Section A

1	$f(0)=\cos 0-0=1>0$ $f(1)=\cos 1-1=-0.459<0$ By Linear Interpolation, there is at least 1 root in the interval $(0,1)$. $f'(x)=-\sin x-1<0$ in the interval $(0,1)$. $\therefore f(x)$ is strictly decreasing in the interval $(0,1)$. Hence there is only one root in the interval $(0,1)$. Given $x_{n+1}=\cos x_n$, and $x_1=1$, $x_2=0.5403$ $x_3=0.8575$ $x_4=0.6542$ $x_5=0.7934$ $x_6=0.7013$ $x_7=0.7639$ $x_8=0.7221$ $x_9=0.7504$ $x_{10}=0.7314$ $x_{11}=0.7442$ $x_{12}=0.7356$ $f(0.745)=-0.0099<0$ $f(0.735)=0.0068>0$ Hence, $0.735<\alpha<0.745$ $\therefore \alpha=0.74$

2	$\tan\theta = \dfrac{y}{x}$ $= \dfrac{a\sin t - 2a\sin t\cos t}{a\cos t - 2a\cos^2 t}$ $= \tan t$ $\therefore \theta = t$ $r^2 = x^2 + y^2$ $= (a\cos\theta)^2(1-2\cos\theta)^2 + (a\sin\theta)^2(1-2\cos\theta)^2$ $= (1-2\cos\theta)^2\left[(a\cos\theta)^2 + (a\sin\theta)^2\right]$ $= a^2(1-2\cos\theta)^2$ $\therefore r = a(1-2\cos\theta)$ When $\theta = 0$, $r = -a$. When $\theta = \dfrac{\pi}{2}$, $r = a$. When $\theta = \pi$, $r = 3a$. When $r = 0$, $\theta = \dfrac{\pi}{3}, -\dfrac{\pi}{3}$. $\text{Area} = 2\left[\left\|\int_{\frac{\pi}{3}}^{\pi} \frac{1}{2}r^2\, d\theta\right\| - \left\|\int_0^{\frac{\pi}{3}} \frac{1}{2}r^2\, d\theta\right\|\right]$ Note that $\int_{\frac{\pi}{3}}^{\pi} \frac{1}{2}r^2\, d\theta > 0$ and $\int_0^{\frac{\pi}{3}} \frac{1}{2}r^2\, d\theta < 0$ Hence $\text{area} = 2\left[\int_{\frac{\pi}{3}}^{\pi} \frac{1}{2}r^2\, d\theta + \int_0^{\frac{\pi}{3}} \frac{1}{2}r^2\, d\theta\right]$ $= 2\int_0^{\pi} \frac{1}{2}r^2\, d\theta$ $= a^2\int_0^{\pi} (1-2\cos\theta)^2\, d\theta$ $= a^2\int_0^{\pi} 1 - 4\cos\theta + 4\cos^2\theta\, d\theta$ $= a^2\int_0^{\pi} 1 - 4\cos\theta + 4\left(\frac{1+\cos 2\theta}{2}\right) d\theta$ $= a^2\int_0^{\pi} 3 - 4\cos\theta + 2\cos 2\theta\, d\theta$ $= a^2\left[3\theta - 4\sin\theta + \sin 2\theta\right]_0^{\pi}$ $= 3\pi a^2$

3(a) (i)

Using Euler's Method, we have

$$y_{n+1} = y_n + 0.5\left(e^{\sin t_n} + e^{\cos y_n}\right)$$

Given $t_0 = 0$, $y_0 = 0$

$t_1 = 0.5$

$y_1 = 0 + 0.5\left(e^{\sin 0} + e^{\cos 0}\right) = 1.85914$

$t_2 = 1$

$y_2 = 1.85914 + 0.5\left(e^{\sin(0.5)} + e^{\cos(1.85914)}\right)$

$= 3.04295$

$= 3.043$

3(a) (ii)

Using the improved Euler's Method

$$y^*_{n+1} = y_n + 0.5\left(e^{\sin t_n} + e^{\cos y_n}\right)$$

$$y_{n+1} = y_n + 0.5\left[\frac{\left(e^{\sin t_n} + e^{\cos y_n}\right) + \left(e^{\sin t_{n+1}} + e^{\cos y^*_{n+1}}\right)}{2}\right]$$

Given $t_0 = 0$, $y_0 = 0$

$t_1 = 0.5$

$y^*_1 = 0 + 0.5\left(e^{\sin 0} + e^{\cos 0}\right) = 1.85914$

$$y_1 = 0 + 0.5\left[\frac{\left(e^{\sin 0} + e^{\cos 0}\right) + \left(e^{\sin(0.5)} + e^{\cos(1.85914)}\right)}{2}\right] = 1.52148$$

