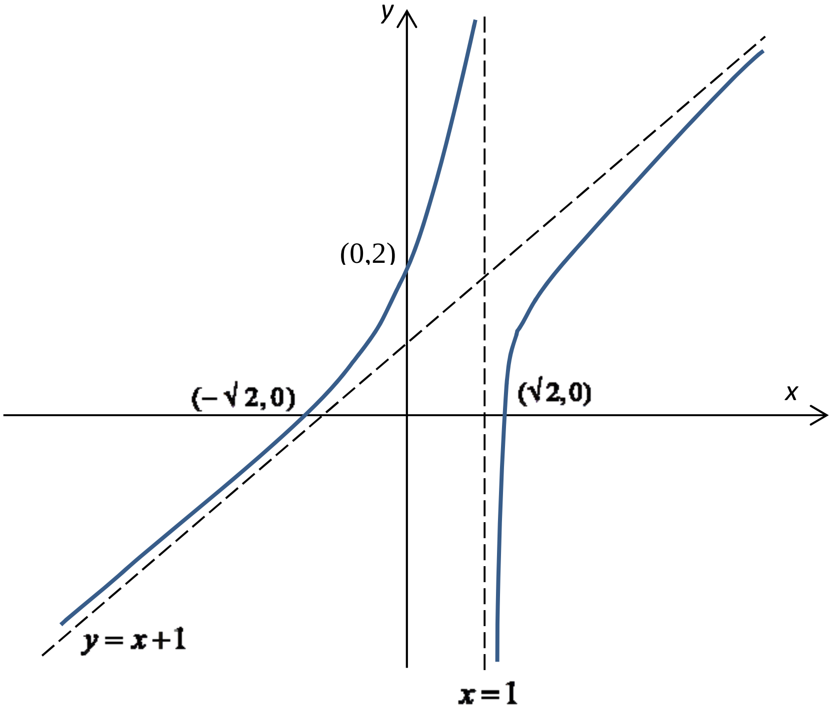
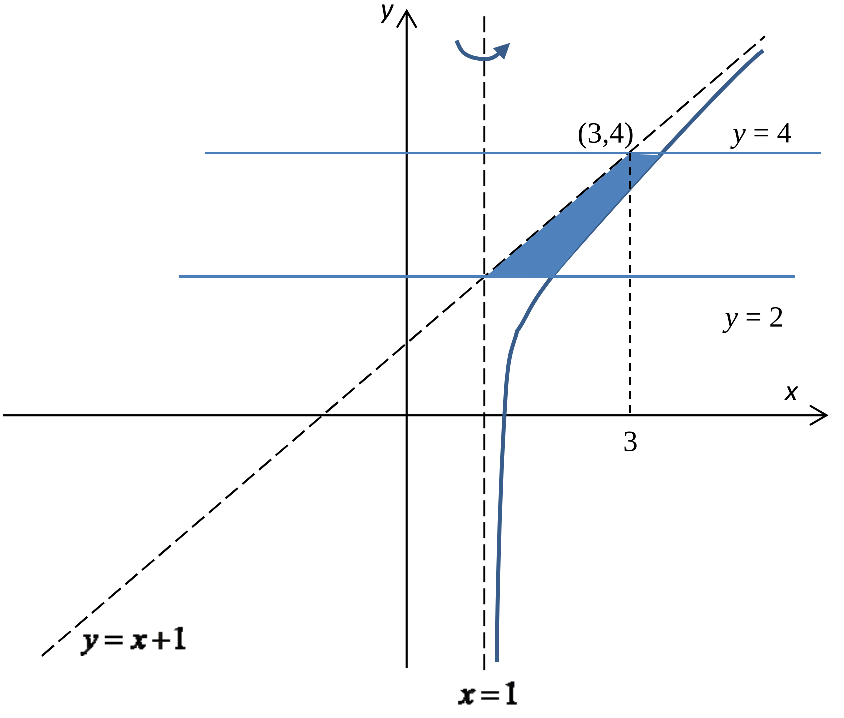
2017 Year 6 H2 Math Prelim Exam Paper 2 Mark Scheme

