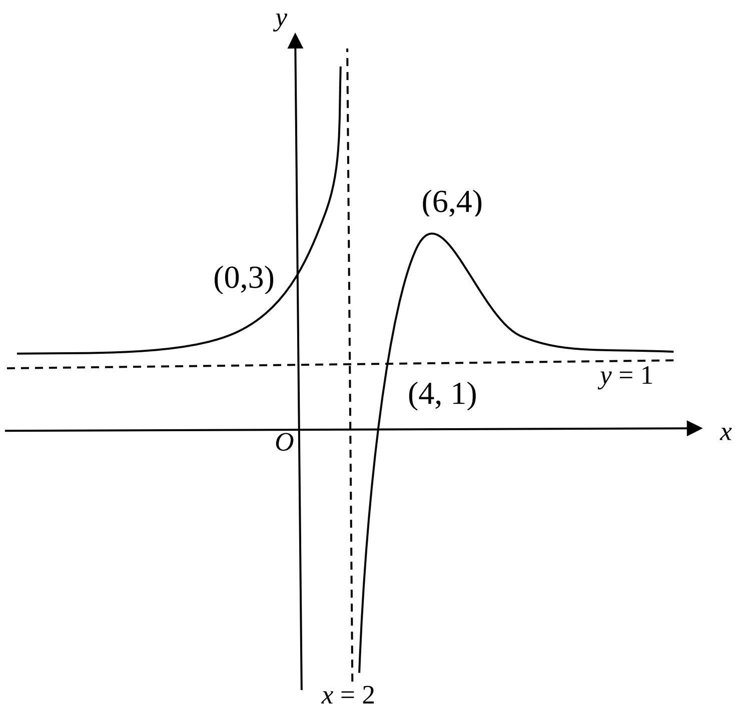
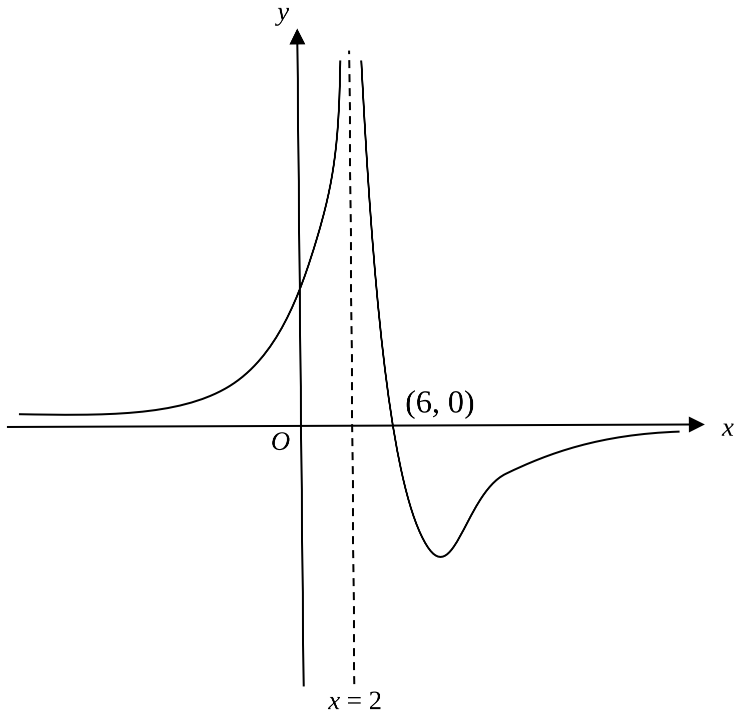
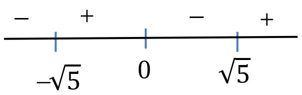
YISHUN JUNIOR COLLEG5E