$y^*_2 = 1.52148 + 0.5\left(e^{\sin(0.5)} + e^{\cos(1.52148)}\right) = 2.85431$

$$y_2 = 1.52148 + 0.5\left[\frac{\left(e^{\sin(0.5)} + e^{\cos(1.85914)}\right) + \left(e^{\sin(1)} + e^{\cos(2.85431)}\right)}{2}\right]$$

$= 2.86366$

$= 2.864$

3(b)	Given $h=\dfrac{2-0}{5-1}=\dfrac{1}{2}$ When $t=0$, $v=e^{\sin 0}=1$ $t=0.5$, $v=e^{\sin(0.5)}=1.61514$ $t=1$, $v=e^{\sin(1)}=2.31977$ $t=1.5$, $v=e^{\sin(1.5)}=2.71148$ $t=2$, $v=e^{\sin(2)}=2.48257$ Distanced Travelled $=\int_0^2 v\,dt$ $\approx \dfrac{0.5}{3}\left[1+4(1.61514)+2(2.31977)+4(2.71148)+2.48257\right]$ $=4.238$

4(i)	$b^2=a^2\left(1-e^2\right)$ $4=a^2\left(1-0.85^2\right)$ $a^2=14.4144$ $a=3.7966$ $p=2a=7.59$
4(ii)	$\dfrac{PF}{Pd}=e$ $\dfrac{r}{r\cos\theta+d}=e$ $r=er\cos\theta+ed$ $r=\dfrac{ed}{1-e\cos\theta}=\dfrac{k}{1-0.85\cos\theta}$ Sub $(0, a+ae)$. $a+ae=\dfrac{k}{1-0.85}$ $k=(1-0.85)(3.796631)(1.85)$ $=1.05303$
4(iii)	$\int_0^{\pi}\sqrt{r^2+\left(\dfrac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta=9.32060$ Cost $=9.32060\times 6\times 28=\$1566$

5(i)	$\lvert M-\lambda I\rvert=0$ $(a-\lambda)^3+2(-2)^3-3(-2)^2(a-\lambda)=0$ $(a-\lambda)^3-12(a-\lambda)-16=0$ Since $\lambda=-3$ is a solution, $(a+3)^3-12(a+3)-16=0$ $a^3+9a^2+15a-25=0$ Using GC, $a=1$, $a=-5$(rej) $\therefore (1-\lambda)^3-12(1-\lambda)-16=0$ $-\lambda^3+3\lambda^2+9\lambda-27=0$ Using GC, $\lambda=-3,3,3$
5(ii)	When $\lambda=-3$, $M-\lambda I=\begin{pmatrix}4&-2&-2\\-2&4&-2\\-2&-2&4\end{pmatrix}$ $\begin{pmatrix}4\\-2\\2\end{pmatrix}\times\begin{pmatrix}-2\\4\\-2\end{pmatrix}=2\begin{pmatrix}3\\3\\3\end{pmatrix}$ $\underset{\sim}{v}_1=\begin{pmatrix}1\\1\\1\end{pmatrix}$
5(iii)	When $\lambda=-3$, $M-\lambda I=\begin{pmatrix}-2&-2&-2\\-2&-2&-2\\-2&-2&-2\end{pmatrix}\to\begin{pmatrix}1&1&1\\0&0&0\\0&0&0\end{pmatrix}$ $\underset{\sim}{x}=\begin{pmatrix}-s-t\\s\\t\end{pmatrix}=s\begin{pmatrix}-1\\1\\0\end{pmatrix}+t\begin{pmatrix}-1\\0\\1\end{pmatrix}$ $\underset{\sim}{v}_2=\begin{pmatrix}-1\\1\\0\end{pmatrix}$, $\underset{\sim}{v}_3=\begin{pmatrix}-1\\0\\1\end{pmatrix}$ $l+m=n$ $l+m-n=0$ $p+r=q$ $p-q+r=0$

