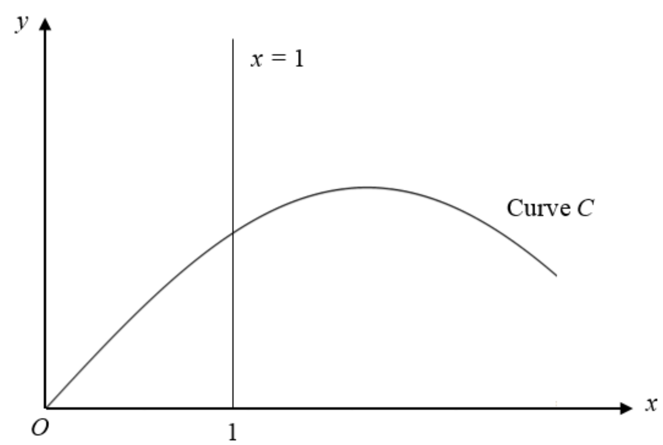
Millennia Institute 2017 PU3 H2 Prelim II Paper 1 Suggested Solutions

<table>
<tr><th>Qn. No.</th><th>Question</th></tr>
<tr><td>1</td><td>The sum of the first n terms of a sequence is denoted by S_n. The first term of the sequence is 3 and it is known that $S_3 = 21$ and $S_{10} = 210$. Given that S_n is a quadratic polynomial in n, find S_n in terms of n.</td></tr>
<tr><td></td><td>Let $S_n = an^2 + bn + c$ where $a,\ b$ and c are constants

$S_1 = 3 \quad \Rightarrow a+b+c=3$
$S_3 = 21 \quad \Rightarrow 9a+3b+c=21$
$S_{10} = 210 \quad \Rightarrow 100a+10b+c=210$

Using GC, $a=2,\ b=1,\ c=0 \quad \Rightarrow \quad S_n = 2n^2+n$</td></tr>
<tr><td>2</td><td>Using the substitution $v=\sqrt{x}+1$, find $\displaystyle\int \frac{1}{x+\sqrt{x}}\,\mathrm{d}x$, where $x>0$.</td></tr>
<tr><td></td><td>$\dfrac{\mathrm{d}v}{\mathrm{d}x} = \dfrac{1}{2x^{\frac{1}{2}}} \Rightarrow \dfrac{\mathrm{d}x}{\mathrm{d}v} = 2x^{\frac{1}{2}}$
$\displaystyle\int \frac{1}{x+\sqrt{x}}\,\mathrm{d}x = \int \frac{1}{x+x^{\frac{1}{2}}}\,\frac{\mathrm{d}x}{\mathrm{d}v}\mathrm{d}v$
where $\dfrac{1}{x+x^{\frac{1}{2}}}\,\dfrac{\mathrm{d}x}{\mathrm{d}v} = \dfrac{1}{x+x^{\frac{1}{2}}}\left(2x^{\frac{1}{2}}\right) = \dfrac{2}{\left(x+x^{\frac{1}{2}}\right)\left(x^{-\frac{1}{2}}\right)} = \dfrac{2}{\left(x^{\frac{1}{2}}+1\right)} = \dfrac{2}{v}$
$\displaystyle\int \frac{1}{x+\sqrt{x}}\,\mathrm{d}x = \int \frac{2}{v}\mathrm{d}v$
$\displaystyle = 2\int \frac{1}{v}\,\mathrm{d}v$
$= 2\ln v + c$
$= 2\ln\left(\sqrt{x}+1\right)+c$</td></tr>
<tr><td>3</td><td></td></tr>
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The diagram shows the curve C with equation $y = \sin x$ and the line $x = 1$. With reference to the diagram, a student wrote down the following series

$$S = \frac{1}{n}\left[\sin\left(\frac{1}{n}\right) + \sin\left(\frac{2}{n}\right) + \sin\left(\frac{3}{n}\right) + \ldots + \sin\left(\frac{n}{n}\right)\right].$$

(i) State what the series represents.

(ii) When $n \to \infty$, $S \to L$. State the geometrical meaning of L. Determine the ex

(iii) What can be said about the value of S in relation to the value of L?