Qn	Suggested Solutions
1(i)	$\frac{\mathrm{d}}{\mathrm{d}x}\tan^2 x = 2\tan x \cdot \sec^2 x$ $\int_0^{\frac{\pi}{4}} \sec^2 x \tan x\, \mathrm{e}^{\tan^2 x}\,\mathrm{d}x = \frac{1}{2}\int_0^{\frac{\pi}{4}} 2\sec^2 x \tan x\, \mathrm{e}^{\tan^2 x}\,\mathrm{d}x$ $= \frac{1}{2}\left[\mathrm{e}^{\tan^2 x}\right]_0^{\frac{\pi}{4}}$ $= \frac{1}{2}\left(\mathrm{e}^{\tan^2 \frac{\pi}{4}} - \mathrm{e}^{\tan^2 0}\right)$ $= \frac{1}{2}(\mathrm{e}-1)$
1(ii)	$\int \frac{1+72x-32x^3}{\sqrt{9-4x^2}}\,\mathrm{d}x = \int \frac{1+8x\left(9-4x^2\right)}{\sqrt{9-4x^2}}\,\mathrm{d}x$ $= \int \frac{1}{\sqrt{9-4x^2}} + 8x\left(9-4x^2\right)^{\frac{1}{2}}\,\mathrm{d}x$ $= \frac{1}{2}\sin^{-1}\left(\frac{2x}{3}\right) - \frac{2}{3}\left(9-4x^2\right)^{\frac{3}{2}} + C$

Qn	Suggested Solution

2i)	$y=\frac{x^2+(a-1)x-a-1}{x-1}=(x+a)-\frac{1}{x-1}$ $\therefore a=1$ (shown) **<u>Alternative</u>** Let $\frac{x^2+(a-1)x-a-1}{x-1}=(x+1)+\frac{b}{x-1}$ $\Rightarrow x^2+(a-1)x-a-1=x^2-1+b$ Comparing coeff of x: $a-1=0$ $\therefore a=1$ (shown) and $b=-1$ $\therefore y=(x+1)-\frac{1}{x-1}=\frac{x^2-2}{x-1}$ HA: $x=1$ OA: $y=x+1$(given) 

(ii)



$$y=\frac{x^2-2}{x-1}\xrightarrow{\text{replace } x \text{ with } (x+1)} y=\frac{(x+1)^2-2}{x}$$

$$y=\frac{(x+1)^2-2}{x}$$

$$xy=x^2+2x-1$$

$$x^2+(2-y)x-1=0$$

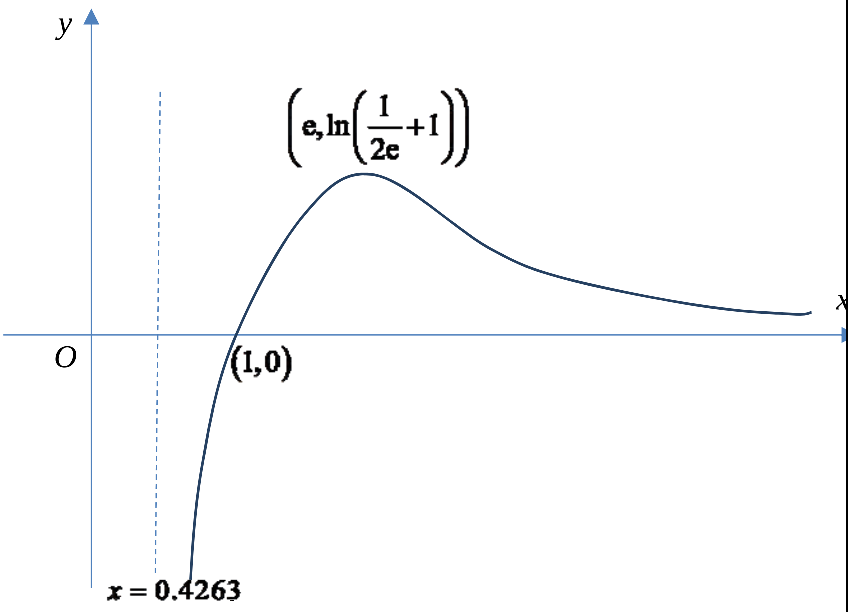
$$x=\frac{-(2-y)\pm\sqrt{(2-y)^2+4(1)(1)}}{2}$$

