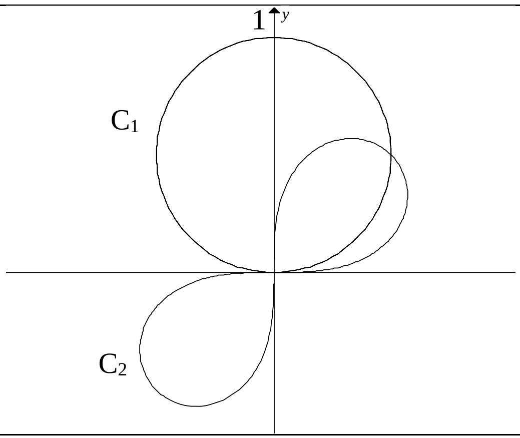
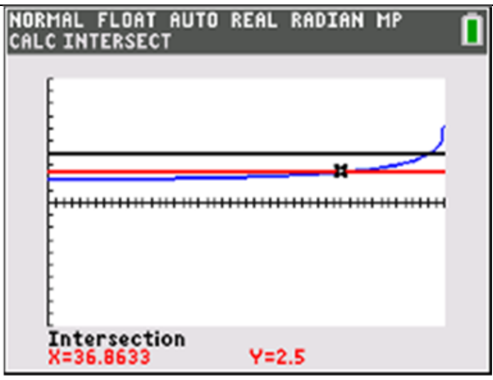
Further Maths Prelim Paper 1 Solution:

1	$\left(x+1\right)\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{y}{\ln\left(x+1\right)}=x^2+x$ $\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{y}{\left(x+1\right)\ln\left(x+1\right)}=x$ I.F. $u=\mathrm{e}^{\int\frac{1}{(x+1)\ln(x+1)}\mathrm{d}x}=\mathrm{e}^{\ln\left[\ln\left(x+1\right)\right]}=\ln\left(x+1\right)$ $\therefore y=\frac{1}{\ln\left(x+1\right)}\int x\ln\left(x+1\right)\mathrm{d}x$ $u=\ln\left(x+1\right),v'=x$ $u'=\frac{1}{x+1},v=\frac{x^2}{2}$ $=\frac{1}{\ln\left(x+1\right)}\left[\frac{x^2}{2}\ln\left(x+1\right)-\int\frac{x^2}{2\left(x+1\right)}\mathrm{d}x\right]$ $=\frac{x^2}{2}-\frac{1}{2\ln\left(x+1\right)}\int\left(x-1+\frac{1}{x+1}\right)\mathrm{d}x$ $=\frac{x^2}{2}-\frac{1}{2\ln\left(x+1\right)}\left[\frac{x^2}{2}-x+\ln\left(x+1\right)+C\right]$ $=\frac{x^2}{2}-\frac{x^2-2x}{4\ln\left(x+1\right)}-\frac{1}{2}+\frac{C}{\ln\left(x+1\right)}$
2(i)	$u_1=-1,\ u_2=-1,\ u_3=3$
(ii)	$v_1=\left(-1\right)^0\left\|1-v_0\right\|-v_0$ $=\left(v_0-1\right)-v_0\qquad\left[\text{iff } v_0\geq 1\right]$ $=-1$ Since $v_1=u_1$ for all $v_0\geq 1$, all subsequent terms will be identical as well. $\therefore v_n=u_n$ for all $n\in\mathbb{Z}$, $n\geq 1$ when $v_0\geq 1$. (shown)
(iii)	$v_2=\left(-1\right)^1\left\|1-v_1\right\|-v_1$ $=\left(v_1-1\right)-v_1\qquad\left[\text{iff } v_1\leq 1\right]$ $=-1$ If $v_0\geq 1,\ \ v_1=u_1$ (reject) If $v_0<1,$ $v_1=\left(-1\right)^0\left\|1-v_0\right\|-v_0$ $=\left(1-v_0\right)-v_0=1-2v_0$ If $v_1\leq 1,\ 1-2v_0\leq 1\Rightarrow v_0\geq 0$ Set of values is $\left\{v_0\in\mathbb{R}:0\leq v_0<1\right\}$