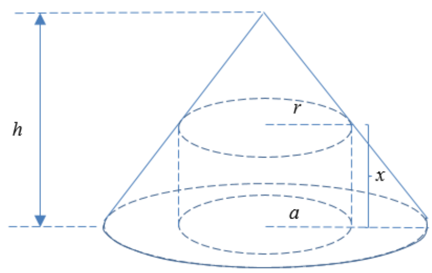
(i)
The sum of the areas of n rectangles with equal width from $x = 0$ to $x = 1$, where the top right vertex of each rectangle lies on the curve.

(ii)
L is the actual area under C from $x = 0$ to $x = 1$.

$$\begin{aligned} L &= \int_0^1 \sin x \, \mathrm{d}x \\ &= \left[-\cos x\right]_0^1 \\ &= -\cos 1 + \cos 0 \\ &= 1 - \cos 1 \qquad \text{i.e. } a = 1,\ b = 1 \end{aligned}$$

(iii)
Since the sum of the areas of the rectangles in part (i) is larger than the actual area under the curve C, $S > L$

4 [It is given that the volume of a circular cone with base radius r and height h is $\frac{1}{3}\pi r^2 h$.]



The diagram above shows a right circular cone with fixed radius a and fixed height h. A cylinder of radius r and height x is removed from the cone.

(i) Show that the volume of the remaining shape, V, is $\frac{\pi h}{3}\left(a^2 - 3r^2 + \frac{3r^3}{a}\right)$.

	(ii) As r varies, use differentiation to find the value of r that gives the minimum value of V, leaving your answer in terms of a.
	(i) $\frac{r}{a}=\frac{h-x}{h}$ $\therefore x=h-\frac{hr}{a}$ $V=\frac{1}{3}\pi a^2h-\pi r^2\left(h-\frac{hr}{a}\right)$ $\quad=\frac{\pi h}{3}\left(a^2-3r^2+\frac{3r^3}{a}\right)$ (shown) (ii) $\frac{\mathrm{d}V}{\mathrm{d}r}=\frac{\pi h}{3}\left(-6r+\frac{9r^2}{a}\right)$ For max/min volume: $\frac{\mathrm{d}V}{\mathrm{d}r}=0$ $\frac{\pi h}{3}\left(-6r+\frac{9r^2}{a}\right)=0$ $-6r+\frac{9r^2}{a}=0$ $r\left(-6+\frac{9r}{a}\right)=0$ $r=0$ (reject as $r>0$) or $r=\frac{2}{3}a$ **<u>Method 1 (2nd derivative test)</u>** $\frac{\mathrm{d}^2V}{\mathrm{d}r^2}=\frac{\pi h}{3}\left(-6+\frac{18r}{a}\right)$ At $r=\frac{2}{3}a$: $\frac{\mathrm{d}^2V}{\mathrm{d}r^2}=\frac{\pi h}{3}\left(-6+\frac{18}{a}\left(\frac{2}{3}a\right)\right)=2\pi h>0$ Therefore the volume is a minimum when $r=\frac{2}{3}a$. **<u>Method 2 (1st derivative test)</u>**

r	$\frac{2}{3}a^-$	$\frac{2}{3}a$	$\frac{2}{3}a^+$
$\frac{\mathrm{d}V}{\mathrm{d}r}$	Negative	Zero	Positive

Therefore the volume is a minimum when $r = \frac{2}{3}a$.

5 A line L passes through the points $A(3, -1,\ 0)$ and $B(11,\ 11,\ 4)$.

(i) Find the angle between L and the y-axis.

(ii) State the geometrical meaning of $\left|\overrightarrow{OB} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right|$.

The point $F(2a+1,\ a,\ a-1)$ is a point on L, where a is a positive constant. The point P is such that $\overrightarrow{PF} = \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}$ and the area of the triangle AFP is $\sqrt{\frac{59}{2}}$ units^2.

(iii) Determine the value of a.

(iv) The point C on L is such that the ratio of the area of triangle AFP to the area of triangle FCP is $2:1$. State the ratio $AF : CF$, justifying your answer.

(i)

$$\overrightarrow{AB} = \begin{pmatrix} 8 \\ 12 \\ 4 \end{pmatrix} = 4\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$$

The required angle, $\theta = \cos^{-1} \dfrac{\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}{\sqrt{14}\sqrt{1}} = 36.7^{\mathrm{o}}$ (1 d.p)

(ii)

The length of projection of $\overrightarrow{OB}$ onto the $z-$axis .

