2017 H2 Further Math Prelim Paper 1 Solutions

1	$A(x_1, y_1) \quad B(x_2, y_2)$ grad AF = grad AB $\dfrac{y_1}{x_1 - c} = \dfrac{y_2 - y_1}{x_2 - x_1}$ $\dfrac{y_1}{\frac{y_1^2}{4c} - c} = \dfrac{y_2 - y_1}{\frac{y_2^2}{4c} - \frac{y_1^2}{4c}}$ $\dfrac{4cy_1}{y_1^2 - 4c^2} = \dfrac{4c(y_2 - y_1)}{y_2^2 - y_1^2}$ $\dfrac{4cy_1}{y_1^2 - 4c^2} = \dfrac{4c}{y_2 + y_1}$ $y_1 y_2 + y_1^2 = y_1^2 - 4c^2$ $y_1 y_2 = -4c^2 \qquad --(1)$
	Midpoint of AB = $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$ To show the locus has equation $y^2 = 2c(x - c)$, i.e. $\left(\dfrac{y_1 + y_2}{2}\right)^2 = 2c\left(\dfrac{x_1 + x_2}{2} - c\right)$. $\text{LHS} = \dfrac{y_1^2 + 2y_1 y_2 + y_2^2}{4}$ $= \dfrac{y_1^2 + y_2^2 - 8c^2}{4} \quad \text{(from (1))}$ $= \dfrac{4c(x_1 + x_2) - 8c^2}{4}$ $= c(x_1 + x_2) - 2c^2$ $= \text{RHS}$

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Let P_n be the proposition $\binom{2n}{n} \le 2^{2n-1}$ for $n \in \mathbb{N}$

For P_1 : $\quad$ $\text{LHS} = \binom{2}{1} = 2$

$\text{RHS} = 2^{2-1} = 2 = \text{LHS}$

Hence P_1 is true.

Assume P_k is true for some $k \in \mathbb{N}$.

For P_{k+1}, consider the term

$$\binom{2(n+1)}{n+1} - 2^{2(n+1)-1}$$

$$= \frac{(2n+2)(2n+1)\cdots(n+2)}{(n+1)!} - 2^{2n+1}$$

$$= \frac{(2n+2)(2n+1)}{(n+1)(n+1)}\left[\frac{(2n)(2n-1)\cdots(n+1)}{n!}\right] - 2^{2n+1}$$

$$= \frac{2(2n+1)}{(n+1)}\binom{2n}{n} - 2^{2n+1}$$

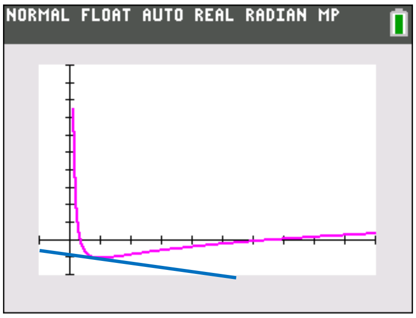
$$\le \frac{2(2n+1)}{(n+1)}\left(2^{2n-1}\right) - 2^{2n+1} \quad \text{(using } \mathrm{P}_k\text{)}$$

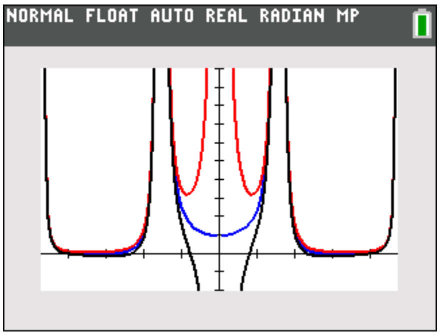
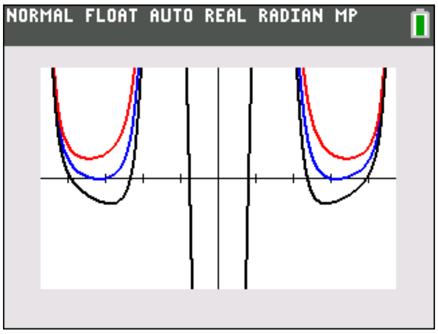
$$= 2^{2n}\left(\frac{2n+1}{n+1} - 2\right) < 0$$

Hence $\binom{2(n+1)}{n+1} \le 2^{2(n+1)-1}$.