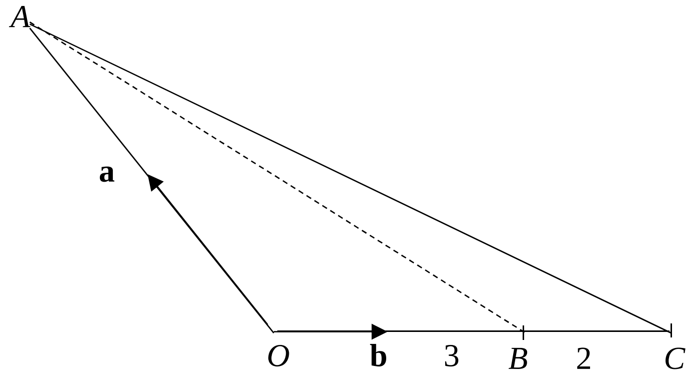
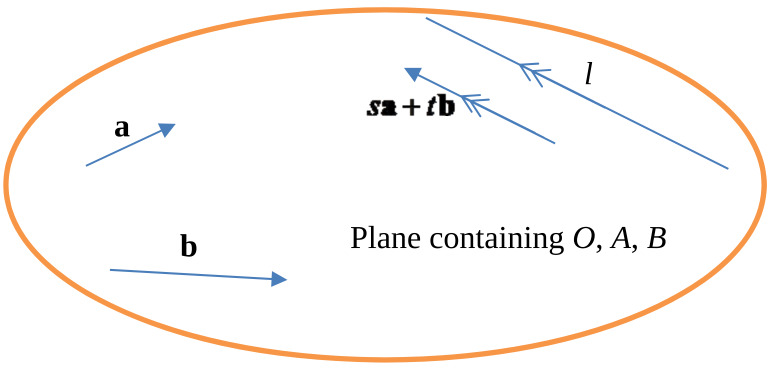
$$\therefore x=\frac{(y-2)+\sqrt{y^2-4y+8}}{2}$$ **(reject -ve root)**

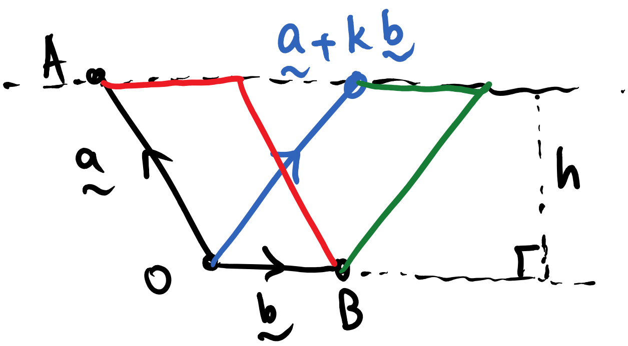
$$\text{Volume} = \pi\int_2^4\left(\frac{(y-2)+\sqrt{y^2-4y+8}}{2}\right)^2 \mathrm{d}y-\frac{1}{3}\pi(2)^2(2)$$

$= 9.75$ units^3 (3 s.f)