(iii)

	$\frac{1}{2}\left\|\overrightarrow{AF}\times\overrightarrow{PF}\right\|=\sqrt{\frac{59}{2}}$ $\frac{1}{2}\left\|\begin{pmatrix}2a-2\\a+1\\a-1\end{pmatrix}\times\begin{pmatrix}3\\0\\2\end{pmatrix}\right\|=\sqrt{\frac{59}{2}}$ $\left\|\begin{pmatrix}2(a+1)\\1-a\\-3(a+1)\end{pmatrix}\right\|=2\sqrt{\frac{59}{2}}$ $\sqrt{4(a+1)^2+(1-a)^2+9(a+1)^2}=2\sqrt{\frac{59}{2}}$ $13(a+1)^2+(1-a)^2=118$ $14a^2+24a-104=0$ $7a^2+12a-52=0$ $(7a+26)(a-2)=0$ $a=-\frac{26}{7}$ (rejected as $a>0$) or $a=2$ Accept: Using GC, $a=2$ or $a=-3.7143$ (rejected as $a>0$) (iv) Both triangles have the same height (h). $AF:CF$ = Area of triangle AFP : Area of triangle FCP $=2:1$
6	**(i)** Show that $\int e^{2x}\cos x\,dx=\frac{2}{5}e^{2x}\cos x+\frac{1}{5}e^{2x}\sin x+C$. **(ii)** Find the volume of the solid generated when the region bounded by $y=e^x\sqrt{(\cos x)}$ and $y=-\frac{2}{\pi}x+1$ between $x=0$ and $x=\frac{\pi}{2}$ is rotated through 2π radians about the x-axis, leaving your answer in exact form. [4]
6(i) **[3]**	(i) $\int e^{2x}\cos x\,dx=\frac{1}{2}e^{2x}\cos x+\int\frac{1}{2}e^{2x}\sin x\,dx$

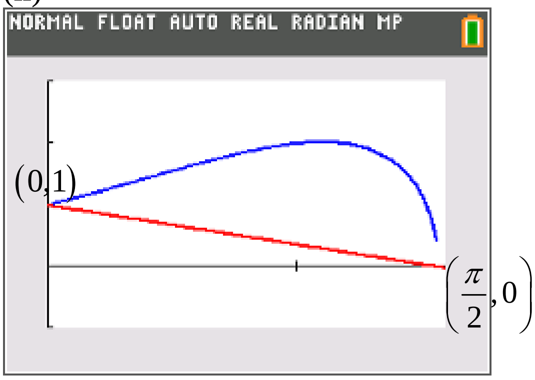
$$=\frac{1}{2}e^{2x}\cos x+\frac{1}{2}\left[\frac{1}{2}e^{2x}\sin x-\frac{1}{2}\int e^{2x}\cos x\,dx\right]$$

$$=\frac{1}{2}e^{2x}\cos x+\frac{1}{4}e^{2x}\sin x-\frac{1}{4}\int e^{2x}\cos x\,dx$$

$$\frac{5}{4}\int e^{2x}\cos x\,dx=\frac{1}{2}e^{2x}\cos x+\frac{1}{4}e^{2x}\sin x+C_1$$

$$\int e^{2x}\cos x\,dx=\frac{2}{5}e^{2x}\cos x+\frac{1}{5}e^{2x}\sin x+C$$

(ii)



Volume = $\pi\int y^2 dx$

$$=\pi\int_0^{\frac{\pi}{2}} e^{2x}\cos x\,dx-\frac{1}{3}\pi(1)^2\left(\frac{\pi}{2}\right)$$

$$=\pi\left[\frac{2}{5}e^{2x}\cos x+\frac{1}{5}e^{2x}\sin x\right]_0^{\frac{\pi}{2}}-\frac{\pi^2}{6}$$

$$=\pi\left[\frac{1}{5}e^{\pi}\sin\frac{\pi}{2}-\frac{2}{5}e^{0}\cos 0\right]-\frac{\pi^2}{6}$$

$$=\frac{1}{5}\pi e^{\pi}-\frac{2}{5}\pi-\frac{\pi^2}{6}$$

7

(i) Prove by the method of mathematical induction that

$$\sum_{r=1}^{n}\frac{2}{r(r+2)}=\frac{3}{2}-\frac{2n+3}{(n+1)(n+2)}$$

for all positive integers of *n*.

