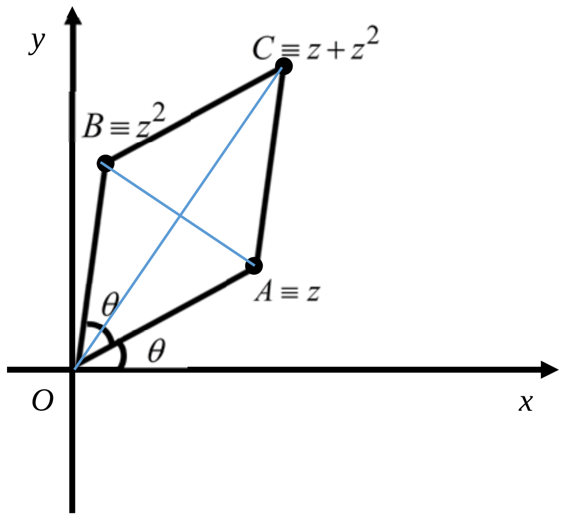
1

(i)



(ii)

Since *OACB* is a parallelogram with 4 equal sides, it is a **rhombus**.

(iii)

$$
\begin{aligned}
&z + z^2 \\
&= \cos\theta + \mathrm{i}\sin\theta + (\cos\theta + \mathrm{i}\sin\theta)^2 \\
&= \cos\theta + \mathrm{i}\sin\theta + \cos^2\theta + 2\mathrm{i}\cos\theta\sin\theta - \sin^2\theta \\
&= (\cos\theta + \cos 2\theta) + \mathrm{i}(\sin\theta + \sin 2\theta) \\
&= 2\cos\frac{3\theta}{2}\cos\frac{\theta}{2} + 2\mathrm{i}\sin\frac{3\theta}{2}\cos\frac{\theta}{2} \\
&= 2\cos\frac{\theta}{2}\left[\cos\frac{3\theta}{2} + \mathrm{i}\sin\frac{3\theta}{2}\right]
\end{aligned}
$$

<u>Alternative</u>

$$\arg\left(z + z^2\right) = \theta + \frac{\theta}{2} = \frac{3}{2}\theta$$

$$\left|z + z^2\right| = 2OM = 2\cos\left(\frac{\theta}{2}\right)$$

$$z + z^2 = 2\cos\left(\frac{\theta}{2}\right)\left[\cos\left(\frac{3}{2}\theta\right) + \mathrm{i}\sin\left(\frac{3}{2}\theta\right)\right]$$

$$\therefore p = 2,\ q = \frac{1}{2},\ k = \frac{3}{2}$$

2

(i)

$$\mathrm{f} : x \mapsto 3 + \frac{1}{x-2},\quad x \in \mathbb{R},\quad x > 2$$

Let $y = \mathrm{f}(x)$.

$$y = 3 + \frac{1}{x-2}$$

$$x - 2 = \frac{1}{y-3}$$

$$x = 2 + \frac{1}{y-3}$$

$$\therefore \quad \mathrm{f}^{-1}(x) = 2 + \frac{1}{x-3}, \ x \in \mathbb{R}, \ x > 3$$

(ii)

$$\mathrm{D_f} = (2, \infty)$$

$$\mathrm{R_f} = (3, \infty)$$

Since $\mathrm{R_f} \subseteq \mathrm{D_f}$, the composite function f^2 exists.

(iii)

