2017 H2 Further Mathematics (9649) Paper 1 Solution **100 marks**

1 The harmonic numbers H_k, $k = 1, 2, 3, \ldots$, are defined by

$$H_k = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{k}.$$

Prove by mathematical induction that

$$H_{2^n} \geq 1 + \frac{n}{2}, \quad \text{where } n \text{ is a non-negative integer.}$$ [6]

Solution

Let P_n be the statement that $H_{2^n} \geq 1 + \frac{n}{2}$ for all non-negative integer.

When $n = 0$, LHS $= H_{2^0} = H_1 = 1$

and RHS $= 1 + \frac{0}{2} = 1$ = LHS. $\therefore$ P_0 is true.

Assume that P_k is true for some non-negative integer k.

ie. $H_{2^k} \geq 1 + \frac{k}{2}$

Need to show that P_{k+1} is true. i.e. $H_{2^{k+1}} \geq 1 + \frac{k+1}{2}$

$$\begin{aligned}
\text{LHS} = H_{2^{k+1}} &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2^k} + \frac{1}{2^k+1} + \frac{1}{2^k+2} + \cdots + \frac{1}{2^k+2^k} \\
&= H_{2^k} + \left(\frac{1}{2^k+1} + \frac{1}{2^k+2} + \cdots + \frac{1}{2^k+2^k} \right) \\
&\geq 1 + \frac{k}{2} + \left(\frac{1}{2^k+1} + \frac{1}{2^k+2} + \cdots + \frac{1}{2^{k+1}} \right) \qquad \because H_{2^k} \geq 1 + \frac{k}{2} \\
&\geq 1 + \frac{k}{2} + \left(\underbrace{\frac{1}{2^{k+1}} + \frac{1}{2^{k+1}} + \cdots + \frac{1}{2^{k+1}}}_{2^k \, terms} \right) \\
&= 1 + \frac{k}{2} + 2^k \left(\frac{1}{2^{k+1}} \right) \\
&= 1 + \frac{k}{2} + \frac{1}{2} \\
&= 1 + \frac{k+1}{2} \quad \Rightarrow P_{k+1} \text{ true}
\end{aligned}$$

Thus P_k true $\Rightarrow P_{k+1}$ true.

Since P_0 is true, and P_k true $\Rightarrow P_{k+1}$ true, by mathematical induction, P_n is true for all non-negative integer n.

2 Show, by means of the substitution $u = y^{1-n}$, that the Bernoulli's differential equation of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \mathrm{f}(x)\,y = \mathrm{g}(x)\,y^n, \quad \text{where } n \text{ is a non-zero integer and } n \neq 1$$

can be reduced to the form $\dfrac{\mathrm{d}u}{\mathrm{d}x} + \mathrm{P}(x)u = \mathrm{Q}(x)$. [2]

A cardiac pacemaker is designed to provide electrical impulses I amps such that as time t increases, I oscillates with a fixed amplitude of one amp. It is proposed that the following differential equation

$$\frac{\mathrm{d}I}{\mathrm{d}t} + (\tan t)\,I = (I \sin t)^2$$

can be used to describe how I changes with t.

By using a substitution of the form $u = I^{1-n}$, find I in terms of t. [5]

State one limitation of this model. [1]

Solution

Differentiate $u = y^{1-n}$ with respect to x, we get:

$$\frac{\mathrm{d}u}{\mathrm{d}x} = (1-n)\,y^{-n}\frac{\mathrm{d}y}{\mathrm{d}x} \Rightarrow y^{-n}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{1-n}\frac{\mathrm{d}u}{\mathrm{d}x}$$

From the given DE, $\dfrac{\mathrm{d}y}{\mathrm{d}x} + \mathrm{f}(x)\,y = \mathrm{g}(x)\,y^n$

$$y^{-n}\frac{\mathrm{d}y}{\mathrm{d}x} + \mathrm{f}(x)\,y^{1-n} = \mathrm{g}(x)$$

$$\frac{1}{1-n}\frac{\mathrm{d}u}{\mathrm{d}x} + \mathrm{f}(x)u = \mathrm{g}(x)$$

