H2 Mathematics 2017 Prelim Exam Paper 1 Question
Answer all questions [100 marks].

1

$\mathrm{i}z + w = 2 + \mathrm{i} - - - - - (1)$

$2w - 1 - \mathrm{i}z = \dfrac{20}{2-\mathrm{i}} - - - (2)$

Let $w = 2 + \mathrm{i} - \mathrm{i}z - - - (3)$

Substitute eq (3) into eq (2)

$2(2 + \mathrm{i} - \mathrm{i}z) - z - \mathrm{i}z = 8 + 4\mathrm{i}$

$4 + 2\mathrm{i} - 3\mathrm{i}z - z = 8 + 4\mathrm{i} - - - - (5)$

Let $z = a + b\mathrm{i}$

Substitute $z = a + b\mathrm{i}$ into eq(5)

$4 + 2\mathrm{i} - 3\mathrm{i}(a + b\mathrm{i}) - (a + b\mathrm{i}) = 8 + 4\mathrm{i}$

$4 + 2\mathrm{i} - 3a\mathrm{i} + 3b - a - b\mathrm{i} = 8 + 4\mathrm{i}$

Comparing real and imaginary parts:

$4 + 3b - a = 8(\text{real parts}) - - - (6)$

$2 - 3a - b = 4(\text{imaginary parts}) - - - (7)$

Eq$(6) \times 3$ $-$ eq(7)

