MERIDIAN JUNIOR COLLEGE
JC2 Preliminary Examination
Higher 2

H2 Mathematics 9758/02

Paper 2 **20 September 2017**

3 Hours

Additional Materials: Writing paper
Graph Paper
List of Formulae (MF 26)

READ THESE INSTRUCTIONS FIRST

Write your name and civics group on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

Section A: Pure Mathematics [40 marks]

1 The complex number z has modulus 3 and argument $\frac{2\pi}{3}$.

(i) Find the modulus and argument of $\frac{-2\mathrm{i}}{z^*}$, where z^* is the complex conjugate of z, leaving your answers in the exact form. [3]

(ii) Hence express $\frac{-2\mathrm{i}}{z^*}$ in the form of $x + \mathrm{i}y$, where x and y are real constants, giving the exact values of x and y in non-trigonometrical form. [2]

(iii) The complex number w is defined such that $w = 1 + \mathrm{i}k$, where k is a non-zero real constant. Given that $\frac{-2\mathrm{i}w}{z^*}$ is purely imaginary, find the exact value of k. [2]

2 Two students are investigating the rate of change of the amount of water in a reservoir, x million cubic metres, at time t hour during a rainfall.

Student A suggests that x and t are related by the differential equation $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \frac{2}{(t+1)^3}$.

(i) Find the general solution of this differential equation. [3]

Student B assumes that the amount of water flowing into the reservoir depends only on the rainfall and is at a constant rate of k million cubic metres per hour. The rate at which water flows out from the reservoir is proportional to the square of the amount of water in the reservoir.

(ii) If the amount of water in the reservoir stabilizes at 0.5 million cubic metres, show that the rate of change of the amount of water in the reservoir can be modelled by the differential equation $\frac{\mathrm{d}x}{\mathrm{d}t} = k\left(1 - 4x^2\right)$. [2]

(iii) Find x in terms of k and t, given that there are initially 1 million cubic metres of water in the reservoir. [5]

3 The function f is defined by

$$\mathrm{f} : x \mapsto \ln\left(x^2 - 1\right),\ x \in \mathbb{R},\ x > 1.$$

(i) Find f^{-1} in similar form. [3]

(ii) Sketch f, f^{-1} and $\mathrm{f}^{-1}\mathrm{f}$ on the same diagram, indicating clearly all asymptotes and axial intercepts. [3]

The functions g and h are defined by

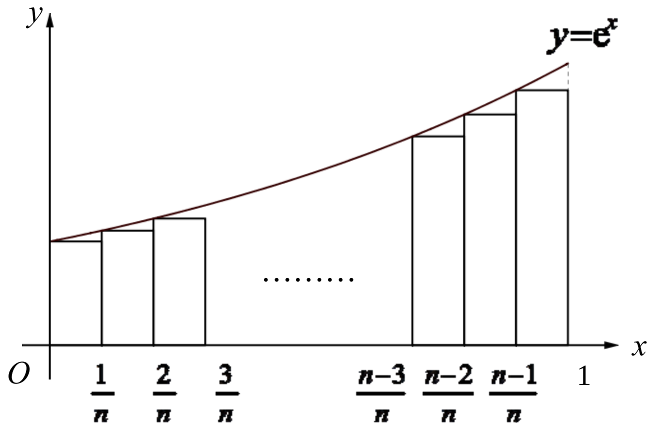
$$\mathrm{g} : x \mapsto \begin{cases} 4\left(x-1\right)^2 & \text{for } 0 \le x < 2, \\ 8 - \left|2x - 8\right| & \text{for } 2 \le x < 8, \end{cases}$$

$$\mathrm{h} : x \mapsto 3\sin x, \qquad 0 \le x \le \pi.$$

(iii) Sketch the graph of $y = \mathrm{g}\left(x\right)$. [3]

(iv) Prove that the function gh exists and find the range of gh. [2]

4 The graph of $y = \mathrm{e}^x$, for $0 \le x \le 1$, is shown in the diagram below. Rectangles, each of width $\dfrac{1}{n}$ where n is an integer, are drawn under the curve.