$\therefore \dfrac{\mathrm{d}u}{\mathrm{d}x} + (1-n)\mathrm{f}(x)u = (1-n)\mathrm{g}(x)$ which is of the form $\dfrac{\mathrm{d}u}{\mathrm{d}x} + \mathrm{P}(x)u = \mathrm{Q}(x)$

where $\mathrm{P}(x) = (1-n)\mathrm{f}(x)$ and $\mathrm{Q}(x) = (1-n)\mathrm{g}(x)$

$$\frac{\mathrm{d}I}{\mathrm{d}t} + (\tan t)\,I = (I \sin t)^2$$

Let $u = I^{-1}$ where $n = 2$, $\mathrm{f}(t) = \tan t$, $\mathrm{g}(t) = \sin^2 t$

$$\therefore \frac{\mathrm{d}u}{\mathrm{d}t} - u\tan t = -\sin^2 t$$

Integrating factor $= \mathrm{e}^{\int -\tan t \, \mathrm{d}t} = \mathrm{e}^{\ln|\cos t|} = \cos t$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(u\cos t) &= -\cos t \sin^2 t \\
u\cos t &= -\int \cos t \sin^2 t \; \mathrm{d}t \\
&= -\frac{\sin^3 t}{3} + C \\
u &= -\frac{\sin^3 t}{3\cos t} + \frac{C}{\cos t}
\end{aligned}$$

$$I = \frac{3\cos t}{A - \sin^3 t} \quad \text{where } A = 3C$$

I is maximum when $t = 0$, $\dfrac{3}{A} = 1 \Rightarrow A = 3$

$$I = \frac{3\cos t}{3 - \sin^3 t}$$

One limitation of the DE $\dfrac{\mathrm{d}I}{\mathrm{d}t} + (\tan t)I = (I\sin t)^2$ is that $\tan t$ is undefined when t takes the values of $(2n-1)\dfrac{\pi}{2}$, $n \in \mathbb{Z}^+$

3 **(a)** A computer system considers a string of digits a valid codeword if it contains an even number of 0 digits. For example, 123040826 and 14947 are valid codewords, whereas 9038040 is not. Let a_n be the number of valid *n*-digit codewords.

(i) Find the values of a_1 and a_2. [2]

(ii) By considering the number of valid and invalid $(n-1)$-digit codewords, find a recurrence relation for a_n. [2]

(b) Solve the recurrence relation $b_n = 4b_{n-1} - 4b_{n-2}$ for $n \geq 2$ with initial conditions $b_0 = 6$ and $b_1 = \sqrt{5}$. [6]

Solution

(a)

(i) $a_1 = 9$ (Since there are 10 1-digit strings, and only one, the string 0, is not valid.)

$a_2 = 10^2 - 9(2) = 82$

(or) $a_2 = 9a_1 + (10 - a_1) = 10 + 8a_1 = 82$

(ii) There are two ways to form a valid string with *n*-digits from a string with $(n-1)$-digits:

(1) By appending a non-zero digit to a valid string of $(n-1)$-digits, this can be done in $9a_{n-1}$ ways.

(2) By appending a 0 to a invalid string of $(n-1)$-digits, this can be done in $(10^{n-1} - a_{n-1})$ ways

Hence, $a_n = 9a_{n-1} + (10^{n-1} - a_{n-1})$

$= 8a_{n-1} + 10^{n-1}$

(b) $b_n = 4b_{n-1} - 4b_{n-2}$

has characteristic equation: $m^2 - 4m + 4 = 0$

$\Rightarrow \ (m-2)^2 = 0$

$\Rightarrow \ m = 2 \ \ $ (Repeated roots)

Hence, the **general solution** is $b_n = 2^n (A + Bn)$

Using the initial conditions,

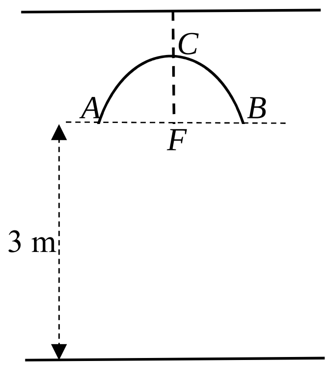
$b_0 = 6 \Rightarrow 6 = 2^0(A) \Rightarrow A = 6$

and $b_1 = \sqrt{5} \Rightarrow \sqrt{5} = 2^1(A + B) \Rightarrow B = \dfrac{\sqrt{5}}{2} - 6$

The **particular solution** to the given recurrence relation is

$$b_n = 2^n \left[6 + \left(\frac{\sqrt{5}}{2} - 6 \right) n \right]$$

4



The diagram shows the vertical cross-section of a specially-designed sound reflector hanging from the ceiling of a room. The arc *ACB* forms part of an ellipse with *F* as one of the foci. *CF* is a vertical line of symmetry and *AFB* is a horizontal straight line 3 m above the ground.