$10 + 10b = 20$

$10b = 10$

$b = 1$

Since $b = 1$, $4 + 3(1) - a = 8 \Rightarrow a = -1$

$\therefore z = -1 + \mathrm{i}$

Substituting $z = -1 + \mathrm{i}$ into eq(3) to solve for w

$w = 2 + \mathrm{i} + \mathrm{i} + 1 = 3 + 2\mathrm{i}$

Answer: $z = -1 + \mathrm{i}$ and $w = 3 + 2\mathrm{i}$

2

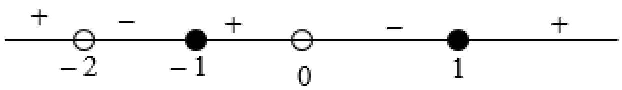
$\dfrac{2x^2 + 2x - 1}{x^2 + 2x} \le 1$

$\dfrac{2x^2 + 2x - 1}{x^2 + 2x} - 1 \le 0$

$\dfrac{2x^2 + 2x - 1 - x^2 - 2x}{x^2 + 2x} \le 0$

$\Rightarrow \dfrac{x^2 - 1}{x(x+2)} \le 0$

$\Rightarrow \dfrac{(x+1)(x-1)}{x(x+2)} \le 0$



Thus, $-2 < x \le -1$ or $0 < x \le 1$

Replacing x with $|x|$,

$-2 < |x| \le -1$ or $0 < |x| \le 1$

$-2 < |x| \le -1 \implies$ no solution

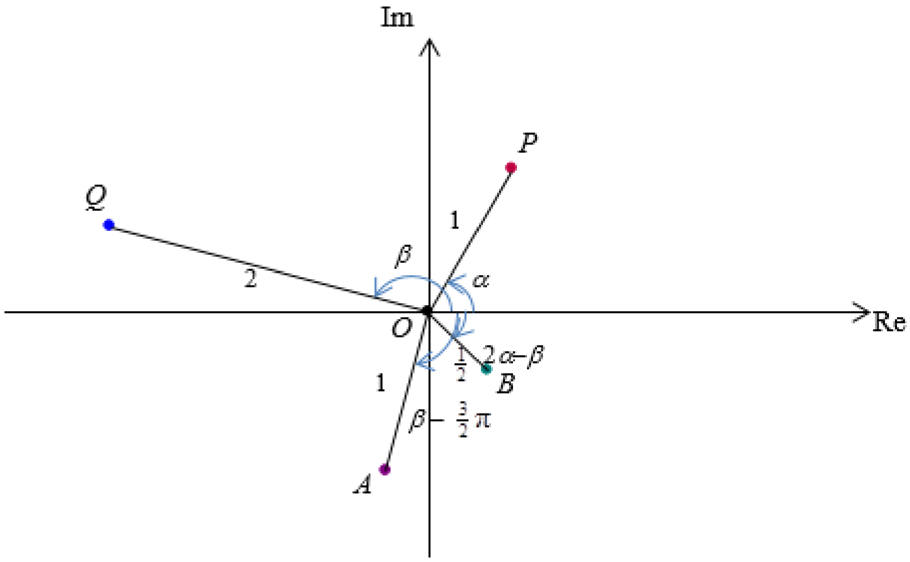
For $0 < |x| \le 1$,

$0 < |x|$ and $|x| \le 1$

$x \in \mathbb{R}, x \neq 0$ and $-1 \le x \le 1$

Thus, range of values: $-1 \le x \le 1,\ x \neq 0$

3



$$\frac{\mathrm{i}}{2} z_2 = \left(\frac{1}{2}\mathrm{e}^{\mathrm{i}\frac{\pi}{2}}\right)\left(2\mathrm{e}^{\mathrm{i}\beta}\right) = \mathrm{e}^{\mathrm{i}\left(\beta + \frac{\pi}{2}\right)}$$

Modulus = 1

Argument = $\beta + \frac{\pi}{2} - 2\pi = \beta - \frac{3\pi}{2}$

(i) Point A correctly plotted

$$\frac{z_1^{\,2}}{z_2} = \frac{\mathrm{e}^{\mathrm{i}\alpha}\mathrm{e}^{\mathrm{i}\alpha}}{2\mathrm{e}^{\mathrm{i}\beta}} = \frac{1}{2}\mathrm{e}^{\mathrm{i}(2\alpha - \beta)}$$