Thus, P_k is true $\Rightarrow$ P_{k+1} is true.

Since P_1 is true, and P_k is true $\Rightarrow$ P_{k+1} is true, by mathematical induction, P_n is true for all $n \in \mathbb{Z}^+$. (Shown)

3	 When $x_1 = 0.9$, $x_2 < 0$ which is undefined, hence Newton-Raphson method fails. The curve has a turning point at $x = 1.0$, hence if $x_1 = 1.0$, then x_2 will be undefined. Let $f(x) = \frac{1}{x} - 2 + \ln x$. $f'(x) = -\frac{1}{x^2} + \frac{1}{x}$ Let $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ $x_1 = 0.3$ $x_2 = 0.31663$ $x_3 = 0.31783$ $x_4 = 0.31784$ $f(0.31775) = 6.37 \times 10^{-4} > 0$ $f(0.31785) = -3.75 \times 10^{-5} < 0$ Hence, $0.31775 < \alpha < 0.31785$ $\therefore \alpha = 0.3178$

4	$\frac{dy}{dx}+\frac{2}{x}y=\frac{2\sin x}{x^2\cos^3 x}$ Integrating Factor: $e^{\int \frac{2}{x}dx}=x^2$ Multiply IF to the de, $x^2\frac{dy}{dx}+2xy=\frac{2\sin x}{\cos^3 x}$ $\frac{d}{dx}\left(x^2 y\right)=2\tan x\sec^2 x$ $x^2 y=2\int \tan x\sec^2 x\,dx$ $=\tan^2 x+C$ $y=\frac{1}{x^2}\left(\tan^2 x+C\right)$  

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$$
\begin{aligned}
z+z^{3}+z^{5}+\cdots+z^{2n-1} &= \frac{z\left(1-z^{2n}\right)}{1-z^{2}} \\
&= \frac{e^{\frac{i\pi}{4}}(e^{\frac{in\pi}{2}}-1)}{e^{\frac{i\pi}{2}}-1} \\
&= \left(\frac{1}{e^{\frac{i\pi}{4}}-e^{-\frac{i\pi}{4}}}\right)\left[e^{\frac{in\pi}{4}}\left(e^{\frac{in\pi}{4}}-e^{-\frac{in\pi}{4}}\right)\right] \\
&= \left(\frac{1}{2i\sin\frac{\pi}{4}}\right)\left(2i\sin\frac{n\pi}{4}\right)\left(e^{\frac{in\pi}{4}}\right) \\
&= \sqrt{2}\left(\sin\frac{n\pi}{4}\right)\left(\cos\frac{n\pi}{4}+i\sin\frac{n\pi}{4}\right)
\end{aligned}
$$

$$z+z^{3}+z^{5}+\cdots+z^{2n-1}=\sqrt{2}\left(\sin\frac{n\pi}{4}\right)\left(\cos\frac{n\pi}{4}+i\sin\frac{n\pi}{4}\right)$$

$$\left(z+z^{3}+z^{5}+\cdots+z^{2n-1}\right)\left(e^{\theta i}\right)=\sqrt{2}\left(\sin\frac{n\pi}{4}\right)\left(\cos\frac{n\pi}{4}+i\sin\frac{n\pi}{4}\right)\left(e^{\theta i}\right)$$

