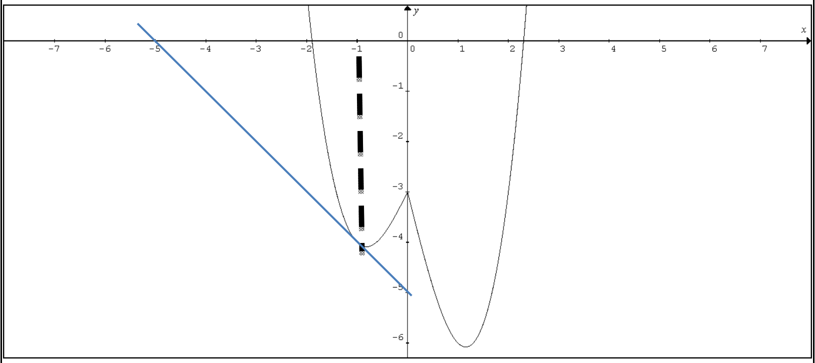
2017 Year 6 FM Prelim II Paper 2 Solution

Question 1 [6 Marks] numerical Analysis		
	$\lvert x\rvert\left(x^2-3\right)-x-3=0$ Let $f(x)=\lvert x\rvert\left(x^2-3\right)-x-3$. $f(0)=-3<0$ $f(-1)=-4<0$ $f(-2)=1>0$ Since y = f(x) is continuous, there is a root is in the interval $(-2,\ -1)$.	
	 $n+1=-1$ is not appropriate because the next approximation it gives is too far away from the actual root, resulting in a slow rate of convergence,	
	Since x is negative, $\lvert x\rvert=-x$. $f(x)=\lvert x\rvert\left(x^2-3\right)-x-3=-x\left(x^2-3\right)-x-3=-x^3+2x-3$ $f'(x)=-3x^2+2$ $x_{n+1}=x_n-\dfrac{-x_n^3+2x_n-3}{-3x_n^2+2}$ $x_0=-2$ $x_1=-1.9$ $x_2=-1.893318233\approx-1.893$ $x_2=-1.893289197\approx-1.893$ $\therefore x=-1.893$ (to 3 dec pl)	

Question 2 [10 Marks] Conics		
	Differentiate $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ wrt x: $\dfrac{2x}{a^2}-\dfrac{2y}{b^2}\dfrac{\mathrm{d}y}{\mathrm{d}x}=0\Rightarrow\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{b^2x}{a^2y}$ Thus, gradient at point P = $\dfrac{b^2x_1}{a^2y_1}$	
	$\tan\angle MPN=\tan\left(90^\circ-\angle PMN\right)$ $=\dfrac{1}{\tan\angle PMN}$ $=\dfrac{a^2y_1}{b^2x_1}$ (shown)	
	In ΔF_1PN , $\tan\angle F_1PN=\dfrac{x_1+c}{y_1}$ In ΔF_2PN , $\tan\angle F_2PN=\dfrac{x_1-c}{y_1}$	
	$\tan\angle F_1PM$ $=\tan(\angle F_1PN-\angle MPN)$ $=\dfrac{\tan\angle F_1PN-\tan\angle MPN}{1+\tan\angle F_1PN\times\tan\angle MPN}$ $=\dfrac{\dfrac{x_1+c}{y_1}-\dfrac{a^2y_1}{b^2x_1}}{1+\left(\dfrac{x_1+c}{y_1}\right)\left(\dfrac{a^2y_1}{b^2x_1}\right)}$ $=\dfrac{b^2x_1^{\ 2}+b^2cx_1-a^2y_1^{\ 2}}{b^2x_1y_1}\times\dfrac{b^2x_1}{b^2x_1+a^2x_1+a^2c}$ $=\dfrac{\left(b^2x_1^{\ 2}-a^2y_1^{\ 2}\right)+b^2cx_1}{y_1\left((a^2+b^2)x_1+a^2c\right)}$ $=\dfrac{a^2b^2+b^2cx_1}{y_1\left(c^2x_1+a^2c\right)}$ Since $b^2x_1^{\ 2}-a^2y_1^{\ 2}=a^2b^2$ and $a^2+b^2=c^2$ $=\dfrac{b^2(a^2+cx_1)}{cy_1(cx_1+a^2)}=\dfrac{b^2}{cy_1}$	

