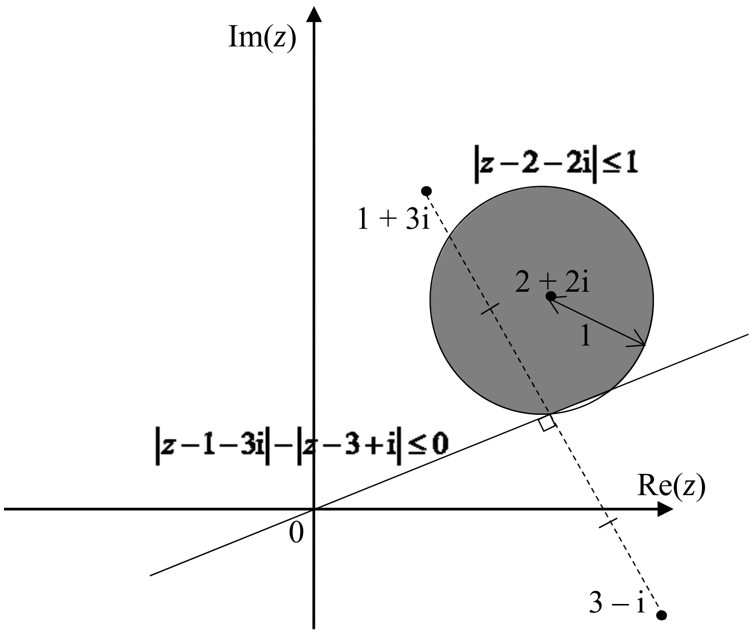
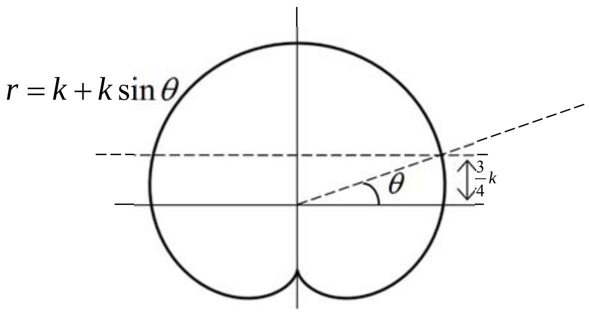
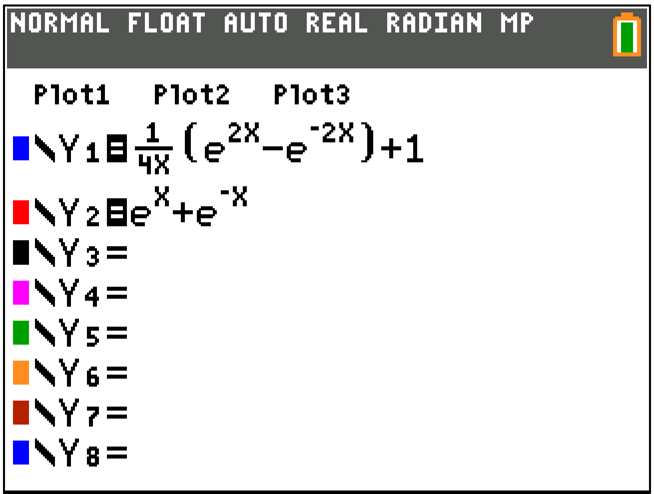
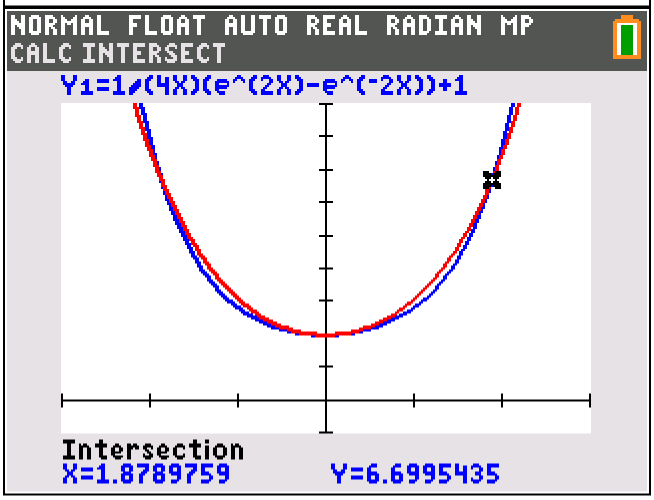
2017 JC2 Further Mathematics Prelim Paper 2