Qn	Solution
3i)	$u=\frac{\ln x}{x}\Rightarrow\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{1-\ln x}{x^2}$ $\frac{\mathrm{d}u}{\mathrm{d}y}=\frac{\mathrm{d}u}{\mathrm{d}x}\times\frac{\mathrm{d}x}{\mathrm{d}y}$ $=\frac{1-\ln x}{x^2}\times\frac{x\ln x+2x^2}{1-\ln x}$ $=\frac{\ln x+2x}{x}$ $=u+2$ (shown) $\frac{1}{u+2}\frac{\mathrm{d}u}{\mathrm{d}y}=1\Rightarrow\ln\lvert u+2\rvert=y+c$, c is an arbitrary constant $u+2=A\mathrm{e}^y$, A is an arbitrary constant $\frac{\ln x}{x}+2=A\mathrm{e}^y$ $y=0, x=1$: $A=2$ $y=\ln\left(\frac{\ln x}{2x}+1\right)$ (shown)
ii)	$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1-\ln x}{x\ln x+2x^2}$ for $x>0$. When $\frac{\mathrm{d}y}{\mathrm{d}x}=0$, $1-\ln x=0\Rightarrow x=\mathrm{e}$, $y=\ln\left(\frac{1}{2\mathrm{e}}+1\right)$ Therefore the maximum point is at $\left(\mathrm{e}, \ln\left(\frac{1}{2\mathrm{e}}+1\right)\right)$.
iii)	Graph of $y=\ln\left(\frac{\ln x}{2x}+1\right)$ When $x\to\infty, \frac{\ln x}{2x}\to 0$, $y=\ln\left(\frac{\ln x}{2x}+1\right)\to\ln 1=0$. Thus $a=0$ (shown)
iv)	For $y\to\infty$, $\frac{\ln x}{2x}+1\to\infty$ $\frac{\ln x}{2x}\to\infty$ $x\to 0$ But $\ln x\to-\infty$, thus y cannot approach positive infinity. For $y\to-\infty$, $\frac{\ln x}{2x}+1\to 0$ $\ln x+2x\to 0$ $x\to 0.4263$ $\therefore b=0.4263$

v)	y $\left(e, \ln\left(\frac{1}{2e}+1\right)\right)$ x O $(1,0)$ $x = 0.4263$

Qn	Suggested Solution
4(a) **(i)**	 $\overrightarrow{OC}=\dfrac{5}{3}\mathbf{b}\quad \therefore \overrightarrow{AC}=\dfrac{5}{3}\mathbf{b}-\mathbf{a}=\dfrac{1}{3}\left(5\mathbf{b}-3\mathbf{a}\right)//5\mathbf{b}-3\mathbf{a}$ Equation of line *AC*: $\mathbf{r}=\mathbf{a}+\lambda\left(3\mathbf{a}-5\mathbf{b}\right),\quad \lambda\in\mathbb{R}$
(ii)	Since *l* lies on the plane containing *O*, *A* and *B*, its direction vector is coplanar with **a** and **b**, thus it will be a linear combination of **a** and **b**, i.e. $s\mathbf{a}+t\mathbf{b}$ is a direction vector for *l*. <u>Alternative 1</u> $s\mathbf{a}+t\mathbf{b}$ is a linear combination of **a** and **b**, so $s\mathbf{a}+t\mathbf{b}$ is coplanar with **a** and **b** on the plane containing *O*, *A* and *B*. Thus the direction vector of any line on the plane containing *O*, *A* and *B* can be expressed as $s\mathbf{a}+t\mathbf{b}$ for some real value of *s* and *t* by construction of a parallel coplanar direction vector.  <u>Alternative 2</u> Points on the plane containing *O*, *A*, *B* has form $\alpha\mathbf{a}+\beta\mathbf{b}$ since the vector equation of the plane is $\mathbf{r}=\alpha\mathbf{a}+\beta\mathbf{b}$. Since the line *l* lies on this plane, points on the line *l* must also have the form $\alpha\mathbf{a}+\beta\mathbf{b}$. Taking 2 points on the line *l* with position vector $\alpha_1\mathbf{a}+\beta_1\mathbf{b}$ and $\alpha_2\mathbf{a}+\beta_2\mathbf{b}$, the direction vector of the line *l* is $\left(\alpha_1\mathbf{a}+\beta_1\mathbf{b}\right)-\left(\alpha_1\mathbf{a}+\beta_1\mathbf{b}\right)=\left(\alpha_1-\alpha_2\right)\mathbf{a}+\left(\beta_1-\beta_2\right)\mathbf{b}=s\mathbf{a}+t\mathbf{b}$ for some real value of *s* and *t*.

