Suggested Solution to 2017 SH2 H2 Mathematics Prelim Exam P1

No.	Suggested Solutions
1(i)	Method 1 **p.q = 0** $\mathbf{p}.\mathbf{q} = 0$ $\begin{pmatrix} 2 \\ \alpha \\ 1 \end{pmatrix} . \begin{pmatrix} \alpha \\ 1 \\ 6 \end{pmatrix} = 0$ $2\alpha + \alpha + 6 = 0$ $\alpha = -2$ Method 2 (for marking reference) Let $\mathbf{w} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. $\mathbf{w} \times \mathbf{p} = \mathbf{q}$ $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \times \begin{pmatrix} 2 \\ \alpha \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha \\ 1 \\ 6 \end{pmatrix}$ $\begin{pmatrix} y - \alpha z \\ 2z - x \\ \alpha x - 2y \end{pmatrix} = \begin{pmatrix} \alpha \\ 1 \\ 6 \end{pmatrix}$ Thus, $y - \alpha z = \alpha$ ------(1) $2z - x = 1$ ------(2) $\alpha x - 2y = 6$ ------(3) $(2) \times \alpha + (3)$: $2\alpha z - 2y = \alpha + 6$ $\Rightarrow 2(\alpha z - y) = \alpha + 6$ $\Rightarrow 2(-\alpha) = \alpha + 6$ (from (1)) $\Rightarrow \alpha = -2$
1(ii)	Let $\mathbf{w} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

	$\mathbf{w}\times\mathbf{p}=\mathbf{q}$ $\begin{pmatrix} x \\ y \\ z \end{pmatrix}\times\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}=\begin{pmatrix} -2 \\ 1 \\ 6 \end{pmatrix}$ $\begin{pmatrix} y+2z \\ 2z-x \\ -2x-2y \end{pmatrix}=\begin{pmatrix} -2 \\ 1 \\ 6 \end{pmatrix}$ $y+2z=-2$ ------(1) $2z-x=1$ ------(2) $-2x-2y=6$ ------(3) Let $z=\lambda, \lambda\in\mathbb{R}$. From (2): $x=-1+2\lambda$ From (1): $y+2\lambda=-2\Rightarrow y=-2-2\lambda$ Thus, $\mathbf{w}=\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} -1+2\lambda \\ -2-2\lambda \\ \lambda \end{pmatrix}=\begin{pmatrix} -1 \\ -2 \\ 0 \end{pmatrix}+\lambda\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$, $\lambda\in\mathbb{R}$, which is the vector equation of the straight line. The set of vectors is $\left\{\mathbf{w}:\mathbf{w}=\begin{pmatrix} -1 \\ -2 \\ 0 \end{pmatrix}+\lambda\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}, \lambda\in\mathbb{R}\right\}$
2 (a) (i)	By GC, sum to infinity is 3.
2(a) (ii)	$u_n=S_n-S_{n-1}$ $=3+7^{-2n}\left(n^2\right)-\left[3+7^{-2(n-1)}(n-1)^2\right]$ $=3-3+7^{-2n}\left(n^2\right)-7^{-2n+2}\left(n^2-2n+1\right)$ $=7^{-2n}\left(n^2-49n^2+98n-49\right)$ $=7^{-2n}\left(-48n^2+98n-49\right)$ $=7^{-2n}(8n-7)(7-6n)$ where $\mathrm{g}(n)=-48n^2+98n-49$

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<tr><td>2(b)</td><td>

$\sum_{r=1}^{n}\left(\int_0^r \mathrm{e}^x-\mathrm{e}^{x-1}\mathrm{d}x\right)$
$=\sum_{r=1}^{n}\left[\mathrm{e}^x-\mathrm{e}^{x-1}\right]_0^r$
$=\sum_{r=1}^{n}\left(\mathrm{e}^r-\mathrm{e}^{r-1}-\mathrm{e}^0+\mathrm{e}^{-1}\right)$
$=\mathrm{e}^1-\mathrm{e}^0-\mathrm{e}^0+\mathrm{e}^{-1}$
$+\mathrm{e}^2-\mathrm{e}^1-\mathrm{e}^0+\mathrm{e}^{-1}$
$+\mathrm{e}^3-\mathrm{e}^2-\mathrm{e}^0+\mathrm{e}^{-1}$
$\ldots$
$+\mathrm{e}^n-\mathrm{e}^{n-1}-\mathrm{e}^0+\mathrm{e}^{-1}$
$=\mathrm{e}^n-1-n(1)+n\mathrm{e}^{-1}$
$=\mathrm{e}^n+n\mathrm{e}^{-1}-(n+1)$

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$\sum_{r=10}^{20}\left(\int_0^r \mathrm{e}^{x+2}-\mathrm{e}^{x+1}\ \mathrm{d}x\right)$
$=\mathrm{e}^2\sum_{r=1}^{20}\left(\int_0^r \mathrm{e}^x-\mathrm{e}^{x-1}\ \mathrm{d}x\right)-\mathrm{e}^2\sum_{r=1}^{9}\left(\int_0^r \mathrm{e}^x-\mathrm{e}^{x-1}\ \mathrm{d}x\right)$
$=\mathrm{e}^2\left[\mathrm{e}^{20}+20\mathrm{e}^{-1}-(20+1)-\left(\mathrm{e}^9+9\mathrm{e}^{-1}-10\right)\right]$
$=\mathrm{e}^{22}-\mathrm{e}^{11}-11\mathrm{e}^2+11\mathrm{e}$

