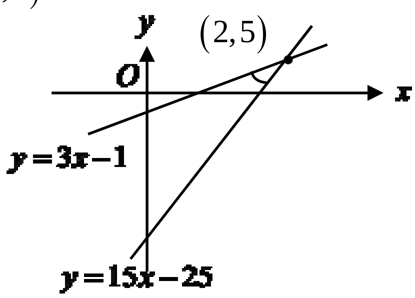
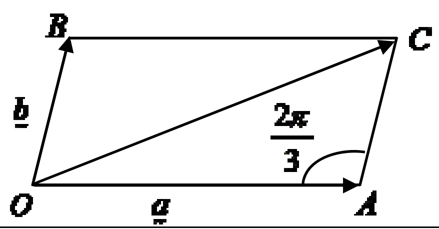
VJC H2 Maths Prelim P2 2017 Solutions/Mark Scheme

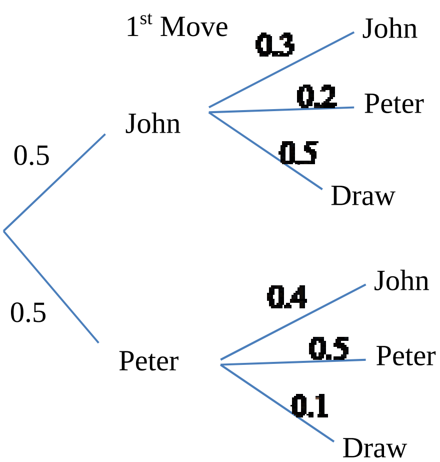
Q	Solution	
Section A: Pure Mathematics [40 marks]		
1	$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}t}\div\frac{\mathrm{d}x}{\mathrm{d}t}$ $=\frac{(1+t)2t-t^2}{(1+t)^2}\div\frac{(1+t)(1)-t}{(1+t)^2}$ $=t^2+2t$	
1i	At point $\left(\frac{p}{1+p},\frac{p^2}{1+p}\right)$, $t=p$ Equation of tangent at point $\left(\frac{p}{1+p},\frac{p^2}{1+p}\right)$, $y-\frac{p^2}{1+p}=\left(p^2+2p\right)\left(x-\frac{p}{1+p}\right)$ $y=p(p+2)x+\frac{p^2}{1+p}-\frac{p^3}{1+p}-\frac{2p^2}{1+p}$ $y=p(p+2)x-\frac{p^2(p+1)}{1+p}$ $y=p(p+2)x-p^2$	
1ii	Tangents pass through $(2,5)$ $\Rightarrow 5=p(p+2)(2)-p^2$ $p^2+4p-5=0$ $p=-5$ or $p=1$  Equations of tangents are $y=3x-1$ and $y=15x-25$ Required acute angle between the 2 tangents $=\tan^{-1}(15)-\tan^{-1}(3)$ $=0.255$ rad or $14.6°$	
2	$\overrightarrow{OC}=\underset{\sim}{a}+\underset{\sim}{b}$  	