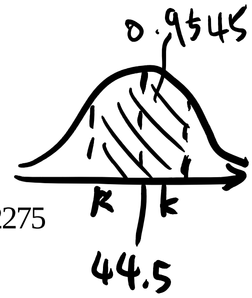
	Since l perpendicular to AB, Consider $(s\mathbf{a}+t\mathbf{b}).(\mathbf{a}-\mathbf{b})=0$ $s\mathbf{a}.\mathbf{a}-s\mathbf{a}.\mathbf{b}+t\mathbf{b}.\mathbf{a}-t\mathbf{b}.\mathbf{b}=0$ $s\lvert\mathbf{a}\rvert^2+(t-s)\mathbf{a}.\mathbf{b}-t\lvert\mathbf{b}\rvert^2=0$ $4s\lvert\mathbf{b}\rvert^2+(t-s)\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\left(-\frac{1}{4}\right)-t\lvert\mathbf{b}\rvert^2=0$ $4s-\frac{1}{2}(t-s)-t=0$ $\frac{9}{2}s=\frac{3}{2}t$ $\therefore t=3s$ (shown) Thus, the direction vector of l can be expressed as $(s\mathbf{a}+3s\mathbf{b})$ or $(\mathbf{a}+3\mathbf{b})$. Equation of l : $\mathbf{r}=\mathbf{b}+\mu(\mathbf{a}+3\mathbf{b}),\quad \mu\in\mathbb{R}$
(iii)	Let intersection point be D. At D, $\mathbf{a}+\lambda(3\mathbf{a}-5\mathbf{b})=\mathbf{b}+\mu(\mathbf{a}+3\mathbf{b})$ Since **a** and **b** are non-zero, non-parallel vectors, $1+3\lambda=\mu$ - - - - - (1) $-5\lambda=1+3\mu$ - - - - (2) Solving, $-5\lambda=1+3(1+3\lambda)$ $14\lambda=-4$ $\therefore \lambda=-\frac{2}{7}, \mu=\frac{1}{7}$ $\qquad \therefore \overrightarrow{OD}=\frac{1}{7}\mathbf{a}+\frac{10}{7}\mathbf{b}$
4(b)	**Version One**  Since the base length (OB) and perpendicular height remain the same, the area of parallelograms formed by different k remains the same as the area of the parallelogram with sides OA and OB. **Version Two**

	$\lvert(\mathbf{a}+k\mathbf{b})\times\mathbf{b}\rvert=\lvert\mathbf{a}\times\mathbf{b}+k\mathbf{b}\times\mathbf{b}\rvert=\lvert\mathbf{a}\times\mathbf{b}+\mathbf{0}\rvert=\lvert\mathbf{a}\times\mathbf{b}\rvert$
	Area of parallelogram $=\lvert\mathbf{a}\times\mathbf{b}\rvert$ $=\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\sin\theta$ $=\lvert\mathbf{a}\rvert\left(\frac{1}{2}\lvert\mathbf{a}\rvert\right)\sqrt{1-\left(-\frac{1}{4}\right)^2}$ $=\frac{\sqrt{15}}{8}\lvert\mathbf{a}\rvert^2$