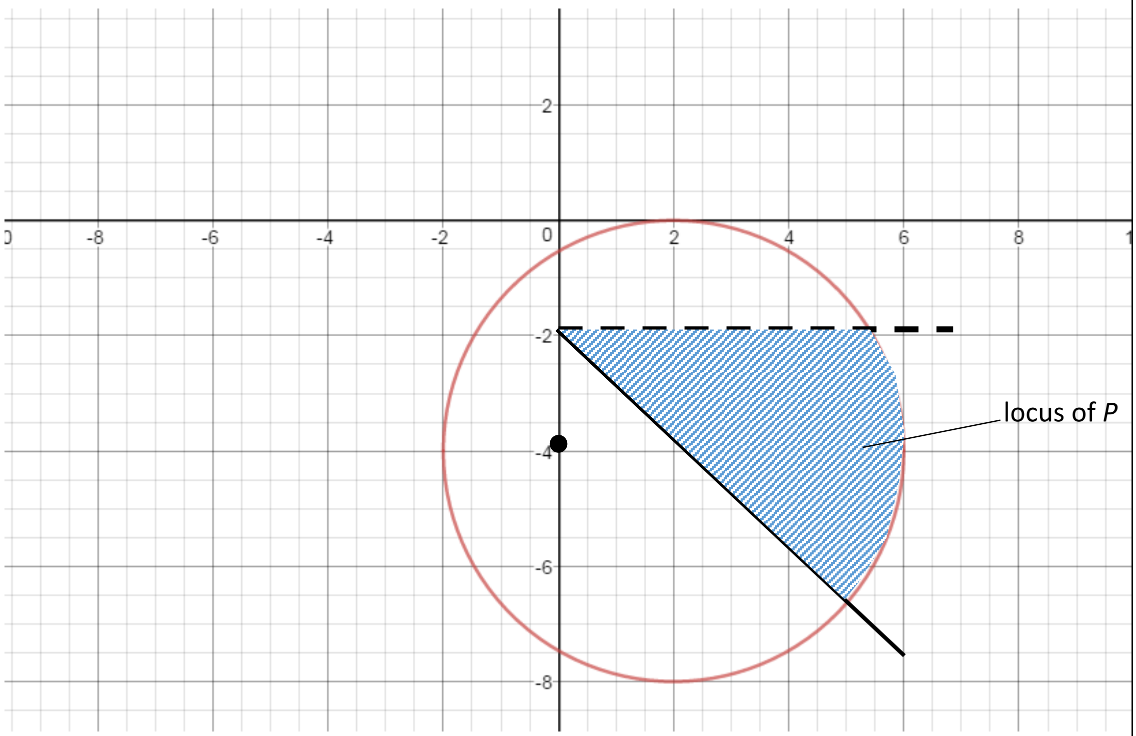
$$\left(e^{\frac{\pi i}{4}}+e^{\frac{3\pi i}{4}}+\cdots+e^{\frac{(2n-1)\pi i}{4}}\right)\left(e^{\theta i}\right)=\sqrt{2}\left(\sin\frac{n\pi}{4}\right)\left(e^{\frac{n\pi i}{4}}\right)\left(e^{\theta i}\right)$$

$$\left(e^{\left(\theta+\frac{\pi}{4}\right)i}+e^{\left(\theta+\frac{3\pi}{4}\right)i}+\cdots+e^{\left(\theta+\frac{(2n-1)\pi}{4}\right)i}\right)=\sqrt{2}\left(\sin\frac{n\pi}{4}\right)\left(e^{\left(\theta+\frac{n\pi}{4}\right)i}\right)$$

Equating real part, we get

$$
\begin{aligned}
&\cos\left(\theta+\frac{\pi}{4}\right)+\cos\left(\theta+\frac{3\pi}{4}\right)+\cos\left(\theta+\frac{5\pi}{4}\right)+\cdots\cos\left(\theta+\frac{(2n-1)\pi}{4}\right) \\
&=\sqrt{2}\left(\sin\frac{n\pi}{4}\right)\left(\cos\left(\theta+\frac{n\pi}{4}\right)\right)
\end{aligned}
$$