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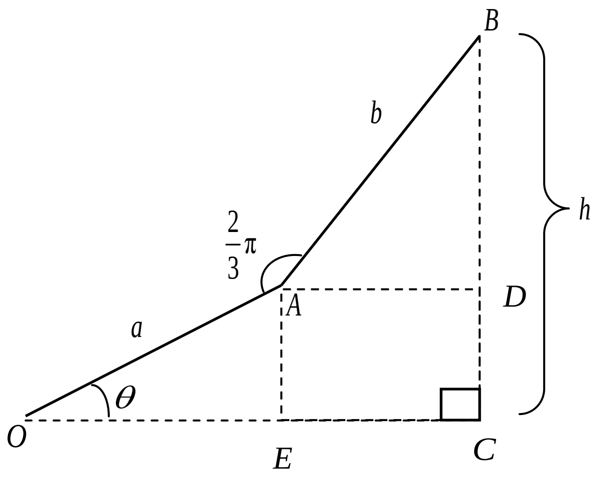
<tr><td>3(i)</td><td>

<u>Method 1</u>



$h=BD+DC,\ DC=a\sin\theta,\ BD=b\sin\angle BAD$
$\angle BAD+\angle DAE+\angle OAE=2\pi-\frac{2\pi}{3}=\frac{4\pi}{3}$
$\Rightarrow\angle BAD+\frac{\pi}{2}+\left(\frac{\pi}{2}-\theta\right)=\frac{4\pi}{3}$
$\Rightarrow\angle BAD=\theta+\frac{\pi}{3}$

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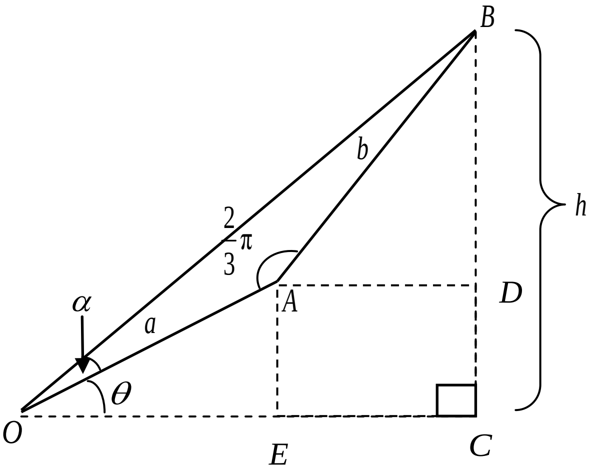
	$\Rightarrow BD = b\sin\left(\theta+\frac{\pi}{3}\right)$ $\therefore h = a\sin\theta + b\sin\left(\theta+\frac{\pi}{3}\right)$ Using $\sin(A+B) = \sin A\cos B + \cos A\sin B$, we have $h = a\sin\theta + b\sin\left(\theta+\frac{\pi}{3}\right)$ $= a\sin\theta + b\sin\theta\cos\left(\frac{\pi}{3}\right) + b\cos\theta\sin\left(\frac{\pi}{3}\right)$ $= \left(a+\frac{b}{2}\right)\sin\theta + \frac{b\sqrt{3}}{2}\cos\theta$ $= R\sin(\theta+\alpha)$, where $R = \sqrt{\left(a+\frac{b}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}b\right)^2} = \sqrt{a^2+ab+\frac{b^2}{4}+\frac{3b^2}{4}} = \sqrt{a^2+ab+b^2}$ $\tan\alpha = \frac{\frac{\sqrt{3}}{2}b}{a+\frac{b}{2}} = \frac{\sqrt{3}b}{2a+b} \Rightarrow \alpha = \tan^{-1}\left(\frac{\sqrt{3}b}{2a+b}\right)$ $\therefore h = \sqrt{a^2+ab+b^2}\,\sin\left[\theta+\tan^{-1}\left(\frac{\sqrt{3}b}{2a+b}\right)\right]$ <u>Method 2</u>  $\sin(\theta+\alpha) = \frac{h}{OB}$ $OB^2 = a^2+b^2-2ab\cos\frac{2\pi}{3}$ $OB = \sqrt{a^2+b^2+ab}$

