left(\frac{\pi}{12}\right) = -0.518$$

4



Let the parametric equations be $x = ct^2$, $y = 2ct$

$\therefore \dfrac{dy}{dx} = \dfrac{dy}{dt} \div \dfrac{dx}{dt} = \dfrac{2c}{2ct} = \dfrac{1}{t}$

$\therefore$ at P, $\dfrac{dy}{dx} = \dfrac{1}{p} \qquad \Rightarrow \tan\theta = \dfrac{1}{p}$

Gradient of $CP = \dfrac{2cp - 0}{cp^2 - c} = \dfrac{2p}{p^2 - 1} = \dfrac{\frac{2}{p}}{1 - \left(\frac{1}{p}\right)^2}$

$$= \frac{2\tan\theta}{1 - \tan^2\theta} = \tan 2\theta$$

$\therefore CP$ makes an angle of 2θ with positive x-axis.

$\angle DPB = \theta$ (vertically opposite angles)

$\angle DPC = 2\theta$ (alternate angles)

$\therefore \angle BPC = 2\theta - \theta = \theta$

$\therefore$ both CP and a line parallel to the axis of the parabola make the same angle with the tangent to the parabola.
ie. we have proven any line parallel to the axis of the parabola, after reflection in the parabola at the point of the intersection, will pass through the focus of the parabola.

Gradient of normal at $P = -p$

Let Q have coordinates $(q, 0)$

$\dfrac{2cp - 0}{cp^2 - q} = -p \Rightarrow q = cp^2 + 2c$

$\therefore M$ has coordinates $\underline{\underline{\left(cp^2 + c, cp\right)}}$

$y = cp \Rightarrow p = \dfrac{y}{c} \qquad \therefore x = c\left(\dfrac{y}{c}\right)^2 + c = \dfrac{y^2}{c} + c$

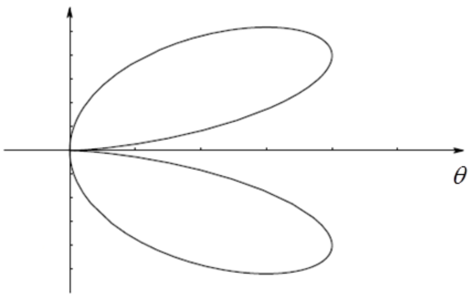
$\therefore$ locus of M is another parabola $\underline{\underline{y^2 = c(x - c)}}$.

Translate in the negative x direction by c units to obtain standard parabola:

$y^2 = cx, \quad$ ie. $y^2 = 4\left(\dfrac{c}{4}\right)x$

$\therefore$ Original parabola has directrix $x = -\dfrac{c}{4} + c, \quad$ ie. $\underline{\underline{x = \dfrac{3c}{4}}}$

5



(i)

(ii) $y = r\sin\theta = 4\cos\theta\sin^2\theta\sin\theta$

$= 4\cos\theta\sin^3\theta$

When the $y-$coordinate has a stationary value,

$$\frac{\mathrm{d}y}{\mathrm{d}\theta} = 4\cos\theta\left(3\sin^2\theta\cos\theta\right) - 4\sin\theta\sin^3\theta$$

$$= 12\sin^2\theta\cos^2\theta - 4\sin^4\theta$$

$$\frac{\mathrm{d}y}{\mathrm{d}\theta} = 0 \Rightarrow 12\sin^2\theta\cos^2\theta - 4\sin^4\theta = 0$$

$$4\sin^2\theta\left(3\cos^2\theta - \sin^2\theta\right) = 0$$

$$4\sin^2\theta = 0 \text{ or } 3\cos^2\theta - \sin^2\theta = 0$$

$\theta = \underline{\underline{0 \text{ or } \pi}}$ or $\tan^2\theta = 3$

$$\tan\theta = \pm\sqrt{3} \Rightarrow \theta = \underline{\underline{\frac{\pi}{3} \text{ or } \frac{2\pi}{3}}}$$

(iii) At $\theta = 0$ or π, coordinates are $(0,0)$.