Mathematics Department

PRELIM Solutions

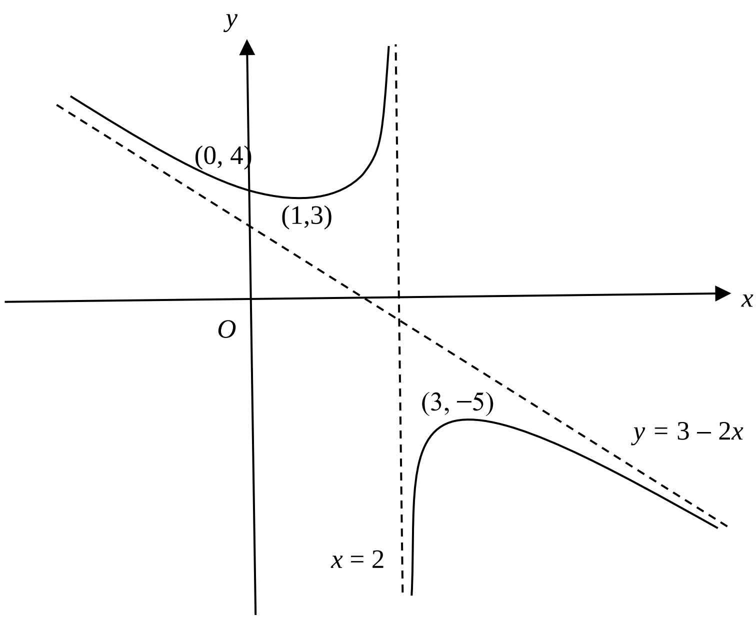
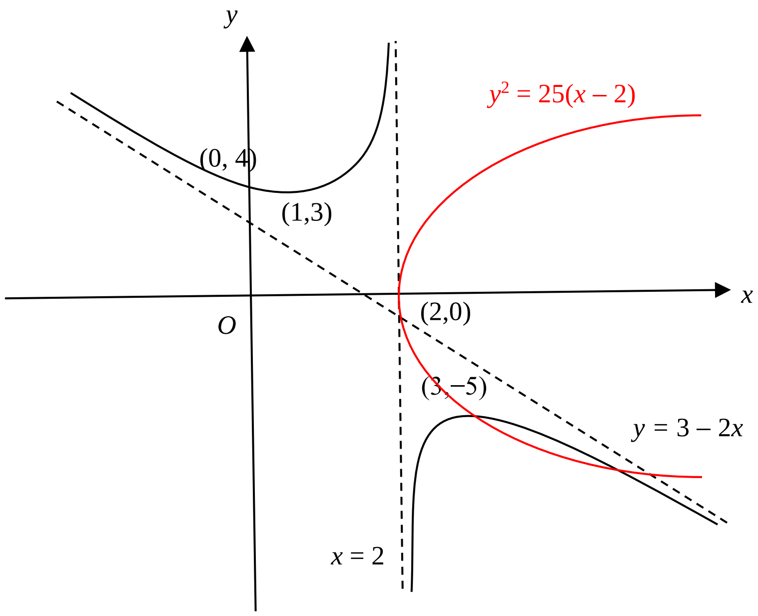
Subject : JC2 H2 MATHEMATICS 9740 P1 **Date** :

Qn	Solution
1(i)	
(ii)	

Qn	Solution
2(i) (ii)	$\dfrac{x}{x^2-5}\le 0$ $\dfrac{x}{\left(x-\sqrt{5}\right)\left(x+\sqrt{5}\right)}\le 0$  $\therefore\ x<-\sqrt{5}$ or $0\le x<\sqrt{5}$ $\dfrac{\sqrt{x}}{x-5}\le 0$ $\dfrac{\sqrt{x}}{\left(\sqrt{x}\right)^2-5}\le 0$ Replace x by $\sqrt{x}$ in the result from (i), $\sqrt{x}<-\sqrt{5}$ or $0\le\sqrt{x}<\sqrt{5}$ (Reject $\because \sqrt{x}\ge 0$) or $0\le x<5$ Required set = $\left\{x\in\mathbf{R}:\ 0\le x<5\right\}$
3	$\overrightarrow{OP}=\dfrac{2}{5}\mathbf{a}$ $\overrightarrow{OQ}=\dfrac{1}{3}\mathbf{b}$ $\overrightarrow{OM}=\dfrac{1}{2}\left(\dfrac{2}{5}\mathbf{a}+\dfrac{1}{3}\mathbf{b}\right)$ Area of triangle *OMP* $=\dfrac{1}{2}\left\|\left(\dfrac{1}{2}\left(\dfrac{2}{5}\mathbf{a}+\dfrac{1}{3}\mathbf{b}\right)\right)\times\dfrac{2}{5}\mathbf{a}\right\|$ $=\dfrac{1}{2}\left\|\left(\left(\dfrac{1}{5}\mathbf{a}+\dfrac{1}{6}\mathbf{b}\right)\right)\times\dfrac{2}{5}\mathbf{a}\right\|$ $=\dfrac{1}{2}\left\|\dfrac{2}{25}\mathbf{a}\times\mathbf{a}+\dfrac{1}{15}\mathbf{b}\times\mathbf{a}\right\|$ $=\dfrac{1}{2}\left\|\dfrac{1}{15}\mathbf{b}\times\mathbf{a}\right\|$ $=\dfrac{1}{30}\left\|-\mathbf{a}\times\mathbf{b}\right\|$ $=\dfrac{1}{30}\left\|\mathbf{a}\times\mathbf{b}\right\|$