Modulus = $\frac{1}{2}$

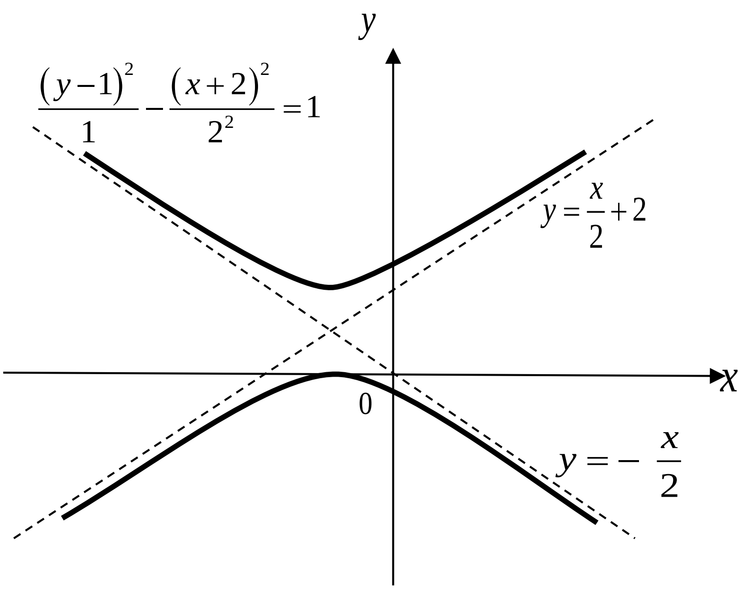
Argument = $2\alpha - \beta$

(ii) Point B correctly plotted

(b) $(z_2)^n = 2^n \mathrm{e}^{\mathrm{i}\frac{11\pi}{12}n}$

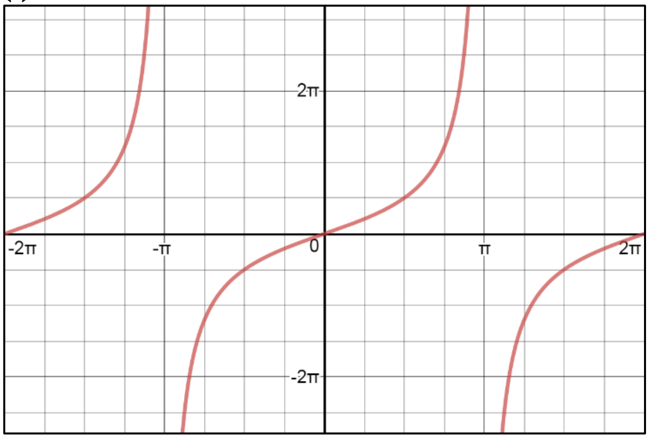
Since the point lies on the negative real axis, $\arg(z_2)^n = (2k + 1)\pi$ for $k \in \mathbb{Z}$.

$\therefore \frac{11}{12} n\pi = (2k + 1)\pi$

		$\Rightarrow n=\frac{12}{11}(2k+1)$ $\Rightarrow$ Smallest n required = 12
4	**(i)**	$-x^2-4x+\left(4y^2-8y-4\right)=0$ For values that y cannot take, there are no real solutions for x and discriminant<0. Therefore, $(-4)^2-4(-1)\left(4y^2-8y-4\right)<0$ $16+16y^2-32y-16<0$ $y^2-2y<0$ $y(y-2)<0$ $\therefore 0<y<2$ Set of values that y cannot take is $\{y\in\mathbb{R}:0<y<2\}$.
	(ii)	$4y^2-8y-x^2-4x-4=0$ $4\left[(y-1)^2-1\right]-\left[(x+2)^2-4\right]-4=0$ $4(y-1)^2-4-(x+2)^2=0$ $\frac{(y-1)^2}{1}-\frac{(x+2)^2}{2^2}=1$ 

5

(i)



$y = \mathrm{f}(x)$

The horizontal line y = 1 cuts the graph of y = f(x) at **2 points**. Thus, f(x) is not a one-one function and the inverse of $\mathrm{f}(x)$ does not exist for the domain $[-2\pi, 2\pi\ ]$.

OR

Any horizontal line $y = k$ $(k \in \mathbb{R})$ cuts the graph at more than one point. Thus, $\mathrm{f}(x)$ is not a one-one function and the inverse of $\mathrm{f}(x)$ does not exist for the domain $[-2\pi, 2\pi\ ]$.

(ii)

$a = \pi$

To make x the subject of y

$$y = \frac{\pi}{2}\tan\left(\frac{x}{2}\right)$$

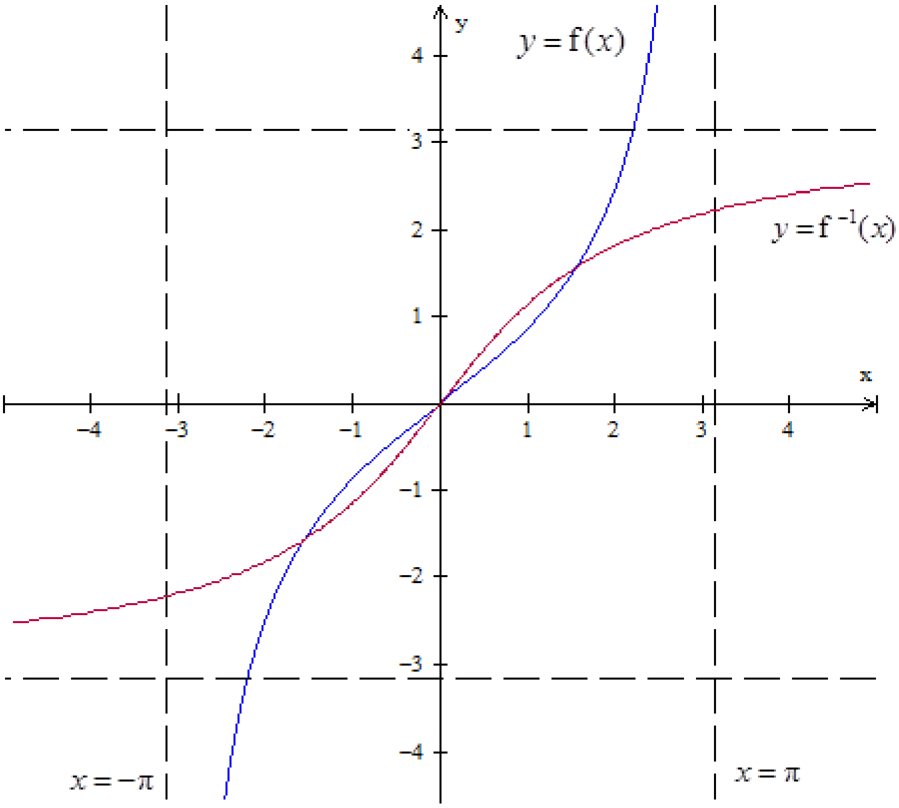
$$\frac{2y}{\pi} = \tan\left(\frac{x}{2}\right)$$

