INNOVA JUNIOR COLLEGE
JC 2 PRELIMINARY EXAMINATION
in preparation for General Certificate of Education Advanced Level
Higher 2

CANDIDATE NAME			
CLASS		INDEX NUMBER	

MATHEMATICS 9740/01

Paper 1

28 August 2017

3 hours

Additional Materials: Answer Paper
Cover Page
List of Formulae (MF 15)

READ THESE INSTRUCTIONS FIRST

Do not open this booklet until you are told to do so.

Write your name, class and index number on all the work you hand in.

Write in dark blue or black pen on both sides of the paper. You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

You are expected to use a graphic calculator.

Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.

You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.

The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **6** printed pages.

1 Prove by the method of mathematical induction that

$$\sum_{r=0}^{n} r\left(r!\right) = \left(n+1\right)! - 1.$$ [5]

2 **(i)** Find $\int \mathrm{e}^{\frac{x}{n}} \cos\left(nx\right) \mathrm{d}x$, where n is a positive constant. [3]

(ii) Hence find the exact value of $\int_{\pi}^{2\pi} \mathrm{e}^{\frac{x}{n}} \cos\left(nx\right) \mathrm{d}x$. [2]

3 The vectors **p** and **q** are given by $\mathbf{p} = 2\mathbf{i} + \mathbf{j} + a\mathbf{k}$ and $\mathbf{q} = b\mathbf{i} + \mathbf{j}$, where a and b are non-zero constants.

(i) Find $(2\mathbf{p} - 5\mathbf{q}) \times (2\mathbf{p} + 5\mathbf{q})$ in terms of a and b. [2]

Given that the **i**- and **j**- components of the answer to part **(i)** are equal, find the value of b. [1]

Use the value of b you have found to solve parts **(ii)** and **(iii)**.

(ii) Given that the magnitude of $(2\mathbf{p} - 5\mathbf{q}) \times (2\mathbf{p} + 5\mathbf{q})$ is 80, find the possible exact values of a. [2]

(iii) Given instead that $2\mathbf{p} - 5\mathbf{q}$ and $2\mathbf{p} + 5\mathbf{q}$ are perpendicular, find the exact value of $|\mathbf{p}|$. [3]

4 A graphic calculator is **not** to be used in answering this question.

(a) The equation $w^3 + pw^2 + qw + 30 = 0$, where p and q are real constants, has a root $w = 2 - \mathrm{i}$. Find the values of p and q, showing your working. [3]

(b) The equation $z^2 + (-5 + 2\mathrm{i})z + (21 - \mathrm{i}) = 0$ has a root $z = 3 + u\mathrm{i}$, where u is real constant. Find the value of u and hence find the second root of the equation in cartesian form, $a + b\mathrm{i}$, showing your working. [5]

5 A sequence $u_1,\ u_2,\ u_3,...$ is such that

$$u_n=\frac{1}{2n^2(n-1)^2} \text{ and } u_{n+1}=u_n-\frac{2}{n(n-1)^2(n+1)^2}, \text{ for all } n\geq 2 .$$

(i) Find $\displaystyle\sum_{n=2}^{N}\frac{2}{n(n-1)^2(n+1)^2}$. [3]

(ii) Explain why $\displaystyle\sum_{n=2}^{\infty}\frac{2}{n(n-1)^2(n+1)^2}$ is a convergent series, and state the value of the sum to infinity. [2]

(iii) Using your answer in part **(i)**, find $\displaystyle\sum_{n=1}^{N}\frac{2N}{(n+1)n^2(n+2)^2}$. [2]

6 **(i)** The variables x and y are related by

$$(x+y)\frac{\mathrm{d}y}{\mathrm{d}x}+ky=2 \text{ and } y=1 \text{ at } x=0,$$

where k is a constant. Show that $(x+y)\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+(1+k)\dfrac{\mathrm{d}y}{\mathrm{d}x}+\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2=0$. [1]

By further differentiation of this result, find the Maclaurin series for y, up to and including the term in x^3, giving the coefficients in terms of k. [4]

(ii) Given that x is small, find the series expansion of $\mathrm{g}(x)=\dfrac{1}{\sin^2\left(2x+\dfrac{\pi}{2}\right)}$ in ascending powers of x, up to and including the term in x^2.