(ii) Explain why $\sum_{r=1}^{n}\frac{2}{r(r+2)}$ is a convergent series, and state the value of the sur

(iii) Using the result in part **(i)**, find $\sum_{r=5}^{N}\frac{2}{(r-2)(r-4)}$.

(i)

Let P_n be the statement $\sum_{r=1}^{n}\frac{2}{r(r+2)}=\frac{3}{2}-\frac{2n+3}{(n+1)(n+2)}$ for $n\in Z^{+}$.

Prove P_1 is true.

LHS = $\frac{2}{(1)(1+2)}=\frac{2}{3}$

RHS = $\frac{3}{2}-\frac{2(1)+3}{(1+1)(1+2)}=\frac{4}{6}=\frac{2}{3}=\text{LHS}$

P_1 is true.

Assume that P_k is true for some $k\in Z^{+}$ i.e. $\sum_{r=1}^{k}\frac{2}{r(r+2)}=\frac{3}{2}-\frac{}{(k}$

Prove P_{k+1} is true i.e. $\sum_{r=1}^{k+1}\frac{2}{r(r+2)}=\frac{3}{2}-\frac{2k+5}{(k+2)(k+3)}$.

LHS

$=\sum_{r=1}^{k}\frac{2}{r(r+2)}+T_{k+1}$

$=\frac{3}{2}-\frac{2k+3}{(k+1)(k+2)}+\frac{2}{(k+1)(k+3)}$

$=\frac{3}{2}-\frac{(2k+3)(k+3)-2(k+2)}{(k+1)(k+2)(k+3)}$

$=\frac{3}{2}-\frac{2k^2+7k+5}{(k+1)(k+2)(k+3)}$

$=\frac{3}{2}-\frac{(2k+5)(k+1)}{(k+1)(k+2)(k+3)}$

$=\frac{2k+5}{(k+2)(k+3)}=\text{RHS}$

P_{k+1} is true

Since P_1 is true, and P_k is true implies P_{k+1} is true,

by Mathematical Induction, $\sum_{r=1}^{n}\frac{2}{r(r+2)}=\frac{3}{2}-\frac{2n+3}{(n+1)(n+2)}$ for $n\in Z^{+}$.

(ii)

$n\to\infty,\frac{2n+3}{(n+1)(n+2)}\to 0,\sum_{r=1}^{n}\frac{2}{r(r+2)}\to\frac{3}{2}$

The series converges to a value. $\therefore$ the series is a convergent series.

$\sum_{r=1}^{\infty}\frac{2}{r(r+2)}=\frac{3}{2}$

(iii)

	$\sum_{r=5}^{N}\frac{2}{(r-2)(r-4)}$ $\xrightarrow[r=l+4]{\text{letting}}\sum_{l+4=5}^{l+4=N}\frac{2}{(l+4-2)(l+4-4)}$ $=\sum_{l=1}^{N-4}\frac{2}{l(l+2)}$ $=\frac{3}{2}-\frac{2(N-4)+3}{(N-4+1)(N-4+2)}$ $=\frac{3}{2}-\frac{2N-5}{(N-3)(N-2)}$
8	Using the substitution $y=ux$, show that the differential equation $$x\frac{\mathrm{d}y}{\mathrm{d}x}=3x+y-2$$ can be reduced to the form $$x^2\frac{\mathrm{d}u}{\mathrm{d}x}=3x-2.$$ Hence, find the general solution to the differential equation $x\frac{\mathrm{d}y}{\mathrm{d}x}=3x+y-2$. [5] **(i)** State the equation of the locus where the stationary points of the solution **(ii)** Sketch, on a single diagram, the graph of the locus found in part **(i)** and two members of the family of solution curves, where the arbitrary constant in the
	$y=ux$ $\frac{\mathrm{d}y}{\mathrm{d}x}=u+x\frac{\mathrm{d}u}{\mathrm{d}x}$ $x\left(u+x\frac{\mathrm{d}u}{\mathrm{d}x}\right)=3x+ux-2$ $ux+x^2\frac{\mathrm{d}u}{\mathrm{d}x}=3x+ux-2$ $x^2\frac{\mathrm{d}u}{\mathrm{d}x}=3x-2$ (shown)