$$\tan^{-1}\left(\frac{2y}{\pi}\right) = \frac{x}{2}$$

$$\Rightarrow x = 2\tan^{-1}\left(\frac{2y}{\pi}\right)$$

$$\mathrm{f}^{-1} : x \mapsto 2\tan^{-1}\left(\frac{2x}{\pi}\right), \quad x \in R.$$

(iii)



The line required is $y = x$.

(iv) $f\left(\frac{\pi}{2}\right) = \frac{\pi}{2}\tan\left(\frac{\pi}{4}\right) = \frac{\pi}{2}$

Thus, $x = \frac{\pi}{2}$ is a root of the equation $x = f(x)$.

Since the graphs of $y = f(x)$ and $y = f^{-1}(x)$ intersect along the line $y = x$, and since $x = \frac{\pi}{2}$ is a root of the equation $x = f(x)$, thus, the graphs of $y = f(x)$ and $y = f^{-1}(x)$ must also intersect at the point $x = \frac{\pi}{2}$.

6

(i) $\mathbf{a}.\mathbf{b} = |a||b|\cos\theta \Rightarrow |1|\left|\sqrt{2}\right|\cos\theta$

$\mathbf{a}.\mathbf{b} = 1 \quad \therefore \cos\theta = \dfrac{1}{\sqrt{2}} \Rightarrow \theta = 45°$ (by inspection)

(ii) $\overline{DE} = \overline{OE} - \overline{OD} = 2\mathbf{a} + \lambda(\mathbf{a}+2\mathbf{b}) - (2\mathbf{a} - \mathbf{b}) = \mathbf{b} + \lambda(\mathbf{a}+2\mathbf{b}),\ \lambda \in R$

To find the square of the distance DE

$DE^2 = [\mathbf{b} + \lambda(\mathbf{a}+2\mathbf{b})].[\mathbf{b} + \lambda(\mathbf{a}+2\mathbf{b})]$

$= \mathbf{b}.\mathbf{b} + \lambda^2(\mathbf{a}+2\mathbf{b}).(\mathbf{a}+2\mathbf{b}) + 2\lambda\mathbf{b}.(\mathbf{a}+2\mathbf{b})$

$= \mathbf{b}.\mathbf{b} + \lambda^2(\mathbf{a}.\mathbf{a} + 4\mathbf{a}.\mathbf{b} + 4\mathbf{b}.\mathbf{b}) + 2\lambda(\mathbf{b}.\mathbf{a} + 2\mathbf{b}.\mathbf{b})$

$= 2 + \lambda^2(1 + 4(1) + 4(2)) + 2\lambda(1 + 2(2))$ as $\mathbf{a}.\mathbf{a} = 1$, $\mathbf{b}.\mathbf{b} = 2$ and $\mathbf{a}.\mathbf{b} = 1$

