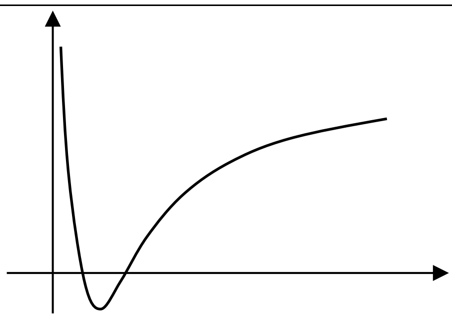
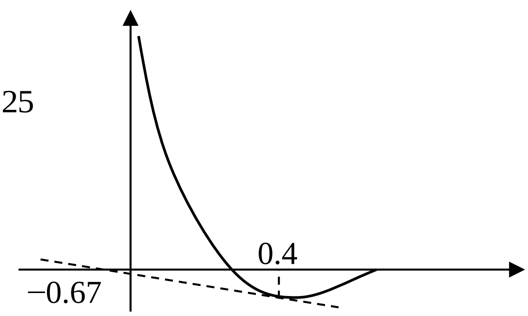
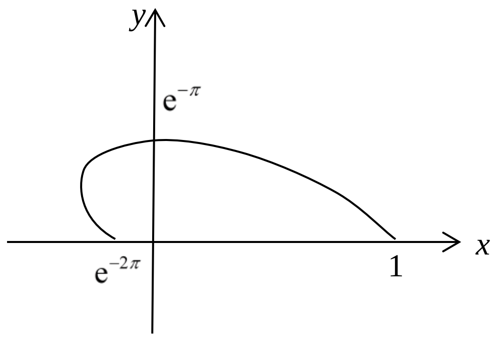
2017 SRJC H2 FM Prelim Paper 1 Solution
Answer all questions [100 marks].

1	Suggested Solution (i) $U_n = 0.6U_{n-1} + 150$, $U_0 = 250$. (ii) $U_n = 0.6(0.6U_{n-2} + 150) + 150$ $= 0.6^2 U_{n-2} + 0.6(150) + 150$ $= 0.6^2 (0.6U_{n-3} + 150) + 0.6(150) + 150$ $= 0.6^3 U_{n-3} + 0.6^2 (150) + 0.6(150) + 150$ $= \cdots$ $= 0.6^n U_0 + 0.6^{n-1}(150) + 0.6^{n-2}(150) + ... + 150$ $= 0.6^n (250) + 150(0.6^{n-1} + 0.6^{n-2} + \cdots + 1)$ $= 0.6^n (250) + 150\left(\frac{1-0.6^n}{1-0.6}\right)$ $= 250(0.6^n) + 375(1-0.6^n)$ $= 375 - 125(0.6^n)$ $\therefore a = -125, \ b = 375$
2	**Suggested Solution**  $\mathrm{f}(x) = \frac{1}{x} - 2 + 2\ln x, \quad x > 0$ $\mathrm{f}(0.1) = \frac{1}{0.1} - 2 + 2\ln 0.1 = 3.394 > 0$ $\mathrm{f}(1) = \frac{1}{1} - 2 + 2\ln 1 = -1 < 0$ Since $\mathrm{f}(0.1).\mathrm{f}(1) < 0$ and f is continuous, there is a root in the interval (0, 1). Taking $x_0 = 0.4$, $\mathrm{f}'(x) = -\frac{1}{x^2} + \frac{2}{x}$ $\mathrm{f}(0.4) = -1.3326$ (4 d.p.), $\mathrm{f}'(0.4) = -1.25$ By Newton-Raphson Method, $x_1 = x_0 - \frac{\mathrm{f}(x_0)}{\mathrm{f}'(x_0)}$ $= 0.4 - \frac{(-1.3326)}{(-1.25)} = -0.67$ (to 2 d.p.) 