	$\tan \angle F_2PM$ $= \tan(\angle MPN - \angle F_2PN)$ $= \dfrac{\tan \angle MPN - \tan \angle F_2PN}{1 + \tan \tan \angle MPN \times \tan F_2PN}$ $= \dfrac{\dfrac{a^2 y_1}{b^2 x_1} - \dfrac{x_1 - c}{y_1}}{1 + \left(\dfrac{a^2 y_1}{b^2 x_1}\right)\left(\dfrac{x_1 - c}{y_1}\right)}$ $= \ldots\ldots$ $= \dfrac{b^2(cx_1 - a^2)}{cy_1(cx_1 - a^2)} = \dfrac{b^2}{cy_1}$ (shown) Using $b^2x_1{}^2 - a^2y_1{}^2 = a^2b^2$ and $a^2 + b^2 = c^2$ $\tan \angle F_1PM = \tan \angle F_2PM$ $\therefore\ \angle F_1PM = \angle F_2PM$	

<table>
<tr><td colspan="3">Question 3 [11 Marks] Eigenvalues & Eigenvectors</td></tr>
<tr><td>i</td><td>Given that $\mathbf{Ax} = \lambda\mathbf{x}$ and $\mathbf{Bx} = \mu\mathbf{x}$
$\mathbf{ABx} = \mathbf{A}(\mu\mathbf{x})$
$\mathbf{ABx} = \mu(\mathbf{Ax})$
$\mathbf{ABx} = \mu(\lambda\mathbf{x}) = \lambda\mu\mathbf{x}$
$\mathbf{AB}$ has eigenvalue $\lambda\mu$ with eigenvector $\mathbf{x}$.</td><td></td></tr>
<tr><td>ii</td><td>$\mathbf{A} = \begin{pmatrix} 1 & -2 & 1 \\ -2 & 2 & 2 \\ 1 & 2 & 1 \end{pmatrix}$ Consider $\mathbf{Ax} = \lambda\mathbf{x}$
$\mathbf{Ax} - \lambda\mathbf{Ix} = \mathbf{0}$
$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$ ----(*)
Since there are non-zero vector(s) $\mathbf{x}$ that satisfy (*)
$\det(\mathbf{A} - \lambda\mathbf{I}) = 0$
$\det\begin{pmatrix} 1-\lambda & -2 & 1 \\ -2 & 2-\lambda & 2 \\ 1 & 2 & 1-\lambda \end{pmatrix} = 0$
$\begin{pmatrix} 1-\lambda \\ -2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 2-\lambda \\ 2 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ 1-\lambda \end{pmatrix} = 0$
$\begin{pmatrix} 1-\lambda \\ -2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} (2-\lambda)(1-\lambda)-4 \\ 2(1-\lambda)+2 \\ -4-(2-\lambda) \end{pmatrix} = 0$
$-\lambda^3 + 4\lambda^2 + 4\lambda - 16 = 0$
$(\lambda+2)(\lambda-2)(\lambda-4) = 0$
$\lambda = -2,\ 2$ or 4

When $\lambda = -2$
$\begin{pmatrix} 3 & -2 & 1 \\ -2 & 4 & 2 \\ 1 & 2 & 3 \end{pmatrix}\mathbf{x} = \mathbf{0}$
Note that $\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ -4 \end{pmatrix}$
Therefore $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$
When $\lambda = 2$</td><td></td></tr>
</table>