A sound transmitter is positioned at *F* and a sound receiver is to be placed at the other focus *X* of the ellipse so as to receive the maximum intensity of the sound from the transmitter.

(i) Given that *CF* = 12 cm and *AB* = 40 cm, determine the position of *X*. [3]

(ii) Find the polar equation of the ellipse if *F* is positioned at the pole and *FB* is in the direction of the line $\theta = 0$. [4]

Solution:

Let *X* be the other focus of the ellipse. Let *x* cm be the distance directly below *F*.

By geometric property of an ellipse,

$$AX + AF = CX + CF$$
$$20 + \sqrt{20^2 + x^2} = (12 + x) + 12$$
$$\sqrt{400 + x^2} = 4 + x$$
$$400 + x^2 = (4 + x)^2$$
$$400 + x^2 = 16 + 8x + x^2$$
$$x = 48$$

Since directrix of the ellipse is a horizontal line above the x-axis, therefore the polar equation of C takes the form of $r = \dfrac{ep}{1 + e\sin\theta}$ where e is the eccentricity and d is the distance between the directrix and the pole.

$$e = \frac{c}{a} = \frac{24}{24 + 12} = \frac{2}{3}$$

$$p = \frac{a}{e} - c = \frac{36}{\frac{2}{3}} - 24 = 30$$

Therefore, the polar equation of *C* is $r = \dfrac{20}{1 + \frac{2}{3}\sin\theta}$, i.e. $r = \dfrac{60}{3 + 2\sin\theta}$

5 Let V and W be subspaces of $\mathbb{R}^n$. Show that $V \cap W$ is also a subspace of $\mathbb{R}^n$. [3]

The transformation $T_1 : \mathbb{R}^4 \to \mathbb{R}^3$ and $T_2 : \mathbb{R}^3 \to \mathbb{R}^3$ are represented by the matrices $\mathbf{M} = \begin{pmatrix} 1 & 0 & -1 & 2 \\ 2 & 1 & -1 & 3 \\ 1 & 2 & 1 & 0 \end{pmatrix}$ and $\mathbf{N} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ respectively. The range space of T_1 is denoted by A and the range space of T_2 is denoted by B.

(i) Find the rank of **M** and find a basis for A. [3]

(ii) Write down a basis for B. [1]

(iii) Find a basis for $A \cap B$. [3]

(iv) Determine whether $A \cup B$ is a subspace of $\mathbb{R}^3$. Justify your answer. [4]

Solution:

Let $\mathbf{x}, \mathbf{y} \in V \cap W$.

Then, $\mathbf{x}, \mathbf{y} \in V$ and $\mathbf{x}, \mathbf{y} \in W$

As V and W are subspaces,

$\mathbf{x} + \mathbf{y} \in V$ and $\mathbf{x} + \mathbf{y} \in W$

$\therefore \mathbf{x} + \mathbf{y} \in V \cap W$

Also, $k\mathbf{x} \in V$ and $k\mathbf{x} \in W$

$\therefore k\mathbf{x} \in V \cap W$

Therefore $V \cap W$ is a subspace of $\mathbb{R}^n$.

(i) $\mathbf{M} = \begin{pmatrix} 1 & 0 & -1 & 2 \\ 2 & 1 & -1 & 3 \\ 1 & 2 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Since there are 2 non-zero rows in the echelon form, rank **M** = 2.

A basis for A is $\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right\}$,

(ii) $\mathbf{N} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

Since **N** is already in echelon form, a basis for B is $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$

(iii) Let $\mathbf{x} \in A \cap B$.

$$\mathbf{x} = a\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = c\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + d\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

$$\Rightarrow \begin{cases} a = c \\ 2a + b = d \\ a + 2b = 0 \end{cases} \Rightarrow \begin{cases} 3c - 2d = 0 \\ a + 2b = 0 \end{cases}$$

$$\mathbf{x} = -2b\begin{pmatrix}1\\2\\1\end{pmatrix} + b\begin{pmatrix}0\\1\\2\end{pmatrix} = -b\begin{pmatrix}2\\3\\0\end{pmatrix}.$$

$\therefore A \cap B = \left\{ \lambda\begin{pmatrix}2\\3\\0\end{pmatrix}, \lambda \in \mathbb{R} \right\}$ and a basis for $A \cap B$ is $\left\{ \begin{pmatrix}2\\3\\0\end{pmatrix} \right\}$.