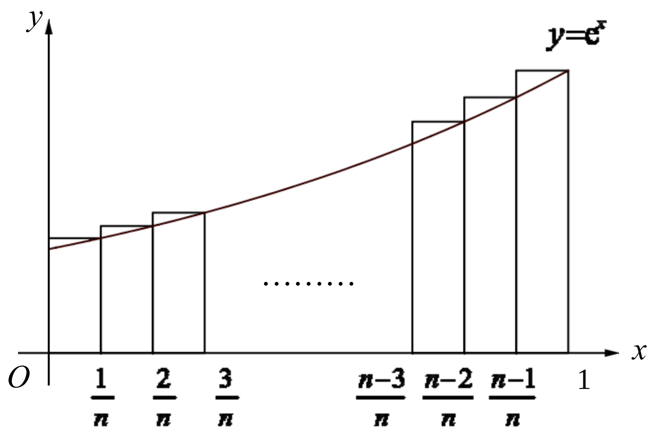
(i) Show that the total area of all the n rectangles, A_n, is $\dfrac{c}{n\left(\mathrm{e}^{\frac{1}{n}}-1\right)}$, where c is an exact constant to be found. [3]

(ii) By considering the Maclaurin Series for $\mathrm{e}^x - 1$, or otherwise, find the value of $\lim\limits_{x \to 0} \dfrac{1}{x}\left(\mathrm{e}^x - 1\right)$. [3]

(iii) Hence, without using integration, find the exact value of $\lim\limits_{n \to \infty} A_n$. [2]

(iv) Give a geometrical interpretation of the value you found in part **(iii)**, and verify your answer in part **(iii)** using integration. [2]

Another set of n rectangles are drawn, as shown in the diagram below.



The total area of all the n rectangles in the second diagram is denoted by B_n. By considering the concavity of the graph of $y = \mathrm{e}^x$, or otherwise, show that

$$\frac{A_n + B_n}{2} > \int_0^1 \mathrm{e}^x \,\mathrm{d}x$$

for any positive integer n. [2]

Section B: Statistics [60 marks]

5 Andy needs two passcodes to open a treasure box. Both passcodes consist of three letters and four digits. Each of the three letters can be any of the twenty-six letters of the alphabet A-Z. Each of the four digits can be any of the ten digits 0-9.

(a) The first passcode consists of three letters followed by four digits. It is also known that no letters and digits are repeated. An example of the code is ABC1234.

(i) Find the total number of possible first passcodes. [2]

(ii) An additional hint is given to Andy to break the first passcode. The four digits of the passcode form a number which is odd and greater than 3000. Find the total number of possible first passcodes. [3]

(b) The second passcode has no fixed arrangement for the letters and digits. Given that the letters and digits can be repeated (i.e. 1AA3C34 can be a possible passcode), find the total number of possible second passcodes. [3]

6 The probability function of X is given by

$$\mathrm{P}(X=x)=\begin{cases}(2x-1)\theta & \text{if } x=1,2,3\\ k & \text{if } x=4\\ 0 & \text{otherwise}\end{cases}$$

where $0<\theta<\frac{1}{9}$.

(i) Show that $k=1-9\theta$. Find, in terms of θ, the probability distribution of X. [2]

(ii) Find $\mathrm{E}(X)$ in terms of θ and hence show that $\mathrm{Var}(X)=26\theta-196\theta^2$. [3]

(iii) The random variable Y is related to X by the formula $Y=a+bX$, where a and b are non-zero constants. Given that $\mathrm{Var}(Y)=\frac{1}{3}b^2$, find the value of θ. [3]

7 Coloured lego pieces are packed into boxes of 20 pieces by a particular manufacturer. Each box is made up of randomly chosen coloured lego pieces. The manufacturer produces a large number of lego pieces every day. On average, 15% of lego pieces are red. Explain why binomial distribution is appropriate for modelling the number of red lego pieces in a box. [2]

(i) Find the probability that a randomly chosen box of lego pieces contains at least 4 red lego pieces. [2]

(ii) A customer buys 50 randomly chosen boxes containing lego pieces. Find the probability that no more than 19 of these boxes contain at least 4 red lego pieces. [2]

It is given instead that the proportion of lego pieces that are red is now p. The probability that there is at least one red lego piece but fewer than four red lego pieces in a box, is 0.22198, correct to 5 significant figures. Write down an equation involving p and hence find the value of p, given that $p > 0.2$. [4]

8 In an assembly line, a machine is programmed to dispense shampoo into empty bottles and the volume of shampoo dispensed into each bottle is a normally distributed continuous random variable X. Under ordinary conditions, the expected value of X is 325 ml.