<table>
<tr><td>1</td><td>Characteristic equation: $m^2-6m+12=0$
Solutions are $m=3\pm\sqrt{3}\mathrm{i}=2\sqrt{3}\left(\cos\frac{\pi}{6}\pm\mathrm{i}\sin\frac{\pi}{6}\right)$
General solution:
$x_n=A\left(2\sqrt{3}\right)^n\left(\cos\frac{\pi}{6}+\mathrm{i}\sin\frac{\pi}{6}\right)^n+B\left(2\sqrt{3}\right)^n\left(\cos\frac{\pi}{6}-\mathrm{i}\sin\frac{\pi}{6}\right)^n$
$=\left(2\sqrt{3}\right)^n\left(A\left(\cos\frac{\pi}{6}+\mathrm{i}\sin\frac{\pi}{6}\right)^n+B\left(\cos\frac{\pi}{6}-\mathrm{i}\sin\frac{\pi}{6}\right)^n\right)$
$=\left(2\sqrt{3}\right)^n\left(A\left(\cos\frac{n\pi}{6}+\mathrm{i}\sin\frac{n\pi}{6}\right)+B\left(\cos\frac{n\pi}{6}-\mathrm{i}\sin\frac{n\pi}{6}\right)\right)$
$=\left(2\sqrt{3}\right)^n\left(C\cos\frac{n\pi}{6}+D\sin\frac{n\pi}{6}\right)$
Where $C=A+B, D=\left(A-B\right)\mathrm{i}$
When $n=0$: $x_0=C\Rightarrow C=1$
When $n=1$: $x_1=2\sqrt{3}\left(C\left(\frac{\sqrt{3}}{2}\right)+D\left(\frac{1}{2}\right)\right)\Rightarrow D=\sqrt{3}$
Hence $x_n=\left(2\sqrt{3}\right)^n\left(\cos\frac{n\pi}{6}+\sqrt{3}\sin\frac{n\pi}{6}\right)$.</td><td></td></tr>
<tr><td>2(i)</td><td></td><td></td></tr>
<tr><td>2(ii)</td><td>Minimum:
$\arg(z)=\arg(2+\mathrm{i})=\tan^{-1}\left(\frac{1}{2}\right)=0.464$ rad