(iv) Let $\mathbf{u} = 1\begin{pmatrix}1\\2\\1\end{pmatrix} + 0\begin{pmatrix}0\\1\\2\end{pmatrix} = \begin{pmatrix}1\\2\\1\end{pmatrix} \in A$ and

$$\mathbf{v} = 0\begin{pmatrix}1\\0\\0\end{pmatrix} + 1\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix} \in B$$

$\therefore \mathbf{u}, \mathbf{v} \in A \cup B$.

$\mathbf{u} + \mathbf{v} = \begin{pmatrix}1\\3\\1\end{pmatrix} \notin A$ and $\notin B$

$\therefore \mathbf{u} + \mathbf{v} \notin A \cup B$.

$\therefore A \cup B$ is not closed under vector addition.

$\therefore A \cup B$ is not a subspace of $\mathbb{R}^3$

6 **(a)** **(i)** Solve the equation

$$z^5 - 32\mathrm{i} = 0\,,$$

giving the roots in the form $r\mathrm{e}^{\mathrm{i}\alpha}$, where $r > 0$ and $-\pi < \alpha \le \pi$.

Draw and represent these roots in an Argand diagram. [4]

(ii) Given that $w = \frac{1}{2} z\mathrm{e}^{-\mathrm{i}\frac{\pi}{10}}$, describe geometrically how the point representing w can be obtained from z. [1]

Find, in simplified form, an equation for which the values of w are the roots. [1]

(b) The complex number z satisfies the relations $|z - 2\mathrm{i}| \le 1$ and $\arg(z+1) \le \frac{\pi}{4}$.

(i) Illustrate the locus of the point representing the complex number z on an Argand diagram. [2]

(ii) Label on your Argand diagram the point P for which arg(z) has the least value. Find this least value and hence find the coordinates of P. [5]

Solution

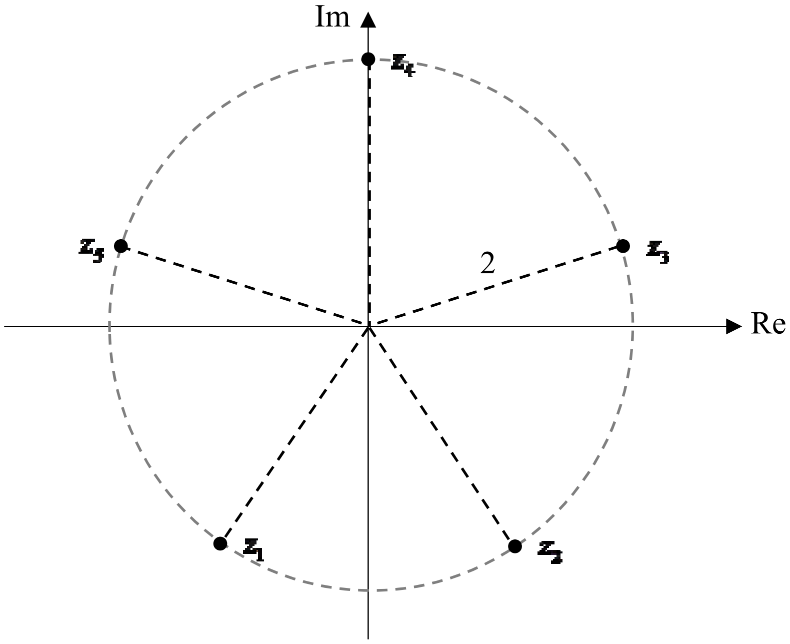
6

(a)(i) $z^5 - 32\mathrm{i} = 0$

$$z^5 = 32\mathrm{e}^{\frac{\pi}{2}}$$

$$z = 2\mathrm{e}^{\mathrm{i}\frac{1}{5}\left(\frac{\pi}{2}+2k\pi\right)},\ k = 0, \pm 1, \pm 2$$

$$z_1 = 2\mathrm{e}^{-\mathrm{i}\frac{7\pi}{10}}, z_2 = 2\mathrm{e}^{-\mathrm{i}\frac{3\pi}{10}}, z_3 = 2\mathrm{e}^{\mathrm{i}\frac{\pi}{10}}, z_4 = 2\mathrm{e}^{\mathrm{i}\frac{\pi}{2}}, z_5 = 2\mathrm{e}^{\mathrm{i}\frac{9\pi}{10}}$$



(ii) Each w is obtained from each z by a clockwise rotation of 18° ($\frac{\pi}{10}$ radians) about the origin and the modulus of w is half of the modulus of z.