	$\begin{pmatrix} -1 & -2 & 1 \\ -2 & 0 & 2 \\ 1 & 2 & -1 \end{pmatrix}\mathbf{x} = \mathbf{0}$ Note that $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ -2 \end{pmatrix}$ Therefore $\mathbf{e}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ When $\lambda = 4$ $\begin{pmatrix} -3 & -2 & 1 \\ -2 & -2 & 2 \\ 1 & 2 & -3 \end{pmatrix}\mathbf{x} = \mathbf{0}$ Note that $\begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix} \times \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$ Therefore $\mathbf{e}_3 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$ $\mathbf{A} = \mathbf{Q}^{-1}\mathbf{D}\mathbf{Q}$ $\mathbf{Q}^{-1} = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 0 & 2 \\ -1 & 1 & 1 \end{pmatrix} \Rightarrow \mathbf{Q} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{6} & \frac{1}{3} & \frac{1}{6} \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$	
iii	(table below)	

Matrix	Eigenvalues	Eigenvectors
$\mathbf{A}$	$-2, 2, 4$	$\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$
$\mathbf{A}+\mathbf{I}$	$-1, 3, 5$	$\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$
$\mathbf{A}-\mathbf{I}$	$-3, 1, 3$	$\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$
$\mathbf{N}$ $= (\mathbf{A}+\mathbf{I})(\mathbf{A}-\mathbf{I})^{-1}$	$(-1)\left(-\frac{1}{3}\right) = \frac{1}{3}$ $(3)\left(\frac{1}{1}\right) = 3$ $(5)\left(\frac{1}{3}\right) = \frac{5}{3}$	$\mathbf{e}_1$ $\mathbf{e}_2$ $\mathbf{e}_3$

$\mathbf{N}$ has eigenvalues $\frac{1}{3}$, 3, $\frac{5}{3}$ with corresponding eigenvectors

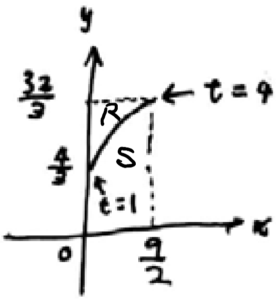
$$\begin{pmatrix}1\\1\\-1\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}-1\\2\\1\end{pmatrix}$$

Question 4 [11 Marks] Application Of Integration

i

$C:\ x=\frac{1}{2}(t-1)^2, \quad y=\frac{4}{3}t^{\frac{3}{2}} \quad \text{for} \quad t\geq 1$

$\frac{dx}{dt}=t-1, \qquad \frac{dy}{dt}=2t^{\frac{1}{2}}$



When $y=\frac{32}{3}$, $t=4$, $x=\frac{9}{2}$

When $x=0$, $t=1$, $y=\frac{4}{3}$

<u>Method 1 (Disc Method)</u>

Volume required when R revolved about x-axis

$=\pi\left(\frac{32}{3}\right)^2\left(\frac{9}{2}\right)-\pi\int_0^{\frac{9}{2}} y^2\ dx$

$=512\pi-\pi\int_1^4 y^2\frac{dx}{dt}\ dt$

$=512\pi-\pi\int_1^4\left(\frac{4}{3}t^{\frac{3}{2}}\right)^2(t-1)\ dt$

$=512\pi-\pi\frac{16}{9}\int_1^4 t^3(t-1)\ dt$

$=512\pi-\frac{16}{9}\pi\left[\frac{1}{5}t^5-\frac{1}{4}t^4\right]_1^4$

$=\frac{1308}{5}\pi$ units^3

<u>Method 2 (Shell Method)</u>

Volume required when R revolved about x-axis

$=\int_{\frac{4}{3}}^{\frac{34}{3}}(2\pi y)x\ dy$

$=2\pi\int_1^4 yx\frac{dy}{dt}\ dt$

$=2\pi\int_1^4\left(\frac{4}{3}t^{\frac{3}{2}}\right)\left(\frac{1}{2}(t-1)^2\right)\left(2t^{\frac{1}{2}}\right)\ dt$ ----(1)