If the coefficient of x^2 in the expansion of $\mathrm{g}(x)$ is equal to twice the coefficient of x^2 in the Maclaurin series for y found in part **(i)**, find the value of k. [4]

7 A population of a certain organism grows from an initial size of 5. After 5 days, the size of the population is 20, and after t days, the size of the population is M. The rate of growth of the population is modelled as being proportional to $\left(100^2 - M^2\right)$.

(i) Write down a differential equation modelling the population growth and find M in terms of t. [6]

(ii) Find the size of the population after 15 days, giving your answer correct to the nearest whole number. [2]

(iii) Find the least number of days required for the population to exceed 80. [2]

8 It is given that $\mathrm{f}(x) = \begin{cases} 2x-1 & 0 \le x \le 2, \\ 2-(x-3)^3 & 2 < x \le 4, \\ 1 & \text{otherwise.} \end{cases}$

Sketch, on separate diagrams, for $0 \le x \le 8$, the graphs of

(i) $y = \mathrm{f}(x)$ and state the range of f, [5]

(ii) $y = \dfrac{1}{\mathrm{f}(x)}$. [4]

In each graph, indicate clearly the coordinates of the end points, points of intersection with the axes and stationary point, if any. State clearly the equation of any asymptote.

(iii) Deduce the value of $\displaystyle\int_{-6}^{-4} \mathrm{f}(-x)\,\mathrm{d}x$. [1]

9 Given that $\mathrm{f}(x)=\sin 2x+\cos 2x$, express $\mathrm{f}(x)$ as $R\sin(2x+\alpha)$, where $R>0$, $0<\alpha<\frac{\pi}{2}$ and R and α are constants to be found. [2]

(i) Describe a sequence of transformations involved that transformed $y=\sin x$ to $y=\mathrm{f}(x)$. [3]

(ii) Sketch the graph of $y=\mathrm{f}(x)$ for $0\le x\le\frac{3\pi}{8}$, indicating clearly the exact coordinates of the maximum point and the end points of the graph. [3]

(iii) The region bounded by the curve $y=\mathrm{f}(x)$, the line $x=\frac{\pi}{8}$ and both axes is rotated about the y-axis through 2π radians. Find the volume of the solid of revolution correct to 4 decimal places. [4]

10 The plane p_1 and p_2 have equations $x+2y+z=8$ and $3x+3y-2z=4$ respectively. It is given that the point A has coordinates $(6c, 0, 2)$, where c is a constant.

(i) Find the coordinates of the foot of perpendicular from A to p_1. Express your answer in terms of c. [4]

(ii) The point B is the mirror image of A in p_1. If B lies in p_2, find the value of c. [3]

(iii) p_1 and p_2 intersect in a line l. Find a vector equation of l. [1]

Another plane p_3 has equation $mx+z=n$, where m and n are constants.

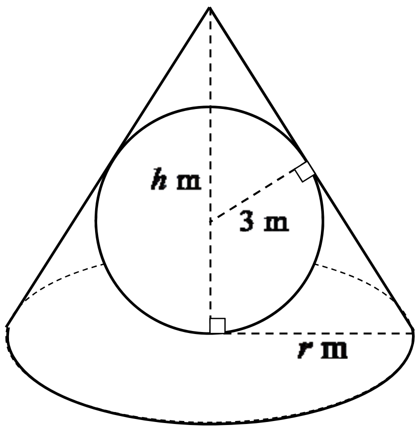
(iv) Given that all three planes meet in the line l, find m and n. [2]

(v) Given instead that the three planes have no point in common, what can be said about the values of m and n? [2]

[Turn over

11 [It is given that the volume of a circular cone with base radius r and height h is $\frac{1}{3}\pi r^2 h$ and the volume and surface area of a sphere of radius r are $\frac{4}{3}\pi r^3$ and $4\pi r^2$ respectively.]

In a distant Northern kingdom of Drivenbell, Elsanna builds a spherical snowball with radius 3 m. The snowball is inscribed in a right conical container of base radius r m and height h m. The container is specially designed to allow the snowball to remain intact with fixed radius 3 m (see diagram).



(i) By considering the slant height of the cone, show that $r = \dfrac{3h}{\sqrt{\left(h^2 - 6h\right)}}$. [3]

(ii) Use differentiation to find the values of h and r that give a minimum volume for the container. Find the value of the minimum volume. [6]

The snowball is being removed from the container and it starts to melt under room temperature.

(iii) Assuming that the snowball remains spherical as it melts, find the rate of decrease of its volume at the instant when the radius of the sphere is 2.5 m, given that the surface area is decreasing at 0.75 m^2 per minute at this instant. [5]