(i) After a routine servicing of the machine, the assembly manager suspects that the machine is dispensing more shampoo than expected. A random sample of 60 bottles is taken and the data is as follows:

Volume of shampoo (correct to nearest ml)	324	325	326	327	328	329	330
Number of bottles	16	20	9	8	4	1	2

Find unbiased estimates of the population mean and variance, giving your answers to 2 decimal places. [2]

Test, at the 5% significance level, whether the assembly manager's suspicion is valid. [4]

Explain what it meant by the phrase 'at 5% significance level' in the context of the question. [1]

(ii) Due to the assembly manager's suspicion, the machine is being recalibrated to dispense 325 ml of shampoo. Another random sample of 50 is taken and a two-tailed test, at the 5% significance level, concluded that the recalibration is done accurately. Given that the volume of shampoo dispensed into each bottle is normally distributed with standard deviation 1.2 ml, find the set of values the mean volume of the 50 bottles can take, giving your answers to 2 decimal places. [4]

9 The consumer price index measures the average price changes in a fixed basket of consumption goods and services commonly purchased by resident households over time. It is commonly used as a measure of consumer price inflation. In the 2013 Singapore household expenditure survey, housing and food made up about half of the average monthly expenditure of an average household.

The table below shows the housing and food price index from 2005 to 2012, where 2005 is the base period, i.e. in 2005, the price index is 100. For example, the food price index of 104.6 in 2007 means that average food prices increased by 4.6% from 2005 to 2007.

Year	2005	2006	2007	2008	2009	2010	2011	2012
Housing Price Index, x	100	100.7	102.3	116.8	123.1	124.3	[redacted]	148
Food Price Index, y	100	101.6	104.6	112.6	115.2	116.8	120.3	123.1

(i) Show that the value of the missing housing price index for 2011 is 136 (nearest integer), given that the regression line of y on x is $y = 54.271 + 0.48363x$, correct to 5 significant figures. [2]

(ii) Draw the scatter diagram for these values, labelling the axes clearly. Comment on the suitability of the linear model. [3]

(iii) It is required to estimate the housing price index in 2016 where the food price index in 2016 is 134.6. Find the equation of an appropriate regression line for y and $\sqrt{x}$ and use it to find the required estimate. Explain why this estimate might not be reliable. [4]

(iv) Find the product moment correlation coefficient between y and $\sqrt{x}$. [1]

(v) To simplify recordings and calculations, it would be more convenient to tabulate $\frac{x}{100}$ and $\frac{y}{100}$ instead. Without any further calculations, explain if the product moment correlation coefficient between $\sqrt{\frac{x}{100}}$ and $\frac{y}{100}$ would differ from the value obtained in part **(iv)**. [1]

10 **(i)** Factory A produces nuts whose mass may be assumed to be normally distributed with mean μ grams and standard deviation σ grams. A random sample of 50 nuts is taken. It is given that the probability that the mean mass is less than 247 grams is 0.018079, correct to 5 significant figures. It is also given that the probability that the total mass exceeds 12600 grams is 0.78397, correct to 5 significant figures. Find the values of μ and σ, giving your answers to the nearest grams. [5]

(ii) (For this question, you should state clearly the values of the parameters of any normal distribution you use.)

Factory B produces bolts and nuts. The masses, in grams, of bolts and nuts produced are modelled as having independent normal distributions with means and standard deviations as shown in the table:

	Mean Mass (in grams)	Standard Deviation (in grams)
Bolts	745	7.3
Nuts	250	5

(a) Find the probability that the mass of a randomly chosen bolt differs from 3 times the mass of a randomly chosen nut by at least 40 grams. [4]

(b) This factory introduces a new process which is able to reduce the mass of each nut by 10%. Find the probability that the total mass, after the introduction of this process, of 10 randomly chosen nuts is less than 2.24 kg. [3]

End of Paper