$=\frac{8}{3}\pi\int_1^4(t-1)^2t^2\ dt$

	$=\frac{8}{3}\pi\int_1^4 t^4-2t^3+t^2\ \mathrm{d}t$ $=\frac{8}{3}\pi\left[\frac{1}{5}t^5-\frac{2}{4}t^4+\frac{1}{3}t^3\right]_1^4$ $=\frac{1308}{5}\pi\quad\text{units}^3$	
ii	Volume required when S revolved about $x=\frac{9}{2}$ (Disc Method) $=\pi\left(\frac{9}{2}\right)^2\left(\frac{4}{3}\right)+\pi\int_{\frac{4}{3}}^{\frac{32}{3}}\left(\frac{9}{2}-x\right)^2\ \mathrm{dy}$ $=\pi\left(\frac{9}{2}\right)^2\left(\frac{4}{3}\right)+\pi\int_{\frac{4}{3}}^{\frac{32}{3}}\left(\frac{9}{2}-(\frac{1}{2}(t-1)^2)\right)^2\ \mathrm{dy}$ $=\pi\left(\frac{9}{2}\right)^2\left(\frac{4}{3}\right)+\pi\int_1^4\left(\frac{9}{2}-\frac{1}{2}(t-1)^2\right)^2(2t^{\frac{1}{2}})\ \mathrm{d}t$ ---(2) $\approx 364.421\ \text{units}^3$	
iii	$D:\ x=\frac{1}{2}(t-1)^2,\quad y=\frac{1}{\sqrt{2}}\left(\frac{4}{3}t^{\frac{3}{2}}\right)\quad$ for $\ t\geq 1$ $\frac{\mathrm{d}x}{\mathrm{d}t}=t-1,\qquad \frac{\mathrm{d}y}{\mathrm{d}t}=(\sqrt{2})t^{\frac{1}{2}}$ $A=\int_1^4\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\mathrm{d}t$ $A=\int_1^4\sqrt{(t-1)^2+2t}\ \ \mathrm{d}t$ $A=\int_1^4\sqrt{(t-1)^2+2t}\ \ \mathrm{d}t$ $A=\int_1^4\sqrt{t^2+1}\ \ \mathrm{d}t$ Let $h=\frac{4-1}{6}=\frac{1}{2}$ Let $\mathrm{f}(t)=\sqrt{t^2+1}$ (table below) $A\approx\frac{h}{3}[y_0+4y_1+2y_2+4y_3+2y_4+4y_5+y_6]$ $A\approx\frac{1}{3}\left(\frac{1}{2}\right)[y_0+4y_1+2y_2+4y_3+2y_4+4y_5+y_6]$ $A\approx 8.1458\ $ units (4 d.p)	

n	t_n	$y_n=\mathrm{f}(t_n)$
0	1	1.4142135
1	1.5	1.8027756
2	2	2.2360679
3	2.5	2.6925824
4	3	3.1622776
5	3.5	3.6400549
6	4	4.1231056

<table>
<tr><td colspan="3">Question 5 [12 Marks] DE</td></tr>
<tr><td>(i)</td><td>$-c\dot{x}$ is the damping force,
$-2cx$ is the restoring force due to the spring, and
$4\sin 2t$ is the external driving force.</td><td></td></tr>
<tr><td>(ii)</td><td>Consider $5\ddot{x}+c\dot{x}+2cx=0$
Discriminant $= c^2-4(5)(2c)=c^2-40c$
For system to be critically damped,
$c^2-40c=0 \Rightarrow c=40$ or $c=0$ (rej. since c>0)</td><td></td></tr>
<tr><td>(ii)</td><td>DE: $5\ddot{x}+\dot{x}+2x=4\sin 2t$