3(a)	$1.5+0.0167\sin E-E=0$ let $\mathrm{f}(E)=1.5+0.0167\sin E-E$ $\mathrm{f}(0)=1.5>0$ $\mathrm{f}(2)=1.5+0.0167\sin 2-2=-0.48481<0$ Since f(E) is continuous on [0, 2] and f(0)f(2) < 0, by Intermediate Value Theorem, there is a root, α, in the interval (0, 2). $\mathrm{f}'(E)=e\cos E-1$ $x_{n+1}=x_n-\dfrac{\mathrm{f}(x_n)}{\mathrm{f}'(x_n)},\quad x_0=0.5$ By G.C., $\alpha=1.517$ (3 d.p.)
(b)	Let $\mathrm{f}(x)=\mathrm{e}^{-x}(5\cos x-10\sin x)$ Using Simpson's rule, $s(0.4)$ $=\dfrac{0.1}{3}\left(\mathrm{f}(0)+4\mathrm{f}(0.1)+2\mathrm{f}(0.2)+4\mathrm{f}(0.3)+\mathrm{f}(0.4)\right)$ $=1.001$ (3 d.p.)
4(i)	
	$\sin\theta=\sqrt{\sin\theta\cos\theta}$ $\sin^2\theta=\sin\theta\cos\theta$ $\sin^2\theta-\sin\theta\cos\theta=0$ $\sin\theta(\sin\theta-\cos\theta)=0$ $\sin\theta=0$ or $\tan\theta=1$ $\theta=0,\pi$ $\quad\theta=\dfrac{\pi}{4},-\dfrac{3\pi}{4}$ (reject) $r=0$ $\quad r=\dfrac{1}{\sqrt{2}}$ Coordinates $=(0,0),\left(\dfrac{1}{\sqrt{2}},\dfrac{\pi}{4}\right)$

(ii)	$r^2 = \sin\theta\cos\theta = \dfrac{\sin 2\theta}{2}$ $2r\dfrac{\mathrm{d}r}{\mathrm{d}\theta} = \cos 2\theta$
	Arc length $= 2\int_0^{\pi/2}\sqrt{r^2 + \left(\dfrac{\cos 2\theta}{2r}\right)^2}\ \mathrm{d}\theta$ $= 2\int_0^{\pi/2}\sqrt{\dfrac{\sin 2\theta}{2} + \dfrac{\cos^2 2\theta}{2\sin 2\theta}}\ \mathrm{d}\theta$ $= \sqrt{2}\int_0^{\pi/2}\sqrt{\dfrac{\sin^2 2\theta + \cos^2 2\theta}{\sin 2\theta}}\ \mathrm{d}\theta$ $= \sqrt{2}\int_0^{\pi/2}\sqrt{\text{cosec}2\theta}\ \mathrm{d}\theta$ $\therefore k = \sqrt{2}$
(iii)	Area $= \dfrac{1}{2}\int_0^{\pi/4}\sin^2\theta\ \mathrm{d}\theta + \dfrac{1}{2}\int_{\pi/4}^{\pi/2}\sin\theta\cos\theta\ \mathrm{d}\theta$ $= \dfrac{1}{2}\int_0^{\pi/4}\dfrac{1-\cos 2\theta}{2}\ \mathrm{d}\theta + \dfrac{1}{2}\left[\dfrac{\sin^2\theta}{2}\right]_{\pi/4}^{\pi/2}$ $= \dfrac{1}{4}\left[\theta - \dfrac{\sin 2\theta}{2}\right]_0^{\pi/4} + \dfrac{1}{4}\left[1 - \dfrac{1}{2}\right]$ $= \dfrac{1}{4}\left[\dfrac{\pi}{4} - \dfrac{1}{2}\right] + \dfrac{1}{8}$ $= \dfrac{\pi}{16}$
5(i)	Let the equation of the hyperbola be $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ When $y = 0$, $x = 40$. Hence $a = 40$. When $x = 50, \dfrac{50^2}{40^2} - \dfrac{{y_1}^2}{b^2} = 1$ $\Rightarrow \dfrac{{y_1}^2}{b^2} = \left(\dfrac{5}{4}\right)^2 - 1 = \dfrac{9}{16}$ $\Rightarrow y_1 = \dfrac{3b}{4}$ When $x = 58$, $\dfrac{{y_2}^2}{b^2} = \left(\dfrac{58}{40}\right)^2 - 1 = \dfrac{441}{400}$ $\Rightarrow y_2 = -\dfrac{21b}{20}$ Since height of tower = $y_1 + \lvert y_2 \rvert = 180$,

