Raffles Institution

H2 Further Mathematics 2017 Preliminary Exam Paper 2 Question

1

(i) By sketching the graphs of $y=\tan^{-1}\left(\frac{x}{5}\right)$ and $y=\sin^{-1}\left(\frac{x}{6}\right)$ on the same diagram, show that the equation $\tan^{-1}\left(\frac{x}{5}\right)=\sin^{-1}\left(\frac{x}{6}\right)$ has a positive real root, α. [2]

(ii) Find the integer N such that $N<\alpha<N+1$. [2]

(iii) Illustrate graphically, using $x_0 = N$, why the iterative procedure

$$x_{n+1}=5\tan\left(\sin^{-1}\left(\frac{x_n}{6}\right)\right)$$

will not give a good approximate value of α. [2]

(iv) Taking the initial approximation to be N, use the Newton-Raphson method to find α to 2 decimal places. [3]

2

Let $\mathrm{T}:\mathbb{R}^3\to\mathbb{R}^3$ be a linear transformation such that

$$\mathrm{T}(\mathbf{k})=2\mathbf{i}-3\mathbf{j}-7\mathbf{k},$$
$$\mathrm{T}(\mathbf{j}+\mathbf{k})=\mathbf{i}-2\mathbf{k},$$
$$\mathrm{T}(\mathbf{i}+\mathbf{j}+\mathbf{k})=-\mathbf{j}-\mathbf{k}.$$

(i) Find $\mathrm{T}(\mathbf{i}+2\mathbf{j}+3\mathbf{k})$. [2]

(ii) Show that the matrix that represents T is given by $\begin{pmatrix}-1 & -1 & 2\\ -1 & 3 & -3\\ 1 & 5 & -7\end{pmatrix}$. [4]

(iii) Find a basis for the null space of T. Hence write down the rank of T. [3]

3

A parabola C has parametric equations

$$x=at^2,\qquad y=2at,$$

where $a>0$ and $t\in\mathbb{R}$.

(i) Points P and Q on C have parameters p and q respectively such that $pq=-1$. Show that PQ is a focal chord. [3]

(ii) The normal at P intersects C again at the point R with parameter r. Show that

$$p^2+pr+2=0.$$ [3]

(iii) The tangent at P meets the tangent at R at the point S. Show that the line through S parallel to the axis of the parabola C meets C at Q. [4]

4

(i) Sketch on the same diagram, the curves C_1 and C_2 whose respective equations are

$$x=2a\cos t,\ \ y=a\sin t,\qquad \text{for } 0\le t\le 2\pi,$$

and

$$r=a\sin 2\theta,\qquad \text{for } 0\le\theta\le 2\pi,$$

where a is a positive constant. [3]

(ii) Show that the length of C_1 is given by

	$$L = 4a\int_0^{\frac{\pi}{2}} \sqrt{\left(4\sin^2 t + \cos^2 t\right)}\ \mathrm{d}t\,.$$ [2] **(iii)** By using the substitution $\theta = \dfrac{\pi}{4} - \dfrac{t}{2}$, show that the length of C_2 has the same expression as L in **(ii)**. [3] **(iv)** Use Simpson's rule with five ordinates to estimate the value of L, leaving your answer in the form ka, giving the value of k to 4 decimal places. [3]	
5	The charge on a capacitor, Q coulombs, at time t seconds, in an electrical circuit is governed by *Kirchoff's Second Law*, which gives the differential equation $$L\frac{\mathrm{d}^2Q}{\mathrm{d}t^2} + R\frac{\mathrm{d}Q}{\mathrm{d}t} + \frac{1}{C}Q = E(t),$$ where L is the inductance (in henries), R the resistance (in ohms), C the capacitance (in farads) and $E(t)$ the electromotive force (in volts). A circuit with L = 0.1 henries, R = 11 ohms, C = 0.01 farads, is connected to a source of voltage such that $E(t) = 12\sin 5t$. **(i)** Find the charge on the capacitor, in terms of t, given that initially there is no charge in the capacitor and that the rate of change of charge with respect to time is zero. [9] **(ii)** Describe the charge on the capacitor for large values of t. [2]	
6	A sample of 250 men had their heights, in centimetres, measured and the symmetric 99% confidence interval for the population mean height of men based on this sample is (172.31, 173.33). Find the sample variance, stating any assumption(s) needed for your calculations. [6]	
7	The lengths, x cm, of one hundred plastic rods are summarised in the table below.	