An equation satisfying w:

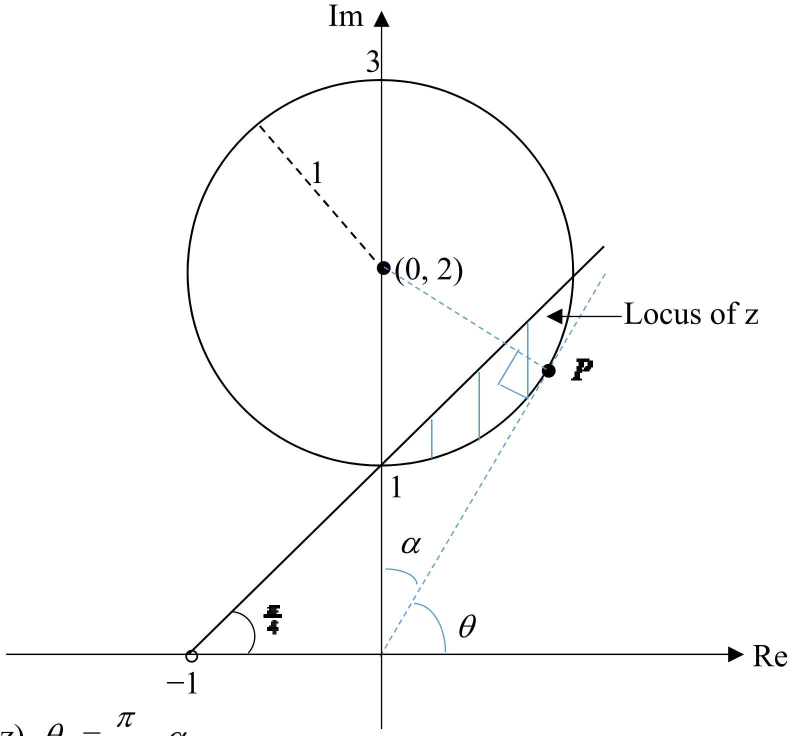
$$w^5 = \left(\frac{1}{2}ze^{-i\frac{\pi}{10}}\right)^5$$

$$w^5 = \frac{1}{32}z^5e^{-i\frac{\pi}{2}}$$

$$w^5 = \frac{1}{32}(32i)(-i)$$

$$w^5 = 1$$

(b) (i)



Least arg(z), $\theta = \frac{\pi}{2} - \alpha$

$$= \frac{\pi}{2} - \sin^{-1}\frac{1}{2}$$

$$= \frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3}$$

Let coordinates of P be (x, y)

$$x = \sqrt{2^2 - 1^2}\cos\theta = \sqrt{3}\left(\frac{1}{2}\right) = \frac{\sqrt{3}}{2}$$

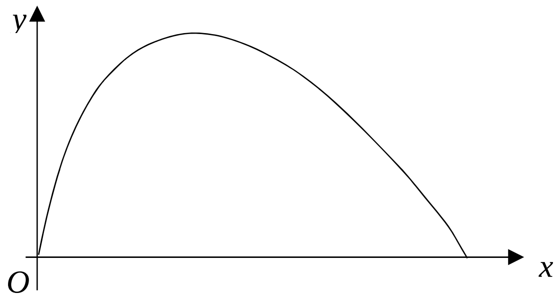
$$x = \sqrt{2^2 - 1^2}\sin\theta = \sqrt{3}\left(\frac{\sqrt{3}}{2}\right) = \frac{3}{2}$$

Coordinates of $P = \left(\frac{\sqrt{3}}{2}, \frac{3}{2}\right)$.

7 The figure below shows the cross-section of a proposed underground tunnel. The equation of the curved section is

$$y = \begin{cases} x - x\ln x, & x > 0 \\ 0 & , x = 0 \end{cases}$$

where x and y are in metres.



The engineers need to estimate the area of the cross section A m^2 so that they can work out the volume of the soil that will need to be removed when a straight tunnel is built.

(i) Engineer Yang uses the trapezium rule to estimate A with 6 strips. Find his estimate correct to 3 decimal places. [2]

(ii) Engineer Tan and Engineer Liu estimate A with 8 strips and 16 strips respectively using trapezium rule. The estimates obtained are 1.799 and 1.833. Without any calculation, identify the estimate that is obtained by engineer Tan and state your reasons clearly. [1]

In order to account for the error when using trapezium rule for the estimation of A, it is proposed to refine the formula using

$$A \approx A(n) + \frac{a}{n^2}$$

where $A(n)$ is the estimation based on n strips and a is a positive constant.