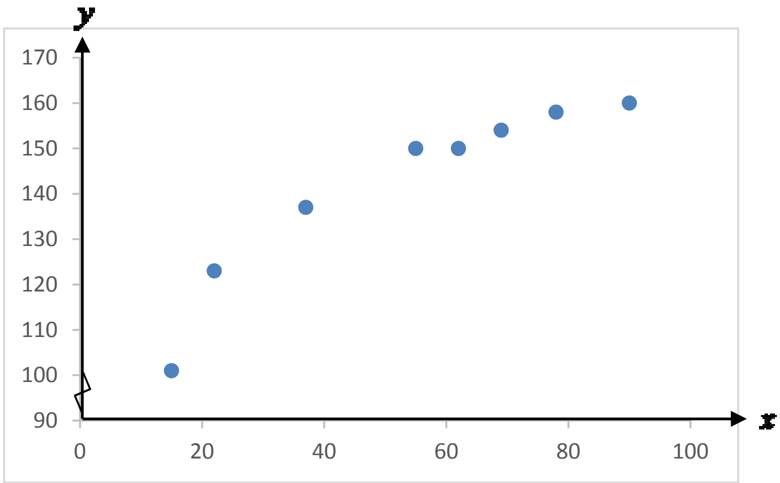
Q	Solution	
2i	Length of projection of $\overrightarrow{OC}$ onto $\overrightarrow{OA}$ $= \left\|(\underset{\sim}{a}+\underset{\sim}{b})\cdot\hat{\underset{\sim}{a}}\right\|$ $= \left\|\underset{\sim}{a}\cdot\hat{\underset{\sim}{a}}+\underset{\sim}{b}\cdot\hat{\underset{\sim}{a}}\right\| = \left\|\underset{\sim}{a}\cdot\underset{\sim}{a}+\underset{\sim}{b}\cdot\underset{\sim}{a}\right\| \quad \because \underset{\sim}{a}=\hat{\underset{\sim}{a}}$ $= \left\| \|\underset{\sim}{a}\|^2 + \|\underset{\sim}{b}\|\|\underset{\sim}{a}\|\cos\left(\pi-\frac{2\pi}{3}\right)\right\|$ $= \left\|1+4\left(\frac{1}{2}\right)\right\|$ $=3$	
2ii	$\lambda\underset{\sim}{a}+\mu\underset{\sim}{b}+\underset{\sim}{d}=\underset{\sim}{0} \qquad ---(1)$ $\lambda+\mu+1=0 \qquad ---(2)$ Sub (2) into (1): $(-1-\mu)\underset{\sim}{a}+\mu\underset{\sim}{b}+\underset{\sim}{d}=\underset{\sim}{0}$ $\mu(\underset{\sim}{b}-\underset{\sim}{a})=\underset{\sim}{a}-\underset{\sim}{d}$ $\mu\overrightarrow{AB}=\overrightarrow{DA}$ Since $AB\,//\,DA$ and A is a common point, A, B and D are collinear	
	Given $\mu=4$, $\underset{\sim}{d}=5\underset{\sim}{a}-4\underset{\sim}{b}$ Area of triangle OBD $=\frac{1}{2}\left\|\underset{\sim}{b}\times\underset{\sim}{d}\right\|$ $=\frac{1}{2}\left\|\underset{\sim}{b}\times(5\underset{\sim}{a}-4\underset{\sim}{b})\right\|$ $=\frac{1}{2}\left\|5\underset{\sim}{b}\times\underset{\sim}{a}-4\underset{\sim}{b}\times\underset{\sim}{b}\right\|$ $=\frac{5}{2}\left\|\underset{\sim}{b}\times\underset{\sim}{a}\right\| \qquad (\because \underset{\sim}{b}\times\underset{\sim}{b}=\underset{\sim}{0})$ $=\frac{5}{2}\left\|\underset{\sim}{a}\times\underset{\sim}{b}\right\|$ $\therefore k=\frac{5}{2}$	
3i	$ar=a+(8-1)d \Rightarrow d=\frac{ar-a}{7}$ $ar^2=a+(13-1)d \Rightarrow d=\frac{ar^2-a}{12}$ $\frac{ar-a}{7}=\frac{ar^2-a}{12}$ $12r-12=7r^2-7$ $7r^2-12r+5=0$	

Q	Solution	
3ii	From the GC, $r=\frac{5}{7}$ or $r=1$. Since $d\neq 0$, the terms of the geometric series are distinct we conclude that $r\neq 1$. Hence, $r=\frac{5}{7}$. As $\lvert r\rvert=\left\lvert\frac{5}{7}\right\rvert<1$, the geometric series is convergent.	
3iii	$\lvert S_\infty - S_n\rvert < 0.001S_\infty$ $\left\lvert \frac{a}{1-\frac{5}{7}} - \frac{a\left(1-\left(\frac{5}{7}\right)^n\right)}{1-\frac{5}{7}} \right\rvert < 0.001\left(\frac{a}{1-\frac{5}{7}}\right)$ $\left\lvert \frac{a}{1-\frac{5}{7}} \right\rvert \left\lvert 1-\left(1-\left(\frac{5}{7}\right)^n\right)\right\rvert < 0.001\left(\frac{a}{1-\frac{5}{7}}\right)$ $\left(\frac{5}{7}\right)^n < 0.001 \quad (\because a>0)$ $n\ln\left(\frac{5}{7}\right) < \ln 0.001$ $n > \frac{\ln 0.001}{\ln\frac{5}{7}}$ $n > 20.53$ Smallest value of n is 21.	
3iv	$d=\frac{ar-a}{7}=\frac{a\left(\frac{5}{7}\right)-a}{7}=-\frac{2}{49}a$ The sum of the first 2017 terms of the arithmetic series $=\frac{2017}{2}\left[2a+(2017-1)\left(-\frac{2}{49}a\right)\right]$ $=-\frac{566777}{7}a$	
4ai	$\mathrm{f}(x)=1+\frac{1}{3}x+nx^2+\ldots$ Comparing with $\mathrm{f}(x)=\mathrm{f}(0)+\mathrm{f}'(0)x+\frac{\mathrm{f}''(0)}{2!}x^2+\ldots$ $\Rightarrow \mathrm{f}(0)=1$	