At $\theta = \frac{\pi}{3}$, x-coordinate $= 4\cos^2\frac{\pi}{3}\sin^2\frac{\pi}{3} = \frac{3}{4}$

y-coordinate $= 4\cos\frac{\pi}{3}\sin^3\frac{\pi}{3} = \frac{3\sqrt{3}}{4}$

At $\theta = -\frac{\pi}{3}$, x-coordinate $= 4\cos^2\left(-\frac{\pi}{3}\right)\sin^2\left(-\frac{\pi}{3}\right) = \frac{3}{4}$

y-coordinate $= 4\cos\left(-\frac{\pi}{3}\right)\sin^3\left(-\frac{\pi}{3}\right) = -\frac{3\sqrt{3}}{4}$

Area of $PQR = \frac{1}{2}\left(\frac{3}{4}\right)\left(\frac{3\sqrt{3}}{2}\right) = \underline{\underline{\frac{9\sqrt{3}}{16}}}$

(iv) $\frac{\mathrm{d}r}{\mathrm{d}\theta} = 4(-\sin\theta)\sin^2\theta + 4\cos\theta\left(2\sin\theta\cos\theta\right)$

$= -4\sin^3\theta + 8\sin\theta\cos^2\theta$

Arc length $= 2\int_0^{\frac{\pi}{2}} \sqrt{\left(4\cos\theta\sin^2\theta\right)^2 + \left(-4\sin^3\theta + 8\sin\theta\cos^2\theta\right)^2}\,\mathrm{d}\theta$

= 7.00

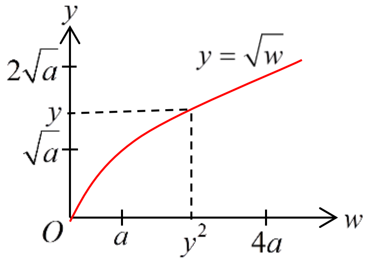
6

$$f(w)=\begin{cases}\dfrac{1}{3a} & a\le w\le 4a,\\ 0 & \text{otherwise.}\end{cases}$$

The p.d.f of W is given by

$W=Y^2 \Rightarrow Y=\sqrt{W}$

$a\le w\le 4a \Leftrightarrow \sqrt{a}\le y\le 2\sqrt{a}$

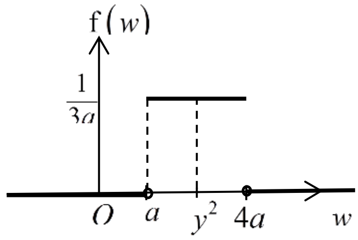


For $\sqrt{a}\le y\le 2\sqrt{a}$,

$G(y)=P(Y\le y)$

$=P\left(X\le y^2\right)$

$=\dfrac{1}{3a}\left(y^2-a\right)$



The p.d.f of Y is given by

$$g(y)=\frac{d}{dy}G(y)=\begin{cases}\dfrac{2y}{3a} & \sqrt{a}\le y\le 2\sqrt{a},\\ 0 & \text{otherwise .}\end{cases}$$

P(a success)

$=P\left(Y>\dfrac{3}{2}\sqrt{a}\right)$

$=1-G\left(\dfrac{3}{2}\sqrt{a}\right)$

$=1-\dfrac{1}{3a}\left(\dfrac{9}{4}a-a\right)=\dfrac{7}{12}$

Required probability

$=qp+q^3p+q^5p+\ldots$ where $p=\dfrac{7}{12}$ and $q=\dfrac{5}{12}$

$=qp+q^3p+q^5p+\ldots=\dfrac{qp}{1-q^2}=\dfrac{qp}{(1+q)(1-q)}$

$=\dfrac{q}{1+q}=\dfrac{5}{12}\times\dfrac{12}{17}=\dfrac{5}{17}$

7

Let X be the number of system failures in a period of t hours. Then $X\sim \text{Po}(0.25t)$