$5m^2+m+2=0$
$m=\dfrac{-1\pm\sqrt{1-4(5)(2)}}{2(5)}=-\dfrac{1}{10}\pm\dfrac{\sqrt{39}}{10}\mathrm{i}$
$\Rightarrow x_c=e^{-\frac{1}{10}t}\left(A\cos\dfrac{\sqrt{39}}{10}t+B\sin\dfrac{\sqrt{39}}{10}t\right)$

Let $x_p=a\sin 2t+b\cos 2t$
$\Rightarrow x_p'=2a\cos 2t-2b\sin 2t$ and
$x_p''=-4a\sin 2t-4b\cos 2t$
Substituting back to DE:
$5(-4a\sin 2t-4b\cos 2t)+(2a\cos 2t-2b\sin 2t)$
$+2(a\sin 2t+b\cos 2t)=4\sin 2t$
$(-18a-2b)\sin 2t+(2a-18b)\cos 2t=4\sin 2t$
$\therefore a=-\dfrac{9}{41}$ and $b=-\dfrac{1}{41}$
Thus $x_p=-\dfrac{9}{41}\sin 2t-\dfrac{1}{41}\cos 2t$
General solution:
$x=e^{-\frac{1}{10}t}\left(A\cos\dfrac{\sqrt{39}}{10}t+B\sin\dfrac{\sqrt{39}}{10}t\right)-\dfrac{9}{41}\sin 2t-\dfrac{1}{41}\cos 2t$

As $t\to\infty$, $e^{-\frac{1}{10}t}\to 0$
Hence, $x\to-\dfrac{9}{41}\sin 2t-\dfrac{1}{41}\cos 2t$
$=-\left[\sqrt{\left(\frac{9}{41}\right)^2+\left(\frac{1}{41}\right)^2}\right]\sin(2t+\alpha)$
As $t\to\infty$, the object oscillates about the equilibrium position with an amplitude of $\dfrac{\sqrt{82}}{41}$ cm.</td><td></td></tr>
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<table>
<tr><td colspan="3">Question 6 [9 Marks] Hypothesis Test</td></tr>
<tr><td></td><td>A 90% confidence interval for the mean number of bicycle mis-use cases in a week
$=\left(\overline{x}-t_{(7,0.95)}\frac{s}{\sqrt{n}}, \overline{x}+t_{(7,0.95)}\frac{s}{\sqrt{n}}\right)$
$=\left(20.625-1.894578584\frac{6.9062601}{\sqrt{8}}, 20.625+1.894578584\frac{6.9062601}{\sqrt{8}}\right)$
$=(15.999, 25.251)=(16.0, 25.3)$ (3 s.f.)</td><td></td></tr>
<tr><td></td><td>A "90% confidence interval" means that there is a 90% chance that the interval contains the true mean number of bicycle mis-use cases in a week.</td><td></td></tr>
<tr><td>(i)</td><td>Let μ_1 and μ_2 be the mean number of cases reported before and after implementation of measures respectively.

Test $H_0: \mu_1 = \mu_2$ against $H_1: \mu_1 > \mu_2$

Test Statistic : Under H_0, $T = \dfrac{\overline{X}_1-\overline{X}_2}{s_p\sqrt{\frac{1}{8}+\frac{1}{8}}} \sim t_{(14)}$

$s_p^2 = \dfrac{(8-1)s_1^2+(8-1)s_2^2}{14} = 7.85300216^2 = 61.66964292$

Value of test statistic
$=\dfrac{20.625-18.75}{s_p\sqrt{\frac{1}{8}+\frac{1}{8}}}=0.4775243818=0.478$ (3 s.f.)

Assumptions:
1. The number of cases before and after implementation of preventive measures have (unknown) common variance.
2. The number of cases before and after implementation of preventive measures are normally distributed.</td><td></td></tr>
<tr><td>(ii)</td><td>$P(T > 0.4775243818)$ where $T \sim t(14)$
$= 0.320$</td><td></td></tr>
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<table>
<tr><td colspan="3">Question 7 [12 Marks] Special Discrete Random Variable</td></tr>
<tr><td>(i)</td><td>Let X be the r.v. for the demand per day. $X \sim Po(2.3)$
For booking to be successful,
$P(X \le 3) = 0.7993470513 = 0.799$ (to 3 s.f.)</td><td></td></tr>
<tr><td>(ii)</td><td>Let C be the number of courts booked.