Find a better estimation of A using the values of $A(8)$ and $A(16)$. [2]

(iii) Engineer Shin uses Simpson's rule with 7 ordinates to estimate A. Find his estimate correct to 3 decimal places. [2]

Without further calculation, explain whether Engineer Yang or Engineer Shin gives a better approximation to A. [1]

Another tunnel with the same cross-section is built by rotating the cross-section by 2π radians about the y-axis.

(iv) Find the exact volume of soil that need to be removed to form the loop. [4]

Solution

7

(i) x-intercept: $x - \ln x = 0$

$x(1-\ln x)=0$

$x = 0$ or $x = \mathrm{e}$

x	0	$\frac{\mathrm{e}}{6}$	$\frac{2\mathrm{e}}{6}$	$\frac{3\mathrm{e}}{6}$	$\frac{4\mathrm{e}}{6}$	$\frac{5\mathrm{e}}{6}$	e
y	0	0.81175	0.99545	0.94208	0.73478	0.41300	0

Engineer Yang's estimate (using trapezium rule)

$=\frac{1}{2}\left(\frac{\mathrm{e}}{6}\right)\left[0+0+2\left(0.81175+0.99545+0.94208+0.73478+0.41300\right)\right]$

$= 1.76555 \approx 1.766$ (to 3 d.p)

(ii) As the curve is concave downwards, trapezium rule will give an underestimate. The estimate should get bigger and more accurate as the number of strips increases. As Engineer Tan uses less strips, his estimate should be the smaller one which is 1.799.

$A \approx A(8)+\frac{a}{8^2} \Rightarrow 64A \approx 115.136+a$ ----- (1)

$A \approx A(16)+\frac{a}{16^2} \Rightarrow 256A \approx 469.248+a$ ----- (2)

Solving (1) and (2) gives $A \approx 1.844$

(iii) Engineer Shin's estimate (using Simpson's Rule)

$=\frac{1}{3}\left(\frac{\mathrm{e}}{6}\right)\left(y_0+4y_1+2y_2+4y_3+2y_4+4y_5+y_6\right)$

$=\frac{1}{3}\left(\frac{\mathrm{e}}{6}\right)\left(0+4(0.81175)+2(0.99545)+4(0.94208)+2(0.73478)+4(0.41300)+0\right)$

$=1.83148 \approx 1.831$ (to 3 d.p)

Simpson's rule uses a quadratic curve to interpolate the points on the curve (ordinates) so it usually gives a better approximate to the actual curve than trapezium rule which uses a straight line to interpolate the ordinates. Hence Engineer Shin's estimate obtained by Simpson's rule gives a better approximation.

(iv) Using "Cylindrical Shell" Method,

Volume $=2\pi\int_0^{\mathrm{e}} x\left(x-x\ln x\right)\,\mathrm{d}x$

$=2\pi\int_0^{\mathrm{e}} x^2\,\mathrm{d}x-2\pi\int_0^{\mathrm{e}} x^2\ln x\,\mathrm{d}x$

$=2\pi\left[\frac{x^3}{3}\right]_0^{\mathrm{e}}-2\pi\left(\left[\frac{x^3}{3}\ln x\right]_0^{\mathrm{e}}-\int_0^{\mathrm{e}}\frac{x^3}{3}\left(\frac{1}{x}\right)\mathrm{d}x\right)$

$$= \frac{2\pi e^3}{3} - 2\pi\left(\frac{e^3}{3} - 0\right) + \frac{2\pi}{3}\int_0^e x^2 \, dx$$

$$= \frac{2\pi}{3}\left[\frac{x^3}{3}\right]_0^e = \frac{2\pi e^3}{9}$$

8 A study is done to find out how the population of a plant species changes under different environmental conditions. Let P denote the size of the population t months after the study began. It is given that

$$\frac{dP}{dt} = kP(N - P)$$

where k is a positive constant and N is the maximum sustainable population of the plant species in the region.

Using an appropriate sketch of P against t for each of the following on a single diagram, explain without solving the differential equation, how P changes with t when

(i) $p_\circ > N$ **(ii)** $0 < p_\circ < N$

where $P = p_0$ is the initial population size. [4]

It is given that $p_\circ = 20$ and $k = 0.01$.