	Since $x_1 = -0.67 < 0$, $\ln x_1$ will not be defined and subsequent iterations using the Newton-Raphson method to obtain subsequent estimates will not be possible. Hence x_0 = 0.4 is not a suitable approximation to find α using the Newton-Raphson Method Taking x_0 = 0.1, (any value below 0.315)
3	(i)  (ii) Area of the region enclosed $= \frac{1}{2}\int_{\theta_1}^{\theta_2} \left(e^{-2\theta}\right)^2 \, d\theta$ $= \frac{1}{2}\left[\frac{e^{-4\theta}}{-4}\right]_{\theta_1}^{\theta_2}$ $= \frac{1}{8}\left(e^{-4\theta_1} - e^{-4\theta_2}\right)$ $= \frac{1}{8}\left(\frac{1}{4} - a^2\right)$ $= \frac{1}{32}\left(1-4a^2\right)$ where $k = \frac{1}{32}$
4	Solution (i) Since $\mathbf{A}^2 = \mathbf{A}$, det $\mathbf{A}^2$ = det $\mathbf{A}$ ⇒ (det $\mathbf{A}$)^2 = det $\mathbf{A}$ ⇒ (det $\mathbf{A}$)^2 – det $\mathbf{A}$ = 0 ⇒ (det $\mathbf{A}$ – 1)(det $\mathbf{A}$) = 0 ⇒ det $\mathbf{A}$ = 1 or 0 (ii)(a) When det $\mathbf{A}$ = 1, then $\mathbf{A}$ is invertible (non-singular). There is an inverse $\mathbf{A}^{-1}$, such that $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$. From $\mathbf{A}^2 = \mathbf{A}$, multiply $\mathbf{A}^{-1}$ to both sides of the equation, $(\mathbf{A}^{-1}\mathbf{A})\mathbf{A} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$ ⇒ $\mathbf{I}\mathbf{A} = \mathbf{I}$ ⇒ $\mathbf{A} = \mathbf{I}$

	(ii)(b) Suppose det $\mathbf{A} = 0$, then $xw = yz$ ------- (1) $\begin{pmatrix} x & y \\ z & w \end{pmatrix}^2 = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$ $\Rightarrow \begin{pmatrix} x^2+yz & xy+yw \\ xz+wz & yz+w^2 \end{pmatrix} = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$ Comparing the (1,1) entry, $\Rightarrow x^2 + yz = x$ ----- (2) Substituting (1) into (2), $x^2 + xw = x$ $\Rightarrow x + w = 1$ (since $x \neq 0$) (shown)
5	Suggested Solution: (i) $z - z^* = 2\text{i} \sin\theta$. $\Rightarrow (z - z^*)^3 = -8\text{i} \sin^3 \theta$ $\Rightarrow z^3 - 3z^2z^* + 3z(z^*)^2 - (z^*)^3 = -8\text{i} \sin^3 \theta$ $\Rightarrow (z^3 - (z^3)^*) - 3zz^*(z - z^*) = -8\text{i} \sin^3\theta$ $\Rightarrow (2\text{i}\sin 3\theta) - 3(2\text{i} \sin \theta) = -8\text{i} \sin^3\theta$ $\Rightarrow 2\sin 3\theta - 6\sin \theta = -8\sin^3\theta$ $\Rightarrow \sin 3\theta = 3\sin \theta - 4\sin^3 \theta$ (shown) (ii) $(X^2-2)\sum_{r=0}^{k} a_r X^{2r} + \sum_{r=0}^{k-1} b_r X^{2r} = \sum_{r=0}^{k} a_r X^{2r+2} - 2\sum_{r=0}^{k} a_r X^{2r} + \sum_{r=0}^{k-1} b_r X^{2r}$ $= \sum_{r=1}^{k+1} a_{r-1} X^{2r} - 2\sum_{r=0}^{k} a_r X^{2r} + \sum_{r=0}^{k-1} b_r X^{2r}$ $= a_{k-1}X^{2k} + a_k X^{2k+2} - 2a_k X^{2k} - 2a_0 + b_0 + \sum_{r=1}^{k-1}\left(-2a_r + a_{r-1} + b_r\right) X^{2r}$ $A = -2a_0 + b_0,\ B = a_{k-1} - 2a_k,\ C = a_k$ (iii) $U_2(x) = \dfrac{\sin 3\theta}{\sin\theta}$ $= \dfrac{3\sin\theta - 4\sin^3\theta}{\sin\theta}$ $= 3 - 4(1-\cos^2\theta)$ $= 4x^2 - 1$ (iv) Let P_n be the statement $U_{2n} = \sum_{r=0}^{n} (-1)^{n-r} \binom{n+r}{n-r} (2x)^{2r}$, $n \in \mathbb{Z}$, $n \geqslant 0$ For $n = 0$, LHS = $U_0 = 1$ (given) RHS = $(-1)^0 \binom{0}{0} (2x)^0$ = 1 = LHS $\therefore$ P_0 is true. For $n = 1$, LHS = $U_2 = 4x^2 - 1$ (from (iii))