6(i)	$\left\|PF_1\right\|^2=(a\cos\theta-ae)^2+(b\sin\theta)^2$ $=a^2\cos^2\theta-2a^2e\cos\theta+a^2e^2+b^2\sin^2\theta$ $=a^2\cos^2\theta-2a^2e\cos\theta+a^2e^2+a^2\left(1-e^2\right)\sin^2\theta$ $=a^2(1)+a^2e^2\cos^2\theta-2a^2e\cos^2\theta$ $=(ae\cos\theta-a)^2$ $\left\|PF_1\right\|\left\|PF_2\right\|=(ae\cos\theta-a)(ae\cos\theta+a)$ $=a^2\left(1-e^2\cos^2\theta\right)$
(ii)	$\left\|PC\right\|=\dfrac{a\cos\theta+a}{\cos\phi}$ $\left\|PD\right\|=\dfrac{a-a\cos\theta}{\cos\phi}$ $\left\|PC\right\|\left\|PD\right\|=\dfrac{a^2-a^2\cos^2\theta}{\cos^2\phi}$ $=\left(a^2\sin^2\theta\right)\left(\dfrac{a^2\sin^2\theta+b^2\cos^2\theta}{a^2\sin^2\theta}\right)$ $=a^2\sin^2\theta+b^2\cos^2\theta$ $=a^2\sin^2\theta+a^2\left(1-e^2\right)\cos^2\theta$ $=a^2(1)-a^2e^2\cos^2\theta$ $=a^2\left(1-e^2\cos^2\theta\right)$ $(\therefore \text{shown})$ $\tan(\pi-\phi)=-\dfrac{b\cos\theta}{a\sin\theta}$ $\sec^2\phi=1+\tan^2\phi$ $=1+\dfrac{b^2\cos^2\theta}{a^2\sin^2\theta}$ $=\dfrac{a^2\sin^2\theta+b^2\cos^2\theta}{a^2\sin^2\theta}$

7(i)	
(ii)	$\sin\frac{\pi}{4} = \frac{l}{2}$ $l = \frac{2}{\sqrt{2}} = \sqrt{2}$ $\sqrt{2} \le \lvert z+4\mathrm{i}\rvert \le 6$
(iii)	$a^2 = 2^2 + 4^2 - 2(2)(4)\cos\frac{3\pi}{4}$ $a^2 = 31.3137$ $\frac{a}{\sin\frac{3\pi}{4}} = \frac{4}{\sin\theta}$ $\theta = 0.530$ $\therefore 0.530 \le \arg(z+4\mathrm{i}) < \frac{\pi}{2}$

8	$\frac{\mathrm{d}^2Q}{\mathrm{d}t^2}+4\frac{\mathrm{d}Q}{\mathrm{d}t}+13Q=145\cos 2t$ Auxiliary equation: $r^2+4r+13=0$ $r=\frac{-4\pm\sqrt{4^2-4(13)}}{2}$ $=-2\pm 3i$ The general solution is $Q=e^{-2t}\left(A\cos 3t+B\sin 3t\right)$. To find the P.I., we use the trial solution $Q=c_1\cos 2t+c_2\sin 2t$. $\frac{\mathrm{d}Q}{\mathrm{d}t}=-2c_1\sin 2t+2c_2\cos 2t$ $\frac{\mathrm{d}^2Q}{\mathrm{d}t^2}=-4c_1\cos 2t-4c_2\sin 2t$. Substituting into the d.e. to solve for c_1, c_2, we have $-4\left(c_1\cos 2t+c_2\sin 2t\right)+8\left(c_2\cos 2t-c_1\sin 2t\right)+13\left(c_1\cos 2t+c_2\sin 2t\right)=145\cos 2t$ Comparing coefficients of $\cos 2t:\ 9c_1+8c_2=145$ $\sin 2t:\ -8c_1+9c_2=0$ Solving with GC, we have $c_1=9$, $c_2=8$. General solution is $Q=e^{-2t}\left(A\cos 3t+B\sin 3t\right)+9\cos 2t+8\sin 2t$. $\frac{dQ}{dt}=e^{-2t}\left(-3A\sin 3t+3B\cos 3t\right)-2e^{-2t}\left(A\cos 3t+B\sin 3t\right)-18\sin 2t+16\cos 2t$ When $t=0$, $Q=0$ $0=A+9$ $A=-9$ When $t=0$, $\frac{dQ}{dt}=0$ $3B-2A+16=0$ $B=-\frac{34}{3}$ $\therefore Q=-e^{-2t}\left(9\cos 3t+\frac{34}{3}\sin 3t\right)+9\cos 2t+8\sin 2t$