	$\frac{\sin\alpha}{b}=\frac{\sin\frac{2\pi}{3}}{\sqrt{a^2+b^2+ab}}$ $\sin\alpha=\frac{b\sqrt{3}}{2\sqrt{a^2+b^2+ab}}$ $\alpha=\sin^{-1}\frac{b\sqrt{3}}{2\sqrt{a^2+b^2+ab}}$ $h=\sqrt{a^2+ab+b^2}\sin\left(\theta+\sin^{-1}\frac{b\sqrt{3}}{2\sqrt{a^2+b^2+ab}}\right).$
3(ii)	Since $a=1, b=2, \alpha=\tan^{-1}\left(\frac{2\sqrt{3}}{2+2}\right)=\tan^{-1}\left(\frac{\sqrt{3}}{2}\right)$ $h=\sqrt{7}\sin\left[\theta+\tan^{-1}\left(\frac{\sqrt{3}}{2}\right)\right], \frac{\mathrm{d}h}{\mathrm{d}\theta}=\sqrt{7}\cos\left[\theta+\tan^{-1}\left(\frac{\sqrt{3}}{2}\right)\right]$ At $\theta=\frac{\pi}{12}$, $\frac{\mathrm{d}\theta}{\mathrm{d}t}=\frac{\mathrm{d}\theta}{\mathrm{d}h}\times\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{1}{\sqrt{7}\cos\left[\frac{\pi}{12}+\tan^{-1}\left(\frac{\sqrt{3}}{2}\right)\right]}\times(-0.5)$ $=-0.337$ radians per minute
3(iii)	$\alpha=\tan^{-1}\left(\frac{\sqrt{3}}{2}\right)\Rightarrow\sin\alpha=\sqrt{\frac{3}{7}}, \cos\alpha=\frac{2}{\sqrt{7}}$ $h=\left(a+\frac{b}{2}\right)\sin\theta+\frac{b\sqrt{3}}{2}\cos\theta$ If θ is small, $h=\sqrt{7}\sin(\theta+\alpha)$ $=\sqrt{7}\sin\theta\cos\alpha+\sqrt{7}\cos\theta\sin\alpha$ $\approx\sqrt{7}\theta\left(\frac{2}{\sqrt{7}}\right)+\sqrt{7}\left(1-\frac{\theta^2}{2}\right)\left(\sqrt{\frac{3}{7}}\right)$ $=2\theta+\sqrt{3}-\frac{\sqrt{3}\theta^2}{2}$ $=-\frac{\sqrt{3}\theta^2}{2}+2\theta+\sqrt{3}$

4(i)

Stretch count, n	Length of string before stretch	Elongation after stretch, u_n	Contraction after stretch, t_n	Final length of string
1	30	10	0.1	$30+10-0.1=39.9$
2	39.9	$10\left(\frac{10}{11}\right)$	$0.1-0.001$ $=0.099$	$39.9+10\left(\frac{10}{11}\right)-0.099$ $=48.8919$

$$30+10-0.1+10\left(\frac{10}{11}\right)-0.099$$

$$=48.8919$$

$$=48.892\ (3\text{ dp})$$

4(ii)

Total length of string

$$=30+u_1+u_2+...+u_n-\left(t_1+t_2+...+t_n\right)$$

$$=30+10+10\left(\frac{10}{11}\right)+...+10\left(\frac{10}{11}\right)^{n-1}$$

$$-\left(0.1+\left(0.1-0.001(1)\right)+...+\left(0.1-0.001(n)\right)\right)$$

$$\sum_{i=1}^{n}u_i=\frac{10\left[1-\left(\frac{10}{11}\right)^n\right]}{1-\frac{10}{11}}=110\left(1-\left(\frac{10}{11}\right)^n\right)$$

$$\sum_{i=1}^{n}t_i=\frac{n}{2}\left[2(0.1)+(n-1)(-0.001)\right]=\frac{n}{2000}(201-n)$$

Length of string after n stretches

$$=30+110\left(1-\left(\frac{10}{11}\right)^n\right)-\frac{n}{2000}(201-n)$$

4(iii)

$$t_n>u_n$$

$$0.1+(n-1)(-0.001)>(10)\left(\frac{10}{11}\right)^{n-1}$$

$$0.1+(n-1)(-0.001)-(10)\left(\frac{10}{11}\right)^{n-1}>0$$

Using GC,

when $n=58$, $0.1+(n-1)(-0.001)-(10)\left(\frac{10}{11}\right)^{n-1}=-7.1364\times10^{-4}$

when $n=59$, $0.1+(n-1)(-0.001)-(10)\left(\frac{10}{11}\right)^{n-1}=0.00226$

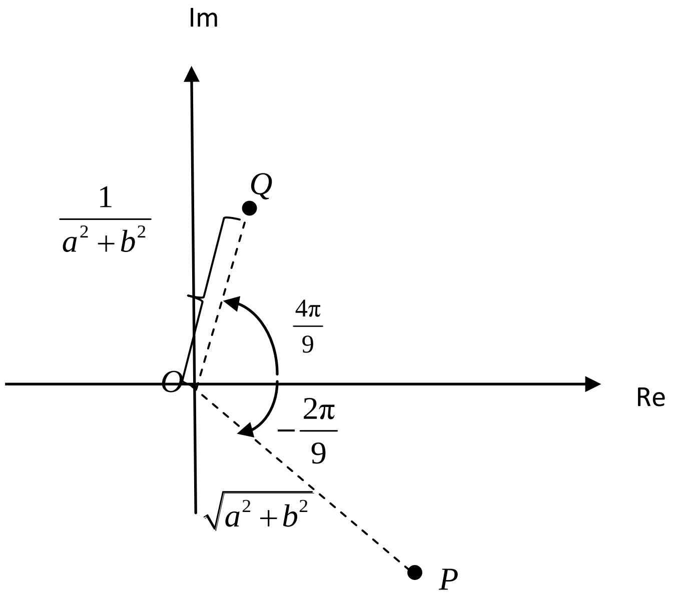
Therefore, the minimum number of stretches is 59.