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{3x-2}{x^2}$$

$$\int \frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x = \int \frac{3x-2}{x^2}\,\mathrm{d}x$$

$$\int \mathrm{d}u = \int \frac{3}{x} - \frac{2}{x^2}\,\mathrm{d}x$$

$$u = 3\ln|x| + \frac{2}{x} + C$$

$$\frac{y}{x} = 3\ln|x| + \frac{2}{x} + C$$

$$y = 3x\ln|x| + Cx + 2$$

(i)

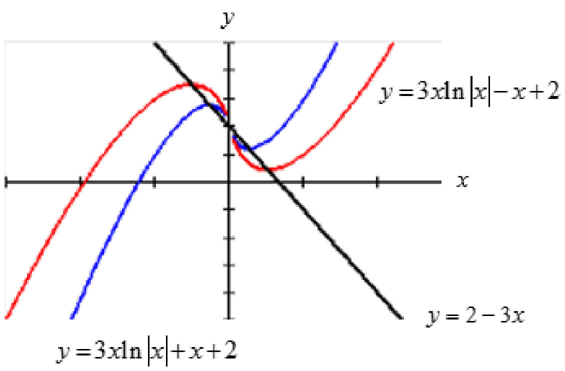
For stationary points, $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$

$\Rightarrow x(0) = 3x + y - 2$

$\Rightarrow y = 2 - 3x$

The equation of the locus is $y = 2 - 3x$.

(ii)



9 It is given that

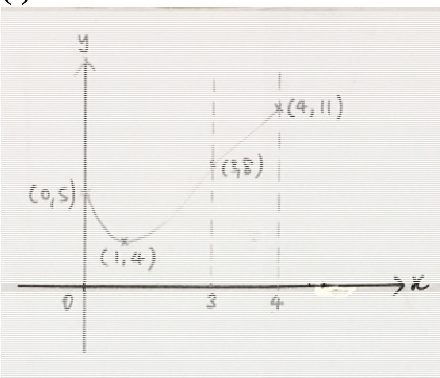
$$\mathrm{f}(x) = \begin{cases} (x-1)^2 + 4 & , \quad k \le x < 3, \\ 3x - 1 & , \quad 3 \le x \le 4, \end{cases}$$

where $k \in \mathbb{R}$, $k < 3$.

(i) Sketch, for $k = 0$, the graph of $y = \mathrm{f}(x)$, stating the coordinates of the turning point. Write down the range of f.

(ii) Explain why f^{-1} does not exist. State the smallest value of k for f^{-1} to exist.

(iii) Using the value of k in part **(ii)**, find f^{-1} in similar form.

	(iv) State the geometrical relationship between f and f^{-1}. The point $P(a,\ b)$, where a and b are constants, lies on the graph $y=\mathrm{f}(x)$. The point Q on the graph $y=\mathrm{f}^{-1}(x)$ is the point corresponding to P. State the coord
	(i)  Range of f, $\mathrm{R_f}=[4,11]$ (ii) A horizontal line, $y=k,\ 4<k\le 5$ intersects the graph of $y=\mathrm{f}(x)$ at 2 points. f is not a one-one function. Hence, f^{-1} does not exist. For f^{-1} to exist, the minimum value of k is 1. (iii) Let $y=\mathrm{f}(x)$ For $1\le x<3$: Let $y=(x-1)^2+4$ $x=1\pm\sqrt{y-4}$ $x=1+\sqrt{y-4}$ since $1\le x<3$ $\mathrm{f}^{-1}(x)=1+\sqrt{x-4},\ 4\le x<8$ For $3\le x\le 4$: Let $y=3x-1$ $x=\frac{1}{3}y+\frac{1}{3}$ $\mathrm{f}^{-1}(x)=\frac{1}{3}x+\frac{1}{3},\ 8\le x\le 11$ $\mathrm{f}^{-1}(x)=\begin{cases}1+\sqrt{x-4} & ,\quad 4\le x<8,\\ \frac{1}{3}x+\frac{1}{3} & ,\quad 8\le x\le 11,\end{cases}$ (iv)