9(a) (i)	$I_n = S_n + P_n + G_n$ $= \frac{1}{6} I_{n-1} + \frac{1}{3}\left(\frac{1}{6} I_{n-1}\right) + b$ $I_n = \frac{2}{9} I_{n-1} + b$ Auxiliary Equation: $r = \frac{2}{9}$ Complementary Solution: $I_n = \left(\frac{2}{9}\right)^n$ For Particular Solution, try $I_n = c$ Subst in the Recurrence Relation, $c = \frac{2}{9}c + b$ $c = \frac{9}{7}b$ $\therefore\ I_n = A\left(\frac{2}{9}\right)^n + \frac{9}{7}b$ Using the initial conditions $I_1 = b$, we have $b = A\left(\frac{2}{9}\right) + \frac{9}{7}b$ $A = -\frac{9}{7}b$ $I_n = -\frac{9b}{7}\left(\frac{2}{9}\right)^n + \frac{9}{7}b$ $= \frac{9}{7}b\left(1 - \left(\frac{2}{9}\right)^n\right)$
9(a) (ii)	As $n \to \infty$, $I_n \to \frac{9}{7}b$. The long term economy of the country will approach $\frac{9}{7}b$

9(b)	$I_n = S_n + P_n + G_n$ $= S_n + S_n - S_{n-1} + \frac{1}{2} I_{n-1}$ $= 2\left(\frac{1}{6} I_{n-1}\right) - \frac{1}{6} I_{n-2} + \frac{1}{2} I_{n-1}$ $I_n = \frac{5}{6} I_{n-1} - \frac{1}{6} I_{n-2}$ Auxiliary Equation: $r^2 - \frac{5}{6} r + \frac{1}{6} = 0$ Solving, we get $r = \frac{1}{2}$, $r = \frac{1}{3}$ Hence, $I_n = A\left(\frac{1}{2}\right)^n + B\left(\frac{1}{3}\right)^n$ When $n = 1$, $\frac{1}{2} A + \frac{1}{3} B = b$ When $n = 2$, $\left(\frac{1}{4}\right) A + \left(\frac{1}{9}\right) B = \frac{3}{8} b$ Solving, we get $A = b$, $B = \frac{9}{4} b$ $I_n = b\left(\frac{1}{2}\right)^n + \frac{9}{4} b\left(\frac{1}{3}\right)^n$

10(i)	$$\begin{pmatrix} 4 & 2 & 1 & -2 \\ 0 & -1 & 2 & 4 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & \frac{5}{4} & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$ A basis for R_1 is $\left\{ \begin{pmatrix} 4 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 4 \\ 1 \\ 3 \end{pmatrix} \right\}$.
10(ii)	Let $x_3 = t$, $\therefore x_1 = -\frac{5}{4}t, \quad x_2 = 2t, \quad x_4 = 0$ $$\underset{\sim}{x} = t\begin{pmatrix} -\frac{5}{4} \\ 2 \\ 1 \\ 0 \end{pmatrix}$$ A basis for $K_1 = \left\{ \begin{pmatrix} -\frac{5}{4} \\ 2 \\ 1 \\ 0 \end{pmatrix} \right\}$
10(iii)	Let $\underset{\sim}{v} \in R_1 \cap K_1$ $\underset{\sim}{v} \in R_1$ and $\underset{\sim}{v} \in K_1$ $$\underset{\sim}{v} = \begin{pmatrix} 4 \\ 0 \\ 0 \\ 0 \end{pmatrix} x_1 + \begin{pmatrix} 2 \\ -1 \\ 0 \\ 0 \end{pmatrix} x_2 + \begin{pmatrix} -2 \\ 4 \\ 1 \\ 3 \end{pmatrix} x_3 = x_4 \begin{pmatrix} -\frac{5}{4} \\ 2 \\ 1 \\ 0 \end{pmatrix}$$

