<u>**2017 NYJC JC2 Prelim Exam 9649/2 Solution**</u>

<table>
<tr><th>Qn</th><th></th></tr>
<tr><td>1(i)</td><td>By GC, $\alpha = -5.690$, $\beta = 1.351$</td></tr>
<tr><td>1(ii)</td><td>

$x_n = 1 + \tan\left(\frac{1}{4}x_{n+1}\right)$

$\Rightarrow x_{n+1} = 4\tan^{-1}(x_n - 1)$

If $\alpha < x_n < \beta$,

$\alpha - 1 < x_n - 1 < \beta - 1$

$\tan^{-1}(\alpha - 1) < \tan^{-1}(x_n - 1) < \tan^{-1}(\beta - 1)$ (since $\tan^{-1}(x)$ is an increasing function on $\mathbb{R}$)

$4\tan^{-1}(\alpha - 1) < 4\tan^{-1}(x_n - 1) < 4\tan^{-1}(\beta - 1)$

$\alpha < x_{n+1} < \beta$

(Since if sequences converges to L as $n \to \infty$, $x_n \to L$, $x_{n+1} \to L$, therefore $\alpha = 4\tan^{-1}(\alpha - 1)$, $\beta = 4\tan^{-1}(\beta - 1)$)

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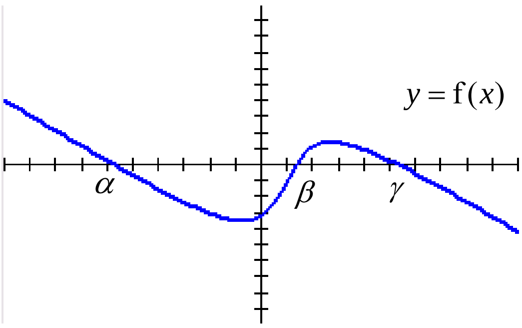
<tr><td>1(iii)</td><td>

Consider $x_{n+1} - x_n = 4\tan^{-1}(x_n - 1) - x_n$

Consider the graph of $\mathrm{f}(x) = 4\tan^{-1}(x - 1) - x$.



From the above graph,

when $\alpha < x_n < \beta$, $\mathrm{f}(x_n) = x_{n+1} - x_n < 0 \Rightarrow x_{n+1} < x_n$

It will converge to $\alpha = -5.690$ as the sequence is decreasing.

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<tr><td>2(i)</td><td>

$\frac{a}{1000} = 0.053744 + 0.5\tan 6.3°$

$a = 108.94$

$\frac{b}{1000} = 0.3601 + 0.5\tan 16°$

$b = 503.47$

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Qn

2(ii)

$$\text{Area under the mountain} = \frac{1}{3}(0.5)\left(0.010 + 4(0.053744) + 2(0.10894) + 4(0.17743)\right.$$
$$\left. + 2(0.26110) + 4(0.36010) + 0.50347\right)$$
$$= 0.60310 \text{ km}^2$$

$$\text{Mean Altitude } = \frac{0.60310}{3.0}$$
$$\approx 0.20103$$
$$= 0.201 \text{ km (3sf)}$$

2(iii)

Since gradient is increasing, slope of mountain is concave upwards. Approximation will be an overestimate.

2(iv)

$h = 0.5$

w	A	$\frac{dA}{dw}$	A^* (Euler)
0	6.5	-0.8125	6.09375
0.5	6.2122		

Predicted gradient at $w = 0.5$ is $-\frac{6.09375}{8(1.5)^2} = -0.33854$

Geographer's altitude $= 6.21$ km (3sf)

3(a)

Let $\mathbf{0}$ be the zero $n \times n$ matrix. Thus

$T(\mathbf{0}) = \mathbf{A}^{-1}\mathbf{0}\mathbf{A} = \mathbf{0}$

Let $\mathbf{B}$, $\mathbf{C} \in \mathrm{M}_{nn}$ and $\alpha \in \mathbb{R}$.

$$T(\mathbf{B}+\mathbf{C}) = \mathbf{A}^{-1}(\mathbf{B}+\mathbf{C})\mathbf{A}$$
$$= \mathbf{A}^{-1}\mathbf{B}\mathbf{A} + \mathbf{A}^{-1}\mathbf{C}\mathbf{A}$$
$$= T(\mathbf{B}) + T(\mathbf{C})$$

$$T(\alpha\mathbf{B}) = \mathbf{A}^{-1}(\alpha\mathbf{B})\mathbf{A}$$
$$= \alpha\mathbf{A}^{-1}\mathbf{B}\mathbf{A}$$
$$= \alpha T(\mathbf{B})$$

Thus T is a linear transformation.