	$\frac{3b}{4}+\frac{21b}{20}=180$ $\frac{9b}{5}=180$ $b=100$ Cartesian equation is $\frac{x^2}{40^2}-\frac{y^2}{100^2}=1$
(ii)	When x = 50, $y=100\sqrt{\left(\frac{50}{40}\right)^2-1}=75$. When x = 58, $y=-100\sqrt{\left(\frac{58}{40}\right)^2-1}=-105$. Volume $=\pi\int_{-105}^{75}40^2\left(1+\frac{y^2}{100^2}\right)\,\mathrm{d}y$ $=1169426.449$ $=1170000\ \ \mathrm{m}^3$
(iii)	Equation of hyperbola: $\frac{x^2}{a^2}-\frac{(y-k)^2}{b^2}=1$. When x = 50, $\frac{(y-k)^2}{b^2}=\frac{50^2}{a^2}-1\Rightarrow y=b\sqrt{\frac{50^2}{a^2}-1}+k$ When x = 58, $y=-b\sqrt{\frac{58^2}{a^2}-1}+k$ Height $=b\sqrt{\frac{50^2}{a^2}-1}+b\sqrt{\frac{58^2}{a^2}-1}=180$ $\frac{b}{a}\left(\sqrt{50^2-a^2}+\sqrt{58^2-a^2}\right)=180$ Eccentricity $e=\frac{\sqrt{a^2+b^2}}{a}$ $=\sqrt{1+\frac{b^2}{a^2}}$ $=\sqrt{1+\frac{180^2}{\left(\sqrt{50^2-a^2}+\sqrt{58^2-a^2}\right)^2}}$



When e = 2.5, a = 36.863
When e = 4, a = 48.017
Range of possible values of a: $36.9 \le a \le 48.0$

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AE: $m^2 + 2m + 10 = 0$

$$\Rightarrow m = \frac{-2 \pm \sqrt{4-40}}{2} = -1 \pm 3\mathrm{i}$$

So CF: $\mathrm{f}(x) = \mathrm{e}^{-x}(A\cos 3x + B\sin 3x)$

PI: $\mathrm{f}(x) = \lambda \mathrm{e}^{-3x} + \mu$

$$\Rightarrow \mathrm{f}'(x) = -3\lambda \mathrm{e}^{-3x}, \mathrm{f}''(x) = 9\lambda \mathrm{e}^{-3x}$$

$$LHS = 9\lambda \mathrm{e}^{-3x} + 2\left(-3\lambda \mathrm{e}^{-3x}\right) + 10\left(\lambda \mathrm{e}^{-3x} + \mu\right)$$

$$= 13\lambda \mathrm{e}^{-3x} + 10\mu$$

$$\therefore \lambda = 1, \mu = 1$$

General solution: $\mathrm{f}(x) = \mathrm{e}^{-3x} + 1 + \mathrm{e}^{-x}(A\cos 3x + B\sin 3x)$

$$\mathrm{f}'(x) = -3\mathrm{e}^{-3x} + \left(-\mathrm{e}^{-x}\right)(A\cos 3x + B\sin 3x) + \mathrm{e}^{-x}(-3A\sin 3x + 3B\cos 3x)$$

$$= -3\mathrm{e}^{-3x} + \mathrm{e}^{-x}\left[(3B - A)\cos 3x - (B + 3A)\sin 3x\right]$$