<table>
<tr><td>C</td><td>0</td><td>1</td><td>2</td><td>3</td></tr>
<tr><td>P(C = c)</td><td>P(X = 0)=
0.1002588</td><td>P(X = 1)=
0.2305953</td><td>P(X = 2)=
0.2651846</td><td>P(X ≥ 3)=
0.4039612</td></tr>
</table>

$E(C) = \sum cP(C = c) = 1.972848 = 1.97$ (3 s.f.)</td><td></td></tr>
<tr><td>(iii)</td><td>Let n be the number of courts needed to meet the demand.
$P(X \le n) \ge 0.99$

By GC,
$P(X \le 5) = 0.97002 < 0.99$
$P(X \le 6) = 0.99064 > 0.99$
$P(X \le 7) = 0.99741 > 0.99$

To meet 99% of the demand per day, the least number of courts needed is 6. So, the least number of courts to be built is 3.</td><td></td></tr>
<tr><td>(iv)</td><td>The mean of 2.3 per week is not likely to remain constant throughout the year since there might be peak period when demand will increase (e.g during school holidays) and period when demand will decrease (e.g. during school exam period).</td><td></td></tr>
<tr><td>(v)</td><td>P(Ben succeeded in booking)
$= P(A_{fail}B_{succeed}) + P(A_fB_fA_fB_s) + P(A_fB_fA_fB_fA_fB_s) + \ldots$
$= (0.201)(0.799) + (0.201)^3(0.799) + (0.201)^5(0.799) + \ldots$
$= (0.201)(0.799)\left(1 + (0.201)^2 + (0.201)^4 + \ldots\right)$
$= (0.201)(0.799)\dfrac{1}{1-(0.201)^2}$
$= 0.1673605329$
$= 0.167$ (3 s.f)</td><td></td></tr>
<tr><td>(vi)</td><td>Let Y be the number of successful days booked, out of 20.
$Y \sim B(20, 0.799)$
$P(Y \ge 16)$
$= 1 - P(Y \le 15) = 0.625$ (3 s.f.)</td><td></td></tr>
</table>

Question 8 [12 Marks] Non Parametric Test		
(i)	Precautions: • Use the same route for the journey in calculating the flat and metered fares. • Same period of travel for each journey in calculating the flat and metered fares. • Use the same start point and same end point in calculating the flat and metered fares. Reason: To ensure same distance is travelled and same surcharges for each pair of fares.	
(ii)	Let X_1 and X_2 denote the fares calculated by the flat-fare and metered-fare respectively. (table below) Test $H_o : m = 0$ against $H_1 : m < 0$ where m = median (flat-fare – metered-fare) $t_+ = 6$ $T = T_+$ Reject H_0 if $T \le 5$. Since $t = 6 > 5$, we do not reject H_0 and conclude that at the 5% significance level, there is insufficient evidence that the flat-fare is cheaper than the metered-fare.	
(iii)	Another method is to construct a 95% confidence interval for the mean difference in fares between the two systems. (table below) A 95% confidence interval for the mean difference in fares between the two systems $=\left(\bar{d}-t_{(8,0.975)}\frac{s_d}{\sqrt{9}}, \bar{d}+t_{(8,0.975)}\frac{s_d}{\sqrt{9}}\right)$ $=\left(-2.7777778-2.30600413\frac{4.352521619}{\sqrt{9}}, -2.7777778+2.30600413\frac{4.352521619}{\sqrt{9}}\right)$ $= (-6.123, 0.56787) = (-6.12, 0.568)$ (3 s.f.) Since the CI contains 0, there is insufficient evidence at the 5% level of significance that there is a difference between the mean fares of the two systems. Assumptions: 1. The differences $D_1, D_2, \ldots, D_n$ are normally distributed. 2. Data within each pair of fares are dependent on each other, but the pairs are independent of one another.	