	RHS = $\sum_{r=0}^{1}(-1)^{1-r}\binom{1+r}{1-r}(2x)^{2r}$ $= -\binom{1}{1}(2x)^0+\binom{2}{0}(2x)^2$ $= 4x^2-1$ = LHS $\therefore$ P_1 is true. Assume P_k and P_{k-1} are true for some $k\in\mathbb{Z}$, $k\geqslant 1$, ie $U_{2k}=\sum_{r=0}^{k}(-1)^{k-r}\binom{k+r}{k-r}(2x)^{2r}$ $U_{2k-2}=\sum_{r=0}^{k-1}(-1)^{k-r-1}\binom{k+r-1}{k-r-1}(2x)^{2r}$ To prove P_{k+1}: $U_{2k+2}=\sum_{r=0}^{k+1}(-1)^{k-r+1}\binom{k+r+1}{k-r+1}(2x)^{2r}$ LHS = U_{2k+2} $= (4x^2-2)U_{2k}(x)-U_{2k-2}(x)$ (from (**)) $=\left(4x^2-2\right)\sum_{r=0}^{k}(-1)^{k-r}\binom{k+r}{k-r}(2x)^{2r}-\sum_{r=0}^{k-1}(-1)^{k-r-1}\binom{k+r-1}{k-r-1}(2x)^{2r}$ By taking $X = 2x$, $a_r=(-1)^{k-r}\binom{k+r}{k-r}$ and $b_r=(-1)^{k-r}\binom{k+r-1}{k-r-1}$ $\left(4x^2-2\right)\sum_{r=0}^{k}(-1)^{k-r}\binom{k+r}{k-r}(2x)^{2r}-\sum_{r=0}^{k-1}(-1)^{k-r-1}\binom{k+r-1}{k-r-1}(2x)^{2r}$ $=(X^2-2)\sum_{r=0}^{k}a_rX^{2r}+\sum_{r=0}^{k-1}b_rX^{2r}$ $=a_{k-1}X^{2k}+a_kX^{2k+2}-2a_kX^{2k}-2a_0+b_0+\sum_{r=1}^{k-1}\left(-2a_r+a_{r-1}+b_r\right)X^{2r}$ $=(-1)^{k+1}-\left(2k+1\right)(2x)^{2k}+(2x)^{2k+2}+\sum_{r=1}^{k-1}(-1)^{k-r+1}\binom{k+r+1}{k-r+1}(2x)^{2r}$ $=(-1)^{k+1}\binom{k+1}{k+1}(2x)^0-\binom{2k+1}{1}(2x)^{2k}+(-1)^0\binom{2k+2}{0}(2x)^{2k+2}$ $+\sum_{r=1}^{k-1}(-1)^{k-r+1}\binom{k+r+1}{k-r+1}(2x)^{2r}$ $=\sum_{r=0}^{k+1}(-1)^{k-r+1}\binom{k+r+1}{k-r+1}(2x)^{2r}$ $\therefore$ P_{k-1} and P_k are true $\Rightarrow$ P_{k+1} is true. Since P_0 and P_1 are true and P_{k-1} and P_k are true $\Rightarrow$ P_{k+1} is true, by mathematical induction, P_n is true for all $n\in\mathbb{Z}$, $n\geq 0$.