4(iv)

$$S_\infty=30+\frac{10}{1-\frac{10}{11}}=140\quad(\text{since }0<r<1)$$

Since the sum to infinity, S_∞ is 140, it is impossible for the string to be stretched beyond

	140cm. OR The theoretical maximum is 140 cm so it is impossible for the strong to be stretched beyond 140 cm.
5(a)	$\mathrm{i}z + w = 2$ -------(1) $zw^* = 2 + 4\mathrm{i}$ -------(2) From (1), $z = \frac{2-w}{\mathrm{i}} = -\mathrm{i}(2-w)$ -------(3) Substitute (3) into (2) and let $w = x + \mathrm{i}y$: $-\mathrm{i}(2-w)w^* = 2+4\mathrm{i}$ $-\mathrm{i}(2w^* - ww^*) = 2+4\mathrm{i}$ $-\mathrm{i}\left[2(x-\mathrm{i}y) - (x^2+y^2)\right] = 2+4\mathrm{i}$ $-2y - \mathrm{i}(2x - x^2 - y^2) = 2+4\mathrm{i}$ Comparing real and imaginary parts, $-2y = 2 \Rightarrow y = -1$ $-2x + x^2 + y^2 = 4$ $\Rightarrow -2x + x^2 + (-1)^2 = 4$ $\Rightarrow x^2 - 2x - 3 = 0$ $\Rightarrow (x-3)(x+1) = 0$ $\Rightarrow x = 3$ or $x = -1$ $\therefore w = 3 - \mathrm{i}$ or $w = -1 - \mathrm{i}$ If $w = 3 - \mathrm{i}$, $z = -\mathrm{i}(2-(3-\mathrm{i})) = 1+\mathrm{i}$. If $w = -1-\mathrm{i}$, $z = -\mathrm{i}(2-(-1-\mathrm{i})) = 1-3\mathrm{i}$.
5(b)(i)	$\left\lvert\frac{1}{p^2}\right\rvert = \frac{1}{\lvert p\rvert^2} = \frac{1}{\left(\sqrt{a^2+b^2}\right)^2} = \frac{1}{a^2+b^2}$ $\arg\left(\frac{1}{p^2}\right) = -2\arg(p) = -2\left(\frac{-2\pi}{9}\right) = \frac{4\pi}{9}$ $\therefore \frac{1}{p^2} = \frac{1}{a^2+b^2}\mathrm{e}^{\mathrm{i}\left(\frac{4\pi}{9}\right)}$

(b)(ii)	
(b)(iii)	Given $\angle OPQ = \alpha$, $\dfrac{\sin\alpha}{1/\left(a^2+b^2\right)} = \dfrac{\sin\left(\frac{\pi}{3}-\alpha\right)}{\sqrt{a^2+b^2}}$ $\left(a^2+b^2\right)^{\frac{3}{2}} = \dfrac{\sqrt{3}\cos\alpha-\sin\alpha}{2\sin\alpha}$ $\approx \dfrac{\sqrt{3}\left(1-\frac{x^2}{2}\right)-\left(x-\frac{x^3}{6}\right)}{2\left(x-\frac{x^3}{6}\right)}$ $=\dfrac{1}{2}\left[\dfrac{1}{x}\left(\sqrt{3}-x-\dfrac{\sqrt{3}x^2}{2}+\dfrac{x^3}{6}\right)\left(1-\dfrac{x^2}{6}\right)^{-1}\right]$ $\approx\dfrac{1}{2}\left[\dfrac{1}{x}\left(\sqrt{3}-x-\dfrac{\sqrt{3}x^2}{2}+\dfrac{x^3}{6}\right)\left(1+(-1)\left(-\dfrac{x^2}{6}\right)\right)\right]$ $=\dfrac{1}{2}\left[\left(\dfrac{\sqrt{3}}{x}-1-\dfrac{\sqrt{3}x}{2}+\dfrac{x^2}{6}\right)\left(1+\dfrac{x^2}{6}\right)\right]$ $=\dfrac{1}{2}\left[\dfrac{\sqrt{3}}{x}+\dfrac{\sqrt{3}}{6}x-1-\dfrac{x^2}{6}-\dfrac{\sqrt{3}x}{2}+\dfrac{x^2}{6}\right]$ $=\dfrac{1}{2}\left[\dfrac{\sqrt{3}}{x}-\dfrac{2\sqrt{3}}{6}x-1\right]$ $=\dfrac{\sqrt{3}}{2x}-\dfrac{1}{2}-\dfrac{1}{2\sqrt{3}}x$