$P(X\ge 1)=1-P(X=0)$

$\quad\quad\quad\;=1-e^{-0.25t}$

Distribution function of T is given by

$F(t)=P(T\le t),\ \ t\ge 0$

	$= \mathrm{P}(\text{at least one system failure in a } t\text{-hour period}),\ t \geq 0$ $= \mathrm{P}(X \geq 1) = 1 - \mathrm{e}^{-0.25t},\ t \geq 0$ $\mathrm{F}(t) = \begin{cases} 1 - \mathrm{e}^{-0.25t} & t \geq 0, \\ 0 & t < 0. \end{cases}$ The probability density function of T is $\mathrm{f}(t) = \frac{\mathrm{d}}{\mathrm{dt}}\mathrm{F}(t) = \begin{cases} 0.25\mathrm{e}^{-0.25t} & t \geq 0, \\ 0 & t < 0. \end{cases}$ (i) Let Y be the number of system failures in a period of 1 hours. Then $Y \sim \mathrm{Po}(0.25)$ $\mathrm{P}(Y = 0 \mid Y < 2) = \frac{\mathrm{P}(Y=0)}{\mathrm{P}(Y \leq 1)} = 0.8$ (ii) The technician cannot complete the test before at least 3 more system failures occur if the time between successive 2 system failures is less than 2 hours each. Probability that he cannot complete a full test $= \mathrm{P}(T < 2)$ $= 1 - \mathrm{e}^{-0.25 \times 2} = 1 - \mathrm{e}^{-0.5}$ Required probability $= \left(1 - \mathrm{e}^{-0.5}\right)^3 = 0.060916 \approx 0.0619$
8	**(i)** Let X g and Y g be the yield from a plant using *Sweetgro* and *Fruitplus* respectively. Let μ_1 g and μ_2 g be the mean yield from plants using *Sweetgro* and *Fruitplus* respectively. $\mathrm{H}_0: \mu_1 - \mu_2 = 20$ $\mathrm{H}_1: \mu_1 - \mu_2 > 20$ Level of significance: 5% Test statistic: Use 2-sample t –test, When H_0 is true, $T = \dfrac{(\bar{X} - \bar{Y}) - 20}{S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \sim t_{n_1+n_2-2}$ where $S_p^2 = \dfrac{\sum(X - \bar{X})^2 + \sum(Y - \bar{Y})^2}{n_1 + n_2 - 2}$ Degrees of freedom: $\nu = 10 + 8 - 2 = 16$ Rejection region: $t \geq 1.7459$ Computation: $n_1 = 10,\ \bar{x} = 1377.4,\ s_1 = 18.39203$ $n_2 = 8,\ \bar{y} = 1342.125,\ s_2 = 17.79597$ $s_p^2 = \dfrac{\sum(x - \bar{x})^2 + \sum(y - \bar{y})^2}{n_1 + n_2 - 2} = 18.13366^2$ $t = 1.7758$, $p - \text{value} = 0.047389 \approx 0.0474$

	Conclusion: Since $p-\text{value}=0.0474<0.05$ (or $t=1.7758>1.7459$), $\therefore \mathrm{H}_0$ is rejected at 5% significance level. Hence there is sufficient evidence to support the claim that the mean yield from plants grown using *Sweetgro* is greater than that from plants grown using *Fruitplus* by more than 20 g at the 5% level of significance. **(ii)** If the common variance is 285.75 grams^2, then normal distribution will be used. When H_0 is true, $Z=\dfrac{\left(\bar{X}-\bar{Y}\right)-20}{\sqrt{285.75\left(\dfrac{1}{n_1}+\dfrac{1}{n_2}\right)}} \sim \mathrm{N}(0,1)$ **(iii)** Cost per plant using *Fruitplus* $=\$\dfrac{10}{8}=\$1.25$ Cost per plant using *Sweetgro* $=\$\dfrac{15}{10}=\$1.50$ Average income per plant using *Fruitplus* $=\$3.50\times\dfrac{1342.125}{1000}=\$4.6974375$ Average income per plant using *Sweetgro* $=\$3.50\times\dfrac{1377.4}{1000}=\$4.8209$ Profit per plant using using *Fruitplus* $=\$4.6974375-\$1.25\approx\$3.45$ Profit per plant using using *Sweetgro* $=\$4.8209-\$1.50\approx\$3.32$ Hence Peter should continue with *Fruitplus* since this gives a greater profit per plant.
9	(a) (i) H_0: colour preference is independent of personality type H_1: colour preference is not independent of personality type Level of significance: 5% Test statistic: $\chi^2=\sum_i\sum_j\dfrac{\left(f_{ij}-e_{ij}\right)^2}{e_{ij}}$ Computation: Under H_0, expected frequencies, $e_{ij}=\dfrac{r_i\times c_j}{N}$ (N = grand total) are shown below:

		Colour preference			
		Red	Yellow	Green	Blue
Personality type	Introvert	47	9.4	18.8	18.8
	Extrovert	153	30.6	61.2	61.2

Calculation of contributions to test statistic:

		Colour preference			
		Red	Yellow	Green	Blue
Personality type	Introvert	2.5745	0.20851	1.43830	2.7574
	Extrovert	0.79085	0.064052	0.44183	0.84706