6	Solution: Let $\dfrac{5x^2-3}{x-x^3}=\dfrac{A}{x}+\dfrac{B}{1-x}+\dfrac{C}{1+x}$ By "Cover-up" rule, $A = -3$ $B = 1$ $C = -1$ $\therefore\ \dfrac{5x^2-3}{x-x^3}=-\dfrac{3}{x}+\dfrac{1}{1-x}-\dfrac{1}{1+x}$ (ii) $y = vx \Rightarrow \dfrac{dy}{dx}=v+x\dfrac{dv}{dx}$ $y\left(x^2+y^2\right)+x\left(3x^2-5y^2\right)\dfrac{dy}{dx}=0$ $\Rightarrow vx\left(x^2+v^2x^2\right)+x\left(3x^2-5v^2x^2\right)\left(v+x\dfrac{dv}{dx}\right)=0$ $\Rightarrow vx^3+v^3x^3+3x^3v-5v^3x^3+x^2\left(3x^2-5v^2x^2\right)\dfrac{dv}{dx}=0$ $\Rightarrow 4x^3v-4v^3x^3=-x^4\left(3-5v^2\right)\dfrac{dv}{dx}$ Note that since $v=\dfrac{y}{x}$, then $x \neq 0$. $\Rightarrow \dfrac{4}{x}=\dfrac{5v^2-3}{v-v^3}\dfrac{dv}{dx}$ $\Rightarrow \displaystyle\int\frac{4}{x}dx=\int\frac{5v^2-3}{v-v^3}dv$ From (i), $4\ln\|x\| = \displaystyle\int\frac{-3}{v}+\frac{1}{1-v}-\frac{1}{1+v}dv$ $= -3\ln\|v\| - \ln\|1-v\| - \ln\|1+v\| + \ln c$, where c is a positive constant. $\therefore\ \ln\left\|x^4\right\|=\ln\left\|\dfrac{c}{v^3\left(1-v^2\right)}\right\| = \ln\left\|\dfrac{c}{\left(\frac{y}{x}\right)^3\left[1-\left(\frac{y}{x}\right)^2\right]}\right\|$ $x^4=cx^4\left\|\dfrac{x}{y^3\left(x^2-y^2\right)}\right\| \Rightarrow \left\|y^3\left(x^2-y^2\right)\right\|=c\|x\|$ $\therefore$ general solution required is $y^3\left(x^2-y^2\right)=Cx$ where $C = \pm c$
7	Solution (i) Difference in distance from the foci = 2 $\Rightarrow (6-x)-x = 2$ or $x-(6-x) = 2$ $\Rightarrow x = 2$ or $x = 4$ Hence, two points on x-axis are (2, 0) and (4, 0). Putting into the equation $2=\dfrac{ed}{1+e}$, $-4=\dfrac{ed}{1-e}$ $\Rightarrow 2 + 2e = 4e - 4$

$\Rightarrow$ e = 3

$\Rightarrow$ d = 8/3

$\therefore$ polar equation of H is $r = \dfrac{8}{1+3\cos\theta}$

(ii) $$\frac{dr}{d\theta} = \frac{8(-3\sin\theta)}{(1+3\cos\theta)^2}$$

$$\therefore\ r^2 + \left(\frac{dr}{d\theta}\right)^2 = \frac{64(1+3\cos\theta)^2 + 24^2\sin^2\theta}{(1+3\cos\theta)^4}$$

$$= \frac{64 + 384\cos\theta + 576\cos^2\theta + 576\sin^2\theta}{(1+3\cos\theta)^4}$$

$$= \frac{128(3\cos\theta+5)}{(1+3\cos\theta)^4}$$

Arc length $C = \displaystyle\int_{-\frac{\pi}{6}}^{\frac{\pi}{3}} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$

$$= \int_{-\frac{\pi}{6}}^{\frac{\pi}{3}} \sqrt{\frac{128(3\cos\theta+5)}{(1+3\cos\theta)^4}}\, d\theta$$

$$= 8\sqrt{2}\int_{-\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{3\cos\theta+5}}{(1+3\cos\theta)^2}\, d\theta \text{ (shown)}$$

(iii) Let f(x) = $8\sqrt{2}\,\dfrac{\sqrt{3\cos\theta+5}}{(1+3\cos\theta)^2}$.

For 6 trapezia, the 7 ordinates are $-\frac{\pi}{6}, -\frac{\pi}{12}, 0, \frac{\pi}{12}, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}$

$$\text{Approx. of } C = \frac{\left(\frac{\pi}{3}+\frac{\pi}{6}\right)}{12}\left(f(-\tfrac{\pi}{6}) + 2f(-\tfrac{\pi}{12}) + 2f(0) + 2f(\tfrac{\pi}{12}) + 2f(\tfrac{\pi}{6}) + 2f(\tfrac{\pi}{4}) + f(\tfrac{\pi}{3})\right)$$

= 3.9808 (to 4 d.p.)