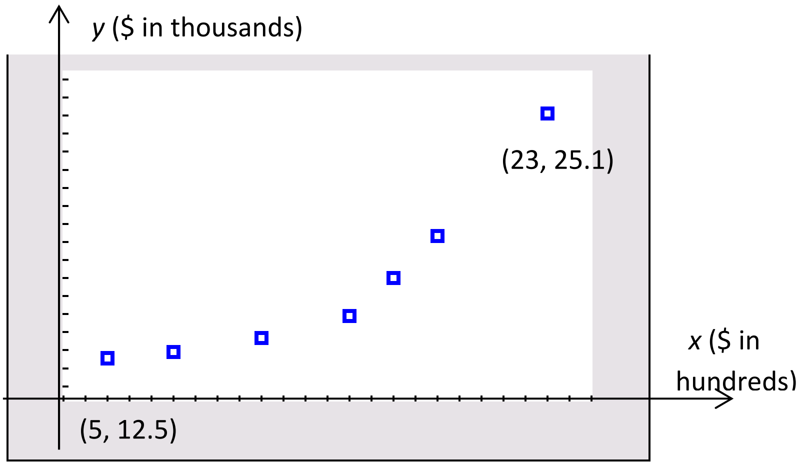
Qn	Suggested Solution
5(i)	<table><tr><td>x</td><td>- 8</td><td>- 5</td><td>- 1</td><td>2</td></tr><tr><td>P(X = x)</td><td>$\frac{2}{36}=\frac{1}{18}$</td><td>$\frac{4}{36}=\frac{2}{18}$</td><td>$\frac{7}{36}$</td><td>$\frac{23}{36}$</td></tr></table> $\mathrm{E}(X)=\frac{46}{36}-\frac{7}{36}-\frac{20}{36}-\frac{16}{36}=\frac{1}{12}$ $\mathrm{Var}(X)=\mathrm{E}(X^2)-\left(\mathrm{E}(X)\right)^2=\frac{92}{36}+\frac{7}{36}+\frac{50}{18}+\frac{64}{18}-\frac{1}{12^2}$ $=\frac{1307}{144}$ or 9.08 (to 3sf)
(ii)	Since n is large, $\overline{X} \sim \mathrm{N}\left(\frac{1}{12},\frac{1307}{144n}\right)$ approximately by Central Limit Theorem.

	For n = 30, $\mathrm{P}\left(\overline{X} > 0\right) = 0.560$ (to 3sf) For n = 50000, $\mathrm{P}\left(\overline{X} > 0\right) = 1.00$ (to 3sf) The **more rounds** this game is played, the **higher the chance of** casino receiving a **positive average winnings.** In other words, it is almost certain that casino will win in the long run.

Qn	Suggested Solution
6(i)	The distribution may become **<u>bimodal</u>** when the data for both groups are combined
(ii)	Let Y be the score of a random student from Group Y. $Y \sim \mathrm{N}(34,\ 25)$ $\mathrm{P}(Y \geq d) \geq 0.6$ $\mathrm{P}(Y \leq d) \leq 0.4$ When $\mathrm{P}(Y \leq d_c) = 0.4$, $d_c = 32.733$. Thus $d < 32.733$. The maximum mark is 32.7
(iii)	$\mathrm{E}\left(\sum_1^4 Y_i - 3X\right) = 4\mathrm{E}(Y) - 3\mathrm{E}(X) = -29$ $\mathrm{Var}\left(\sum_1^4 Y_i - 3X\right) = 4\mathrm{Var}(Y) + 9\mathrm{Var}(X) = 280$ $\therefore \sum_1^4 Y_i - 3X \sim \mathrm{N}(-29, 280)$ $\mathrm{P}(\sum_1^4 Y_i < 3X) = \mathrm{P}(\sum_1^4 Y_i - 3X < 0) = 0.958$ (to 3sf)