Qn	Solution
4(a)	$\int \cos(\ln x)\ \mathrm{d}x = x\cos(\ln x) - \int -x\sin(\ln x).\frac{1}{x}\ \mathrm{d}x$ $= x\cos(\ln x) + \int \sin(\ln x)\ \mathrm{d}x$ $= x\cos(\ln x) + x\sin(\ln x) - \int \cos(\ln x)\ \mathrm{d}x$ $2\int \cos(\ln x)\ \mathrm{d}x = x\cos(\ln x) + x\sin(\ln x) + \text{constant}$ $\int \cos(\ln x)\ \mathrm{d}x = \frac{1}{2}x\left[\cos(\ln x) + \sin(\ln x)\right] + C$
(b)	$\int \frac{1-2x}{2x^2+1}\ \mathrm{d}x$ $= \frac{1}{2}\int \frac{1}{x^2+\frac{1}{2}}\ \mathrm{d}x - \frac{1}{2}\int \frac{4x}{2x^2+1}\ \mathrm{d}x$ $= \frac{\sqrt{2}}{2}\tan^{-1}\sqrt{2}x - \frac{1}{2}\ln\left(2x^2+1\right) + c$
5(i)	$\lvert z\rvert = \left\lvert\sqrt{3}+\mathrm{i}\right\rvert = \sqrt{3+1} = 2$, $\lvert w\rvert = \lvert -1+\mathrm{i}\rvert = \sqrt{1+1} = \sqrt{2}$ $\arg(z) = \arg(\sqrt{3}+\mathrm{i})$ $= \tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6}$ $\arg(w) = \arg(-1+\mathrm{i})$ $= \pi - \tan^{-1}1 = \frac{3\pi}{4}$ $\frac{z^2}{w^*} = \frac{\left(2\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}\right)^2}{\sqrt{2}\mathrm{e}^{\mathrm{i}\left(-\frac{3\pi}{4}\right)}}$ $= 2^{\frac{3}{2}}\mathrm{e}^{\mathrm{i}\frac{13\pi}{12}}$ $= 2^{\frac{3}{2}}\mathrm{e}^{-\mathrm{i}\frac{11\pi}{12}}$
(ii)	$\arg\left(1-\frac{q}{z}\right) = \arg\left(\frac{z-q}{z}\right)$ $= \arg\left(z-q\right) - \arg\left(z\right) = \frac{\pi}{12}$ $\arg\left(z-q\right) = \frac{\pi}{12} + \frac{\pi}{6} = \frac{\pi}{4}$ $\arg\left(\left(\sqrt{3}-q\right)+\mathrm{i}\right) = \frac{\pi}{4}$ $\sqrt{3}-q = 1 \Rightarrow q = \sqrt{3}-1$