Length (cm)	Frequency
$140 \le x < 142$	1
$142 \le x < 144$	6
$144 \le x < 146$	9
$146 \le x < 148$	27
$148 \le x < 150$	21
$150 \le x < 152$	10
$152 \le x < 154$	12
$154 \le x < 156$	8
$156 \le x < 158$	3
$158 \le x < 160$	3

Use a χ^2 goodness of fit test at 5% level of significance to test whether the above data fit a normal distribution. [8]

8	The lifetime, T hours, of an Eneready battery can be modelled by the probability density function

		$$\mathrm{f}(t)=\begin{cases}\dfrac{1}{\alpha}\mathrm{e}^{-\beta t} & \text{for } t\ge 0,\\ 0 & \text{otherwise,}\end{cases}$$ where α and β are positive constants. **(i)** Show that $\alpha\beta=1$ and $\mathrm{E}(T)=\dfrac{1}{\beta}$. [4] **(ii)** Find the median, m, of T in terms of β. [2] An Eneready battery has been in use for 0.5α hours and has not failed. Find the probability that the battery will not fail within $2m$ hours. [4]
9	**(a)**	The discrete random variable X has the geometric distribution, with parameter p, given by $$\mathrm{P}(X=r)=(1-p)^{r-1}p,\qquad r=1,2,3,\ldots.$$ Show that $\mathrm{E}(X)=\dfrac{1}{p}$ and $\mathrm{Var}(X)=\dfrac{1-p}{p^2}$. [3] [You may assume that $\sum_{r=1}^{\infty} rx^{r-1}=\dfrac{1}{(1-x)^2}$ and $\sum_{r=1}^{\infty} r^2x^{r-1}=\dfrac{1+x}{(1-x)^3}$ for $\lvert x\rvert<1$.] **(b)** A bag contains 3 red balls and 7 green balls. The balls are identical apart from colour. Ben and Gwen play the following game: Ben pays Gwen \$1 for every same coloured ball he draws starting from the second draw with replacement. Ben receives \$$r$ from Gwen when the rth ball he draws is a different coloured ball. The game stops when Ben draws a different coloured ball or he completes 5 draws. **(i)** Show that the probability that exactly 5 draws are required for balls of both colours to be obtained is 0.0777. [1] **(ii)** The random variable W is the amount Ben receives in one game. Tabulate the probability distribution of W. Show that Ben is the expected winner. [4] **(iii)** Find the minimum number of red balls to be added into the bag so that Gwen is the expected winner. [4]
10		Ben likes to guess the results of the Barclays Premier League. He decides to analyse the defence of the top 10 teams by studying the number of goals conceded by each team for the past 2 seasons. The data are collated below.

Team	1	2	3	4	5	6	7	8	9	10
x	33	26	39	42	44	29	44	47	67	51
y	53	35	41	50	36	35	55	41	66	48

x : number of goals conceded for the 16/17 season.
y : number of goals conceded for the 15/16 season.
Ben had a glance at the data and concluded that the top 10 teams conceded fewer goals in

the 16/17 season.

(i) Test, at the 5% level of significance, his belief using a paired-sample *t*-test. [5]

(ii) Write down a 95% confidence interval for $\mu_x - \mu_y$. [1]

Ben showed the result of the *t*-test to his colleague Gwen who is a statistics lecturer. Gwen advised him to perform the Wilcoxon matched-pair sign test instead.

(iii) Explain why a non-parametric test may be more appropriate in this context. [1]

(iv) Explain why the Wilcoxon matched-pair sign test is better than the sign test. [1]

It was subsequently found that Team 9 conceded 66 goals instead of 67 for the 16/17 season.

(v) Test Ben's belief again using the Wilcoxon matched-pair sign test at the 5% level of significance. [6]