$\therefore \chi^2=9.1225$

$p-\text{value}=0.0277$

Degrees of freedom: $\nu=(2-1)(4-1)=3$

Rejection region: $\chi^2\geq 7.815$

	Conclusion: Since 9.1225 > 7.815 (or p – value = 0.0277 < 0.05), $\therefore \mathrm{H}_0$ is rejected at the 5% significance level. Hence there is sufficient evidence to suggest that colour preference is not independent of personality type at the 5% significance level. (ii) The greatest contributions for not independent (association) are in the cells introvert/red, introvert/green and introvert/blue. Under H_0, the observed frequencies are much greater than expected for the cells introvert/green and introvert/blue and the observed frequency for the cell introvert/red is much lower than expected. This suggests that introverts are far more likely than expected to prefer blue or green. (or introverts are far less likely to choose red). (Alternatively, extroverts are more likely than expected to prefer red under H_0) (b) (i) Let p be the actual proportion of students who are in favour of the move. Number in favour of move, $x=458$, $n=1250$, $\hat{p}=\frac{x}{n}=\frac{458}{1250}=0.3664$ A 95% confidence interval for p is given by $0.3664-1.9600\sqrt{\frac{0.3664(1-0.3664)}{1250}}<p$ $<0.3664+1.9600\sqrt{\frac{0.3664(1-0.3664)}{1250}}$ $0.33969<p<0.39311$ $0.340<p<0.393$ (iii) Since the interval is entirely above the value $\frac{1}{3}$ (lower limit 0.340 is greater than $\frac{1}{3}$), thus the interval calculated support the claim that more than a third of the students in the institution is in favour of the move.
10	(i) So that any influence of the order of taking drugs does not affect the outcome of the investigation. (ii) Let X_A and X_B hours be the number of hours of relief from pain gained by arthritis sufferers taking drug A and drug B respectively. Let $D=X_\mathrm{A}-X_\mathrm{B}$ (a) Let m hours be the median of differences. $\mathrm{H}_0: m=0$ $\mathrm{H}_1: m<0$ Level of significance: 1% Test Statistic: Let P and Q be the sum of the ranks corresponding to the positive and negative differences respectively. Let T be the smaller of these sums

Computation:

$d = x_A - x_B$	rank of $\lvert d \rvert$	Signed rank negative	Signed rank positive
–1.5	6	–6	
–2.1	7	–7	
–0.2	2	–2	
0.3	3		3
–2.6	8	–8	
0.1	1		1
0.6	4		4
–1.2	5	–5	
–3.0	9	–9	
–3.4	10	–10	
		–47	8

$\therefore p = 8, \quad q = 47$

$\therefore t = 8$

Rejection region: $t \le 5$

Conclusion: Since $t = 8 > 5$, H_0 is not rejected at 1% level of significance. Hence there is insufficient evidence that the data support the belief that drug B is more effective in relieving pain gained by arthritis sufferers than drug A at 1% significance level.

(b) Let μ_A and μ_B hours be the mean number of hours of relief from pain gained by arthritis sufferers taking drug *A* and drug *B* respectively.

$H_0 : \mu_A - \mu_B = 0$

$H_1 : \mu_A - \mu_B < 0$

Level of significance: 1%

Test Statistic: use paired – sample t – test

When H_0 is true, $T = \dfrac{\bar{D}}{S/\sqrt{n}} \sim t_{n-1}$ where $\bar{D} = \dfrac{\sum D}{n}$ and $S^2 = \dfrac{\sum\left(D - \bar{D}\right)^2}{n-1}$

Degrees of freedom: $\nu = 10 - 1 = 9$

Rejection region: $t \le -2.8214$

Computation: $\sum d = -13, \ \sum d^2 = 35.92, \ \ n = 10$

$$\therefore \bar{d} = \frac{\sum d}{n} = -1.30, \quad s^2 = \frac{1}{n-1}\left[\sum d^2 - \frac{\left(\sum d\right)^2}{n}\right] = 1.45373^2$$

$t = -2.8279$

$p - \text{value} = 0.0098956 \approx 0.009896$

Conclusion: Since p-value = 0.009896 < 0.01 (or $t = -2.8279 < -2.8214$), $\therefore H_0$ is rejected at 1% significance level. Hence, there is sufficient evidence that the data support the belief that drug B is more effective in relieving pain gained by arthritis sufferers than drug A at the 1% significance level.

	Given the populations are normally distributed, the t –test is more appropriate as it is a more powerful test because it takes into consideration the size of differences between the paired observations while the Wilcoxon matched-pairs signed rank test only considers the rank of the absolute difference between paired observations. (iii) The conclusion in part (ii) might not apply to all adult arthritis sufferers since it is based on a sample in which adults volunteered to take part in the experiment, the sample is not a random sample. (iv) Let m_{A} hours be the median number of hours of relief from pain gained by arthritis sufferers taking drug A. $\text{H}_0 : m_{\text{A}} = 4.5$ $\text{H}_1 : m_{\text{A}} < 4.5$ Level of significance: 10% Test statistic: Let S be the number of plus signs from $\left(X_{\text{A}} - 4.5\right)$ in a sample of size 10. Under H_0, $S \sim \text{B}\left(10,\ \frac{1}{2}\right)$ Computation: From the sample, there are 4 observations greater than 4.5, $\therefore s = 4$ $p - \text{value} = \text{P}\left(S \leq 4\right)$ $= 0.37695 \approx 0.377$ Conclusion: Since p – value = 0.377 > 0.10, $\therefore \text{H}_0$ is not rejected at 10% level of significance. Hence there is insufficient evidence that the average number of hours is less than 4.5 hours at the 10% significance level.