$$\mathrm{f}^2(x) = x$$

$$\mathrm{f}\left(3 + \frac{1}{x-2}\right) = x$$

$$3 + \frac{1}{3 + \frac{1}{x-2} - 2} = x$$

$$3 + \frac{1}{\left(\frac{x-1}{x-2}\right)} = x$$

$$\frac{3(x-1) + (x-2)}{x-1} = x$$

$$4x - 5 = x(x-1)$$

$$x^2 - 5x + 5 = 0$$

Using GC, $x = 1.38$ (rej $\because 1.38 \notin \mathrm{D_f}$) or $x = 3.62$

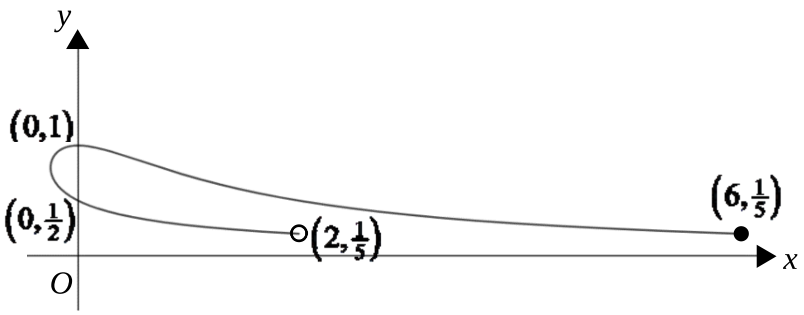
$$\mathrm{ff}(x) = x$$

$$\mathrm{f}^{-1}\mathrm{ff}(x) = \mathrm{f}^{-1}(x)$$

$$\mathrm{f}(x) = \mathrm{f}^{-1}(x)$$

Therefore $x = 3.62$ satisfies $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$.

3	



When $x = 0$, $t(t-1)=0 \quad \Rightarrow t=0$ or $t=1$

$\Rightarrow y=1$ or $y=\frac{1}{2}$

Coordinates are $(0,1)$ and $\left(0,\frac{1}{2}\right)$.

(ii)

$\frac{dx}{dt}=2t-1, \quad \frac{dy}{dt}=\frac{-2t}{\left(t^2+1\right)^2}$

$$\therefore \frac{dy}{dx}=\frac{-2t}{\left(t^2+1\right)^2}\times\frac{1}{2t-1}$$

$$=\frac{-2t}{\left(t^2+1\right)^2(2t-1)}$$

When tangent is parallel to y-axis,

$$\left(t^2+1\right)^2(2t-1)=0 \Rightarrow t=\frac{1}{2} \qquad \left(\because \left(t^2+1\right)^2>0\right)$$

Equation of tangent: $x=-\frac{1}{4}$

(iii)

Area of the required region

$$=\int_{-1/4}^{0} y \, dx$$

$$=\int_{1/2}^{1} \frac{1}{t^2+1}(2t-1)\, dt$$

When $x=-\frac{1}{4}$, $t=\frac{1}{2}$ When $x=0$, $t=1$

$$=\int_{1/2}^{1} \frac{2t}{t^2+1}-\frac{1}{t^2+1}\, dt$$

$$=\left[\ln\left(t^2+1\right)-\tan^{-1}t\right]_{1/2}^{1}$$

$$=\left[\left(\ln 2-\frac{\pi}{4}\right)-\left(\ln\frac{5}{4}-\tan^{-1}\frac{1}{2}\right)\right]$$

$$=\ln\frac{8}{5}-\frac{\pi}{4}+\tan^{-1}\frac{1}{2}$$

4	(a)(i) Area of **unsown** ploughed land $=0.4\left[0.4(300)+100\right]$ $=88\ \text{m}^2$

(a)(ii)

n	Beginning of week	End of week
1	300	$0.4(300)$
2	$0.4(300)+100$	$0.4\left[0.4(300)+100\right]$ $=0.4^2(300)+0.4(100)$
3	$0.4^2(300)+0.4(100)$ $+100$	$0.4\left[0.4^2(300)+0.4(100)+100\right]$ $=0.4^3(300)+0.4^2(100)+0.4(100)$
..	…	…
n	…	$0.4^n(300)+0.4^{n-1}(100)+...$ $+0.4^2(100)+0.4^1(100)$

Area of land **unsown** ploughed land at the end of nth week

$$=0.4^n(300)+100\left[\frac{0.4\left(1-0.4^{n-1}\right)}{1-0.4}\right]$$

$$=\left[0.4^n(300)+\frac{200}{3}\left(1-0.4^{n-1}\right)\right]\ \text{m}^2$$

$\therefore$ the value of k is $\dfrac{200}{3}$.