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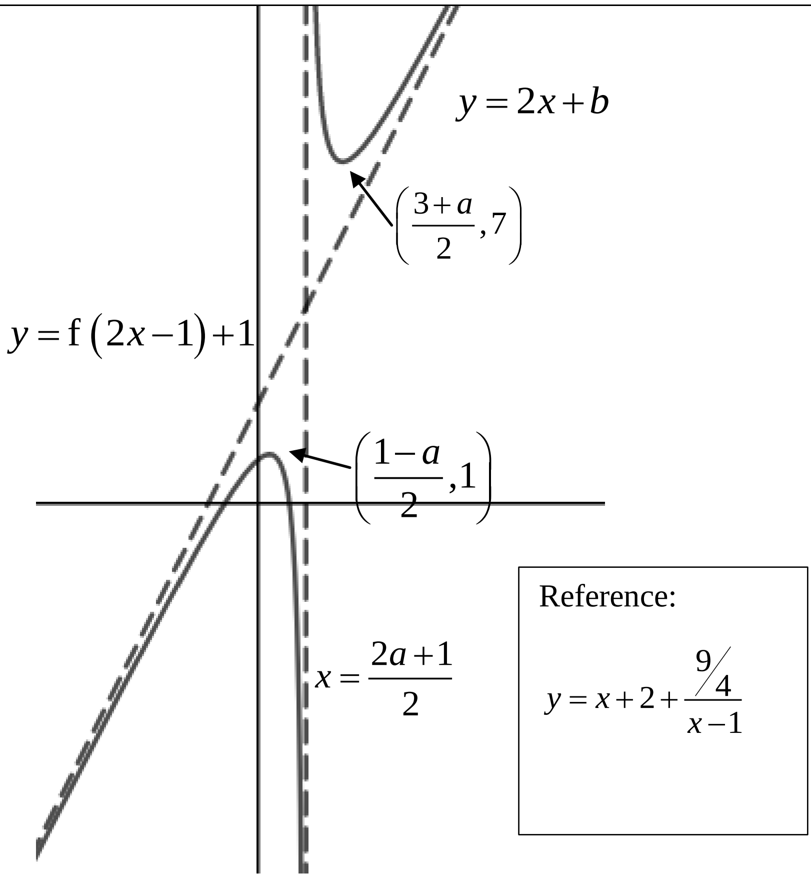
<td>
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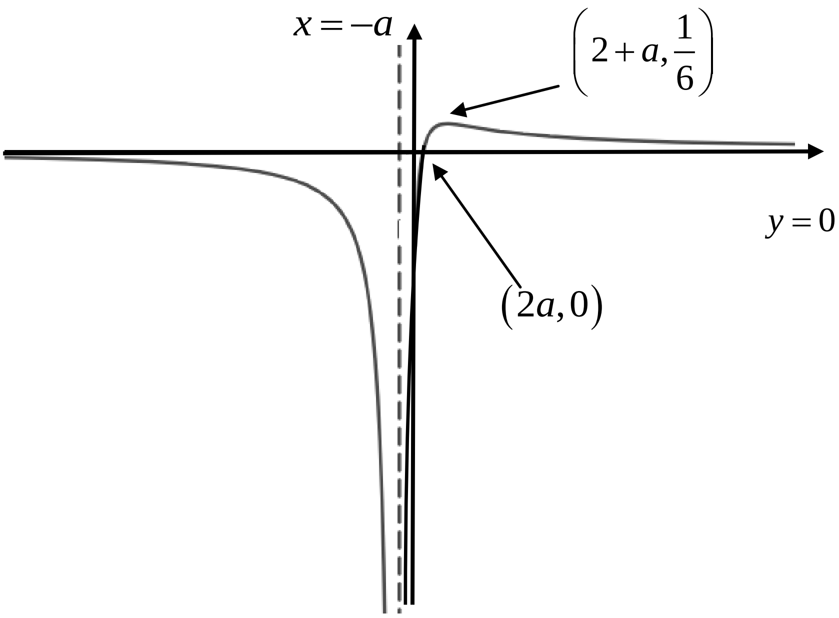
<td>
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<td>(iii)</td>
<td>Point P is (2a, 2a+b)
$\frac{y-\left(2a+b\right)}{x-2a}=m \Rightarrow y=mx-2am+2a+b$
Hence, P lies on the line $y=mx+\left(b+2a-2am\right)$ for $m\in\mathbb{R}$.
From the graph, $m\le 1$ for the line not to cut $y=\mathrm{f}(x)$.</td>
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7(a)

$$\int e^x \cos 2x \, dx = e^x \cos 2x + 2\int e^x \sin 2x \, dx$$

$$= e^x \cos 2x + 2\left(e^x \sin 2x - 2\int e^x \cos 2x \, dx\right)$$

$$5\int e^x \cos 2x \, dx = e^x \cos 2x + 2e^x \sin 2x$$

$$\therefore \int e^x \cos 2x \, dx = \frac{e^x(\cos 2x + 2\sin 2x)}{5} + c$$

(b)(i)

$$x = t - e^t, \quad y = 3\cos^2 t - 1$$

$$\frac{dx}{dt} = 1 - e^t, \quad \frac{dy}{dt} = 6\cos t(-\sin t) = -3\sin(2t)$$

$$\frac{dy}{dx} = \frac{-3\sin(2t)}{1-e^t}$$

$$\frac{dy}{dx} = 0 \Rightarrow \sin(2t) = 0$$

$$\Rightarrow 2t = 0 \text{ (N.A.) or } 2t = \pi \text{ or } 2t = 2\pi \text{(N.A.)}$$

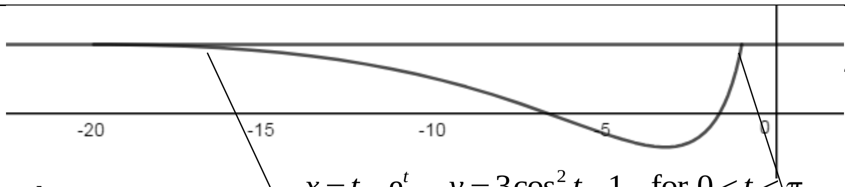
$$\Rightarrow t = \frac{\pi}{2}$$

t	1.6	$\frac{\pi}{2}$	1.5
x	-3.35303	-3.2396811	-2.981689
$\frac{dy}{dx}$	-0.0443	0	0.122

NB: t increases as x decreases.

Hence x-coordinate of the minimum point at is $\frac{\pi}{2} - e^{\frac{\pi}{2}}$.