	$Q=\begin{pmatrix}1&-1&-1\\1&1&0\\1&0&1\end{pmatrix}$
(iv)	$\begin{pmatrix}1&-1&-1\\1&1&0\\1&0&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}3\\2\\1\end{pmatrix}$ $\left(\begin{array}{ccc\|c}1&-1&-1&3\\1&1&0&2\\1&0&1&1\end{array}\right)\to\left(\begin{array}{ccc\|c}1&0&0&2\\0&1&0&0\\0&0&1&1\end{array}\right)$ $\therefore\begin{pmatrix}3\\2\\1\end{pmatrix}=2\begin{pmatrix}1\\1\\1\end{pmatrix}-\begin{pmatrix}-1\\0\\1\end{pmatrix}$ $M^k\begin{pmatrix}3\\2\\1\end{pmatrix}=M^k\left[2\begin{pmatrix}1\\1\\1\end{pmatrix}-\begin{pmatrix}-1\\0\\1\end{pmatrix}\right]$ $=M^{k-1}\left(M\left[\begin{pmatrix}2\\2\\2\end{pmatrix}-\begin{pmatrix}-1\\0\\1\end{pmatrix}\right]\right)$ $=M^{k-1}\left(3\begin{pmatrix}2\\2\\2\end{pmatrix}-(-3)\begin{pmatrix}-1\\0\\1\end{pmatrix}\right)$ $=M^{k-2}\left(3^2\begin{pmatrix}2\\2\\2\end{pmatrix}-(-3)^2\begin{pmatrix}-1\\0\\1\end{pmatrix}\right)$ $\vdots$ $=3^k\begin{pmatrix}2\\2\\2\end{pmatrix}-(-3)^k\begin{pmatrix}-1\\0\\1\end{pmatrix}$ $=3^k\left[\begin{pmatrix}2\\2\\2\end{pmatrix}-(-1)^k\begin{pmatrix}-1\\0\\1\end{pmatrix}\right]$

When k is even,

$$M^k\begin{pmatrix}3\\2\\1\end{pmatrix}=3^k\left[\begin{pmatrix}2\\2\\2\end{pmatrix}-\begin{pmatrix}-1\\0\\1\end{pmatrix}\right]$$

$$=3^k\begin{pmatrix}3\\2\\1\end{pmatrix}$$

When k is odd,

$$M^k\begin{pmatrix}3\\2\\1\end{pmatrix}=3^k\left[\begin{pmatrix}2\\2\\2\end{pmatrix}+\begin{pmatrix}-1\\0\\1\end{pmatrix}\right]$$

$$=3^k\begin{pmatrix}1\\2\\3\end{pmatrix}$$

Section B Marking Scheme

6(i)

The two sample are not independent since they are taken on the same days. e.g. on public holidays, there is a larger crowd, so earnings for both restaurants will be higher.

The assumptions are:
D_i's are independent and normally distributed.

6(ii)

$H_0 : \mu_d = 0$

$H_1 : \mu_d \neq 0$

Test at α% sig level.

Under H_0, test statistic is $T = \dfrac{\bar{D} - \mu_d}{s_d\sqrt{\frac{1}{12}}} \sim t_{11}$

Using GC, p-value = 0.0117

H_0 rejected $\Rightarrow 0.0117 \leq \dfrac{\alpha}{100}$

Least $\alpha = 1.17$

7(i)

H_0: a person's colour of eyes and reaction to UV are independent.
H_1: a person's colour of eyes and reaction to UV are not independent.

Test at 5% sig level.

diffce btw obs'd & expected.

Under H_0, expected freq is

15.675	7.425	9.9
26.125	12.375	16.5
15.2	7.2	9.6

$\dfrac{(O_i - E_i)^2}{E_i}$

Brown eye

Contributions to test statistic:

4.801	0.0445	6.6273
0.3164	0.456	0.015
2.2132	0.45	6.017

bigger the value the more evidence that two factors not indep.

Using GC,

$\chi^2 = 20.9$

p-value = $3.25 \times 10^{-4} < 0.05$

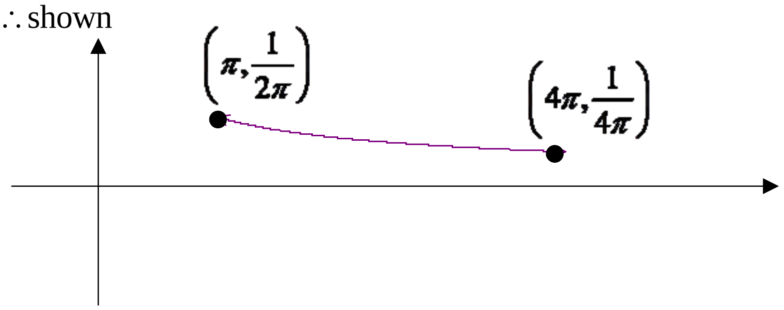
Reject H_0 as there is sufficient evidence at 5% sig level that a person's eye colour and reaction to UV are not independent.