(a) When $N = 1500$, Euler's method of step size $\Delta t = 0.5$ month is used to estimate the population sizes. The following table of result is obtained where $P = p_n$ and $f(t_n, p_n) = kp_n(N - p_n)$ after t_n months.

n	t_n	p_n	$f(t_n, p_n)$
0	0	20	266
1	0.5	168	1831
2	1	1287	2742

Explain, with the aid of an appropriate sketch, whether the population size estimated using Euler's method at the end of one month is an over or under-estimation. [3]

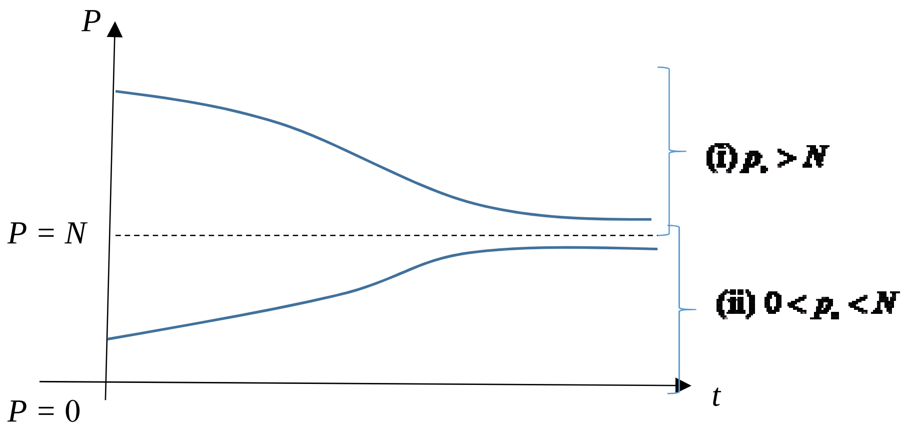
Explain whether Euler's Method can be used to estimate the population size at the end of the second month. [2]

(b) Due to the presence of locust during different seasons, N varies with t and is given by

$$N = 1500\left[1 - \frac{1}{10}\cos\left(\frac{1}{4}t\right)\right]$$

Use Improved Euler's Method once to estimate the population size at the end of one month. [3]

Solution



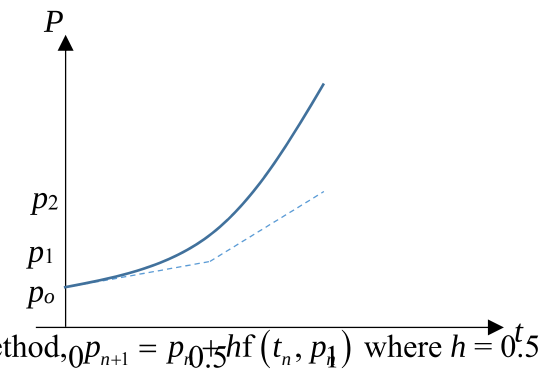
(i) P decreases and stabilise at a population size of N in the long term.

(ii) P increases and stabilise at a population size of N in the long term.

(a) $\frac{\mathrm{d}P}{\mathrm{d}t} = 0.01P(1500 - P) = \mathrm{f}(t, p)$

$\frac{\mathrm{d}P}{\mathrm{d}t} = 0.01P(1500 - P) > 0$ for $0 < P < 1500$ $\Rightarrow$ graph of P against t is increasing.

And since $\mathrm{f}(t_o, p_o) = 266 < \mathrm{f}(t_1, p_1) = 1831.41$ shows that gradient of the tangent lines is increasing $\Rightarrow$ curve is concave upward. Thus 1287 is an under-estimation of the actual population size at the end of one month.



By Euler's method, $p_{n+1} = p_n + h\mathrm{f}(t_n, p_n)$ where $h = 0.5$

$p_3 = 2658$ and $p_4 = -12732 < 0 \because$ once P exceeded 1500,

$\mathrm{f}(t, p) = 0.01P(1500 - P)$ becomes negative and hence the tangent line will end at a point below the t-axis which fails to predict the correct population size at the end of the second month ("jump" to a different solution curve that has $p_o > 1500$). Thus Euler's method fails.

(b)

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1500 - 150\cos\left(\frac{1}{4}t\right) - P\right)$$