(b)(ii)



$y = 2$

$x = t - e^t, \quad y = 3\cos^2 t - 1, \quad \text{for } 0 < t < \pi.$

$\pi - e^{\pi}$ $\qquad$ $x = -1$

When $y = 2$,

$2 = 3\cos^2 t - 1$

$\Rightarrow \cos t = \pm 1$

$\Rightarrow t = 0, \pi$

When $t = 0, x = 0 - e^0 = -1$

When $t = \pi, x = \pi - e^{\pi} = -19.9991$

Area required

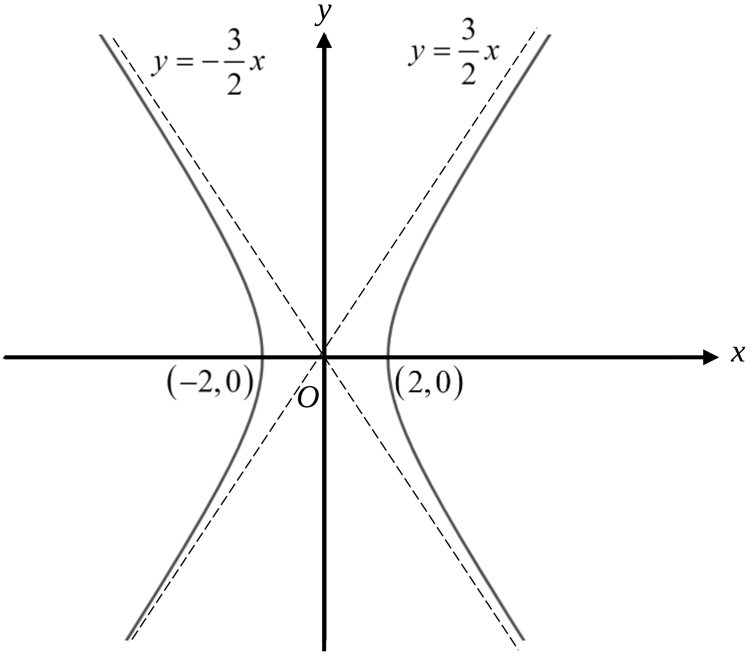
$$= \int_{\pi - e^{\pi}}^{-1} (2 - y) \, dx$$

$$= \int_{\pi}^{0} \left(2 - \left(3\cos^2 t - 1\right)\right)\left(1 - e^t\right) dt$$

$$= \int_{\pi}^{0} \left(3 - 3\cos^2 t\right)\left(1 - e^t\right) dt$$

	$=3\int_{\pi}^{0}\left(1-\cos^2 t\right)\left(1-\mathrm{e}^t\right)\mathrm{d}t$ $=3\int_{\pi}^{0}\left(1-\frac{\cos 2t+1}{2}\right)\left(1-\mathrm{e}^t\right)\mathrm{d}t$ $=3\int_{\pi}^{0}\left(\frac{1}{2}-\frac{\cos 2t}{2}\right)\left(1-\mathrm{e}^t\right)\mathrm{d}t$ $=\frac{3}{2}\int_{\pi}^{0}\left(1-\cos 2t\right)\left(1-\mathrm{e}^t\right)\mathrm{d}t$ $=\frac{3}{2}\int_{\pi}^{0}\left(1-\cos 2t-\mathrm{e}^t+\mathrm{e}^t\cos 2t\right)\mathrm{d}t$ $=\frac{3}{2}\left[t-\frac{\sin 2t}{2}-\mathrm{e}^t+\frac{\mathrm{e}^t\left(\cos 2t+2\sin 2t\right)}{5}\right]_{\pi}^{0}$ $=\frac{3}{2}\left[-\frac{4}{5}-\pi+\frac{4e^{\pi}}{5}\right]$ $=-\frac{3}{2}\pi-\frac{6}{5}+\frac{6}{5}\mathrm{e}^{\pi}$, where $a=-\frac{3}{2},\ b=-\frac{6}{5},\ c=\frac{6}{5}$
8 (i)	$\overrightarrow{BC}=\overrightarrow{OC}-\overrightarrow{OB}=-\mathbf{a}-\mathbf{b}=-\left(\mathbf{a}+\mathbf{b}\right)$
(ii)	Since ΔOCD is similar to ΔACB, OD parallel to AB. $\frac{OD}{AB}=\frac{CO}{CA}=\frac{1}{2}$. $\overrightarrow{OD}=\frac{1\left(-\mathbf{a}\right)+1\left(\mathbf{b}\right)}{2}=\frac{\mathbf{b}-\mathbf{a}}{2}$
(iii)	Let $\overrightarrow{OB}=\begin{pmatrix}x\\0\\z\end{pmatrix}$. $\overrightarrow{CB}=\begin{pmatrix}x\\0\\z\end{pmatrix}-\begin{pmatrix}2\\0\\0\end{pmatrix}=\begin{pmatrix}x-2\\0\\z\end{pmatrix}$ $\overrightarrow{CB}.\overrightarrow{CO}=\begin{pmatrix}x-2\\0\\z\end{pmatrix}.\begin{pmatrix}-2\\0\\0\end{pmatrix}=\left\lvert\overrightarrow{CB}\right\rvert(2)\cos\frac{\pi}{6}---(1)$ $\left\lvert\overrightarrow{CB}\right\rvert=4\cos\frac{\pi}{6}=2\sqrt{3}$ $-2x+4=2\sqrt{3}(2)\frac{\sqrt{3}}{2}$ $x=-1$ $\left\lvert\overrightarrow{OB}\right\rvert=\left\lVert\begin{pmatrix}-1\\0\\z\end{pmatrix}\right\rVert=2$