(a)(iii)

<u>Method 1</u>

$$0.4^n(300)+\frac{200}{3}\left(1-0.4^{n-1}\right)<70$$

$$0.4^n(300)+\frac{200}{3}-\frac{200}{3}(0.4)^{-1}0.4^n<70$$

$$\frac{400}{3}\left(0.4^n\right)<\frac{10}{3}$$

$$0.4^n<\frac{1}{40}$$

$$n>\ln\left(\tfrac{1}{40}\right)\Big/\ln 0.4$$

$$n>4.02588$$

Hence the number of complete weeks required is 5.

Method 2

$$0.4^n(300)+\frac{200}{3}\left(1-0.4^{n-1}\right)<70$$

Using GC,
when n = 4, unsown ploughed land = 70.08 (> 70)
when n = 5, unsown ploughed land = 68.032 (< 70)
when n = 6, unsown ploughed land = 67.213 (< 70)

Hence the number of complete weeks required is 5.

(b)(i)

n	Beginning of week	End of week
1	300	$300-80$
2	$300+(100)-80$	$300+(100)-80-100$
3	$300+2(100)-80-100$	$300+2(100)-80-100-120$
..	$\ldots$	$\ldots$
n	$\ldots$	$300+(n-1)(100)-80-100$ $-\ \cdots\ -\left[80+20(n-1)\right]$

Area of **unsown** ploughed land at the end of nth week

$$=300+100(n-1)-\frac{n}{2}\left[2(80)+20(n-1)\right]$$

$$=300+100n-100-\frac{n}{2}(140+20n)$$

$$=300+100n-100-70n-10n^2$$

$$=-10n^2+30n+200$$

(b)(ii)
For the farmer to finish sowing all the ploughed farmland,

$$-10n^2+30n+200\le 0$$

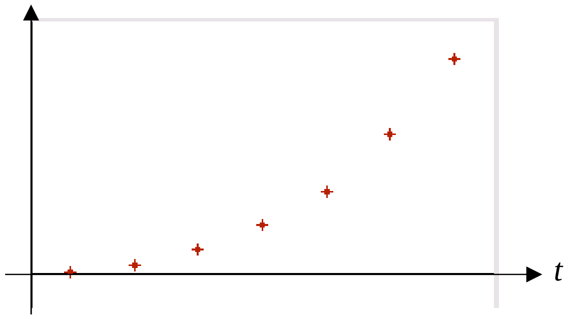
Method 1:
Solving the inequality,
$n\ge 6.21699$ or $n\le -3.21699$ (rejected)
Hence the number of complete weeks is 7.

Method 2:
Using GC to set up a table,
When $n=6$, area unsown $=20$
When $n=7$, area unsown $=-80$
When $n=8$, area unsown $=-200$
Hence the number of complete weeks is 7.

	In week 6, the area of **unsown** ploughed land $=-10(6)^2+30(6)+200=20\ \text{m}^2$ $\therefore$ area of ploughed land to be **sown** in week 7 (the final week) $=20+100=120\ \text{m}^2$
5	(i) Number of arrangements = $6!\times 2^6=46080$ (ii) Required probability $=\dfrac{{}^{6}C_5\times(5-1)!\times 2}{{}^{12}C_{10}\times(10-1)!}$ $=\dfrac{288}{23950080}$ $=0.0000120$ (3 sig fig)
6	(i) P(Clark wins in 3rd draw) $=\dfrac{7}{9}\times\dfrac{7}{9}\times\dfrac{7}{9}\times\dfrac{7}{9}\times\dfrac{2}{9}$ $=0.081322$ $=0.0813$ (ii) P(Kara wins) $=\dfrac{7}{9}\times\dfrac{2}{9}+\left(\dfrac{7}{9}\right)^3\times\dfrac{2}{9}+\left(\dfrac{7}{9}\right)^5\times\dfrac{2}{9}+....$ $=\dfrac{2}{9}\left[\dfrac{7}{9}+\left(\dfrac{7}{9}\right)^3+\left(\dfrac{7}{9}\right)^5+...\right]$ $=\dfrac{2}{9}\left(\dfrac{\frac{7}{9}}{1-\left(\frac{7}{9}\right)^2}\right)$ $=0.4375\ \textit{or}\ \dfrac{7}{16}$
7	(i) Given that X is the number of points scored for one arrow shot. $\text{P}(X=50)=\dfrac{\pi(10)^2}{\pi(60)^2}=\dfrac{1}{36}$ $\text{P}(X=20)=\dfrac{\pi(20)^2-\pi(10)^2}{\pi(60)^2}=\dfrac{1}{12}$ $\text{P}(X=10)=\dfrac{\pi(40)^2-\pi(20)^2}{\pi(60)^2}=\dfrac{1}{3}$