Qn	Solution
6(i)	$y=\ln(3+e^x)$ $\dfrac{dy}{dx}=\dfrac{e^x}{3+e^x}$ $\left(3+e^x\right)\dfrac{dy}{dx}=e^x$ $\dfrac{d^2y}{dx^2}\left(3+e^x\right)+e^x\dfrac{dy}{dx}=e^x$ $\dfrac{d^2y}{dx^2}+\dfrac{e^x}{\left(3+e^x\right)}\dfrac{dy}{dx}=\dfrac{e^x}{\left(3+e^x\right)}$ $\dfrac{d^2y}{dx^2}+\left(\dfrac{dy}{dx}\right)^2=\dfrac{dy}{dx}$ (proved)
(ii)	$\dfrac{d^3y}{dx^3}+2\left(\dfrac{dy}{dx}\right)\dfrac{d^2y}{dx^2}=\dfrac{d^2y}{dx^2}$ When $x=0,\ y=\ln 4,\ \dfrac{dy}{dx}=\dfrac{1}{4},\ \dfrac{d^2y}{dx^2}=\dfrac{3}{16},\ \dfrac{d^3y}{dx^3}=\dfrac{3}{32}$ $y=\ln 4+x\left(\dfrac{1}{4}\right)+\dfrac{x^2}{2!}\left(\dfrac{3}{16}\right)+\dfrac{x^3}{3!}\left(\dfrac{3}{32}\right)+...$ $=\ln 4+\dfrac{1}{4}x+\dfrac{3}{32}x^2+\dfrac{1}{64}x^3+...$
(iii)	$\dfrac{e^x}{3+e^x}=\dfrac{1}{4}+\dfrac{3}{16}x+\dfrac{3}{64}x^2+\dots$ $\dfrac{e^{-2x}}{3+e^{-2x}}=\dfrac{1}{4}+\dfrac{3}{16}(-2x)+\dfrac{3}{64}(-2x)^2+\dots$ $=\dfrac{1}{4}-\dfrac{3}{8}x+\dfrac{3}{16}x^2+\dots$
7(i)	$a=-2$ By long division, $y=(b-4)-2x+\dfrac{2b-16}{x-2}$. $b-4=3\Rightarrow b=7$ (shown)

Qn	Solution
(ii)	The equation is $y=\dfrac{-2x^2+7x-8}{x-2}$ 
(iii)	$(-2x^2 + 7x - 8)^2 - 25(x-2)^3 = 0$ $(-2x^2 + 7x - 8)^2 = 25(x-2)^3$ $\left(\dfrac{-2x^2+7x-8}{x-2}\right)^2 = 25(x-2)$ Add graph of $y^2 = 25(x-2)$  From the graphs, the number of real roots is 2.

<table>
<tr><th>Qn</th><th>Solution</th></tr>
<tr><td>8(i)

(ii)</td><td>

This is an AP with $a = 1$, $d = 1$.

For $S_n \le 1016$

$\frac{n}{2}(1+n) \le 1016$

$n^2 + n - 2032 \le 0$

From GC, $-45.58 \le n \le 44.58$

She can complete a maximum of 44 rows.

$S_{44} = \frac{44}{2}(1+44) = 990$

Number of bricks left = 1016 – 990 = 26

The sequence is a GP with common ratio 2

$S_{2k-1} = 1016$

$\frac{m\left[2^{2k-1}-1\right]}{2-1} = 1016$

$m\left[2^{2k-1}-1\right] = 1016 \quad ---(1)$

$T_k = 64 \Rightarrow m2^{k-1} = 64$ ---- (2)

(1) ÷ (2):

$\frac{2^{2k-1}-1}{2^{k-1}} = \frac{1016}{64}$

From GC, $k = 4$

Sub. into (2): $m2^{4-1} = 64$

$\Rightarrow m = 8$

No. of bags = 2(4) – 1 = 7

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<tr><td>9(a)</td><td>

$\frac{\mathrm{d}x}{\mathrm{d}\theta} = 3\sec\theta\tan\theta$

$\int_{3\sqrt{2}}^{6} \frac{3x+1}{\sqrt{x^2-9}}\,\mathrm{d}x$

$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{9\sec\theta+1}{\sqrt{9\sec^2\theta-9}}\left(3\sec\theta\tan\theta\right)\mathrm{d}\theta$

$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{9\sec\theta+1}{3\tan\theta}\left(3\sec\theta\tan\theta\right)\mathrm{d}\theta$