Maximum:</td><td></td></tr>
</table>

	$\arg(z)=\arg(2+2\mathrm{i})+\sin^{-1}\left(\frac{1}{2\sqrt{2}}\right)=\frac{\pi}{4}+\sin^{-1}\left(\frac{1}{2\sqrt{2}}\right)=1.15 \text{ rad}$	
2(i)	$\frac{\mathrm{d}r}{\mathrm{d}\theta}=0 \Rightarrow k\cos\theta=0 \Rightarrow \theta=\frac{\pi}{2},\frac{3\pi}{2}$ $\frac{\mathrm{d}^2r}{\mathrm{d}\theta^2}=-k\sin\theta$ When $\theta=\frac{\pi}{2}$, $\frac{\mathrm{d}^2r}{\mathrm{d}\theta^2}=-k<0$ $\therefore$ maximum r=2k occurs when $\theta=\frac{\pi}{2}$ Furthest distance, r = 2k	
2(ii)	 When $y=\frac{3}{4}k, \quad r\sin\theta=\frac{3}{4}k$ $\Rightarrow (k+k\sin\theta)\sin\theta=\frac{3}{4}k$ $\Rightarrow 4\sin^2\theta+4\sin\theta-3=0$ $\Rightarrow \sin\theta=\frac{1}{2}$ or $-\frac{3}{2}$ (rejected since $0<\theta<\frac{\pi}{2}$) $\Rightarrow \theta=\frac{\pi}{6}$ Area = $2\left[\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} k^2(1+\sin\theta)^2\,\mathrm{d}x-\frac{1}{2}\left(\frac{3}{4}k\right)\left(\frac{\frac{3}{4}k}{\tan\frac{\pi}{6}}\right)\right]$ $=k^2\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}(1+\sin\theta)^2\,\mathrm{d}x-\frac{9\sqrt{3}}{16}k^2$ $=2.64607k^2$ Given: Area $\geq 0.75\,(13\times5)$ $\Rightarrow 2.64607k^2 \geq 0.75\,(13\times5)$ $\Rightarrow k \geq 4.38$	
4	$y=\frac{k}{2}\left(e^{\frac{x}{k}}+e^{-\frac{x}{k}}\right) \Rightarrow \frac{dy}{dx}=\frac{1}{2}\left(e^{\frac{x}{k}}-e^{-\frac{x}{k}}\right)$	

	$S=2\pi\int_{-a}^{a} y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$ $=2\pi\int_{-a}^{a}\frac{k}{2}\left(e^{\frac{x}{k}}+e^{-\frac{x}{k}}\right)\sqrt{1+\left(\frac{1}{2}\left(e^{\frac{x}{k}}-e^{-\frac{x}{k}}\right)\right)^2}\,dx$ $=k\pi\int_{-a}^{a}\left(e^{\frac{x}{k}}+e^{-\frac{x}{k}}\right)\sqrt{1+\frac{1}{4}\left(e^{\frac{2x}{k}}-2+e^{-\frac{2x}{k}}\right)}dx$ $=k\pi\int_{-a}^{a}\left(e^{\frac{x}{k}}+e^{-\frac{x}{k}}\right)\sqrt{\frac{1}{4}\left(e^{\frac{2x}{k}}+2+e^{-\frac{2x}{k}}\right)}dx$ $=\frac{k\pi}{2}\int_{-a}^{a}\left(e^{\frac{x}{k}}+e^{-\frac{x}{k}}\right)\sqrt{\left(e^{kx}+e^{-kx}\right)^2}\,dx$ $=\frac{k\pi}{2}\int_{-a}^{a}\left(e^{\frac{x}{k}}+e^{-\frac{x}{k}}\right)^2 dx$ $=\frac{k\pi}{2}\int_{-a}^{a}e^{\frac{2x}{k}}+2+e^{-\frac{2x}{k}}\,dx$ $=\frac{k\pi}{2}\left[\frac{k}{2}e^{\frac{2x}{k}}+2x-\frac{k}{2}e^{-\frac{2x}{k}}\right]_{-a}^{a}$ $=\frac{k\pi}{2}\left[\left(\frac{k}{2}e^{\frac{2a}{k}}+2a-\frac{k}{2}e^{-\frac{2a}{k}}\right)-\left(\frac{k}{2}e^{-\frac{2a}{k}}-2a-\frac{k}{2}e^{\frac{2a}{k}}\right)\right]$ $=\frac{k\pi}{2}\left(ke^{\frac{2a}{k}}+4a-ke^{-\frac{2a}{k}}\right)$	
	At $x=a$, $y=\frac{k}{2}\left(e^{\frac{a}{k}}+e^{-\frac{a}{k}}\right)$. $S_1=2\pi y(2a)$ $=2ka\pi\left(e^{\frac{a}{k}}+e^{-\frac{a}{k}}\right)$ If $S=S_1$: $\frac{k\pi}{2}\left(ke^{\frac{2a}{k}}+4a-ke^{-\frac{2a}{k}}\right)=2ka\pi\left(e^{\frac{a}{k}}+e^{-\frac{a}{k}}\right)$ $ke^{\frac{2a}{k}}+4a-ke^{-\frac{2a}{k}}=4a\left(e^{\frac{a}{k}}+e^{-\frac{a}{k}}\right)$ $\left(\frac{k}{4a}\right)\left(e^{\frac{2a}{k}}-e^{-\frac{2a}{k}}\right)+1=e^{\frac{a}{k}}+e^{-\frac{a}{k}}$ Let $w=\frac{a}{k}$:	