(iv)	$\overline{M} = \dfrac{\sum_{i=1}^{20} X_i + \sum_{i=1}^{20} Y_i}{40}$ $\mathrm{E}\left(\overline{M}\right) = \dfrac{20\mathrm{E}(X) + 20\mathrm{E}(Y)}{40} = \dfrac{1}{2}\left(\mathrm{E}(X) + \mathrm{E}(Y)\right) = 44.5$ Let $\sigma^2 = \mathrm{Var}\left(\overline{M}\right)$ $= \dfrac{1}{1600}\left(20\mathrm{Var}(X) + 20\mathrm{Var}(Y)\right)$ $= \dfrac{1}{80}\left(\mathrm{Var}(X) + \mathrm{Var}(Y)\right) = 0.5625$  $\overline{M} \sim N\left(44.5, 0.5625\right)$ Since $\mathrm{P}(\lvert\overline{M} - 44.5\rvert < k) = 0.9545$ $\Rightarrow \mathrm{P}(\overline{M} < 44.5 - k) = \dfrac{1 - 0.9545}{2} = 0.02275$ $\therefore 44.5 - k = 43.000$ $\Rightarrow k = 1.50$ (3 s.f) **Alternative** $\overline{M} \sim N\left(44.5, \sigma^2\right)$ Since $\mathrm{P}(\lvert\overline{M} - 44.5\rvert < 2\sigma) = 0.9545$ $\therefore k = 2\sigma = 2\sqrt{0.5625} = 1.50$ (3sf)
	Marks of students are independent of one another.

Qn
7(i)
(ii)
(iii)

Qn	Suggested Solution
8(i)	**No, because correlation does not imply causation /** The increase in the monthly sales of refrigerators could be due to other factors such as a rise in the income level.
(ii)	 There appears to be a curvilinear/non-linear relationship between x and y, thus a linear model is not likely to be appropriate.
(iii)	(A) $r = 0.9684$ (to 4dp) (B) $r = 0.9495$ (to 4dp) Since the r value between $e^{\sqrt{x}}$ and y has an absolute value closer to 1, $y = a + be^{\sqrt{x}}$ is the better model.
(iv)	New regression line (8 data points) for y on $e^{(\sqrt{x})}$ is $y = 10.876 + 0.09906e^{(\sqrt{x})}$ $\bar{y} = \frac{11.3+12.5+12.9+13.6+14.8+17+19.3+25.1}{8} = 15.813$

	Since $\bar{x}$ and $\bar{y}$ lie on the new regression line y on $e^{(\sqrt{x})}$, and letting $x = m$ when $y = 11.3$, $15.813 = 10.876 + 0.09906\left(\dfrac{\sum_{i=1}^{7} e^{(\sqrt{x_i})} + e^{(\sqrt{m})}}{8}\right)$ Using GC (1-var stats), $\sum_{i=1}^{7} e^{(\sqrt{x_i})} = 390.96$ $\therefore 15.813 = 10.876 + 0.09906\left(\dfrac{390.96 + e^{(\sqrt{m})}}{8}\right)$ $\therefore e^{(\sqrt{m})} = 7.7479 \quad \Rightarrow \quad m = 4.19 \approx 4$ Monthly advertising expenditure = \$400 (nearest hundred)

Qn	Suggested Solution
9(i)	Assume that the: • weights of the 5 people chosen are independent of each other • weights of the people in the village are independent • sample is chosen randomly.
(ii)	$P(X = 1) < P(X = 2)$ and $P(X = 2) > P(X = 3)$ ${}^5C_1 p(1-p)^4 < {}^5C_2 p^2(1-p)^3$ and ${}^5C_2 p^2(1-p)^3 > {}^5C_3 p^3(1-p)^2$ Since $(1-p) > 0$ and $p > 0$, $1 - p < 2p$ and $1 - p > p$ $p > \frac{1}{3}$ and $p < \frac{1}{2}$ $\therefore \frac{1}{3} < p < \frac{1}{2}$ (shown)
(iii)	$\bar{x} = 1.965$ (from GC) Since $n = 5$, $np \approx 1.965 \Rightarrow p \approx 0.393$
(iv)	$X \sim B(5, 0.393)$ $P((X_1 \geq 4) \cap (X_1 > X_2))$ $= P(X_1 = 4)P(X_2 \leq 3) + P(X_1 = 5)P(X_2 \leq 4)$ $= 0.0724(0.91823) + (0.00937)(0.99063)$ $= 0.0758$ (3 sf)