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Qn	Solution
	$=\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} 9\sec^2\theta+\sec\theta \ \mathrm{d}\theta$ $=\left[9\tan\theta+\ln\lvert\sec\theta+\tan\theta\rvert\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$ $=9\tan\frac{\pi}{3}+\ln\left\lvert\sec\frac{\pi}{3}+\tan\frac{\pi}{3}\right\rvert-\left(9\tan\frac{\pi}{4}+\ln\left\lvert\sec\frac{\pi}{4}+\tan\frac{\pi}{4}\right\rvert\right)$ $=9\sqrt{3}+\ln\left\lvert 2+\sqrt{3}\right\rvert-\left(9+\ln\left\lvert\sqrt{2}+1\right\rvert\right)$ $=9\sqrt{3}-9+\ln\frac{2+\sqrt{3}}{\sqrt{2}+1}$
(b)(i)	Consider $y=2\pm 2\sqrt{1-\frac{x^2}{16}}$ Required area $=\int_{-3}^{3} 2+2\sqrt{1-\frac{x^2}{16}} \ \mathrm{d}x-2(6)$ $=10.753\,(3\text{ dp})$ <u>Alternative</u> Consider $\frac{x^2}{16}+\frac{y^2}{4}=1 \Rightarrow y=\pm 2\sqrt{1-\frac{x^2}{16}}$ Required area $=\int_{-3}^{3} 2\sqrt{1-\frac{x^2}{16}} \ \mathrm{d}x$ or $4\int_{0}^{3}\sqrt{1-\frac{x^2}{16}} \ \mathrm{d}x$ $=10.753\,(3\text{ dp})$
(ii)	When $x=3$, $y=2+2\sqrt{1-\frac{9}{16}}=2+\frac{1}{2}\sqrt{7}$ When $x=0$, $y=4$ Required Volume $=\frac{\sqrt{7}}{2}\pi(3^2)+\pi\int_{2+\frac{1}{2}\sqrt{7}}^{4} 16\left(1-\frac{(y-2)^2}{4}\right)\mathrm{d}y$ $=\frac{9\sqrt{7}}{2}\pi+16\pi\int_{2+\frac{1}{2}\sqrt{7}}^{4} 1-\frac{(y-2)^2}{4} \ \mathrm{d}y$ $=\frac{9\sqrt{7}}{2}\pi+16\pi\left[y-\frac{(y-2)^3}{12}\right]_{2+\frac{1}{2}\sqrt{7}}^{4}$ $=\frac{9\sqrt{7}}{2}\pi+16\pi\left[\left(4-\frac{2}{3}\right)-\left(2+\frac{\sqrt{7}}{2}-\frac{7\sqrt{7}}{96}\right)\right]$ $=\frac{1}{3}\left(64-7\sqrt{7}\right)\pi$

Qn	Solution
	Alternative When $x=3$, $y=2\sqrt{1-\frac{9}{16}}=\frac{1}{2}\sqrt{7}$ When $x=0$, $y=2$ Required Volume $=\frac{\sqrt{7}}{2}\pi(3^2)+\pi\int_{\frac{1}{2}\sqrt{7}}^{2}16\left(1-\frac{y^2}{4}\right)\,\mathrm{d}y$ $=\frac{9\sqrt{7}}{2}\pi+16\pi\int_{\frac{1}{2}\sqrt{7}}^{2}1-\frac{y^2}{4}\,\mathrm{d}y$ $=\frac{9\sqrt{7}}{2}\pi+16\pi\left[y-\frac{y^3}{12}\right]_{\frac{1}{2}\sqrt{7}}^{2}$ $=\frac{9\sqrt{7}}{2}\pi+\frac{1}{6}\left[128-41\sqrt{7}\right]\pi$ $=\frac{1}{3}\left(64-7\sqrt{7}\right)\pi$
10(a)	$z=x-y\Rightarrow\frac{\mathrm{d}z}{\mathrm{d}x}=1-\frac{\mathrm{d}y}{\mathrm{d}x}$ $\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=1-\frac{\mathrm{d}z}{\mathrm{d}x}$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{x-y-1}{x-y+1}$. $\Rightarrow 1-\frac{\mathrm{d}z}{\mathrm{d}x}=\frac{z-1}{z+1}$ $\Rightarrow\frac{\mathrm{d}z}{\mathrm{d}x}=1-\frac{z-1}{z+1}$ $\Rightarrow\frac{\mathrm{d}z}{\mathrm{d}x}=\frac{2}{z+1}$ $\int(z+1)\,\mathrm{d}z=\int 2\,\mathrm{d}x$ $\frac{z^2}{2}+z=2x+C$ $\frac{(x-y)^2}{2}+x-y=2x+C$ where C is a constant $\frac{(x-y)^2}{2}-x-y=C$ When $x=1$, $y=1$, $\Rightarrow C=-2$ Therefore $\frac{(x-y)^2}{2}-x-y=-2$