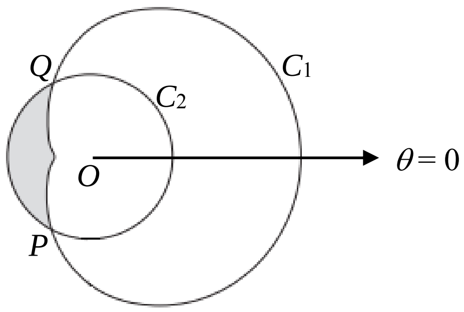
$$\mathrm{f}\left(t_n, P_n\right) = 0.01P_n\left(1500 - 150\cos\left(\frac{1}{4}t_n\right) - P_n\right)$$

$$\begin{aligned} u_1 &= p_\circ + 1 \times \mathrm{f}\left(t_\circ, p_\circ\right) \\ &= 20 + 0.01(20)\left[1500 - 150\cos(0) - 20\right] \\ &= 286 \end{aligned}$$

$$\begin{aligned} p_1 &= p_\circ + \frac{1}{2}\left[\mathrm{f}\left(t_\circ, p_\circ\right) + \mathrm{f}\left(t_1, u_1\right)\right] \\ &= 20 + \frac{1}{2}\left[266 + 0.01(286)\left\{1500 - 150\cos\left(\frac{1}{4}t\right) - 286\right\}\right] \\ &= 20 + \frac{1}{2}\left[266 + 3056.3766\right] \\ &= 1681.188 \approx 1681 \end{aligned}$$

Population size at end of 1 month is 1681.

9



The poloar equation of the curve C_1 is

$$r = 3 + 2\cos\theta, \qquad 0 \le \theta \le 2\pi \,.$$

The circle C_2 with centre at the pole O and radius 2 units intersects C_1 at the points P and Q, as shown in the diagram.

(a) Find the polar coordinates of P and Q. [3]

(b) The straight line PO is extended to intersect the curve again at the point A.

(i) Find the polar coordinates of A. [2]

(ii) Find the exact length of AQ. [3]

(iii) Hence, or otherwise, show that the line AQ is a tangent to the circle C_2. [2]

(c) The shaded region R lies inside C_2 but outside C_1. Show that the area of R can be expressed in the form $\frac{1}{6}\left(a\sqrt{3} + b\pi\right)$, where a and b are integers to be determined.

[8]

Solution:

(a) At intersection of $r = 2$ and $r = 3 + 2\cos\theta$,

$$2 = 3 + 2\cos\theta \;\Rightarrow\; \cos\theta = -\frac{1}{2}$$

$$\Rightarrow\; \theta = \pm\frac{2\pi}{3}$$

Coordinates of P is $\left(2, -\frac{2\pi}{3}\right)$ and coordinates of Q is $\left(2, \frac{2\pi}{3}\right)$

(b)(i) Angle between OA and the line $\theta = 0$ is $\frac{\pi}{3}$

When $\theta = \frac{\pi}{3}$, $r = 3 + 2\cos\frac{\pi}{3} = 4$.

Coordinates of A is $\left(4, \frac{\pi}{3}\right)$.

(ii) $OA = 4, \quad OQ = 2$

$\angle AOQ = \frac{2\pi}{3} - \frac{\pi}{3} = \frac{\pi}{3}$

$AQ^2 = 4^2 + 2^2 - 2(4)(2)\cos\angle AOQ = 12$

$AQ = \sqrt{12}$ [A1]

(iii) Since $\because OA^2 = 4^2 = 2^2 + \left(\sqrt{12}\right)^2 = OQ^2 + AQ^2$, $\therefore \angle OQA = \frac{\pi}{2}$

$\therefore$ AQ is a tangent to the circle C_2.

(c) Area of minor sector OPQ of circle $= \frac{1}{2}(2)^2\left(\frac{2\pi}{3}\right) = \frac{4\pi}{3}$

Area of minor region OPQ of curve

$$\begin{aligned}
&= 2\left(\frac{1}{2}\int_{\frac{2\pi}{3}}^{\pi} (3 + 2\cos\theta)^2 \, d\theta\right) \\
&= \int_{\frac{2\pi}{3}}^{\pi} 9 + 12\cos\theta + 4\cos^2\theta \, d\theta \\
&= \int_{\frac{2\pi}{3}}^{\pi} 9 + 12\cos\theta + 2\cos 2\theta + 2 \, d\theta \\
&= \left[11\theta + 12\sin\theta + \sin 2\theta\right]_{\frac{2\pi}{3}}^{\pi} \\
&= \left[\frac{11\pi}{3} - 6\sqrt{3} + \frac{\sqrt{3}}{2}\right] \\
&= \frac{11\pi}{3} - \frac{11\sqrt{3}}{2}
\end{aligned}$$

Area of shaded region

$$\begin{aligned}
&= \frac{4\pi}{3} - \left(\frac{11\pi}{3} - \frac{11\sqrt{3}}{2}\right) \\
&= \frac{11\sqrt{3}}{2} - \frac{7\pi}{3} \\
&= \frac{1}{6}\left(33\sqrt{3} - 14\pi\right)
\end{aligned}$$