They are similar matrices.

3(b) (i)

$$\det(\mathbf{P}) = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & t \\ 1 & 4 & t^2 \end{vmatrix}$$
$$= \begin{vmatrix} 2 & t \\ 4 & t^2 \end{vmatrix} - \begin{vmatrix} 1 & t \\ 1 & t^2 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix}$$
$$= t^2 - 3t + 2 = (t-2)(t-1)$$

3(b) (ii)

For $\mathbf{P}$ to be singular, det($\mathbf{P}$) = 0. Thus

$(t-2)(t-1) = 0 \quad \Rightarrow \quad t = 1, 2$

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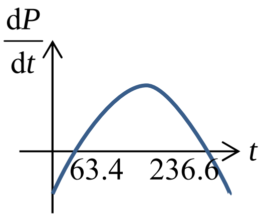
3(b) (iii)	When $t \neq 1$ and $t \neq 2$, nullity(T) = 0 When $t = 1$, $\mathbf{P} \rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, thus nullity($T$) = 1. When $t = 2$, $\mathbf{P} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, thus nullity($T$) = 1 as well. Possible values of nullity(T) is 0 or 1.
4	(a)(i) $d = a + b(\theta + 2\pi) - [a + b\theta] = 2b\pi$ which is a constant. (ii) Parametric equations of S are $x = r\cos\theta = a(1+\theta)\cos\theta$ and $y = r\sin\theta = a(1+\theta)\sin\theta$. $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}\theta} \Big/ \frac{\mathrm{d}x}{\mathrm{d}\theta} = \frac{\sin\theta + (1+\theta)\cos\theta}{\cos\theta - (1+\theta)\sin\theta}$ $= \frac{\tan\theta + (1+\theta)}{1 - (1+\theta)\tan\theta}$ $= \tan(\theta + \phi)$ where $\phi \equiv \tan^{-1}(1+\theta)$ So angle which the tangent to S at the point (r, θ) makes with the initial line is $\theta + \phi = \theta + \tan^{-1}(1+\theta)$. Gradient of the tangent to S at the point where $\theta = 0$ is $\tan(\tan^{-1} 1) = 1$. Also, when $\theta = 0, r = a$. Thus the cartesian equation of the tangent is $y = x - a$. (b) Consider the circle $r = a$. Surface area of sphere $= 2\pi \int_0^{\pi} r\sin\theta \left\{ r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta} \right)^2 \right\}^{\frac{1}{2}} \mathrm{d}\theta$ $= 2\pi \int_0^{\pi} a\sin\theta \left\{ a^2 \right\}^{\frac{1}{2}} \mathrm{d}\theta \quad (\frac{\mathrm{d}r}{\mathrm{d}\theta} = 0)$ $= -2\pi a^2 \left[\cos\theta \right]_0^{\pi}$ $= 4\pi a^2$