	$\frac{1}{4w}\left(e^{2w}-e^{-2w}\right)+1=e^{w}+e^{-w}$   $\frac{a}{k}=w=1.88$ (3 sf)	
	$V=\pi\int_{-a}^{a}y^2dx$ $=\pi\int_{-a}^{a}\frac{k^2}{4}\left(e^{\frac{x}{k}}+e^{-\frac{x}{k}}\right)^2dx$ $=\frac{k^2\pi}{4}\int_{-a}^{a}\left(e^{\frac{x}{k}}+e^{-\frac{x}{k}}\right)^2dx$ $=\frac{k^2\pi}{4}\left(\frac{2S}{k\pi}\right)$ $=\frac{Sk}{2}$	
5(i)	$z=xy\ \Rightarrow \frac{\mathrm{d}z}{\mathrm{d}x}=y+x\frac{\mathrm{d}y}{\mathrm{d}x}\ \Rightarrow \frac{\mathrm{d}^2z}{\mathrm{d}x^2}=2\frac{\mathrm{d}y}{\mathrm{d}x}+x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ Substituting,	

	$x\dfrac{d^2y}{dx^2}+(2-4x)\dfrac{dy}{dx}+4y(x-1)=0$ $\dfrac{d^2z}{dx^2}-4x\dfrac{dy}{dx}+4xy-4y=0$ $\dfrac{d^2z}{dx^2}-4\left(\dfrac{dz}{dx}-y\right)+4xy-4y=0$ $\dfrac{d^2z}{dx^2}-4\dfrac{dz}{dx}+4z=0$ (shown) Characteristic equation: $m^2-4m+4=(m-2)^2=0 \Rightarrow m=2$. Hence the general solution: $z=(Ax+B)e^{2x}$ $y=\left(A+\dfrac{B}{x}\right)e^{2x}$, where A and B are constants.	
5(ii)	Complementary solution of $\dfrac{d^2s}{dt^2}-4\dfrac{ds}{dt}+4s=\cos t$ -(1) is $s=(Ct+D)e^{2t}$, where C and D are constants. Particular solution: Try $s=E\cos t+F\sin t$, where E and F are constants. $\dfrac{ds}{dt}=-E\sin t+F\cos t$ $\dfrac{d^2s}{dt^2}=-s$ Substituting into (1): $-4(-E\sin t+F\cos t)+3(E\cos t+F\sin t)=\cos t$ $(3E-4F)\cos t+(4E+3F)\sin t=\cos t$ Comparing coefficients of cosine and sine: $\begin{cases}3E-4F=1\\4E+3F=0\end{cases} \Rightarrow E=\dfrac{3}{25},\ F=-\dfrac{4}{25}$ General solution of (1) is: $s=(Ct+D)e^{2t}+\dfrac{1}{25}(3\cos t-4\sin t)$ $\dfrac{ds}{dt}=Ce^{2t}+2(Ct+D)e^{2t}-\dfrac{1}{25}(3\sin t+4\cos t)$ -(2) $t=0,\ s=0 \Rightarrow 0=D+\dfrac{3}{25} \Rightarrow D=-\dfrac{3}{25}$ t=0, $\dfrac{ds}{dt}=0 \Rightarrow 0=C+2D-\dfrac{4}{25} \Rightarrow C=\dfrac{10}{25}$	