	$(-1)^2+z^2=2^2$ $z^2=3$ $z=\pm\sqrt{3}$ $\overrightarrow{OB}=\begin{pmatrix}-1\\0\\\sqrt{3}\end{pmatrix}$ or $\begin{pmatrix}-1\\0\\-\sqrt{3}\end{pmatrix}$ $(\text{rejected} \because z\text{-component}>0)$.
(iv)	Equation of line passing through *OB*: $\overrightarrow{OB}=\lambda\begin{pmatrix}-1\\0\\\sqrt{3}\end{pmatrix},\ \lambda\in\mathbb{R}$ $\frac{x-2}{3}=\mu\Rightarrow x=2+3\mu$ $\frac{y}{3}=\mu\Rightarrow y=3\mu$ $z-1=\mu\Rightarrow z=\mu+1$ Equation of line: $\mathbf{r}=\begin{pmatrix}2\\0\\1\end{pmatrix}+\mu\begin{pmatrix}3\\3\\1\end{pmatrix},\mu\in\mathbb{R}$ Direction vector of line is not parallel to direction vector of line passing through *O* and *B* since direction vectors of both lines are not scalar multiple of each other. Solving equations simultaneously: $\begin{pmatrix}2+3\mu\\3\mu\\\mu+1\end{pmatrix}=\lambda\begin{pmatrix}-1\\0\\\sqrt{3}\end{pmatrix}$ There is no value of λ and μ that satisfy the above equation. Since the lines are not parallel and non-intersecting, the lines are skew.

9(i)	$x = 2\sec t$ and $y = 3\tan t$ $1 + \tan^2 t = \sec^2 t$ $\Rightarrow 1 + \dfrac{y^2}{9} = \dfrac{x^2}{4}$ $\Rightarrow \dfrac{x^2}{4} - \dfrac{y^2}{9} = 1$
9(ii)	
9(iii)	**Method 1** $\dfrac{dy}{dx} = \dfrac{dy}{dt} \cdot \dfrac{dt}{dx} = 3\sec^2 t \cdot \dfrac{1}{2\sec t \tan t} = 1.5\operatorname{cosec} t$ $1.5\operatorname{cosec} t = \sqrt{3}$ $t = \dfrac{\pi}{3}$ or $\dfrac{2\pi}{3}$ When $t = \dfrac{\pi}{3}$, $x = 2\sec\dfrac{\pi}{3} = 4$, $y = 3\tan\dfrac{\pi}{3} = 3\sqrt{3}$. Equation of tangent:

$y-3\sqrt{3}=\sqrt{3}(x-4)$

$\Rightarrow y=\sqrt{3}x-4\sqrt{3}+3\sqrt{3}$

$\Rightarrow y=\sqrt{3}x-\sqrt{3}$

$\therefore k=-\sqrt{3}$ (Shown)

When $t=\dfrac{2\pi}{3}$,

$x=2\sec\dfrac{2\pi}{3}=-4$, $y=3\tan\dfrac{2\pi}{3}=-3\sqrt{3}$.

Equation of tangent:

$y+3\sqrt{3}=\sqrt{3}(x+4)$

$\Rightarrow y=\sqrt{3}x+4\sqrt{3}-3\sqrt{3}$

$\Rightarrow y=\sqrt{3}x+\sqrt{3}$

$\therefore k=\sqrt{3}$ (N.A. $\because k<0$)

Method 2

$\dfrac{x^2}{4}-\dfrac{\left(\sqrt{3}x+k\right)^2}{9}=1$

$\Rightarrow -3x^2-\left(8\sqrt{3}k\right)x-\left(36+4k^2\right)=0$

Since the line $y=\sqrt{3}x+k$, where $k<0$, is a tangent to C, there should be repeated roots.

Thus,

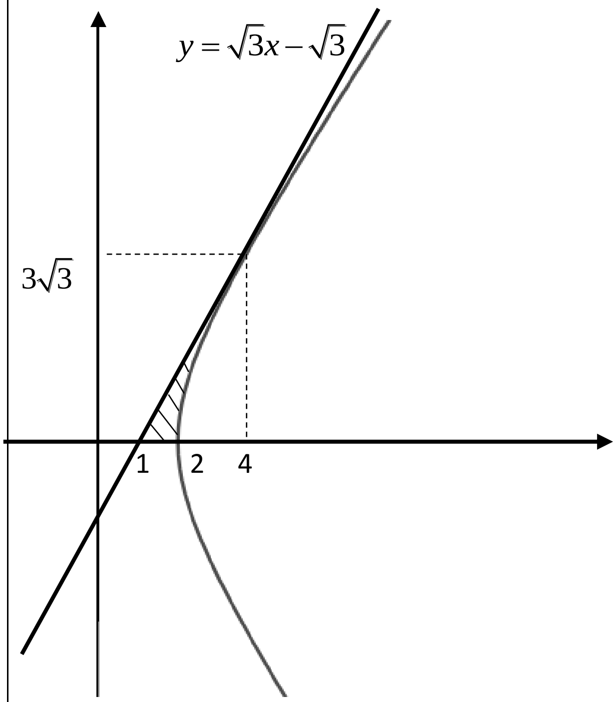
$\left(8\sqrt{3}k\right)^2-4(-3)\left(-36-4k^2\right)=0$

