NATIONAL JUNIOR COLLEGE
SENIOR HIGH 2 Preliminary Examination
Higher 2

FURTHER MATHEMATICS 9649 / 01

28 August 2017

3 hours

Additional Materials: Answer Paper
List of Formulae (MF26)
Cover Sheet

READ THESE INSTRUCTIONS FIRST

Write your name, registration number, subject tutorial group, on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use an HB pencil for diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in the brackets [ ] at the end of each question or part question.

1 Prove by induction that $7^{2n}+2^{3n-3}\cdot 3^{n-1}$ is divisible by 25 for every positive integer n. [5]

2 Use De Moivre's Theorem to show that

$$\tan 5\theta = \frac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1-10\tan^2\theta+5\tan^4\theta}.$$ [3]

Hence find the exact value of $\tan^2\frac{\pi}{10}$. [3]

3 A curve C has polar equation $r=\dfrac{3}{2+\cos\theta}$.

Find a Cartesian equation of C, showing your steps clearly. Describe the curve completely.

The curve C is the locus of a variable point P. Prove that for some fixed point F with coordinates $(c, 0)$ and for some fixed line d with equation $x=k$, the ratio $\dfrac{PF}{Pd}$ is a constant. [6]

4 It is given that the general solution of the differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+a\frac{\mathrm{d}x}{\mathrm{d}t}+bx=0,$$

where a and b are real constants, is

$x=(At+B)\mathrm{e}^{-0.5t}$, where A and B are arbitrary constants.

(i) Find the values of a and b. [2]

(ii) Hence, find the particular solution of

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+a\frac{\mathrm{d}x}{\mathrm{d}t}+bx=\mathrm{e}^{-0.5t},$$

given that $x=\dfrac{\mathrm{d}x}{\mathrm{d}t}=1$ when $t=0$. [5]

`[Turn Over

5 Solve the equation $z^3 = -4\sqrt{3} + 4\text{i}$, expressing the roots in the form $r\text{e}^{\text{i}\theta}$, where $r > 0$ and $-\pi < \theta \leq \pi$. [3]

Let α, β and γ be these roots.

(i) Show that $\alpha + \beta + \gamma = 0$. [2]

(ii) Using the identity $zz^* = |z|^2$ or otherwise, show that

$$|w-\alpha|^2 + |w-\beta|^2 + |w-\gamma|^2$$

is a constant for any complex number *w* with modulus 2. [3]

6 **(a)** Find the exact length of arc *PQ*, where $P\left(\frac{2}{3},1\right)$ and $Q\left(\frac{14}{3},3\right)$ are two points that lie on the curve $6xy = y^4 + 3$. [5]

(b) The area of the region bounded by the curve $y = \frac{4-x^2}{4+x^2}$, the lines $x = 0$ and $x = 4$, and the *x*-axis is denoted by *A*. Use the trapezium rule with 7 ordinates to estimate the value of *A*. [3]

Express *A* in the form $\int_a^b \text{g}(x)\,\text{d}x$. Hence or otherwise, determine whether the value obtained by the trapezium rule is an overestimate or underestimate of *A*. [2]

7 The matrix **A** is given by

$$\mathbf{A} = \begin{pmatrix} a & -3 & -3 \\ 2 & a-5 & -3 \\ -2 & 8 & a+6 \end{pmatrix},$$

where *a* is a real constant.

Given that **A** has an eigenvalue of 2, find the possible values of *a* exactly. [3]

For the negative value of *a* that you have obtained, determine all the eigenvalues and corresponding eigenvectors of

(i) **A**, [5]

(ii) 2**A** – 3**I**, where **I** is the 3 × 3 identity matrix. [2]

8 It is given that the equation $5x-18x^{\frac{2}{3}}-36x^{\frac{1}{3}}-24=0$ has only one real root α.

(i) Without finding the value of α, determine the integer k such that α lies in $\left[50k, 50(k+1)\right]$. [2]

An iteration formula (*) for finding α is given by

$$x_n=\frac{1}{8}\left[3x_{n-1}+18\left(x_{n-1}\right)^{\frac{2}{3}}+36\left(x_{n-1}\right)^{\frac{1}{3}}+24\right],\ n\geq 1.$$

(ii) Taking the upper bound in part **(i)** as the initial value x_0, show that the value of α, correct to 1 decimal place, <u>cannot</u> be obtained on the 15th iteration. [2]

(iii) By using the substitution $x_n=y_n{}^3$, show that the iteration formula (*) can be reduced to a first order linear recurrence relation in terms of y_n and y_{n-1}. [2]

(iv) By considering the general solution to the recurrence relation in part **(iii)**, explain why $\{x_n\}$ converges for any initial value x_0.

Deduce the exact value of α. [4]

9 In $\mathbb{R}^2$, a horizontal shear is a mapping that takes generic point with position vector $\begin{pmatrix} x \\ y \end{pmatrix}$ to the point with position vector $\begin{pmatrix} x+my \\ y \end{pmatrix}$, where m is a fixed real parameter called the *shear factor*.