$= 2 + 13\lambda^2 + 10\lambda$

$= 13\lambda^2 + 10\lambda + 2$

(iii) **<u>Method One:</u>**

$DE^2 = 13\left[\lambda^2 + \dfrac{10}{13}\lambda\right] + 2$

$= 13\left(\lambda + \dfrac{10}{26}\right)^2 + 2 - \dfrac{25}{13} = 13\left(\lambda + \dfrac{5}{13}\right)^2 + \dfrac{1}{13}$

$DE = \sqrt{13\left(\lambda + \dfrac{5}{13}\right)^2 + \dfrac{1}{13}}$

The perpendicular distance from E to l occurs when D is closest to l, that is when DE is minimum or $\lambda = -\dfrac{5}{13}$.

Exact shortest distance from D to l is $\dfrac{1}{\sqrt{13}}$ units.

	Method Two: DE is minimum when DE^2 is minimum: $\frac{\mathrm{d}}{\mathrm{d}x}(DE^2) = 26\lambda + 10$ To find stationary point: When $\frac{\mathrm{d}}{\mathrm{d}x}(DE^2) = 0,\ 26\lambda + 10 = 0$ $\therefore \lambda = -\frac{5}{13}$. Since DE^2 is quadratic and coefficient of $\lambda^2 > 0$, DE^2 is minimum at $\lambda = -\frac{5}{13}$ $\therefore$ perpendicular distance from D to l occur when $\lambda = -\frac{5}{13}$. $DE^2 = 13\lambda^2 + 10\lambda + 2 = 13\left(-\frac{5}{13}\right)^2 + 10\left(-\frac{5}{13}\right) + 2 = \frac{1}{13}$ Exact shortest distance from D to l is $\frac{1}{\sqrt{13}}$ units. (iv) Let F be the foot of the perpendicular from D to l. $\overrightarrow{OF} = 2\mathbf{a} - \frac{5}{13}(\mathbf{a} + 2\mathbf{b}) = \frac{21}{13}\mathbf{a} - \frac{10}{13}\mathbf{b}$
7	**(a)** $\sec x = \frac{1}{\cos x}$ $= \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + \ldots\right)^{-1}$ $= 1 + (-1)\left[-\frac{1}{2}x^2 + \frac{1}{24}x^4\right] + \frac{(-1)(-2)}{2!}\left[-\frac{1}{2}x^2 + \frac{1}{24}x^4\right]^2 + \ldots$ $= 1 + \frac{1}{2}x^2 - \frac{1}{24}x^4 + \frac{1}{4}x^4 + \ldots$ $= 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4$ (up to x^4) (shown) $\ln(\sec x) \approx \ln\left[1 + \frac{1}{2}x^2 + \frac{5}{24}x^4\right]$ $= \left[\frac{1}{2}x^2 + \frac{5}{24}x^4 + \ldots\right] - \frac{1}{2}\left[\frac{1}{2}x^2 + \frac{5}{24}x^4 + \ldots\right]^2$