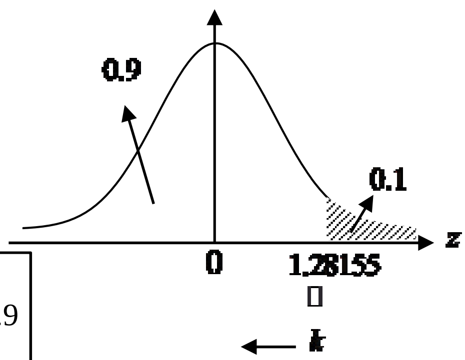
Q	Solution	
	$\Rightarrow \mathrm{f}'(0)=\frac{1}{3}$	
4aii	Given $my^2\frac{\mathrm{d}y}{\mathrm{d}x}-y^3=-\mathrm{e}^x\sin x\ ---(2)$ When $x=0$, $m(1)^2\left(\frac{1}{3}\right)-(1)^3=-\mathrm{e}^0\sin 0$ $\frac{1}{3}m=1$ $m=3$ Differentiate (2) w.r.t. x: $3y^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+6\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2-3y^2\frac{\mathrm{d}y}{\mathrm{d}x}=-\mathrm{e}^x\sin x-\mathrm{e}^x\cos x$ When $x=0$, $3(1)^2(2n)+6\left(\frac{1}{3}\right)^2-3(1)^2\left(\frac{1}{3}\right)=-1$ $6n=-\frac{2}{9}$ $n=-\frac{1}{9}$	
4bi	<u>**Method 1**</u> $\angle ACD=\frac{\pi}{2}+\frac{\pi}{6}=\frac{2\pi}{3}\quad$ (ext angle of a triangle) Using Sine Rule in $\triangle ACD$ $\frac{AD}{\sin\frac{2\pi}{3}}=\frac{AC}{\sin\left(\pi-\frac{2\pi}{3}-\theta\right)}$ $AD=\frac{\sqrt{3}/2}{\sin\left(\frac{\pi}{3}-\theta\right)}$	

Q	Solution	
	$= \dfrac{\sqrt{3}/2}{\sin\frac{\pi}{3}\cos\theta - \cos\frac{\pi}{3}\sin\theta}$ $= \dfrac{\sqrt{3}/2}{\left(\sqrt{3}/2\right)\cos\theta - \left(1/2\right)\sin\theta}$ $= \dfrac{\sqrt{3}}{\sqrt{3}\cos\theta - \sin\theta}$ **<u>Method 2</u>** In right-angled $\triangle ABC$, $AB = \dfrac{1}{\tan\frac{\pi}{6}} = \sqrt{3}$. $\angle ADB = \pi - \dfrac{\pi}{6} - \left(\dfrac{\pi}{2} + \theta\right) = \dfrac{\pi}{3} - \theta$ (angle sum of a triangle) Using Sine Rule in $\triangle ABD$ $\dfrac{AD}{\sin\frac{\pi}{6}} = \dfrac{AB}{\sin\left(\frac{\pi}{3} - \theta\right)}$ $AD = \dfrac{\sqrt{3}\sin\frac{\pi}{6}}{\sin\left(\frac{\pi}{3} - \theta\right)}$ $= \dfrac{\sqrt{3}/2}{\sin\frac{\pi}{3}\cos\theta - \cos\frac{\pi}{3}\sin\theta}$ $= \dfrac{\sqrt{3}/2}{\left(\sqrt{3}/2\right)\cos\theta - \left(1/2\right)\sin\theta}$ $= \dfrac{\sqrt{3}}{\sqrt{3}\cos\theta - \sin\theta}$	
4bii	When θ is a sufficiently small angle,	