Qn	Solution
(b)	$\frac{\mathrm{d}v}{\mathrm{d}t}=10-kv^2$, where $k>0$ When $v=50, \frac{\mathrm{d}v}{\mathrm{d}t}=7.5$ $7.5=10-k(50)^2$ $\Rightarrow k=0.001$ $\therefore \frac{\mathrm{d}v}{\mathrm{d}t}=10-0.001v^2$ $\int \frac{1}{10-0.001v^2}\ \mathrm{d}v=\int 1\ \mathrm{d}t$ $\frac{1}{0.001}\int \frac{1}{10000-v^2}\ \mathrm{d}v=\int 1\ \mathrm{d}t$ $1000\int \frac{1}{100^2-v^2}\ \mathrm{d}v=\int 1\ \mathrm{d}t$ $\frac{1000}{200}\ln\left\|\frac{100+v}{100-v}\right\|=t+C$ $\ln\left\|\frac{100+v}{100-v}\right\|=\frac{1}{5}t+\frac{1}{5}C$ $\left\|\frac{100+v}{100-v}\right\|=\mathrm{e}^{\frac{1}{5}t+\frac{1}{5}C}$ $\frac{100+v}{100-v}=\pm\mathrm{e}^{\frac{1}{5}t+\frac{1}{5}C}$ $\frac{100+v}{100-v}=A\mathrm{e}^{0.2t}$ where $A=\pm\mathrm{e}^{0.2C}$ When $t=0$, $v=0$ then $A=1$ $\frac{100+v}{100-v}=\mathrm{e}^{0.2t} \Rightarrow \frac{100-v}{100+v}=\mathrm{e}^{-0.2t}$ $\mathrm{e}^{-0.2t}(100+v)=100-v$ $v\left(1+\mathrm{e}^{-0.2t}\right)=100\left(1-\mathrm{e}^{-0.2t}\right)$ $v=\frac{100\left(1-\mathrm{e}^{-0.2t}\right)}{1+\mathrm{e}^{-0.2t}}$ As $t\to\infty$, $\mathrm{e}^{-0.2t}\to 0$ and $v\to 100$ The sky diver's speed would increase to a limit of 100 m/s long after he has descended and before he deployed his parachute.

Qn	Solution
11(i)	$V=2\left[\pi(3r)^2 y\right]+\left(\pi r^2\times 15\right)$ $k=18\pi yr^2+15\pi r^2$ $y=\frac{1}{18\pi r^2}\left(k-15\pi r^2\right)$ $A=4\left[\pi(3r)^2\right]-2\pi r^2+15(2\pi r)+2y\left[2\pi(3r)\right]$ $=34\pi r^2+30\pi r+2\left[\frac{1}{18\pi r^2}\left(k-15\pi r^2\right)\right]\left[2\pi(3r)\right]$ $=34\pi r^2+30\pi r+\frac{2}{3r}\left(k-15\pi r^2\right)$ $=34\pi r^2+20\pi r+\frac{2k}{3r}$ $\frac{\mathrm{d}A}{\mathrm{d}r}=68\pi r+20\pi-\frac{2k}{3r^2}$ At minimum area, $\frac{\mathrm{d}A}{\mathrm{d}r}=0$ $68\pi r+20\pi-\frac{2k}{3r^2}=0$ $204\pi r^3+60\pi r^2-2k=0$ $102\pi r^3+30\pi r^2-k=0$ (shown)
(ii)	$102\pi r^3+30\pi r^2-450=0$ From GC, $r=1.03$ (3 s.f.)
(iii)	Volume of water pumped after 1 min = 15 (60) = 900 cm^3 Volume of a weight = $\pi(3\times 2)^2\times 7=791.68$ cm^3 Volume of the handle = $\pi(2)^2\times 15=188.50$ cm^3 Since $900<791.68+188.50=980.18$, the water level is at the handle at 1 min. Let W = volume of water in the handle and h = depth of water from the base of the handle $W=\pi(2)^2 h=4\pi h$ $\frac{\mathrm{d}W}{\mathrm{d}h}=4\pi$ $\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{\mathrm{d}W}{\mathrm{d}t}\times\frac{\mathrm{d}h}{\mathrm{d}W}$ $=15\times\frac{1}{4\pi}$ Thus the depth of the water is increasing at a rate of $\frac{15}{4\pi}$ cm s^{-1}.