$\Rightarrow 192k^2-432-48k^2=0$

$\Rightarrow 144k^2=432$

$\Rightarrow k^2=3$

$\Rightarrow k=\sqrt{3}$ (N.A. $\because k<0$) or $k=-\sqrt{3}$ (Shown)



For $k=-\sqrt{3}$,

$-3x^2-\left(8\sqrt{3}\left(-\sqrt{3}\right)\right)x-\left(36+4(3)\right)=0$

$3x^2-24x+48=0$

$x^2-8x+16=0$

$(x-4)^2=0$

$\Rightarrow x=4$

$y^2=9\left(\dfrac{x^2}{4}-1\right)$

When $x=4$, $y=\sqrt{27}=3\sqrt{3}\,(y>0)$

Required volume

$=\dfrac{1}{3}\pi\left(3\sqrt{3}\right)^2(3)-\pi\int_2^4 y^2\ \mathrm{d}x$

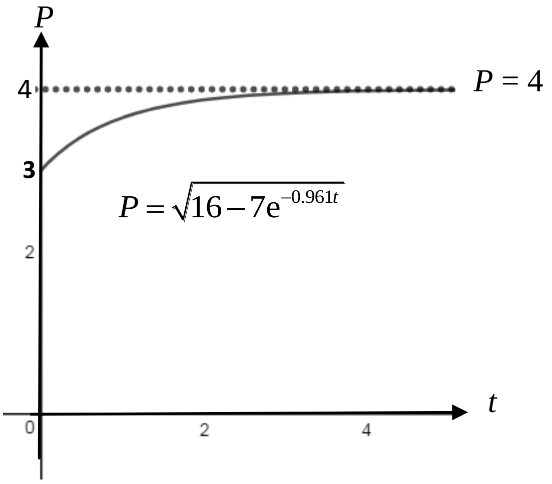
$=27\pi-9\pi\int_2^4\left(\dfrac{x^2}{4}-1\right)\ \mathrm{d}x$

$=27\pi-9\pi\left(\dfrac{8}{3}\right)$

$=3\pi$

$=9.42$ (3 sf)

10(a)	$u=\frac{y}{x}\Rightarrow y=ux$, $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{y^2+xy+x^2}{x^2}=\left(\frac{y}{x}\right)^2+\frac{y}{x}+1$ ---(1) $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}u}{\mathrm{d}x}x+u$ -- (2) Sub (2) into (1): $\frac{\mathrm{d}u}{\mathrm{d}x}x+u=u^2+u+1\Rightarrow\frac{\mathrm{d}u}{\mathrm{d}x}x=u^2+1$ $\frac{1}{u^2+1}\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{1}{x}$ $\int\frac{1}{u^2+1}\mathrm{d}u=\int\frac{1}{x}\mathrm{d}x$ $\tan^{-1}u=\ln\lvert x\rvert+c$, where c is an arbitrary constant. $\tan^{-1}u=\ln x+c$ (since $x>0$) $u=\tan(\ln x+c)$ $y=x\tan(\ln x+c)$
10(b)(i)	Point A shows that at <u>4 dollars per kg, 1 tonne of rice</u> is <u>produced and all of it is bought</u> by the consumers. This is the equilibrium point where the price is <u>4 dollars per kg</u> and the quantity produced/consumed is <u>1 tonne.</u>
10(b) (ii)	$C_1:Q=\frac{k_1}{P}$ $C_2:Q=k_2P$ When $Q=1,P=4$, $k_1=4,k_2=\frac{1}{4}$. $C_1:Q=\frac{4}{P};C_2:Q=\frac{P}{4}$ Hence, $C_1:Q=\frac{4}{P};C_2:Q=\frac{p}{4}$.
10(iii)	$\frac{\mathrm{d}P}{\mathrm{d}t}=k_3\left(\frac{4}{P}-\frac{P}{4}\right)$ $\frac{\mathrm{d}P}{\mathrm{d}t}=k_3\left(\frac{16-P^2}{4P}\right)$ $\int\frac{4P}{16-P^2}\mathrm{d}P=\int k_3\mathrm{d}t$ $-2\int\frac{-2P}{16-P^2}\mathrm{d}P=\int k_3\mathrm{d}t$ $-2\ln\lvert16-P^2\rvert=k_3t+c$ $\ln\lvert16-P^2\rvert=\frac{-k_3}{2}t+\frac{-c}{2}$ $\lvert16-P^2\rvert=\mathrm{e}^{\frac{-k_3}{2}t+\frac{-c}{2}}$

	$16-P^2=\left(\pm e^{\frac{-c}{2}}\right)e^{\frac{-k_3}{2}t}$ $16-P^2=Ae^{Bt}, A=\pm e^{\frac{-c}{2}}, B=\frac{-k_3}{2}$ $\sqrt{16-Ae^{Bt}}=P\ \left(P>0\right)$ When $t=0, P=3$: $\sqrt{16-Ae^{B(0)}}=3$ $16-A=3^2$ $A=7$ When $t=1, P=3.65$: $\sqrt{16-7e^{B}}=3.65$ $B=\ln\frac{16-3.65^2}{7}=-0.96102663=-0.961$ $\therefore P=\sqrt{16-7e^{-0.961t}}$ 
10(iii)	Rice production will only occur if the price is able to at least cover the initial cost of investment.