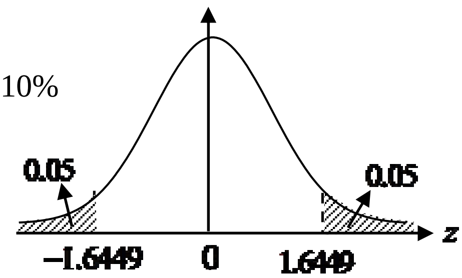
Q	Solution	
	$AD \approx \dfrac{\sqrt{3}}{\sqrt{3}\left(1-\dfrac{\theta^2}{2}\right)-\theta}$ $=\sqrt{3}\left(\sqrt{3}-\theta-\dfrac{\sqrt{3}}{2}\theta^2\right)^{-1}$ $=\left[1+\left(-\dfrac{\theta}{\sqrt{3}}-\dfrac{\theta^2}{2}\right)\right]^{-1}$ $\approx 1-\left(-\dfrac{\theta}{\sqrt{3}}-\dfrac{\theta^2}{2}\right)+\dfrac{(-1)(-2)}{2!}\left(-\dfrac{\theta}{\sqrt{3}}-\dfrac{\theta^2}{2}\right)^2$ $\approx 1+\dfrac{\theta}{\sqrt{3}}+\dfrac{\theta^2}{2}+\dfrac{\theta^2}{3}$ $=1+\dfrac{1}{\sqrt{3}}\theta+\dfrac{5}{6}\theta^2$ $\therefore a=\dfrac{1}{\sqrt{3}}, \quad b=\dfrac{5}{6}$	
Section B: Statistics [60 marks]		
5		
5i	$\mathrm{P}(\text{Peter made first move} \mid \text{Peter won the game})$ $=\dfrac{\mathrm{P}(\text{Peter made first move and Peter won the game})}{\mathrm{P}(\text{Peter won the game})}$ $=\dfrac{0.5\times0.5}{0.5\times0.2+0.5\times0.5}$ $=\dfrac{5}{7}$	
5ii	$\mathrm{P}(\text{John wins})=0.5\times0.3+0.5\times0.4=0.35$	

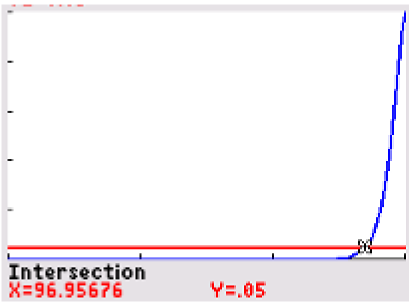
Q	Solution	
	$\mathrm{P}(\text{John wins in exactly 1 game})$ $=(0.35)(0.65)(0.65)\times\frac{3!}{2!}$ $=0.443625$ or $\frac{3549}{8000}$ $=0.444(\text{to 3 s.f.})$ **<u>Alternative</u>** Let X be the number of games won by John out of 3 games. $X \sim B(3, 0.35)$ $\mathrm{P}(\text{John wins in exactly 1 game})$ $=\mathrm{P}(X=1)$ $=0.443625$ or $\frac{3549}{8000}$ $=0.444(\text{to 3 s.f.})$	
6i		
6iia	From GC, r = 0.93639 = 0.936 (3 s.f)	
6iib	From GC, r = 0.98775 = 0.988 (3 s.f)	
6iii	Since 1) the points on the scatter diagram seem to lie close to an increasing curve with decreasing gradient (or close to a curve in which y increases by decreasing amounts as x increases), and 2) the product moment correlation coefficient between ln x and y of 0.988 is closer to 1 than the product moment correlation coefficient between x and y of 0.936, hence $y=c+d\ln x$ is the better model.	
6iv	From (iii), we should use the regression line of y on ln x. From GC, the equation of the regression line of y on ln x is	