	$$\begin{pmatrix}4\\0\\0\\0\end{pmatrix}x_1+\begin{pmatrix}2\\-1\\0\\0\end{pmatrix}x_2+\begin{pmatrix}-2\\4\\1\\3\end{pmatrix}x_3+\begin{pmatrix}\frac{5}{4}\\-2\\-1\\0\end{pmatrix}x_4=\underset{\sim}{0}$$ $$\begin{pmatrix}4&2&-2&\frac{5}{4}\\0&-1&4&-2\\0&0&1&-1\\0&0&3&3\end{pmatrix}\to\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$ $x_1=x_2=x_3=x_4=0$ $\therefore \underset{\sim}{v}=\underset{\sim}{0}$
10(iv)	Let $\underset{\sim}{v}\in\ker(T_2)$ $M^2(\underset{\sim}{v})=\underset{\sim}{0}$ $M(M\underset{\sim}{v})=\underset{\sim}{0}$ Hence, we have $M\underset{\sim}{v}\in K_1$. Also $M\underset{\sim}{v}\in R_1$. Using (iii), we have $M\underset{\sim}{v}=\underset{\sim}{0}$ Hence $\underset{\sim}{v}\in K_1$ Let $\underset{\sim}{v}\in K_1$, since $M^2(\underset{\sim}{v})=M(M\underset{\sim}{v})$ $=M(\underset{\sim}{0})$ $=\underset{\sim}{0}$ Hence $\underset{\sim}{v}\in\ker(T_2)$. $\therefore\ \ker(T_2)=K_1$

10(v)

$$\left\{\begin{pmatrix}4\\0\\0\\0\end{pmatrix},\begin{pmatrix}2\\-1\\0\\0\end{pmatrix},\begin{pmatrix}-2\\4\\1\\3\end{pmatrix}\right\}\ K_1=\left\{\begin{pmatrix}-\frac{5}{4}\\2\\1\\0\end{pmatrix}\right\}$$

Consider $\begin{pmatrix}\frac{5}{4}\\0\\0\\0\end{pmatrix}\in R_1\subseteq R_1\cup K_1$ and $\begin{pmatrix}-\frac{5}{4}\\2\\1\\0\end{pmatrix}\in K_1\subseteq R_1\cup K_1$

$$\begin{pmatrix}\frac{5}{4}\\0\\0\\0\end{pmatrix}+\begin{pmatrix}-\frac{5}{4}\\2\\1\\0\end{pmatrix}=\begin{pmatrix}0\\2\\1\\0\end{pmatrix}$$

Note that $\begin{pmatrix}0\\2\\1\\0\end{pmatrix}\notin K_1$

Let $\begin{pmatrix}4\\0\\0\\0\end{pmatrix}x_1+\begin{pmatrix}2\\-1\\0\\0\end{pmatrix}x_2+\begin{pmatrix}-2\\4\\1\\3\end{pmatrix}x_3=\begin{pmatrix}0\\2\\1\\0\end{pmatrix}$

Augmented Matrix $\begin{pmatrix}4&2&2&0\\0&-1&4&2\\0&0&1&1\\0&0&3&0\end{pmatrix}\to\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$, hence $\begin{pmatrix}0\\2\\1\\0\end{pmatrix}\notin R_1$

Hence $\begin{pmatrix}0\\2\\1\\0\end{pmatrix}\notin R_1\cup K_1$.

$R_1\cup K_1$ is not a vector space.