Show that every horizontal shear mapping in $\mathbb{R}^2$ is a linear transformation. State the matrix that represents the horizontal shear with shear factor m. [3]

The circle $x^2+y^2=1$ is transformed by a horizontal shear with shear factor λ. The resultant curve is denoted by C.

(i) Find a Cartesian equation of the resultant curve in terms of λ. [2]

(ii) Show that the area enclosed by C is independent of λ. [5]

10 Diagram I shows a hyperbola with foci F_1 and F_2.

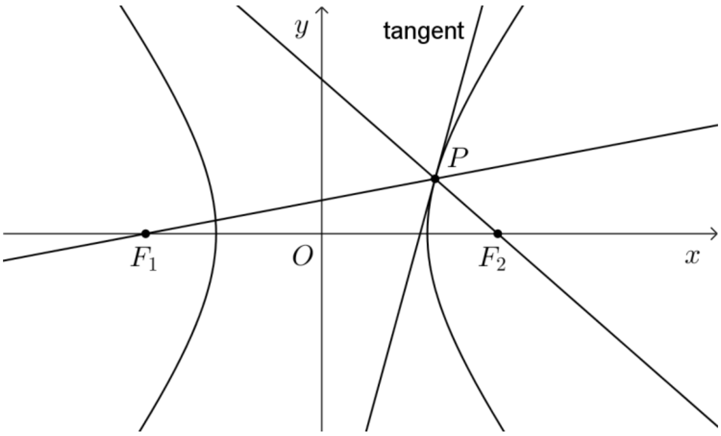

Diagram I

Show that for any point P lying on the hyperbola in the first quadrant, the lines PF_1 and PF_2 make an equal angle with the tangent to the hyperbola at P. [6]

For this property, a hyperbolic mirror is often used as the secondary mirror in a telescope. Diagram II shows the side view of a cylindrical telescope with the following specifications:

- length (AD or BC): 9m
- diameter of its aperture (CD): 2m
- focal length of its secondary hyperbolic mirror (F_1F_2): 10m

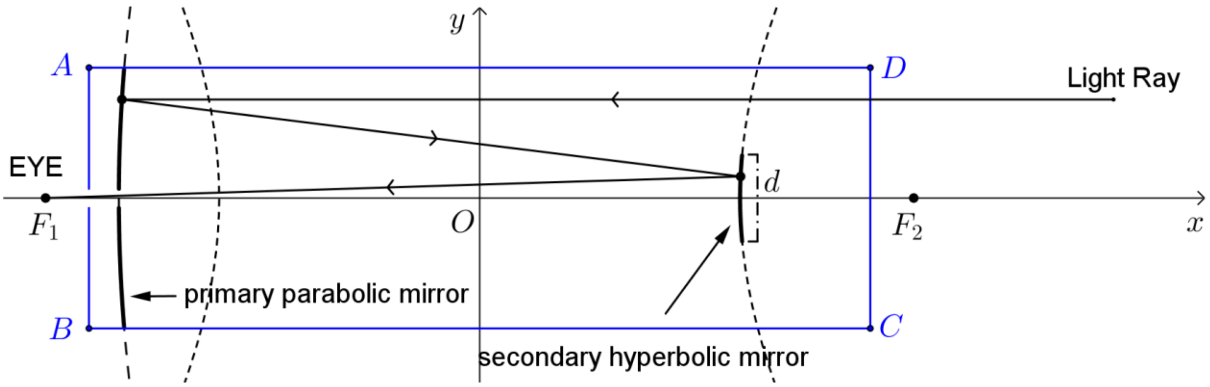

Diagram II

For a light ray parallel to the x-axis and passing through the aperture, it will first be reflected by the primary parabolic mirror with focus F_2, then by the secondary hyperbolic mirror before being observed at F_1.

It is given that the origin O is both the centre of the rectangle $ABCD$ and the hyperbola.

(i) Show that the equation of the parabola modelling the primary mirror can be expressed as $y^2 = 4c(x+c-5)$. State the largest possible value of c. [3]

(ii) It is further given that $c = 9$ and the eccentricity of the hyperbolic mirror is $\frac{5}{3}$. Find the minimum diameter d of the secondary mirror required to capture all the possible light rays reflected by the primary mirror, giving your answer to 3 decimal places. [5]

11 In a conservation project, the number of seals living in a particular stretch of coastline is recorded every year. It was found that the population, $P(t)$, of these seals grew by 20% one year after the start of the project. Assuming that the growth is governed by the exponential growth model, write down a differential equation and solve it to find the time, t (in years), taken for the population to reach twice its original size. [4]

Taking into consideration the competition for limited food resources near the coastline, the above model is replaced by the logistic growth model with equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = mP - mnP^2,$$

where m and n are positive constants.

(i) Explain the significance of the two constants in this model. [2]

Assume for the remainder of this question that $m = 0.2$, $n = 0.002$ and $P(0) = 100$.

(ii) Find an estimated number of the seals one year after the project started. [5]

Suppose a viral epidemic swept through the seal population, killing N seals per month.

(iii) Write down the modified differential equation, and find the least number of seals to be killed by the virus in a month that will eventually result in a complete elimination of the seal population. [3]

--- END OF PAPER ---