	$\mathrm{E}(X)=(10)\left(\frac{1}{3}\right)+(20)\left(\frac{1}{12}\right)+(50)\left(\frac{1}{36}\right)$ $=6.389$ (4 sig fig) (ii) If the archer is to shoot at the target board repeatedly, then in the long run his average score will be 6.389 points. (iii) $\mathrm{Var}(X)=(10)^2\left(\frac{1}{3}\right)+(20)^2\left(\frac{1}{12}\right)+(50)^2\left(\frac{1}{36}\right)-(6.38888)^2$ $=95.2932$ Let $\overline{X}=\frac{X_1+X_2+\ldots+X_{40}}{40}$. Since n = 40 is large, by Central Limit Theorem, $\overline{X}\sim \mathrm{N}\left(6.38888,\frac{95.2932}{40}\right)$ approximately. Required probability $=\mathrm{P}(10<\overline{X}<20)$ $=0.00965$ (3 sig fig)
8	(i) Whether a randomly chosen patient turns up for an appointment is independent of any other patient. (ii) Let X be the number of patients who turn up for their appointments, out of 20 appointments. $X\sim \mathrm{B}(20,0.845)$ $\mathrm{P}(X>15)$ $=1-\mathrm{P}(X\le 15)$ $=0.812$ (3 sig fig) (iii) Required probability $=\mathrm{P}(X\le 17 \mid X\ge 12)$ $=\frac{\mathrm{P}(12\le X\le 17)}{\mathrm{P}(X\ge 12)}$ $=\frac{\mathrm{P}(X\le 17)-\mathrm{P}(X\le 11)}{1-\mathrm{P}(X\le 11)}$ $=0.618$ (3 sig fig)

	(iv) Let Y be the number of patients who turn up for their appointments, out of n appointments. $Y \sim \mathrm{B}(n, 0.845)$ $\mathrm{P}(Y \le 20) \ge 0.85$ --- (*) Using GC, When $n = 21$, $\mathrm{P}(Y \le 20) = 0.9709$ (> 0.85) When $n = 22$, $\mathrm{P}(Y \le 20) = 0.8762$ (> 0.85) When $n = 23$, $\mathrm{P}(Y \le 20) = 0.7146$ (< 0.85) $\therefore$ Largest n is 22.
9	(i)(a) Given: $L \sim \mathrm{N}(35.2, 5.2^2)$ $P \sim \mathrm{N}(24.6, 3.8^2)$ $C \sim \mathrm{N}(29.3, 4.3^2)$ Let $T = 3L + 2P$. $\mathrm{E}(T) = 3 \times 35.2 + 2 \times 24.6 = 154.8$ $\mathrm{Var}(T) = 3^2 \times 5.2^2 + 2^2 \times 3.8^2 = 301.12$ $\therefore T \sim \mathrm{N}(154.8, 301.12)$ Let a be the required score exceed by 1% of the candidates. $\mathrm{P}(T > a) = 0.01$ $\Rightarrow \mathrm{P}(T \le a) = 0.99$ Using GC, $a = 195.2$ (1 dec pl) (i)(b) Required probability $= [\mathrm{P}(T > 150)]^3 [\mathrm{P}(T < 140)]^2 \times \left(\frac{5!}{2!3!}\right)$ $= 0.0875$ (3 sig fig) (ii) Consider $A = 3L + 2P - 5C$ $\mathrm{E}(A) = 154.8 - 5(29.3) = 8.3$ $\mathrm{Var}(A) = 301.12 + 5^2 (4.3^2) = 763.37$ $\therefore A \sim \mathrm{N}(8.3, 763.37)$ Required probability $= \mathrm{P}(\lvert A \rvert < 25)$ $= \mathrm{P}(-25 < A < 25)$ $= 0.613$ (3 sig fig) Required percentage = 61.3%