The strongest evidence for association comes from the brown eye/ People with brown eyes tend to show no or just a slight reaction to UV light.

7(ii)

sample proportion = 0.275

95% CI for $p = (0.1951, 0.3549)$

8(i)	$\mathrm{P}(A \le a) = \mathrm{P}(\pi X^2 \le a)$ $= \mathrm{P}\left(X^2 \le \frac{a}{\pi}\right)$ $= \mathrm{P}\left(-\sqrt{\frac{a}{\pi}} \le X \le \sqrt{\frac{a}{\pi}}\right)$ $= \mathrm{P}\left(1 \le X \le \sqrt{\frac{a}{\pi}}\right)$ $= \sqrt{\frac{a}{\pi}} - 1$ $\mathrm{P}(A = a) = \frac{\mathrm{d}}{\mathrm{d}a}\mathrm{P}(A \le a)$ $= \frac{1}{2}\left(\frac{a}{\pi}\right)^{-\frac{1}{2}}\frac{1}{\pi}$ $= \frac{1}{2\pi}\sqrt{\frac{\pi}{a}} = \frac{1}{2\sqrt{\pi}}\frac{1}{\sqrt{a}}$ $1 \le x \le 2 \;\Rightarrow\; \pi \le \pi x^2 \le 4\pi$ $\therefore$ shown 
8(ii)	$\mathrm{E}(\pi X^2) = \pi\,\mathrm{E}(X^2)$ $= \pi\int_1^2 x^2 \; \mathrm{d}x$ $= \pi\left[\frac{x^3}{3}\right]_1^2$ $= \frac{7\pi}{3}$

9	$X \sim \mathrm{Po}(\lambda)$ $$\mathrm{P}(Y=r)=\sum_{m=0}^{\infty}\left[\mathrm{P}(X=0)\right]^{m}\ \mathrm{P}(X=r)$$ $$=\mathrm{P}(X=r)\sum_{m=0}^{\infty}\left(\mathrm{e}^{-\lambda}\right)^{m}$$ $$=\mathrm{P}(X=r)\frac{1}{1-\mathrm{e}^{-\lambda}}$$ $$\mathrm{E}(Y)=\frac{1}{1-e^{-\lambda}}\sum_{r=1}^{\infty}r\mathrm{P}(X=r)$$ $$=\frac{1}{1-e^{-\lambda}}\sum_{r=0}^{\infty}r\mathrm{P}(X=r)$$ $$=\frac{\lambda}{1-e^{-\lambda}}$$ λ is small. $$\frac{\lambda}{1-e^{-\lambda}}\approx\frac{\lambda}{1-\left(1-\lambda+\frac{\lambda^{2}}{2}\right)}$$ $$=\lambda\left(\lambda-\frac{\lambda^{2}}{2}\right)^{-1}$$ $$=\lambda\lambda^{-1}\left(1-\frac{\lambda}{2}\right)^{-1}$$ $$\approx 1+\frac{\lambda}{2}$$
9(i)	$R \sim \mathrm{Geom}\left(1-\mathrm{P}(X=0)\right)$ $R \sim \mathrm{Geom}\left(\frac{1}{2}\right)$
9(ii)	$\mathrm{E}\left(R_1+R_2\right)=2\mathrm{E}(R)$ $=2(2)=4$

10(a)	Let X and Y be a random candidate's marks in P1 and P2 respectively. H_0 : median of $(X-Y)=0$ H_1 : median of $(X-Y)<0$ Test at 5% sig level. Sign of $(X-Y)$: $- - - - + - - -$ $1'+',\ 7'-'$ Let A be the no. of positive signs out of 8. Under H_0, $A \sim B\left(8, \frac{1}{2}\right)$ $P(A \le 1) = 0.035156 < 0.05$ Hence we reject H_0 as there is suff evidence at 5 % sig level that P2 is easier than P1.
10(b)	(table below) $T_+ = 7$ $T_- = 29$ $T_{calc} = \min(7, 29) = 7 > 5$ Hence do not reject H_0 as there is insuff evidence at 5% sig level that P2 is easier than P1.
	Wilcoxon test is more appropriate. It takes into account both the magnitude and sign of the differences in marks for each candidiate.

Candidate	1	2	3	4	5	6	7	8
$d_i = x_i - y_i$	-28	-4	-40	-28	36	-24	-8	-16
Rank of $\lvert d_i \rvert$	5.5	1	8	5.5	7	4	2	3
Sign	-	-	-	-	+	-	-	-