Part (ii) table:

$X_1 - X_2$	+1	–9	–7	–5	+3	0	+2	–4	–6
Signed rank	+1	–8	–7	–5	+3		+2	–4	–6

Part (iii) table:

$D = X_1 - X_2$	+1	–9	–7	–5	+3	0	+2	–4	–6

Question 9 [17 Marks] CRV + GOF

$$P\left(X > x\right) = \frac{\left(2-k\right)x+31}{2x+31}$$

$$F(x) = P\left(X \le x\right) = 1 - \frac{\left(2-k\right)x+31}{2x+31}$$

$$F(1.5) = 1 \Rightarrow 1 - \frac{\left(2-k\right)1.5+31}{2(1.5)+31} = 1 \Rightarrow \frac{\left(2-k\right)1.5+31}{2(1.5)+31} = 0$$

$$\Rightarrow k = \frac{68}{3}$$

Test H0: Data supports claim that the probability that the bank charges interest rate of more than *x*% is $\frac{\left(2-k\right)x+31}{2x+31}$, where $k = \frac{68}{3}$.

Against H1: Data does not support claim that the probability that the bank charges interest rate of more than *x*% is $\frac{\left(2-k\right)x+31}{2x+31}$, where $k = \frac{68}{3}$.

Under H0, $e_i = 100P(a < X < b) = 100\left[\frac{\left(2-k\right)a+31}{2a+31} - \frac{\left(2-k\right)b+31}{2b+31}\right]$.

x	$0 < x \le 0.3$	$0.3 < x \le 0.6$	$0.6 < x \le 0.9$	$0.9 < x \le 1.2$	$1.2 < x \le 1.5$
f	28	20	16	25	11
e	21.51899	20.71704	19.95910	19.24200	18.56287

Test statistic: Under H0, $\chi^2 = \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i} \sim \chi^2_{(5-1)}$

Critical region: Reject H0 if p-value < 0.05

Since p-value = 0.109 > 0.05, we do not reject H0 and conclude that at the 5% significance level, there is insufficient evidence that the data does not support claim that the probability that the bank charges interest rate of more than *x*% is $\frac{\left(2-k\right)x+31}{2x+31}$, where $k = \frac{68}{3}$.

(i)

$$F(i) = P(I \le i) = P(Le^{-X} \le i) = P\left(X \ge \ln\frac{L}{i}\right)$$

$$= \frac{\left(2 - \frac{68}{3}\right)\ln\frac{L}{i} + 31}{2\ln\frac{L}{i} + 31} = \frac{-\frac{62}{3}\ln\frac{L}{i} + 31}{2\ln\frac{L}{i} + 31} \quad \text{for } Le^{-1.5} \le i < L$$

	Let m be the median interest payable. $F(m)=\frac{1}{2}$ $\frac{-\frac{62}{3}\ln\frac{L}{m}+31}{2\ln\frac{L}{m}+31}=\frac{1}{2}$ $-\frac{62}{3}\ln\frac{L}{m}+31=\frac{1}{2}\left(2\ln\frac{L}{m}+31\right)$ $\ln\frac{L}{m}=\frac{93}{130}$ $m=Le^{-\frac{93}{130}}$	
(ii)	$E(I)=E\left(Le^{-X}\right)=LE\left(e^{-X}\right)$ $=L\int_0^{1.5}e^{-x}f(x)\ dx$ $=L\int_0^{1.5}-e^{-x}\left(\frac{(31+2x)(-\frac{62}{3})-(31-\frac{62}{3}x)(2)}{(31+2x)^2}\right)\ dx$ $=0.530L$ The expected interest payable is \$0.530L.	