Q	Solution	
	$y = 20.8496 + 31.539\ln x$ $y = 20.8 + 31.5\ln x$ (3 s.f) When y = 144, $144 = 20.8496 + 31.539\ln x$ $\therefore x = 49.635 = 50$ (nearest gram)	
6v	Since x = 110 is outside the range of data values ($15 \le x \le 90$), hence the estimated value of y may not be reliable.	
7i	P(no odd digits) = P(all even digits) $= \frac{{}^4C_4}{{}^9C_4} \left(\text{or } \frac{{}^4P_4}{{}^9P_4} \right)$ $= \frac{1}{126}$	
7ii	$\mathrm{P}(X = 1) = \frac{{}^5C_1\,{}^4C_3}{{}^9C_4} = \frac{10}{63}$ x: 0, 1, 2, 3, 4 $\mathrm{P}(X = x)$: $\frac{1}{126}$, $\frac{10}{63}$, $\frac{10}{21}$, $\frac{20}{63}$, $\frac{5}{126}$	
7iii	$\mathrm{E}(X) = \sum_{x=0}^{4} x\mathrm{P}(X = x)$ $= 1\left(\frac{10}{63}\right) + 2\left(\frac{10}{21}\right) + 3\left(\frac{20}{63}\right) + 4\left(\frac{5}{126}\right)$ $= \frac{20}{9}$ $\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$ $= \sum_{x=0}^{4} x^2\mathrm{P}(X = x) - \left(\frac{20}{9}\right)^2$ $= 1\left(\frac{10}{63}\right) + 4\left(\frac{10}{21}\right) + 9\left(\frac{20}{63}\right) + 16\left(\frac{5}{126}\right) - \left(\frac{20}{9}\right)^2$ $= \frac{50}{81}$	

Q	Solution	
7iv	$\mathrm{P}\left(\lvert X_1 - X_2\rvert < 3\right) = \mathrm{P}\left(-3 < X_1 - X_2 < 3\right)$ $= 1 - 2\mathrm{P}\left(X_1 = 0 \quad \& \quad X_2 = 4\right)$ $-2\mathrm{P}\left(X_1 = 0 \quad \& \quad X_2 = 3\right)$ $-2\mathrm{P}\left(X_1 = 1 \quad \& \quad X_2 = 4\right)$ $= 1 - 2\left(\frac{1}{126}\right)\left(\frac{5}{126}\right) - 2\left(\frac{1}{126}\right)\left(\frac{20}{63}\right)$ $-2\left(\frac{10}{63}\right)\left(\frac{5}{126}\right)$ $= \frac{7793}{7938} \quad (\text{or } 0.982)$	
8	Let X kg and Y kg be the mass of a randomly chosen D25 durian and Musang Queen durian respectively. $X \sim \mathrm{N}(1.5, 0.02^2), \qquad Y \sim \mathrm{N}(1.8, 0.035^2)$	
8ii	Let $T = 9\left(X_1 + X_2 + X_3\right) + 18\left(Y_1 + Y_2\right)$ $\mathrm{E}(T) = \mathrm{E}\left[9\left(X_1 + X_2 + X_3\right) + 18\left(Y_1 + Y_2\right)\right]$ $= 9(3)(1.5) + 18(2)(1.8)$ $= 105.3$ $\mathrm{Var}(T) = \mathrm{Var}\left[9\left(X_1 + X_2 + X_3\right) + 18\left(Y_1 + Y_2\right)\right]$ $= (9)^2(3)(0.02)^2 + (18)^2(2)(0.035)^2$ $= 0.891$ $T \sim \mathrm{N}(105.3, 0.891)$ $\mathrm{P}(T > 107) = 0.035852$ $= 0.0359$ (3 s.f)	
8ii	The masses of all the durians are independent of each other.	
8iii	Let $\overline{X} = \frac{X_1 + X_2 + \ldots + X_n}{n}$ $\overline{X} \sim \mathrm{N}\left(1.5, \frac{0.02^2}{n}\right)$ Given $\mathrm{P}\left(\overline{X} > m\right) \ldots 0.1$ $\mathrm{P}\left(Z > \frac{m - 1.5}{0.02/\sqrt{n}}\right) \ldots 0.1$ $\boxed{\Rightarrow \mathrm{P}\left(Z < \frac{m - 1.5}{0.02/\sqrt{n}}\right) „ \ 0.9}$  	