Qn	
5	(i) $\dfrac{\mathrm{d}P}{\mathrm{d}t}=2P-\dfrac{1}{150}P^2=2P\left(1-\dfrac{P}{300}\right)$. The carrying capacity is 300. (ii) $\dfrac{\mathrm{d}P}{\mathrm{d}t}=2P-\dfrac{1}{150}P^2=\dfrac{1}{150}P\left(300-P\right)$ $\displaystyle\int\frac{1}{P\left(300-P\right)}\,\mathrm{d}P=\int\frac{1}{150}\,\mathrm{d}t$ $\displaystyle\int\frac{1}{150^2-\left(P-150\right)^2}\,\mathrm{d}P=\int\frac{1}{150}\,\mathrm{d}t$ (or LHS = $\displaystyle\frac{1}{300}\int\left(\frac{1}{P}+\frac{1}{300-P}\right)\mathrm{d}P$) $\dfrac{1}{300}\ln\left(\dfrac{P}{300-P}\right)=\dfrac{1}{150}t+C$ $\ln\left(\dfrac{P}{300-P}\right)=2t+300C$ $\dfrac{P}{300-P}=A\mathrm{e}^{2t}$ where $A=\mathrm{e}^{300C}$ $P=\dfrac{300A\mathrm{e}^{2t}}{1+A\mathrm{e}^{2t}}$ or $P=\dfrac{300}{B\mathrm{e}^{-2t}+1}$. (iii) $\dfrac{\mathrm{d}P}{\mathrm{d}t}=2P-\dfrac{1}{150}P^2-h$ $=-\dfrac{1}{150}\left(P^2-300P\right)-h$ $=-\dfrac{1}{150}\left[\left(P-150\right)^2-150^2\right]-h$ $=-\dfrac{1}{150}\left(P-150\right)^2+150-h$ $\dfrac{\mathrm{d}P}{\mathrm{d}t}<0$ for all P if $150-h<0 \Rightarrow h>150$. Thus $h\le 150$ and $h_{\max}=150$, and $\dfrac{\mathrm{d}P}{\mathrm{d}t}=0$ when $P=150=\dfrac{300}{2}$. <u>Alternative Solution</u> $\dfrac{\mathrm{d}P}{\mathrm{d}t}=2P-\dfrac{1}{150}P^2-h$ ___(*) $\dfrac{\mathrm{d}}{\mathrm{d}P}\dfrac{\mathrm{d}P}{\mathrm{d}t}=2-\dfrac{1}{75}P$ $\dfrac{\mathrm{d}}{\mathrm{d}P}\dfrac{\mathrm{d}P}{\mathrm{d}t}=0 \Rightarrow P=150$. Substitute $P=150$ into (*): $\dfrac{\mathrm{d}P}{\mathrm{d}t}=2\left(150\right)-\dfrac{1}{150}\left(150\right)^2-h\ =150-h$ $\dfrac{\mathrm{d}P}{\mathrm{d}t}<0$ if $h>150$ So $h\le 150$ and $h_{\max}=150$, and $\dfrac{\mathrm{d}P}{\mathrm{d}t}=0$ when $P=150=\dfrac{300}{2}$.

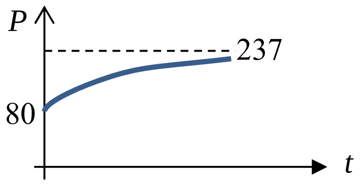
Qn	

(iv) $\dfrac{\mathrm{d}P}{\mathrm{d}t}=2P-\dfrac{1}{150}P^2-100$

$\dfrac{\mathrm{d}P}{\mathrm{d}t}=0 \Rightarrow P=63$ (unstable) or $P=237$ (stable) (nearest integer)



$P_0=80$: Since $63<P_0<237$, $\dfrac{\mathrm{d}P}{\mathrm{d}t}>0$, the population will increase to 237.



(v) Without harvesting,

$P=\dfrac{300}{\mathrm{B}e^{-2t}+1}$

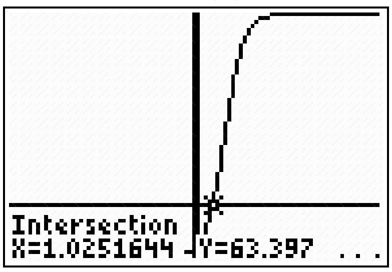
When $t=0$, $P=10$,

$10=\dfrac{300}{\mathrm{B}+1} \Rightarrow B=29$

$P=\dfrac{300}{29e^{-2t}+1}$

For population not to die out, $P>63.397$.

$$\frac{300}{29e^{-2t}+1}>63.397$$



$t>1.025$

Hence, least $t=1.03$ (2 d.p.)