	The graph $y = \mathrm{f}^{-1}(x)$ is the reflection of the graph $y = \mathrm{f}(x)$ in the line $y = x$. The coordinates of Q is (b, a).
10	**(a)** It is given that $-1+\mathrm{i}$ is a root of the equation $2z^3 + az^2 + bz + (3+\mathrm{i}) = 0$. **(i)** Find the values of the real numbers *a* and *b*. **(ii)** Using these values of *a* and *b*, find the other roots of this equation. **(b)** It is given that $w = -1 + (\sqrt{3})\mathrm{i}$. **(i)** Without using a calculator, find an exact expression for w^5. Give your answer in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $0 \le \theta \le 2\pi$. **(ii)** Without using a calculator, find the three smallest positive whole numb
	(a)(i) Since $-1+\mathrm{i}$ is a root of $2z^3 + az^2 + bz + (3+\mathrm{i}) = 0$, $2(-1+\mathrm{i})^3 + a(-1+\mathrm{i})^2 + b(-1+\mathrm{i}) + (3+\mathrm{i}) = 0$ $4 + 4\mathrm{i} + a(-2\mathrm{i}) - b + b\mathrm{i} + 3 + \mathrm{i} = 0$ Comparing real parts: $4 - b + 3 = 0 \Rightarrow b = 7$ Comparing imaginary parts: $4 - 2a + b + 1 = 0 \Rightarrow a = 6$ (a)(ii) $2z^3 + 6z^2 + 7z + (3+\mathrm{i}) = 0$ $\left[z - (-1+\mathrm{i})\right]\left[2z^2 + (4+2\mathrm{i})z + (1+2\mathrm{i})\right] = 0$ $z = -1+\mathrm{i}$ (given) or $z = \dfrac{-(4+2\mathrm{i}) \pm \sqrt{\left[(4+2\mathrm{i})^2 - 4(2)(1+2\mathrm{i})\right]}}{2(2)}$ $= \dfrac{(-4-2\mathrm{i}) \pm 2}{4}$ $z = -\frac{1}{2} - \frac{1}{2}\mathrm{i}$ or $z = -\frac{3}{2} - \frac{1}{2}\mathrm{i}$ (b)(i) $\left\lvert -1 + \mathrm{i}\sqrt{3} \right\rvert = 2$ $\arg\left(-1 + \mathrm{i}\sqrt{3}\right) = \pi - \tan^{-1}\left(\sqrt{3}\right) = \dfrac{2\pi}{3}$

$$w^5 = \left(2\mathrm{e}^{\mathrm{i}\left(\frac{2\pi}{3}\right)}\right)^5 = 32\mathrm{e}^{\mathrm{i}\left(\frac{10\pi}{3}\right)} = 32\mathrm{e}^{\mathrm{i}\left(\frac{4\pi}{3}\right)}$$

(b)(ii)

$$\frac{w^*}{w^n} = \frac{2\mathrm{e}^{\mathrm{i}\left(-\frac{2\pi}{3}\right)}}{\left[2\mathrm{e}^{\mathrm{i}\left(\frac{2\pi}{3}\right)}\right]^n} = 2^{1-n}\mathrm{e}^{\mathrm{i}\left(-\frac{2\pi}{3}-\frac{2n\pi}{3}\right)}$$

Method 1

$$\begin{aligned} 2^{1-n}\mathrm{e}^{\mathrm{i}\left(-\frac{2\pi}{3}-\frac{2n\pi}{3}\right)} &= 2^{1-n}\left[\cos\left(-\frac{2\pi}{3}-\frac{2n\pi}{3}\right)+\mathrm{i}\sin\left(-\frac{2\pi}{3}-\frac{2n\pi}{3}\right)\right] \\ &= 2^{1-n}\left[\cos\left(\frac{2\pi}{3}+\frac{2n\pi}{3}\right)-\mathrm{i}\sin\left(\frac{2\pi}{3}+\frac{2n\pi}{3}\right)\right] \end{aligned}$$

Since $\dfrac{w^*}{w^n}$ is a real number,

$$\sin\left(\frac{2\pi}{3}+\frac{2n\pi}{3}\right) = 0$$

$$\frac{2\pi}{3}+\frac{2n\pi}{3} = \pi, 2\pi, 3\pi, 4\pi, 5\pi, 6\pi \ldots$$

$$2\pi + 2n\pi = 3\pi, 6\pi, 9\pi, 12\pi, 15\pi, 18\pi \ldots$$

$$2n\pi = \pi, 4\pi, 7\pi, 10\pi, 13\pi, 16\pi \ldots$$

$$n = \frac{1}{2}, 2, \frac{7}{2}, 5, \frac{13}{2}, 8, \ldots$$

The 3 smallest positive whole number values of n are 2, 5 and 8.