(iv) For 3 parabolas, there will be 6 strips and 7 ordinates $-\frac{\pi}{6}, -\frac{\pi}{12}, 0, \frac{\pi}{12}, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}$

$$\text{Approx. of } C = \frac{\left(\frac{\pi}{3}+\frac{\pi}{6}\right)}{18}\left(f(-\tfrac{\pi}{6}) + 4f(-\tfrac{\pi}{12}) + 2f(0) + 4f(\tfrac{\pi}{12}) + 2f(\tfrac{\pi}{6}) + 4f(\tfrac{\pi}{4}) + f(\tfrac{\pi}{3})\right)$$

= 3.9252 (to 4 d.p.)

8 Solution

(i) From definition of S,

$MF = eMl$

$\Rightarrow 5 = ce$

$$\Rightarrow\ e=\frac{5}{c}$$

$$\sqrt{(x-3)^2+y^2}=\frac{5}{c}(c-x),$$

$$\Rightarrow c^2(x^2-6x+9)+c^2y^2=25(x^2-2xc+c^2)$$

$$\Rightarrow (c^2-25)x^2+2(25c)x-6c^2x+c^2y^2=16c^2$$

$$\Rightarrow (c^2-25)x^2+2(25-3c)cx+c^2y^2=16c^2$$

(ii) $c = 5$ $\Rightarrow\ e=\frac{5}{c}=1$

$\Rightarrow$ S is a parabola.

(iii) From part (ii), equation of S is $y^2=16-4x$.

For $a = 0$, equation of $y=\frac{1}{2}x-4$. Substituting into equation of S,

$\left(x-8\right)^2=64-16x \Rightarrow x^2=0 \Rightarrow x=0,\ y=-4$ is the only point of intersection.

(iv) For point of intersection,

$$\left(1-a^2\right)(x-8)^2=64-16x$$

$$\Rightarrow\ (1-a^2)x^2+16a^2x-64a^2=0$$

$$\Rightarrow\ x=\frac{-8a^2\pm 8a}{1-a^2}$$

$$\Rightarrow\ x=\frac{8a}{a+1}\ \text{or}\ x=\frac{8a}{a-1}$$

Hence the coordinates of Q and R are $\left(\frac{8a}{a+1},\frac{-4\sqrt{1-a^2}}{a+1}\right)$ and $\left(\frac{8a}{a-1},\frac{4\sqrt{1-a^2}}{a-1}\right)$

For gradient of tangents of S,

$$2y\frac{\mathrm{d}y}{\mathrm{d}x}=-4\ \Rightarrow\ \frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{2}{y}$$

Equations of tangents at Q and R are $y-\frac{-4\sqrt{1-a^2}}{a+1}=\frac{a+1}{2\sqrt{1-a^2}}\left(x-\frac{8a}{a+1}\right)$

$\Rightarrow\ 2y\sqrt{1-a^2}=\left(a+1\right)x-8$, $A = a + 1$, $B = -8$

and $y-\frac{4\sqrt{1-a^2}}{a-1}=\frac{1-a}{2\sqrt{1-a^2}}\left(x-\frac{8a}{a-1}\right)$

$\Rightarrow\ 2y\sqrt{1-a^2}=\left(1-a\right)x-8$, $A = 1 - a$, $B = -8$.

(v) Substituting l_R into l_Q,

$$\left(1-a\right)x-8=\left(a+1\right)x-8$$

$$\Rightarrow\ -ax+x=ax+x$$

$$\Rightarrow 2ax=0$$

$\Rightarrow\ x=0$ for all a, $-1<a<1$, $a\neq 0$ (Proven)

9	Suggested Solution

(i) For point of intersection of $x = t^2 - 2\ln t$, $y = 4(t-1)$ and $y = \frac{1}{x}$,

$$4(t-1) = \frac{1}{t^2 - 2\ln t}$$

$$t - 1 = \frac{1}{4\left(t^2 - 2\ln t\right)}$$

$$t = 1 + \frac{1}{4\left(t^2 - 2\ln t\right)}$$

$$h(t_n) = 4\left(t_n{}^2 - 2\ln t_n\right)$$

(ii) $t_{n+1} = 1 + \dfrac{1}{4\left(t_n{}^2 - 2\ln t_n\right)}$, $t_0 = 1$

$t_1 = 1.25$
$t_2 = 1.22397$
$t_3 = 1.22854$
$t_4 = 1.22776$
$t_5 = 1.22789$
$t_6 = 1.22786$
Hence, $\alpha = 1.228$ to three decimal places.