Q	Solution	
	From the GC, $\mathrm{P}\left(Z<1.28155\right)=0.9$ $\therefore \dfrac{m-1.5}{0.02/\sqrt{n}}$ „ 1.28155 $\Rightarrow \left(m-1.5\right)\sqrt{n}$ „ 0.025631 when $m=1.51$ $\Rightarrow \left(1.51-1.5\right)\sqrt{n}$ „ 0.025631 $\Rightarrow n$ „ 6.5695 Largest value of *n* is 6	
9i	Let $y=x-240$ unbiased estimate of population mean $=\bar{x}$ $=\bar{y}+240$ $=\dfrac{\sum y}{n}+240$ $=\dfrac{120}{50}+240$ $=242.4$ Unbiased estimate of population variance $=s^2$ $=\dfrac{1}{n-1}\left(\sum y^2-\dfrac{\left(\sum y\right)^2}{n}\right)$ $=\dfrac{1}{49}\left(11200-\dfrac{120^2}{50}\right)$ $=222.69=223$ (3 s.f)	
9ii	Let μ be the population mean of *X*. $\mathrm{H}_0: \mu=240$ $\mathrm{H}_1: \mu>240$ Level of significance: 10% Test Statistic : since *n* = 50 is sufficiently large, By Central Limit Theorem, $\bar{X}$ is approximately normal. When H_0 is true, $Z=\dfrac{\bar{X}-240}{\frac{S}{\sqrt{50}}}$ N(0,1) approximately	

Q	Solution	
	Computation : $\bar{x} = 242.4$ $s = \sqrt{222.69} = 14.923$ $p-\text{value} = 0.128$ (3 s.f) Conclusion : Since p-value = 0.128 > 0.10, H_0 is not rejected at the 10% significance level. So there is insufficient evidence that the population mean waiting time is more than 240 seconds.	
9iii	No assumption is needed. Since the sample size is large, by Central Limit Theorem, the distribution of the sample mean ($\bar{X}$) is approximately normal.	
9iv	There is a probability of 0.10 that the test will conclude the population mean waiting time is more than 240 seconds when it is actually 240 seconds.	
9v	$H_0 : \mu = k$ $H_1 : \mu \neq k$ Level of significance: 10% For H_0 to be rejected,  $z \leq -1.6449$ or $z \geq 1.6449$ $\frac{\bar{x}-k}{s/\sqrt{50}} \leq -1.6449$ or $\frac{\bar{x}-k}{s/\sqrt{50}} \geq 1.6449$ $\frac{242.4-k}{14.923/\sqrt{50}} \leq -1.6449$ or $\frac{242.4-k}{14.923/\sqrt{50}} \geq 1.6449$ $242.4-k \leq -3.4714$ or $242.4-k \geq 3.4714$ $k \geq 245.87$ or $k \leq 238.93$ $\{k \in \mathbb{R} : k \leq 239 \text{ (3 s.f)} \text{ or } k \geq 246 \text{ (3 s.f)}\}$	
10i	The assumptions are (1) The probability that a customer turn up for the flight is $\frac{p}{100}$ <u>for all the 154 customers</u>. (2) Customers turn up independently of each other.	
10ii	Customers may be travelling in a group or as a family. Therefore, customers may not turn up independently of the others in their group.	

Q	Solution	
10iii	$X \sim \mathrm{B}\left(154, \frac{p}{100}\right)$ Given $\mathrm{P}(X \geq 153) = 0.05$ $\mathrm{P}(X = 153) + \mathrm{P}(X = 154) = 0.05$ $\binom{154}{153}\left(\frac{p}{100}\right)^{153}\left(1 - \frac{p}{100}\right) + \binom{154}{154}\left(\frac{p}{100}\right)^{154} = 0.05$ $154\left(\frac{p}{100}\right)^{153}\left(1 - \frac{p}{100}\right) + \left(\frac{p}{100}\right)^{154} = 0.05$  From the GC, $p = 96.9568 = 97.0 \quad (\text{to 3 s.f.})$	
10iv a	$X \sim \mathrm{B}(154, 0.94)$ $\mathrm{P}(141 \leq X \leq 148) = \mathrm{P}(X \leq 148) - \mathrm{P}(X \leq 140)$ $= 0.825$	
10iv b	$\mathrm{P}(X \leq 150) = 0.98443 = 0.984 \ (\text{to 3 s.f.})$	
10v	Let Y be the number of days (out of 7) in which every customer who turns up gets a seat on the flight $Y \sim \mathrm{B}(7, 0.98443)$ $\mathrm{P}(Y > 5) = 1 - \mathrm{P}(Y \leq 5)$ $= 0.995$	