Stationary point at $(0,5) \Rightarrow \mathrm{f}(0) = 5, \mathrm{f}'(0) = 0$

$$\therefore 5 = 1 + 1 + A, 0 = -3 + 3B - A$$

$$\Rightarrow A = 3, B = 2$$

Solution is $\mathrm{f}(x) = \mathrm{e}^{-3x} + 1 + \mathrm{e}^{-x}(3\cos 3x + 2\sin 3x)$

When $x \to \infty, \mathrm{e}^{-x} \to 0, \mathrm{e}^{-3x} \to 0 \therefore y \to 1$

7(i)	$x = a\left(\sin t + \dfrac{\sin 2t}{2}\right)$ $\dfrac{dx}{dt} = a(\cos t + \cos 2t)$ $= 2a\cos\dfrac{3t}{2}\cos\dfrac{t}{2} = 0$ $\cos\dfrac{3t}{2} = 0$ or $\cos\dfrac{t}{2} = 0$ $\dfrac{3t}{2} = \dfrac{\pi}{2}$ $\quad\dfrac{t}{2} = \dfrac{\pi}{2}$ $t = \dfrac{\pi}{3}$ $\quad t = \pi$ (reject)
	$x = a\left(1 + \cos\dfrac{\pi}{3}\right)\sin\dfrac{\pi}{3}$ $= a\dfrac{3\sqrt{3}}{4} = \dfrac{15}{2}$ $a = \dfrac{10}{\sqrt{3}}$
	When $t = 0$, $y = 2b$. When $t = \dfrac{3\pi}{4}$, $y = -\sqrt{2}b$. Height of balloon = $(2+\sqrt{2})b = 25$ $b = \dfrac{25}{2+\sqrt{2}}$
(ii)	$x = \dfrac{10}{\sqrt{3}}\left(\sin t + \dfrac{\sin 2t}{2}\right) \Rightarrow \dfrac{dx}{dt} = \dfrac{10}{\sqrt{3}}(\cos t + \cos 2t)$ $y = \dfrac{50}{2+\sqrt{2}}\cos t \Rightarrow \dfrac{dy}{dt} = -\dfrac{50}{2+\sqrt{2}}\sin t$ Surface area = $\int_0^{3\pi/4} 2\pi x\sqrt{\left(\dfrac{dy}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2}\ dt$ $= \dfrac{20\pi}{\sqrt{3}}\int_0^{3\pi/4}\left(\sin t + \dfrac{\sin 2t}{2}\right)\sqrt{\dfrac{10^2(\cos t + \cos 2t)^2}{3} + \dfrac{50^2\sin^2 t}{(2+\sqrt{2})^2}}\ dt$ = 932.568 m^2 $= \dfrac{200\pi}{\sqrt{3}}\int_0^{3\pi/4}\left(\sin t + \dfrac{\sin 2t}{2}\right)\sqrt{\dfrac{(\cos t + \cos 2t)^2}{3} + \dfrac{25\sin^2 t}{6+4\sqrt{2}}}\ dt$ = 932.6 m^2 (to 3 s.f.)