(iii) $A(1,0)$ and $B\left(\frac{1}{4}, 4\right)$

$P(1.097, 0.912)$

(iv) For C_1,

$$\frac{dx}{dt} = 2t - \frac{2}{t}, \quad \frac{dy}{dt} = 4$$

For C_2,

$$\frac{dy}{dx} = -\frac{1}{x^2}$$

Arc length = $\displaystyle\int_1^{1.228} \sqrt{\left(2t - \frac{2}{t}\right)^2 + (4)^2}\, dt + \int_{\frac{1}{4}}^{1.097} \sqrt{1 + \left(-\frac{1}{x^2}\right)^2}\, dx$

= 0.91876 + 3.28148

= 4.20 (3s.f.)

(v) Surface area generated = $\displaystyle\int_1^{1.228} 2\pi\left(t^2 - 2\ln t\right)\sqrt{\left(2t - \frac{2}{t}\right)^2 + (4)^2}\ dt$

$\displaystyle + \int_{\frac{1}{4}}^{1.097} 2\pi x\sqrt{1 + \left(-\frac{1}{x^2}\right)^2}\ dx$

= 5.96380 + 10.28167

	= 16.2 (3s.f.)
10	Solution: (i) $\begin{pmatrix} x_1 & x_2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix} = \begin{pmatrix} x_1u_n + x_2u_{n-1} \\ u_n \end{pmatrix}$ $= \begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix}$ $\Rightarrow u_{n+1} = x_1u_n + x_2u_{n-1} = 2\,a\,u_n + b\,u_{n-1}$ $\Rightarrow x_1 = 2a,\ x_2 = b$ (ii) $\mathbf{A} = \begin{pmatrix} 2a & b \\ 1 & 0 \end{pmatrix}$. Characteristic equation $\begin{vmatrix} 2a-\lambda & b \\ 1 & -\lambda \end{vmatrix} = 0$ $\Rightarrow (-\lambda)(2a-\lambda) - b = 0$ $\Rightarrow \lambda^2 - 2a\lambda - b = 0$ $\Rightarrow \lambda = \dfrac{2a \pm \sqrt{4a^2 - 4(1)(-b)}}{2} = a \pm \sqrt{a^2+b}$ Hence, the eigenvalues are $a+\sqrt{a^2+b}$ or $a-\sqrt{a^2+b}$. For eigenvalue $a+\sqrt{a^2+b}$, $\mathbf{A}-\lambda\mathbf{I} = \begin{pmatrix} a-\sqrt{a^2+b} & b \\ 1 & -a-\sqrt{a^2+b} \end{pmatrix}$ Corresponding eigenvector: $\begin{pmatrix} a+\sqrt{a^2+b} \\ 1 \end{pmatrix}$ For eigenvalue $a-\sqrt{a^2+b}$, $\mathbf{A}-\lambda\mathbf{I} = \begin{pmatrix} a+\sqrt{a^2+b} & b \\ 1 & -a+\sqrt{a^2+b} \end{pmatrix}$ Corresponding eigenvector: $\begin{pmatrix} a-\sqrt{a^2+b} \\ 1 \end{pmatrix}$ (iii) $\mathbf{D} = \begin{pmatrix} a+\sqrt{a^2+b} & 0 \\ 0 & a-\sqrt{a^2+b} \end{pmatrix}$, $\mathbf{P} = \begin{pmatrix} a+\sqrt{a^2+b} & a-\sqrt{a^2+b} \\ 1 & 1 \end{pmatrix}$ (iv) Given that $b = 3a^2$, $\mathbf{D} = \begin{pmatrix} 3a & 0 \\ 0 & -a \end{pmatrix}$, $\mathbf{P} = \begin{pmatrix} 3a & -a \\ 1 & 1 \end{pmatrix}$ $\begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix} = \mathbf{x}_{n+1}$ $= \mathbf{A}\begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix}$ $= \mathbf{A}^2\begin{pmatrix} u_{n-1} \\ u_{n-2} \end{pmatrix}$ $= \ldots$

$$= \mathbf{A}^n \begin{pmatrix} u_1 \\ u_0 \end{pmatrix}$$

$$= \mathbf{PD}^n\mathbf{P}^{-1} \begin{pmatrix} -a \\ a \end{pmatrix}$$

$$= \begin{pmatrix} 3a & -a \\ 1 & 1 \end{pmatrix} \begin{pmatrix} (3a)^n & 0 \\ 0 & (-a)^n \end{pmatrix} \frac{1}{4a} \begin{pmatrix} 1 & a \\ -1 & 3a \end{pmatrix} \begin{pmatrix} -a \\ a \end{pmatrix}$$