	$=\frac{1}{2}x^2+\frac{5}{24}x^4 \quad -\frac{1}{2}\left(\frac{1}{4}x^4 \quad\right)+\ldots$ $=\frac{1}{2}x^2+\frac{1}{12}x^4$ Thus $A=\frac{1}{12}$ **(b)(i)** $\left(1+x^2\right)\frac{\mathrm{d}y}{\mathrm{d}x}=1+y$ $\left(1+x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+2x\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}x}$ $\left(1+x^2\right)\frac{\mathrm{d}^3y}{\mathrm{d}x^3}+2x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\left(1-2x\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-2\frac{\mathrm{d}y}{\mathrm{d}x}$ At $x=0$, $y=0$ $\frac{\mathrm{d}y}{\mathrm{d}x}=1,\ \frac{\mathrm{d}^2y}{\mathrm{d}x^2}=1,\ \frac{\mathrm{d}^3y}{\mathrm{d}x^3}=-1$ Thus, $y=x+\frac{x^2}{2!}-\frac{x^3}{3!}+\ldots$ i.e. $y=x+\frac{x^2}{2}-\frac{x^3}{6}+\ldots$ **(ii)** $\ln\left(1+y\right)=\tan^{-1}x\Rightarrow\frac{1}{1+y}\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{1+x^2}$ $\therefore\left(1+x^2\right)\frac{\mathrm{d}y}{\mathrm{d}x}=1+y$ so condition **(A)** is satisfied. At $x=0$, $\ln\left(1+y\right)=\tan^{-1}0=0\Rightarrow1+y=\mathrm{e}^0$ $\therefore y=0$ **(iii)** $\int_0^{\frac{1}{2}}\left(\mathrm{e}^{\tan^{-1}x}-1\right)\mathrm{d}x\approx\int_0^{\frac{1}{2}}\left(x+\frac{x^2}{2}-\frac{x^3}{6}\right)\mathrm{d}x=\frac{55}{384}$
8	**(a)**Let a denote the first term of the geometric progression. Likewise, let b and d denote the first term and common difference of the arithmetic progression. $\therefore$ $ar^4=b+6d$ …Eq(1) $ar^8=b+24d$ …Eq(2) $ar^{10}=b+48d$ …Eq(3) Eq(2) – Eq(1): $ar^8-ar^4=18d$ …Eq(4) Eq(3) – Eq(2): $ar^{10}-ar^8=24d$ …Eq(5) Eq(5)/Eq(4): $\frac{ar^8\left(r^2-1\right)}{ar^4\left(r^4-1\right)}=\frac{24d}{18d}$ $\frac{r^4}{r^2+1}=\frac{4}{3}$ $3r^4=4r^2+4$ (Shown)

From GC, $r = \pm\sqrt{2}$ so $|r| > 1$
Hence, the geometric progression is not convergent.

(b)
Let a be the 1st term and r be the common ratio of the G.P.

$$S_8 = \frac{A\left(1-r^8\right)}{1-r} = 72\pi \qquad \text{-----}\ (1)$$

$$S_{odd} - S_{even} = 10\pi$$

$$\Rightarrow \frac{A\left(1-(r^2)^4\right)}{1-r^2} - \frac{Ar\left(1-(r^2)^4\right)}{1-r^2} = 10\pi$$

$$\frac{A\left(1-r^8\right)}{(1-r)(1+r)}\left[1-r\right] = 10\pi \quad \text{-----}\ (2)$$

(1) ÷ (2):

$$\frac{1-r}{1+r} = \frac{10}{72}$$

$$72 - 72r = 10 + 10r$$

$$82r = 62$$

$$r = 0.75610$$

Substituting into equation (1), A = 61.8 (to 3 s.f.)
Let the production level in the first year be a.

Total production of the coal mine = $\dfrac{a}{1-0.96}$ = $25a$

Thus, the total production of the coal mine can never exceed 25 times the production in the first year.

9

(a) Given $u = 2x+3 \Rightarrow \dfrac{\mathrm{d}u}{\mathrm{d}x} = 2$

$$\int \frac{x}{(2x+3)^3}\,\mathrm{d}x = \int \frac{\frac{1}{2}(u-3)}{u^3}\cdot\frac{1}{2}\,\mathrm{d}u$$