Method 2

Since $\dfrac{w^*}{w^n}$ is a real number, $\arg\left(\dfrac{w^*}{w^n}\right) = k\pi,\ k \in \mathbb{Z}$

$$-\frac{2n\pi}{3}-\frac{2\pi}{3} = k\pi$$

$$n = -1-\frac{3k}{2}$$

At $k=-2$: $n=2$
At $k=-4$: $n=5$
At $k=-6$: $n=8$

The 3 smallest positive whole number values of n are 2, 5 and 8.

11

A curve C_1 is defined parametrically by the equations $x = t - \frac{1}{t}$, $y = t + \frac{1}{t}$, $t \neq 0$.

(i) Sketch C_1, stating the equation of the asymptotes and coordinates of any poi

(ii) Show that the equation of the normal to C_1 at the point with parameter *p* is given by $y = -\frac{p^2+1}{p^2-1}x + \frac{2(p^2+1)}{p}$.

(iii) The normal in part **(ii)** intersects the *x*-axis at the point *A* and the *y*-axis at the point *B*. Find, in terms of *p*, an expression for the area of the tri

The line *l* is the normal to C_1 when $p = 2$.

(iv) Find the equation of *l*.

A curve C_2 is defined parametrically by the equations $x = 3at$, $y = -t^2 + a$, $t \in$ □ where *a* is a non-zero constant.

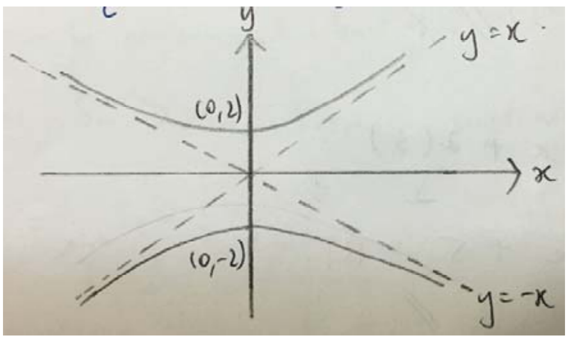
(v) Given that *l* intersects C_2, show that the parameter *q* of the point(s) of intersection satisfies the equation

$$q^2 - 5aq + 5 - a = 0.$$

Hence, determine the range of values of *a* such that *l* intersects C_2 at two distinct points.

(i)

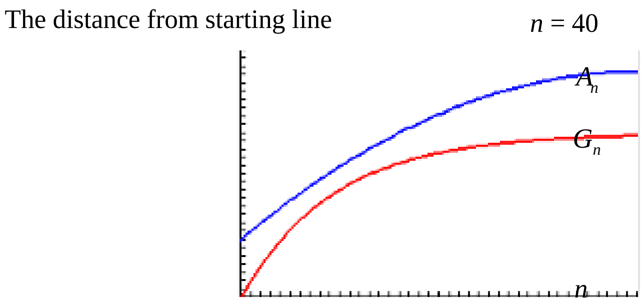
$x = t - \frac{1}{t}$, $y = t + \frac{1}{t}$, $t \neq 0$.



(ii)