$$= \frac{1}{4a} \begin{pmatrix} (3a)^{n+1} & (-a)^{n+1} \\ (3a)^n & (-a)^n \end{pmatrix} \begin{pmatrix} 1 & a \\ -1 & 3a \end{pmatrix} \begin{pmatrix} -a \\ a \end{pmatrix}$$

$$= \frac{1}{4a} \begin{pmatrix} (3a)^{n+1} - (-a)^{n+1} & a(3a)^{n+1} + 3a(-a)^{n+1} \\ (3a)^n - (-a)^n & a(3a)^n + 3a(-a)^n \end{pmatrix} \begin{pmatrix} -a \\ a \end{pmatrix}$$

$$= \frac{1}{4} \begin{pmatrix} -(3a)^{n+1} + (-a)^{n+1} + a(3a)^{n+1} + 3a(-a)^{n+1} \\ -(3a)^n + (-a)^n + a(3a)^n + 3a(-a)^n \end{pmatrix}$$

$$\therefore u_n = \frac{1}{4}\left\{(a-1)(3a)^n + (3a+1)(-a)^n\right\}$$

OR

$$u_n = \frac{a-1}{4}(3a)^n + \frac{3a-1}{4}(-a)^n$$

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11 Solution

(i) General solution of DE is $x = A\mathrm{e}^{kt}$

When t = 0, x = 2

$\Rightarrow A = 2$

When t = 5, x = 64

$\Rightarrow 64 = 2\mathrm{e}^{5k}$

$\Rightarrow k = \ln 2$

Hence, solution is $x = 2\mathrm{e}^{t\ln 2} = 2^{t+1}$

(ii) $$\int \frac{1}{x\left(1 - \frac{x}{2002}\right)} \,\mathrm{d}x = t\ln 2 + C$$

$$\int \frac{1}{x} + \frac{1}{2002 - x} \,\mathrm{d}x = t\ln 2 + C$$

$$\ln|x| - \ln|2002 - x| = t\ln 2 + C$$

$\Rightarrow \dfrac{x}{2002 - x} = B\mathrm{e}^{t\ln 2}$, where $B = \pm\, \mathrm{e}^C$

When t = 0, x = 2

$$\Rightarrow B = \frac{2}{2002 - 2} = 0.001$$

$$\Rightarrow x = 0.001\left(2^t\right)\left(2002 - x\right)$$

$$\Rightarrow x = \frac{2.002\left(2^t\right)}{1 + 0.001(2^t)}$$

When t = 10,

$$x = \frac{2.002\left(2^{10}\right)}{1+0.001(2^{10})} = 1013 \text{ (to nearest whole no.)}$$

(iii) $$\frac{\mathrm{d}x}{\mathrm{d}t} = \mathrm{f}(t,x) = (\ln 2)x\left(1-\frac{x}{2002}\right)+\frac{1}{2}t^2$$

Using Euler's method,

$x_1 = x_0 + h\mathrm{f}(t_0, x_0)$

$$= 2 + 0.5\left[(\ln 2)(2)\left(1-\frac{2}{2002}\right)+\frac{1}{2}(0)^2\right]$$

$= 2.6924$

$x_2 = 3.6868$

$x_3 = 5.2122$

$x_4 = 7.5764 \approx 8$ (to the nearest whole no.)

(iv) As the estimation of the next term depend on the estimation of the earlier term, the estimation becomes poorer after each iteration and is considered unreliable.

(v) Using GC,

n	x_n	$\mathrm{f}(t_n, x_n)$	x_{n+1}*	$\mathrm{f}(t_{n+1}, x_{n+1}*)$	x_{n+1}
0	2	1.3849	2.6924	**1.9887**	2.8434
1	2.8434	2.0931	**3.8899**	3.1910	4.1644
2	4.1644	**3.3805**	5.8547	5.1571	6.3024
3	**6.3024**	5.4797	9.0423	8.2393	**9.7322**
4	**9.7322**	8.7130	14.088	12.821	15.115
5	15.115	13.523	21.877	19.498	23.371