11(a)	$I_{n+2}=\int \sec^{n+2} x\,dx$ $=\int \sec^{n} x\sec^2 x\,dx$ Let $u=\sec x$, $\dfrac{du}{dx}=n\left(\sec^{n-1} x\right)\left(\sec x\tan x\right)$ $\dfrac{dv}{dx}=\sec^2 x$, $v=\tan x$ $I_{n+2}=\tan x\sec^n x-\int n\sec^n x\tan^2 x\,dx$ $=\tan x\sec^n x-n\int \sec^n x\left(\sec^2 x-1\right)dx$ $=\tan x\sec^n x-n\left(I_{n+2}-I_n\right)$ $\left(n+1\right)I_{n+2}=\tan x\sec^n x+nI_n$
11(b) (i)	$\dfrac{dx}{d\theta}=\dfrac{\left(1+\cos\theta\right)\left(-2a\sin\theta\right)-\left(2a\cos\theta\right)\left(-\sin\theta\right)}{\left(1+\cos\theta\right)^2}$ $=\dfrac{-2a\sin\theta}{\left(2\cos^2\dfrac{\theta}{2}\right)^2}$ $=\dfrac{-a\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2}}{\cos^4\dfrac{\theta}{2}}$ $=-a\tan\dfrac{\theta}{2}\sec^2\dfrac{\theta}{2}$ $\dfrac{dy}{d\theta}=a\sec^2\dfrac{\theta}{2}$ $\sqrt{\left(\dfrac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2+\left(\dfrac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2}=\sqrt{\left(-a\tan\dfrac{\theta}{2}\sec^2\dfrac{\theta}{2}\right)^2+\left(a\sec^2\dfrac{\theta}{2}\right)^2}$ $=a\sqrt{\sec^4\dfrac{\theta}{2}\left(\tan^2\dfrac{\theta}{2}+1\right)}$ $=a\sec^3\dfrac{\theta}{2}$

11(b) (ii)	At $A(a,0)$, $y = 2a\tan\left(\frac{\theta}{2}\right) = 0$ $\theta = 0$ At $B(0,2a)$, $y = 2a\tan\left(\frac{\theta}{2}\right) = 2a$ $\tan\left(\frac{\theta}{2}\right) = 2$ $\theta = \frac{\pi}{2}$ Arc length $= \int_0^{\frac{\pi}{2}} \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2}\, d\theta$ $= \int_0^{\frac{\pi}{2}} a\sec^3\left(\frac{\theta}{2}\right) d\theta$ Let $u = \frac{\theta}{2}$, $\frac{du}{d\theta} = \frac{1}{2}$ Arc length $= \alpha\int_0^{\frac{\pi}{4}} \sec^3 u\,(2)\,du$ $= \frac{1}{2}(2a)\left(\left[\tan u \sec u\right]_0^{\frac{\pi}{4}} + \int_0^{\frac{\pi}{4}} \sec u\, du\right)$ $= a\left(\sqrt{2} + \left[\ln\left(\sec x + \tan x\right)\right]_0^{\frac{\pi}{4}}\right)$ $= a\left(\sqrt{2} + \ln\left(\sqrt{2}+1\right)\right)$

11(b) (iii)	Surface Area $= 2\pi\int_0^{\frac{\pi}{2}}\left(2a\tan\dfrac{\theta}{2}\right)\left(a\sec^3\dfrac{\theta}{2}\right)d\theta$ $= 4\pi a^2\int_0^{\frac{\pi}{2}}\left(\tan\dfrac{\theta}{2}\sec\dfrac{\theta}{2}\right)\sec^2\dfrac{\theta}{2}\,d\theta$ $= 4\pi a^2\left[\left(\dfrac{1}{3}\right)(2)\left(\sec^3\dfrac{\theta}{2}\right)\right]_0^{\frac{\pi}{2}}$ $= \dfrac{8}{3}\pi a^2\left(2\sqrt{2}-1\right)$