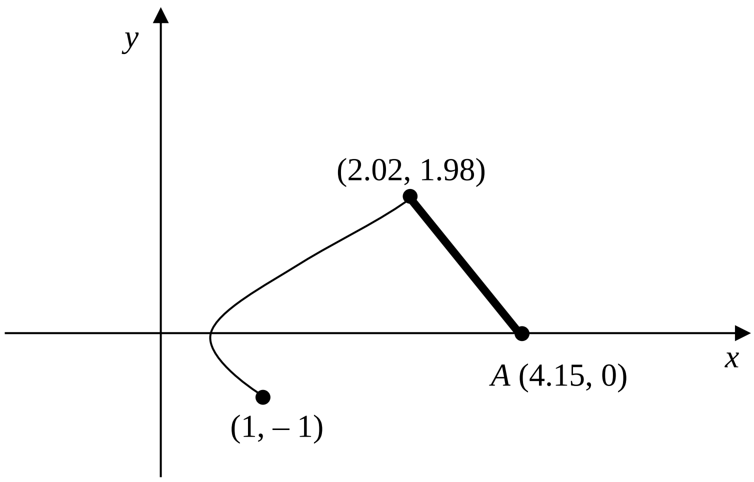
$$= \frac{1}{4}\int\left[u^{-2} - 3u^{-3}\right]\mathrm{d}u$$

$$= \frac{1}{4}\left[-u^{-1} + \frac{3}{2}u^{-2}\right] + C$$

$$= -\frac{1}{4(2x+3)} + \frac{3}{8(2x+3)^2} + C$$

$$= \frac{-2(2x+3)+3}{8(2x+3)^2} + C$$

	$$= -\frac{4x+3}{8(2x+3)^2} + C$$ $P = 4$, $Q = 3$ and $R = 8$ $$\int \frac{\ln(4x+3)^x}{(2x+3)^3}\ \mathrm{d}x$$ $$= \int \frac{x}{(2x+3)^3} \cdot \ln(4x+3)\ \mathrm{d}x \qquad \text{Let } \frac{\mathrm{d}v}{\mathrm{d}x} = \frac{x}{(2x+3)^3}, u = \ln(4x+3)$$ $$= -\frac{(4x+3)\ln(4x+3)}{8(2x+3)^2} - \int -\frac{(4x+3)}{8(2x+3)^2} \cdot \frac{4}{(4x+3)}\mathrm{d}x + C$$ $$= -\frac{(4x+3)\ln(4x+3)}{8(2x+3)^2} + \frac{1}{2}\int (2x+3)^{-2}\ \mathrm{d}x + C$$ $$= -\frac{(4x+3)\ln(4x+3)}{8(2x+3)^2} + \frac{1}{2}(2x+3)^{-1}\left(-\frac{1}{2}\right) + C$$ $$= -\frac{(4x+3)\ln(4x+3)}{8(2x+3)^2} - \frac{1}{4(2x+3)} + C$$ $$= -\frac{(4x+3)\ln(4x+3) + 2(2x+3)}{8(2x+3)^2} + C$$ (b) $$\int \sin 4x \cos 6x\ \mathrm{d}x$$ $$= \frac{1}{2}\int \sin 10x + \sin(-2x)\ \mathrm{d}x$$ $$= \frac{1}{2}\int \sin 10x - \sin 2x\ \mathrm{d}x$$ $$= \frac{1}{2}\left[-\frac{1}{10}\cos 10x + \frac{1}{2}\cos 2x\right] + C$$ $$= -\frac{1}{20}\cos 10x + \frac{1}{4}\cos 2x + C$$ $$\int e^x \sin 4e^x \cos 6e^x\ \mathrm{d}x$$ $$= -\frac{1}{20}\cos 10e^x + \frac{1}{4}\cos 2e^x + C$$
10	**(i)** At the original position, $t = 0$ $x = 0 + \mathrm{e}^0 = 1$ and $y = 0 - \mathrm{e}^0 = -1$ Thus the coordinates are $(1, -1)$. **(ii)** As t tends to infinity, $\mathrm{e}^{-2t} \to 0$ so $x \to t$ and $y \to t$ Thus, the path of the particle **approaches the line $y = x$**