Qn	Solutions
10(a) (i)	Number of ways = $\frac{12!}{3!2!} = 39916800$
(ii)	**<u>Method 1</u>** I S D R I B U I O N I S D R I B U I O N I S D R I B U I O N There are 3 ways to slot in the 2 T's Total number of ways = total number of ways to arrange the remaining ten letters (excluding the 2 T's) $\times 3 = \frac{10!}{3!} \times 3 = 1814400$ **<u>Method 2</u>** **<u>Case 1</u>:** One I included between the two T's Number of ways = $8! \times \frac{3!}{2!} = 120960$ **<u>Case 2</u>**: Two I's included between the two T's Number of ways = ${}^{7}C_{6} \times \frac{8!}{2!} \times 3! = 846720$ **<u>Case 3</u>:** Three I's included between the two T's Number of ways = ${}^{7}C_{5} \times \frac{8!}{3!} \times 3! = 846720$ Total number of ways $= 120960 + 2(846720) = 1814400$
(iii)	<u>Case 1</u>: Both I together but both T separated Number of ways = ${}^{9}C_{2} \times 8! = 1451520$ II D S R B U O N <u>Case 2</u>: Both T together but I separated Number of ways = ${}^{9}C_{2} \times 8! = 1451520$ (same approach as case 1) <u>Case 3:</u> Both I together and both T together

Number of ways = $9! = 362880$

Total no. of ways = $(1451520 \times 2) + 362880 = 3265920$

Alternative

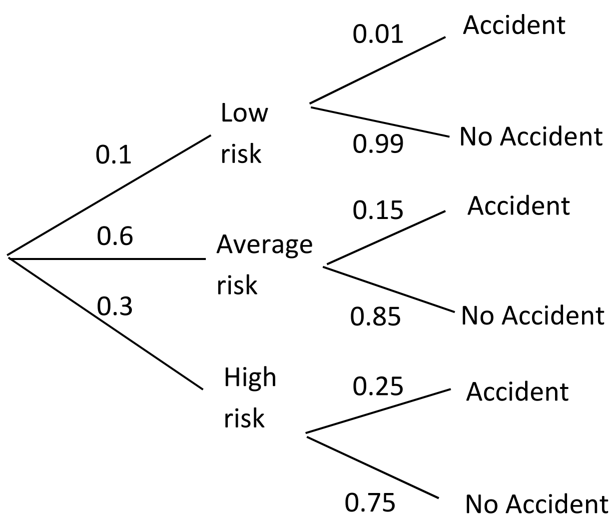
$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

$$= 2 \times \frac{10!}{2!} - 9! = 3265920$$

Required probability = $1 - \dfrac{3265920}{\frac{11!}{2!2!}} = 0.673$

(b)(i) P(holder is not involved in any accident | the holder is classified as ‘average’ risk) = 0.85

(ii)



Probability of a randomly chosen policy holder not involved in any car accident
= (0.1)(0.99) + (0.6)(0.85) + (0.3)(0.75)
= 0.834

(iii) P(policy holder is ‘low risk’ | has met at least one car accident)

$$= \frac{\text{P(holder is classified as 'low' risk and met with at least 1 accident)}}{\text{P(holder meets with at least 1 accident)}}$$

$$= \frac{0.1(0.01)}{1 - 0.834}$$

$= 0.00602$

(iv) Let C be the cost of insuring a randomly chosen ‘low risk’ policy holder (in thousands).

c	0	5	10	50	100
$P(C = c)$	0.99	0.0075	0.0015	0.0008	(0.01)(0.02) = 0.0002

(v)	$\mathrm{E}(C) = 100(0.0002)+50(0.0008)+10(0.0015)+5(0.0075)$ $= 0.1125$ $1000(0.1125) + 200 = 312.5$ The company should charge \$312.50 for a car insurance plan for 'low risk'.