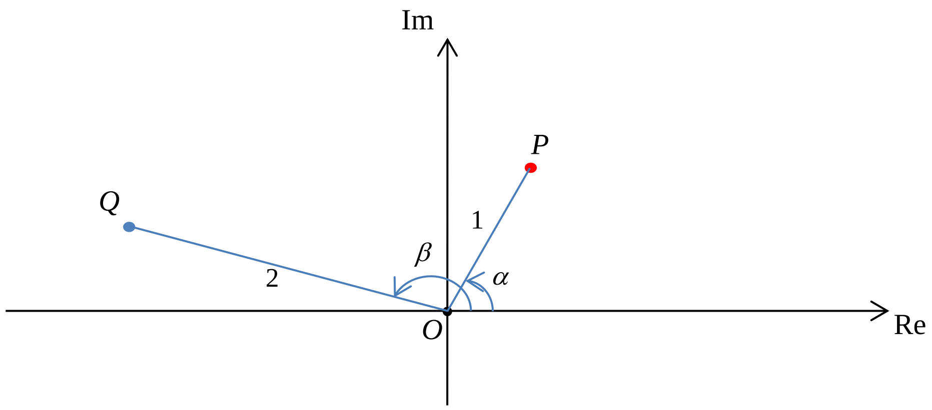
H2 Mathematics 2017 Prelim Exam Paper 1 Question
Answer all questions [100 marks].

1	The complex numbers z and w satisfy the simultaneous equations $\mathrm{i}z + w = 2 + \mathrm{i}$ and $2w - (1+\mathrm{i})z = 8 + 4\mathrm{i}$. Find z and w in the form of $a + \mathrm{i}b$, where a and b are real.	[5]
2	Solve the inequality $\dfrac{2x^2 + 2x - 1}{x^2 + 2x} \le 1$. Hence, solve the inequality $\dfrac{2x^2 + 2\lvert x\rvert - 1}{x^2 + 2\lvert x\rvert} \le 1$.	[6]
3	For $\alpha, \beta \in \mathbb{R}$ such that $2\alpha < \beta$, the complex numbers $z_1 = \mathrm{e}^{\mathrm{i}\alpha}$ and $z_2 = 2\mathrm{e}^{\mathrm{i}\beta}$ are represented by the points P and Q respectively in the Argand diagram below.  Find the modulus and argument of the complex numbers given by $\dfrac{\mathrm{i}}{2}z_2$ and $\dfrac{{z_1}^2}{z_2}$.	[4]
	Copy the given Argand diagram onto your answer script and indicate clearly the following points representing the corresponding complex numbers on your diagram.	
	(i) A: $\dfrac{\mathrm{i}}{2}z_2$	[1]
	(ii) B: $\dfrac{{z_1}^2}{z_2}$	[1]
	You are expected to indicate clearly the relevant moduli and arguments for parts **(i)** and **(ii)** on your Argand diagram. If $\beta = \dfrac{11}{12}\pi$, find the smallest positive integer n such that the point representing the complex number $(z_2)^n$ lies on the negative real axis.	[3]

4 The curve C has equation $4y^2 - 8y - x^2 - 4x - 4 = 0$.

(i) Using an algebraic method, find the set of values that y cannot take. [3]

(ii) Showing any necessary working, sketch C and indicate the equations of the asymptotes. [4]

5 The function f is defined by

$$\mathrm{f} : x \mapsto \frac{\pi}{2}\tan\left(\frac{x}{2}\right), \quad x \in \mathbb{R},\ -2\pi \le x \le 2\pi.$$

(i) Explain why f^{-1} does not exist. [2]

(ii) The domain of f is restricted to $(-\pi, a)$ such that a is the largest value for which the inverse function f^{-1} exists. State the exact value of a and define f^{-1} in a similar form. [3]

In the rest of the question, the domain of f is $(-\pi, a)$, where a takes the value found in part **(ii)**.

(iii) Sketch, in a single diagram, the graphs of $y = \mathrm{f}(x)$ and $y = \mathrm{f}^{-1}(x)$, labelling each graph clearly.Write down the equation of the line in which the graph of $y = \mathrm{f}(x)$ must be reflected in order to obtain the graph of $y = \mathrm{f}^{-1}(x)$ and draw this line on your diagram. [3]

(iv) Verify that $x = \frac{\pi}{2}$ is a root of the equation $x = \mathrm{f}(x)$. Hence, explain why $x = \frac{\pi}{2}$ is also a solution to the equation $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$. [2]

6 Referred to the origin O, the two points A and B have position vectors given by **a** and **b**, where **a** and **b** are non-zero vectors. The line l has equation $\mathbf{r} = 2\mathbf{a} + \lambda(\mathbf{a} + 2\mathbf{b})$, where $\lambda \in \mathbb{R}$. The point E is a general point on l and the point D has position vector $2\mathbf{a} - \mathbf{b}$.