Qn	
6	$\mathrm{P}(R \le r) = \int_2^r \frac{1}{6}x \, \mathrm{d}x = \left[\frac{1}{12}x^2\right]_2^r = \frac{1}{12}\left(r^2 - 4\right)$ Volume of a randomly chosen cone $V = \frac{1}{3}\pi R^3$ $\mathrm{P}(V \le v) = \mathrm{P}\left(\frac{1}{3}\pi R^3 \le v\right) = \mathrm{P}\left(R \le \sqrt[3]{\frac{3v}{\pi}}\right) = \frac{1}{12}\left(\left(\frac{3v}{\pi}\right)^{\frac{2}{3}} - 4\right)$ $\frac{\mathrm{d}}{\mathrm{d}v}\mathrm{P}(V \le v) = \frac{1}{12}\frac{2}{3}\left(\frac{\pi}{3v}\right)^{\frac{1}{3}}\frac{3}{\pi} = \frac{1}{6\pi}\sqrt[3]{\frac{\pi}{3v}}$ $\mathrm{g}(v) = \begin{cases} \frac{1}{6\pi}\sqrt[3]{\frac{\pi}{3v}}, & \text{if } \frac{8}{3}\pi \le v \le \frac{64}{3}\pi, \\ 0, & \text{otherwise.} \end{cases}$ Expected time = $\frac{1}{3}\int_{\frac{8}{3}\pi}^{\frac{64}{3}\pi} \frac{v}{6\pi}\sqrt[3]{\frac{\pi}{3v}} \, \mathrm{d}v = 11.54s$
7	To test H_0 : There is no association between performance and gender. H_1 : There is some association between performance and gender. at 5% level of significance. Expected Frequencies

	Distinction	Pass	Sub-pass	Fail	Total
Male	22.8	82.2	11.4	3.6	120
Female	15.2	54.8	7.6	2.4	80
Total	38	137	19	6	200

Since the expected frequencies for the last column is less than 5, we merge the columns for Sub-pass and Fail.

Qn	
	Under H_0, $\chi^2 = \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i} \sim \chi^2(2)$. Reject H_0 if p-value $\le$ 0.05 (or $\chi^2_{calc} \ge 5.991$) Using GC, p-value = 0.0787 (or $\chi^2_{calc} = 5.085$) Since p-value > 0.05 (or $\chi^2_{calc} > 5.991$), we do not reject H_0 at 5% level of significance. There is insufficient evidence to conclude that there is some association between grades and gender. The value of χ^2 in the larger sample is *n* times the value of χ^2 in the smaller sample. If candidate did not combine columns: Under H_0, $\chi^2 = \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i} \sim \chi^2(3)$. Reject H_0 if p-value $\le$ 0.05 (or $\chi^2_{calc} \ge 7.815$) Using GC, p-value = 0.0328 (or $\chi^2_{calc} = 8.748$) Since p-value < 0.05 (or $\chi^2_{calc} > 7.815$), we reject H_0 at 5% level of significance. There is sufficient evidence to conclude that there is some association between grades and gender.
8(i)	$r_k(p) = \binom{k}{1} p(1-p)^{k-1} + \binom{k}{1} p^{k-1}(1-p)$ $= kp(1-p)^{k-1} + kp^{k-1}(1-p)$
8(ii)	$N_k \sim \text{Geo}(r_k(p))$
8(iii)	Note that $M_k = \sum_{j=3}^{k} N_j$. Thus $\text{E}(M_k) = \text{E}\left(\sum_{j=3}^{k} N_j\right) = \sum_{j=3}^{k} \text{E}(N_j) = \sum_{j=3}^{k} \frac{1}{r_j(p)}$ Since N_j 's are independent, $\text{Var}(M_k) = \text{Var}\left(\sum_{j=3}^{k} N_j\right) = \sum_{j=3}^{k} \text{Var}(N_j) = \sum_{j=3}^{k} \frac{1-r_j(p)}{\left[r_j(p)\right]^2}$

Qn	
9(i)	Sample proportion $\hat{p} = \dfrac{0.884+0.676}{2} = 0.78$ Width of C.I. = $0.208 = 2z_{\frac{\alpha}{2}}\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}$ $z_{\frac{\alpha}{2}} = \dfrac{0.208}{2\sqrt{\dfrac{0.78(1-0.78)}{80}}} = 2.24553$ $\dfrac{\alpha}{2} = \mathrm{P}(Z \geq 2.24553) = 0.012367$ $\alpha = 0.024734$ $k = 100\left(1-0.024734\right) = 97.5$
9(ii)	Let X be the rating of a randomly chosen teacher. Using G.C., $\bar{x} = 6.4875$, $s^2 = (0.37961)^2$ 99% C.I. for μ is (6.02, 6.96)
9(iii)	It means that out of 100 such intervals constructed from a large number of random samples of 8 teachers using the above method, we expect 99 of them to contain the true population mean rating for the new curriculum.
9(iv)	Since $\mu = 7$ does not lie in the 99% confidence interval in (ii) also mean that it will not lie in the narrower 95% confidence interval, the null hypothesis can be rejected at 5% significance level.
10(i)	Let X be the number of flaws in a cable of length 2 km. $X \sim Po(4)$. $\text{Prob. Req'd} = P(X > 4)$ $= 1 - P(X \leq 4) = 0.371$
10(ii)	Let Y and W be the number of flaws for the first 400 m and next 600 m of a cable respectively. $Y \sim Po(0.8)$ and $W \sim Po(1.2)$. $\text{Prob. Req'd} = P(Y = 3 \text{ and } W < 2)$ $= P(Y=3)\cdot P(W<2)$ $= P(Y=3)\cdot P(W \leq 1)$ $= 0.0254$
10(iii)	Let V be the number of flaws in a cable of length 500 m. $V \sim Po(1)$. Note that $V_1 + V_2 \sim Po(2)$. For $r = 0,1,2,\cdots,n$,