(iii)	Volume $= \pi \int_{3\pi/4}^{0} \frac{100}{3} \left(\sin t + \sin t \cos t\right)^2 \frac{-50}{2+\sqrt{2}} \sin t \; dt$ $= \frac{5000\pi}{3(2+\sqrt{2})} \int_{3\pi/4}^{0} \sin^2 t \left(1+\cos t\right)^2 (-\sin t) \; dt$ $= \frac{5000\pi}{3(2+\sqrt{2})} \int_{3\pi/4}^{0} (1-\cos^2 t)\left(1+2\cos t+\cos^2 t\right)(-\sin t) \; dt$ $= \frac{5000\pi}{3(2+\sqrt{2})} \int_{3\pi/4}^{0} (1+2\cos t-2\cos^3 t-\cos^4 t)(-\sin t) \; dt$ $= \frac{5000\pi}{3(2+\sqrt{2})} \left[\cos t + \cos^2 t - \frac{\cos^4 t}{2} - \frac{\cos^5 t}{5} \right]_{3\pi/4}^{0}$ $= \frac{5000\pi}{3(2+\sqrt{2})} \left[1+1-\frac{1}{2}-\frac{1}{5}-\left(-\frac{1}{\sqrt{2}}+\frac{1}{2}-\frac{1}{8}+\frac{1}{20\sqrt{2}}\right)\right]$ $= \frac{5000\pi}{3(2+\sqrt{2})} \left[\frac{37}{40} + \frac{19}{20\sqrt{2}} \right]$ $= \frac{250\pi}{3(2+\sqrt{2})} \left[\frac{37}{2} + \frac{19}{\sqrt{2}} \right]$
8(i)	$\mathbf{A} - I\lambda = \begin{pmatrix} 1-\lambda & c & 3 \\ 4 & 1-\lambda & 0 \\ 3 & 0 & 1-\lambda \end{pmatrix}$ $\det\left(\mathbf{A} - I\lambda\right)$ $= (1-\lambda)\left[(1-\lambda)(1-\lambda)\right] - 4c(1-\lambda) + 3\left[0-3(1-\lambda)\right]$ $= 0$ $(1-\lambda)^3 - 4c(1-\lambda) - 9(1-\lambda) = 0$ When $\lambda = 6$, $-125+20c+45=0$ $20c = 80$ $c = 4$ $(1-\lambda)^3 - 25(1-\lambda) = 0$ $(1-\lambda)\left[(1-\lambda)^2 - 25\right] = 0$ $-(\lambda-1)(\lambda-6)(\lambda+4) = 0$ $\lambda = 1,\ \lambda = 6$ or $\lambda = -4$ The remaining eigenvalues are 1 and -4.

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When $\lambda = 1$,

$$\mathbf{A} - I\lambda = \begin{pmatrix} 0 & 4 & 3 \\ 4 & 0 & 0 \\ 3 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 4 & 3 \\ 4 & 0 & 0 \\ 3 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

By G.C., $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 - \frac{3}{4}z \\ z \end{pmatrix} = \frac{1}{4}z\begin{pmatrix} 0 \\ -3 \\ 4 \end{pmatrix}$

Hence, $\begin{pmatrix} 0 \\ -3 \\ 4 \end{pmatrix}$ is an eigenvector.

When $\lambda = 6$,

$$\mathbf{A} - I\lambda = \begin{pmatrix} -5 & 4 & 3 \\ 4 & -5 & 0 \\ 3 & 0 & -5 \end{pmatrix}$$

$$\begin{pmatrix} -5 & 4 & 3 \\ 4 & -5 & 0 \\ 3 & 0 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

By G.C., $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{3}z\begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix}$

Hence, $\begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix}$ is an eigenvector.

When $\lambda = -4$,

$$\mathbf{A} - I\lambda = \begin{pmatrix} 5 & 4 & 3 \\ 4 & 5 & 0 \\ 3 & 0 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 5 & 4 & 3 \\ 4 & 5 & 0 \\ 3 & 0 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

By G.C., $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{3}z\begin{pmatrix} -5 \\ 4 \\ 3 \end{pmatrix}$