10	(i)  (ii) The product moment correlation coefficient between *t* and *m* is $r = 0.94597$ (5 d.p.). A value of 0.94597 for *r* suggests that there is a strong positive linear correlation between *t* and *m*. However, the points on the scatter diagram <u>**show a curvilinear relationship**</u>. Therefore this value of *r* does not necessarily mean that the linear model is best model for the relationship between *t* and *m*. (iii) $m = at^{b}$ $\ln m = \ln\left(at^{b}\right)$ $\ln m = b\ln t + \ln a$ The product moment correlation coefficient between ln *t* and ln *m* is $r = 0.98967 = 0.990$ (3 sig fig) **Reason 1:** From the scatter diagram, as *t* increases, the **weight of the foetus increases at an increasing rate**. **Reason 2:** The value of *r* between ln *t* and ln *m* is 0.98967, which is closer to 1 as compared to that between *t* and *m*, hence indicating a **stronger positive linear correlation** between ln *t* and ln *m*. Hence $m = at^{b}$ is a better model. (iv) From GC, $\ln m = -8.3764 + 4.5938\ \ln t$ (5 sig fig) $\ln a = -8.3764$ $a = 2.30\times10^{-4}$ and $b = 4.59$ (v) When $t = 26$, $\ln m = -8.3764 + 4.5938\ln 26$ $m = 728$ (nearest grams) Since the value of 26 is within the range of values of *t* and the value of *r* is close to 1, this estimate is reliable.

11	(i) Let X be the random variable denoting the mass of strawberry jam, in grams, in a randomly chosen jar. Unbiased estimate of population mean $\bar{x}=\frac{-66}{30}+200=197.8$ Unbiased estimate of population variance $s^2=\frac{1}{29}\left[958-\frac{(-66)^2}{30}\right]=28.02759$ $H_0 : \mu = 200$ $H_1 : \mu < 200$ Test at 2% significance level Assume H_0 is true. $\bar{X} \sim N\left(200, \frac{28.02759}{30}\right)$ Test statistic: $Z=\frac{\bar{X}-200}{\sqrt{28.02759/30}} \sim N(0,1)$ Using GC, p-value = 0.011420121 < 0.02 Reject H_0 and conclude that there is sufficient evidence at 2% level of significance that the mean mass of strawberry jam in each jar is overstated. Therefore the retailer's suspicion is justifiable. (ii) At 2% significance level means that there is a probability of 0.02 that **<u>the test will indicate</u>** that the mean mass of the strawberry jam in the jar is less than 200 g when in fact it is 200 g. (iii) $H_0 : \mu = 200$ $H_1 : \mu \neq 200$ For a two tailed test, the p-value will be twice of 0.0114 which is 0.0228. This value is now more than the 0.02 where we do not reject H_0 at 2% significance level. As such this will result in a different conclusion. (iv) $H_0 : \mu = 200$ $H_1 : \mu \neq 200$ Test at 2% significance level Assume H_0 is true. $\bar{X} \sim N\left(200, \frac{3.5^2}{20}\right)$. Test statistic: $Z=\frac{\bar{X}-200}{\sqrt{3.5^2/20}} \sim N(0,1)$

	For the retailer's suspicion that the mean mass differs from 200 g to be not justified, **do not reject H_0.** $\Rightarrow$ z-value falls outside the critical region $-2.32635 < z\text{-value} < 2.32635$ $-2.32635 < \dfrac{k-200}{3.5/\sqrt{20}} < 2.32635$ $-1.82066 < k - 200 < 1.82066$ $198.17934 < k < 201.82066$ $\Rightarrow 198.2 < k < 201.8$ (to 1 d.p)