<table>
<tr><th>Qn</th><th></th></tr>
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$$P(V_1=r\mid V_1+V_2=n)=\frac{P(V_1=r\text{ and }V_1+V_2=n)}{P(V_1+V_2=n)}$$
$$=\frac{P(V_1=r\text{ and }V_2=n-r)}{P(V_1+V_2=n)}$$
$$=\frac{P(V_1=r)P(V_2=n-r)}{P(V_1+V_2=n)}$$
$$=\frac{\left(\mathrm{e}^{-1}1^r/r!\right)\cdot\left(\mathrm{e}^{-1}1^{n-r}/(n-r)!\right)}{\mathrm{e}^{-2}2^n/n!}$$
$$=\frac{n!}{r!(n-r)!}\left(\frac{1}{2}\right)^r\left(\frac{1}{2}\right)^{n-r}$$
Thus $V_1\mid V_1+V_2=n\sim \mathrm{B}\left(n,\frac{1}{2}\right)$.
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<tr><td>10(iv)</td><td>
Let U and T be the number of flaws in the first t meters and next $500-t$ meters of a cable. Thus $U\sim Po(t/500)$ and $T\sim Po(1-t/500)$ and $U+T\sim Po(1)$.
$$\text{Prob. Req'd}=P(U=1\mid U+T=1)$$
$$=\frac{P(U=1\text{ and }T=0)}{P(U+T=1)}$$
$$=\frac{P(U=1)P(T=0)}{P(U+T=1)}$$
$$=\frac{1}{\mathrm{e}^{-1}}\left(\mathrm{e}^{-t/500}\cdot\frac{t}{500}\right)\left(\mathrm{e}^{-1+t/500}\right)$$
$$=\frac{t}{500}$$
Thus location of the flaw is uniformly distributed on (0, 500).
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<tr><td>11(i)</td><td>
Assume that X and Y are normally distributed.
Let $D=X-Y$ and μ_D be the population mean of D.
To test $H_0:\mu_D=0$
$H_1:\mu_D>0$ at 5% level of sig.
Under H_0, $T=\dfrac{\bar{D}}{s_d/\sqrt{10}}\sim t(9)$.
Reject H_0 if $p-value\le 0.05$
Calculation: $\bar{d}=0.027,\quad s_D=0.0457$
$p-value=0.0473$
Conclusion: Since $p-value<0.05$, we reject H_0 and conclude that there is sufficient evidence at 5% level of significance that taking Vitamin E daily for 2 years will reduce the average thickness of the plague.
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Qn	
11(ii)	When the distribution of the population cannot be assumed to follow a normal distribution. Assume distribution is symmetric.

D	0.06	0.07	0.06	-0.01	-0.05	0.08	0	-0.03	0.05	0.04
	+	+	+	−	−	+		−	+	+
R	6.5	8	6.5	1	4.5	9		2	4.5	3

To test $H_0 : m_D = 0$

$H_1 : m_D > 0$ at 5% level of sig.

Perform Wilcoxon signed rank test.

Reject H_0 if $W \le 8 \le 0.05$

Calculation: $P = 37.5, \quad Q = 7.5$

$W = \min(P, Q) = 7.5$

Conclusion: Since $W < 8$, we reject H_0 and conclude that there is sufficient evidence at 5% level of significance that taking Vitamin E daily for 2 years will reduce the mean thickness of the plague.