	(iii) $\dfrac{dy}{dt}=1+2e^{-2t}$ and $\dfrac{dx}{dt}=1-2e^{-2t}$ $\dfrac{dy}{dx}=\dfrac{1+2e^{-2t}}{1-2e^{-2t}}$ At t = 2, $x=2+e^{-4}=2.01832$, $y=2-e^{-4}=1.98168$ and $\dfrac{dy}{dx}=\dfrac{1+2e^{-4}}{1-2e^{-4}}$ Gradient of normal = $\dfrac{2e^{-4}-1}{1+2e^{-4}}=-0.92933$ Thus, an equation for C_2 is $y-1.98168=-0.92933\left(x-2.01832\right)$ i.e. $y=-0.92933x+3.85737$ i.e. $y=-0.929x+3.857$ (correct to 3 d.p.) **(iv)** At point A, y = 0 $0=-0.929x+3.857 \Rightarrow x=4.15178$ Coordinates of A are (4.15, 0) Sketch of motion of particle:  **(v)** Consider the curve C_1 when y = 0, $t=e^{-2t}$ and solving by GC, t = 0.4263 Thus, x = 0.85261 Required area $=\int_{0.852}^{2.02} y\ dx+\int_{2.02}^{4.15}\left(-0.929x+3.857\right)\ dx$ $=\int_{0.4263}^{2}\left(t-e^{-2t}\right)\left(1-2e^{-2t}\right)dt+\int_{2.02}^{4.15}\left(-0.929x+3.857\right)\ dx$ = 3.5576 units^2 = 3.56 units^2
11	**(i)** Perimeter of cross-sectional area $=100=\left(2a+4b\right)+\dfrac{1}{2}\left(2\pi a\right)$ $\Rightarrow 100=4b+a\left(\pi+2\right)$

$$\Rightarrow b = \frac{100 - a(\pi + 2)}{4}$$

(ii) $S = (2a)(2b) + \frac{1}{2}(\pi a^2)$

$$= 4a\left[\frac{100 - a(\pi+2)}{4}\right] + \frac{\pi}{2}a^2$$

$$= 100a - a^2(\pi+2) + \frac{\pi}{2}a^2$$

$$= 100a - \frac{a^2}{2}(2\pi + 4 - \pi)$$

$$= 100a - \frac{a^2}{2}(\pi + 4) \quad \text{(shown)}$$

Note that, $a + b = a + \frac{100 - a(\pi+2)}{4}$

$$= \frac{4a + 100 - a(\pi+2)}{4}$$

$$= \frac{1}{4}\left[100 + a(2-\pi)\right]$$

$$V = \left[100a - \frac{a^2}{2}(\pi+4)\right]2(a+b)$$

$$= \left[100a - \frac{a^2}{2}(\pi+4)\right] \cdot \frac{2}{4}\left[100 + a(2-\pi)\right]$$

$$= \frac{a}{2}\left[100 - \frac{a}{2}(\pi+4)\right] \cdot \left[100 + a(2-\pi)\right]$$

$$= 5000a - 75\pi a - \frac{a^3}{4}(\pi^2 + 2\pi - 8)$$

(iii) $\frac{\mathrm{d}V}{\mathrm{d}a} = 5000 - 150\pi a - \frac{3}{4}a^2(\pi^2 + 2\pi - 8)$

When $\frac{\mathrm{d}V}{\mathrm{d}a} = 0$, using the GC, $a = 12.70471$ or $a = 64.36321$

For $a = 12.70471$			
A	a^-	a	a^+
Sign	–	0	+
$\frac{\mathrm{d}V}{\mathrm{d}a}$	/	—	\

For $a = 64.36321$			
a	a^-	a	a^+
sign	–	0	+
$\frac{\mathrm{d}V}{\mathrm{d}a}$	\	—	/

Thus when $a = 12.70471 = 12.7$ (3 s.f.), volume is greatest.
Using the GC, greatest volume is 29671.95154=29671.95 cm^3.

− End Of Paper −