	$x=t-\frac{1}{t}, \quad y=t+\frac{1}{t}$ $\frac{\mathrm{d}x}{\mathrm{d}t}=1+\frac{1}{t^2}=\frac{t^2+1}{t^2}, \quad \frac{\mathrm{d}y}{\mathrm{d}t}=1-\frac{1}{t^2}=\frac{t^2-1}{t^2}$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}t}\times\frac{\mathrm{d}t}{\mathrm{d}x}=\frac{t^2-1}{t^2+1}$ When $t=p,$ the gradient of normal $=-\frac{p^2+1}{p^2-1}$ The required equation of normal: $y-\left(p+\frac{1}{p}\right)=-\frac{p^2+1}{p^2-1}\left[x-\left(p-\frac{1}{p}\right)\right]$ $y=-\frac{p^2+1}{p^2-1}x+\frac{p^2+1}{p^2-1}\left(p-\frac{1}{p}\right)+p+\frac{1}{p}$ $y=-\frac{p^2+1}{p^2-1}x+\frac{p^2+1}{p^2-1}\left(\frac{p^2-1}{p}\right)+\frac{p^2+1}{p}$ $y=-\frac{p^2+1}{p^2-1}x+\frac{2\left(p^2+1\right)}{p}$ (shown) (iii) When $x=0, y=\frac{2\left(p^2+1\right)}{p}\Rightarrow B\left(0,\ \frac{2\left(p^2+1\right)}{p}\right)$ When $y=0,\ -\frac{p^2+1}{p^2-1}x+\frac{2\left(p^2+1\right)}{p}=0$ $x=\frac{2\left(p^2-1\right)}{p}\Rightarrow A\left(\frac{2\left(p^2-1\right)}{p},\ 0\right)$ Area of triangle *OAB* $=\left\|\frac{1}{2}\left[\frac{2\left(p^2+1\right)}{p}\right]\left[\frac{2\left(p^2-1\right)}{p}\right]\right\|$ $=2\left\|\frac{\left(p^2+1\right)\left(p^2-1\right)}{p^2}\right\|$ $=\frac{2}{p^2}\left\|\left(p^2+1\right)\left(p^2-1\right)\right\|$ $=\frac{2\left(p^2+1\right)}{p^2}\left\|p^2-1\right\|$ or $\frac{2}{p^2}\left\|p^4-1\right\|$ units^2

	(iv) When $p = 2$, the equation of the normal is $y = -\frac{2^2+1}{2^2-1}x + \frac{2\left(2^2+1\right)}{2}$ $y = -\frac{5}{3}x + 5$ The equation of l is $y = -\frac{5}{3}x + 5.$ (v) By substitution, $-q^2 + a = -\frac{5}{3}(3aq) + 5$ $q^2 - 5aq + 5 - a = 0$ (shown) For l to intersect C_2 at 2 distinct points, $b^2 - 4ac > 0$ $(-5a)^2 - 4(1)(5-a) > 0$ $25a^2 + 4a - 20 > 0$ $a < -0.978$ or $a >$ 0.818 (3 s.f)
12	As part of a project, a group of engineering students design two robots for a game. One robot is called 'Prey' and the other robot is called 'Predator'. The two robots are designed with the following specifications. 'Prey': It is designed to leap 1 m forward for the first leap. Subsequently, it 'Predator': It is designed to leap 2 m forward for the first leap. Subsequently, it Both robots take each leap at the same time and the number of leaps taken is given by n. 'Predator' starts the game from the starting line while 'Prey' starts the g **(i)** Find the distance of 'Prey' and of 'Predator' from the starting line after n lea **(ii)** Explain why 'Predator' has to catch 'Prey' before 'Predator's distance from the starting line reaches 20 m. **(iii)** Using a graphical method, explain why 'Predator' will not ca **(iv)** 'Prey' now starts the game 4 m in front of 'Predator'. 'Predator' catches 'Pre Calculate the distance of 'Predator' from the starting line after completing th
	(i)

	Distance of 'Prey' from starting line, A_n $= \frac{n}{2}\left[2(1)+(n-1)(-0.025)\right]+7=-0.0125n^2+1.0125n+7$ Distance of 'Predator' from starting line, G_n $= \frac{2\left(1-0.9^n\right)}{1-0.9}=20\left(1-0.9^n\right)$ (ii) The sum to infinity $=\frac{2}{1-0.9}=20$ Hence, 'Predator' has to catch 'Prey' before its distance from the starting line reaches 20 m. (iii) To determine when the leap distance of 'Prey' becomes 0 (if it is not caught): $1+(n-1)(-0.025)>0$ $n<41$ If not caught, 'Prey' will leap 40 times before the leap distance becomes 0. Plot, for $0 \le n \le 40$, the graphs of $A_n=-0.0125n^2+1.0125n+7$ and $G_n=20\left(1-0.9^n\right)$ as follows:  For $0 \le n \le 40$, since the two curves do not intersect, 'Predator' will not catch 'Prey'. (iv) When 'Predator' catches 'Prey', $-0.0125n^2+1.0125n+4=20\left(1-0.9^n\right)$ Using GC, $n=7.2557$ or 12.012 (rejected) 'Predator' catches 'Prey' on the 8th leap $\Rightarrow k=8$. The required distance $=20\left(1-0.9^8\right)=11.4$ m (3 s.f)