Qn	Solution
12(i)	Let θ be the acute angle between the plane and the incident beam. $\sin\theta = \dfrac{\left\| \begin{pmatrix}1\\2\\1\end{pmatrix} \cdot \begin{pmatrix}-1\\2\\3\end{pmatrix} \right\|}{\sqrt{1+4+1}\sqrt{1+4+9}}$ $= \dfrac{6}{\sqrt{84}}$ Therefore $\theta = 40.9°$
(ii)	Let F be the foot of the perpendicular from O to the plane. $\overrightarrow{OF} = \lambda\begin{pmatrix}-1\\2\\3\end{pmatrix}$, for some $\lambda \in \mathbb{R}$ F is on plane $\Rightarrow \overrightarrow{OF} \cdot \begin{pmatrix}-1\\2\\3\end{pmatrix} = 12$ $\Rightarrow \lambda\begin{pmatrix}-1\\2\\3\end{pmatrix} \cdot \begin{pmatrix}-1\\2\\3\end{pmatrix} = 12$ $14\lambda = 12$ $\lambda = \dfrac{6}{7}$ $\overrightarrow{OO'} = \dfrac{12}{7}\begin{pmatrix}-1\\2\\3\end{pmatrix}$ $\overrightarrow{OP} = \mu\begin{pmatrix}1\\2\\1\end{pmatrix}$, for some $\mu \in \mathbb{R}$ P is on plane $\Rightarrow \overrightarrow{OP} \cdot \begin{pmatrix}-1\\2\\3\end{pmatrix} = 12$ $\Rightarrow \mu\begin{pmatrix}1\\2\\1\end{pmatrix} \cdot \begin{pmatrix}-1\\2\\3\end{pmatrix} = 12$ $6\mu = 12$ $\mu = 2$ $\overrightarrow{OP} = \begin{pmatrix}2\\4\\2\end{pmatrix}$

Qn	Solution
	$\overrightarrow{O'P} = \frac{2}{7}\begin{pmatrix}13\\2\\-11\end{pmatrix}$ Hence $l:\mathbf{r} = \begin{pmatrix}2\\4\\2\end{pmatrix} + \gamma\begin{pmatrix}13\\2\\-11\end{pmatrix}, \gamma \in \mathbb{R}$
(iii)	Let $B \equiv (3,1,0)$. Shortest distance of *B* from incident beam $= \frac{\left\|\begin{pmatrix}3\\1\\0\end{pmatrix} \times \begin{pmatrix}1\\2\\1\end{pmatrix}\right\|}{\sqrt{1+4+1}} = \frac{\left\|\begin{pmatrix}1\\-3\\5\end{pmatrix}\right\|}{\sqrt{6}} = \sqrt{\frac{35}{6}} > 2$ $\overrightarrow{PB} = \begin{pmatrix}3\\1\\0\end{pmatrix} - \begin{pmatrix}2\\4\\2\end{pmatrix} = \begin{pmatrix}1\\-3\\-2\end{pmatrix}$ Shortest distance of *B* from reflected beam $= \frac{\left\|\begin{pmatrix}1\\-3\\-2\end{pmatrix} \times \begin{pmatrix}13\\2\\-11\end{pmatrix}\right\|}{\sqrt{169+4+121}} = \frac{\left\|\begin{pmatrix}37\\-15\\41\end{pmatrix}\right\|}{\sqrt{294}} = \sqrt{\frac{3275}{294}} > 2$ Hence sensor will not work properly