Given that vector **a** is a unit vector, vector **b** has a magnitude of $\sqrt{2}$ units and that $\mathbf{a} \cdot \mathbf{b} = 1$,

(i) find the angle between vectors **a** and **b**, and, [2]

(ii) by considering $\overrightarrow{DE} \cdot \overrightarrow{DE}$, find an expression for the square of the distance DE, leaving your answer in terms of λ. [3]

Hence or otherwise, find the exact shortest distance of D to l, and write down the position vector of the foot of the perpendicular from D to l, in the form $p\mathbf{a} + q\mathbf{b}$. [3]

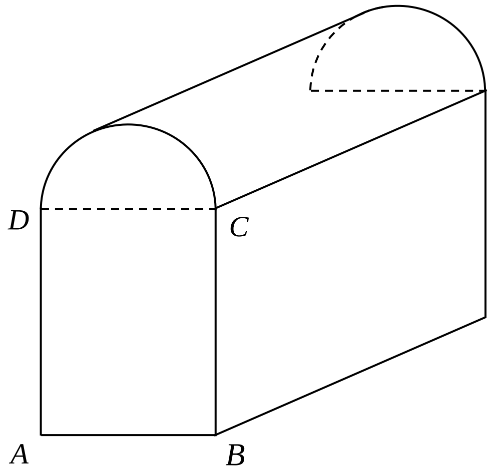
7 **(a)** By considering the Maclaurin expansion for $\cos x$, show that the expansion of $\sec x$ up to and including the term in x^4 is given by $1 + \frac{1}{2}x^2 + \frac{5}{24}x^4$. Hence show that the expansion for $\ln(\sec x)$ up to and including the term in x^4 is given by $\left[\frac{1}{2}x^2 + Ax^4\right]$ where A is an unknown constant to be determined. [4]

(b) The variables x and y satisfy the conditions **(A)** and **(B)** as follows:

$$\left(1 + x^2\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 1 + y \quad \text{---}\mathbf{(A)}$$

$$y = 0 \text{ when } x = 0 \quad \text{---}\mathbf{(B)}$$

	(i) Obtain the Maclaurin expansion of y, up to and including the term in x^3. [4] **(ii)** Verify that both conditions **(A)** and **(B)** hold for the curve $\ln(1+y)=\tan^{-1}x$. [2] **(iii)** Hence, without using a graphing calculator, find an approximation for $\int_0^{\frac{1}{2}}\left(e^{\tan^{-1}x}-1\right)\,dx$. [2]
8	**(a)** The fifth, ninth and eleventh terms of a geometric progression are also the seventh, twenty-fifth and forty-ninth terms of an arithmetic progression with a non-zero common difference respectively. Show that $3R^6-7R^4+4=0$, where R is the common ratio of the geometric progression and determine if the geometric progression is convergent. [4] **(b)** A semicircle with radius 12 cm is cut into 8 sectors whose areas follow a geometric progression. The first sector, which is the largest, has an area of A cm^2. The second sector has an area of Ar cm^2, the third sector has an area of Ar^2 cm^2, and so on, where r is a positive constant. Given also that the total area of the odd-numbered sectors is 10π cm^2 more than that of the even-numbered sectors, find the values of A and r. [5] **(c)** The production levels of a particular coal mine in any year is 4% less than in the previous year. Show that the total production of the coal mine can never exceed 25 times the production in the first year. [2]
9	**(a)** Using the substitution $u=2x+3$, find $\int \frac{x}{(2x+3)^3}\,dx$ in the for $-\frac{Px+Q}{R(2x+3)^2}+c$ where P, Q and R are positive integers to be determined. [3] Hence find $\int \frac{x\ln(4x+3)}{(2x+3)^3}\,dx$. [3] **(b)** Find $\int \sin 4x\cos 6x\,dx$. [2] Hence or otherwise, find $\int e^x \sin 4e^x \cos 6e^x\,dx$. [1]
10	A particle is moving along a curve, C, such that its position at time t seconds after it is set into motion is given by the parametric equations $x=t+e^{-2t},\ y=t-e^{-2t}$. **(i)** State the coordinates of the initial position of the particle. [1] **(ii)** Explain what would happen to the path of the particle after a long time. [1] At the time of 2 seconds after the particle was set into motion, an external force struck the particle resulting in the particle moving in a straight line along the normal to the path at the point of collision. **(iii)** Find an equation for the normal to the curve C at the point t = 2, leaving your answer correct to 3 decimal places. [3]

	After T seconds, where $T > 2$, the particle reaches point A, which lies on the x-axis, and stops moving. **(iv)** Find the coordinates of the point A. Hence, give a sketch of the path traced by the particle, indicating the coordinates of any axial intercepts. [4] **(v)** Find the total area bounded by the path of the particle in the first T seconds and the positive x-axis. [4]
11	 A heavy wooden chest has a cross-sectional area made up of a rectangle and a semi-circle as shown in the diagram above. The wooden chest is constructed such that the perimeter of the cross-sectional area is 100 cm. It is given that the wooden chest is $2(a + b)$ cm long and the lengths of AB and BC are $2a$ cm and $2b$ cm respectively, where $a < 70$. **(i)** Express b in terms of a. [1] **(ii)** Show that the cross-sectional area of the wooden chest is given by $S = 100a - \frac{a^2}{2}(\pi + 4)$ and find the volume of the chest in terms of a and π. [4] **(iii)** As a varies, find the value of a such that the volume of this wooden chest is greatest and find this volume correct to 2 decimal places. [5]

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