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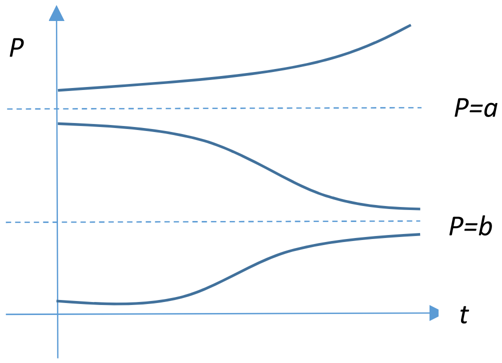
	Hence, $\begin{pmatrix} -5 \\ 4 \\ 3 \end{pmatrix}$ is an eigenvector. $\mathbf{A} = \begin{pmatrix} 0 & 5 & -5 \\ -3 & 4 & 4 \\ 4 & 3 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & -4 \end{pmatrix}\begin{pmatrix} 0 & 5 & -5 \\ -3 & 4 & 4 \\ 4 & 3 & 3 \end{pmatrix}^{-1}$
(iii)	$\mathbf{Y}' = \begin{pmatrix} \frac{dy_1}{dx} \\ \frac{dy_2}{dx} \\ \frac{dy_3}{dx} \end{pmatrix} = \begin{pmatrix} 1 & 4 & 3 \\ 4 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \mathbf{PDP}^{-1}\mathbf{Y}$ $\mathbf{U}' = \mathbf{P}^{-1}\mathbf{Y}' = \mathbf{P}^{-1}\mathbf{PDP}^{-1}\mathbf{Y} = \mathbf{DU}$ Hence, we have $\frac{du_1}{dx} = u_1$ $\frac{du_2}{dx} = 6u_2$ $\frac{du_3}{dx} = -4u_3$ $\int \frac{1}{u_1}\,du_1 = x + C$ $\ln\lvert u_1\rvert = x + C$ $u_1 = \pm e^{x+C} = Ae^{x}$ Similarly, $u_2 = Be^{6x}, u_3 = Ce^{-4x}$. $\mathbf{U} = \mathbf{P}^{-1}\mathbf{Y} \Rightarrow \mathbf{Y} = \mathbf{PU}$ $Y = \begin{pmatrix} 0 & 5 & -5 \\ -3 & 4 & 4 \\ 4 & 3 & 3 \end{pmatrix}\begin{pmatrix} Ae^{x} \\ Be^{6x} \\ \mathrm{C}e^{-4x} \end{pmatrix}$ $= \begin{pmatrix} 5Be^{6x} - 5\mathrm{C}e^{-4x} \\ -3Ae^{x} + 4Be^{6x} + 4\mathrm{C}e^{-4x} \\ 4Ae^{x} + 3Be^{6x} + 3\mathrm{C}e^{-4x} \end{pmatrix}$

9(a)(i)	$\frac{dP}{dt}=0$ $\Rightarrow kP\left(1-\frac{P}{a}\right)-H=0$ $kP(a-P)-aH=0$ $kP^2-akP+aH=0$ Two equilibrium population values $\Rightarrow \Delta>0$ $\therefore (ak)^2-4akH>0$ $ak(ak-4H)>0$ $\because a,k>0 \therefore ak-4H>0$ Condition of $H: H<\frac{ak}{4}$
(ii)	$H=\frac{ak}{4}\Rightarrow$ one equilibrium population $\frac{dP}{dt}=kP\left(1-\frac{P}{a}\right)-\frac{ak}{4}=-\frac{k}{a}\left(P-\frac{a}{2}\right)^2$ So the population will decrease and stabilize at $\frac{a}{2}$
(b)(i)	(a)  For $P_0>a$, population grows infinitely For $b<P_0<a$, population decreases and stabilizes at b For $P_0<b$, population increases and stabilizes at b
(ii)	$P_0\le a$
(iii)	a and b are the two equilibrium population values
(iv)	$\frac{dP}{dt}=\frac{8}{5}P\left(1-\frac{P}{60}\right)\left(1-\frac{P}{20}\right)$ By Euler Method, $\frac{1}{2}P_0=P_0+\frac{8}{5}P_0\left(1-\frac{P_0}{60}\right)\left(1-\frac{P_0}{20}\right)$ $0=\frac{1}{2}+\frac{1}{750}(P_0-60)(P_0-20)$ $\Rightarrow P_0^2-80P_0+1575=0$ $\Rightarrow P_0=35$ or 45

	If $P_0 = 35$, the population will decrease and stabilize at 20, it will not reach $\frac{35}{2} = 17.5$, so we reject 35 If $P_0 = 45$, after one year the population decrease to 22.5, which is still above 20. So $P_0 = 45$, the population will stabilize at 20.