$\therefore s=\left(\frac{2}{5}t-\frac{3}{25}\right)\mathrm{e}^{2t}+\frac{1}{25}\left(3\cos t-4\sin t\right)$

$\frac{\mathrm{d}s}{\mathrm{d}t}=\frac{2}{5}\mathrm{e}^{2t}+2\left(\frac{2}{5}t-\frac{3}{25}\right)\mathrm{e}^{2t}-\frac{1}{25}\left(3\sin t+4\cos t\right)$

From GC, $\frac{\mathrm{d}s}{\mathrm{d}t}>340$ when $t\approx 2.52$ seconds (nearest hundredth).
(Accept 2.53 seconds as well)

6

Let the random variable X denote the number of pay-outs per 10 plays on a machine.
Then $X\sim \mathrm{B}\left(10,0.05\right)$ according to the manufacturer.

Testing $\mathrm{H}_0:\ \mathrm{B}\left(10,0.05\right)$ is an appropriate model for X

against $\mathrm{H}_1:\ \mathrm{B}\left(10,0.05\right)$ is not an appropriate model for X.

X	0	1	2	≥ 3
E_i	59.874	31.512	7.464	1.150
O_i	72	25	1	2
$\frac{\left(O_i-E_i\right)^2}{E_i}$	2.456	1.346	3.659	

$\chi^2=7.46$
Degree of freedom $=3-1=2$.
By GC, p value $=0.0240$.

The p value indicates that there is evidence to reject H_0 at the 5% level of significance (but not at the 1%), i.e., there is sufficient evidence that the slot machines do not pay out 5% of the time.

There is some (but not strong) evidence to suggest that the casino tampered with the probability of the pay-outs.

Based on the contributions to the test statistic of individual cells, it is likely that the casino adjusted the probability of pay-outs downwards as the cell for 0 pay-outs has a frequency much larger than expected and the cell for 2 or more pay-outs has a frequency much lower than expected.

7

Let X and Y be the sales before and after the online advertising respectively.
Let $D=Y-X$ and m be the median of D.

D is unlikely to follow a normal distribution since it represents the difference in sales of different types of products.
A paired sample is used to match the 'sales after' to the 'sales before' for the same product. Different skincare products often target different age groups and may thus differ in their response to online advertising. Paired sample eliminates such differences across different products.

Using a Wilcoxon test:

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Test $\quad$ H₀ : $m = 0$
against $\quad$ H₁ : $m > 0$ at 5% significance level

D	32	3	82	-12	26	8	-28	72	70	123
Rank of D	6	1	9	3	4	2	5	8	7	10
+	6	1	9		4	-2		8	7	10
-				-3			-5			

T = sum of '-' differences = 8
Rejection region: $T \le 10$
Since $T = 8 < 10$, reject H_0
Conclude that there is sufficient evidence that sales have improved at 5% level of significance.

Assume that the samples are drawn from a continuous and symmetrical distribution.

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<tr><td>8(i)</td><td>

P(Claire wins a game on her x^{th} attempt)

$= (0.6)^{r-1}(0.65)^{r-1}(0.4)$

$= (0.39)^{r-1}(0.4), \quad r = 1, 2, 3, \ldots$

P(Claire wins a game)

$= \sum_{r=1}^{\infty} \left(0.39\right)^{r-1}(0.4)$

$= \dfrac{0.4}{1-0.39}$

$= \dfrac{40}{61}$

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<tr><td>8(ii)</td><td>

$P(X > a+b \mid X > a)$

$= \dfrac{P(X > a+b)}{P(X > a)}$

$= \dfrac{(0.6)^{\frac{a+b}{2}}(0.65)^{\frac{a+b}{2}}}{(0.6)^{\frac{a}{2}}(0.65)^{\frac{a}{2}}}$

$= (0.6)^{\frac{b}{2}}(0.65)^{\frac{b}{2}}$

$= (0.39)^{\frac{b}{2}}$

Put $a =$ 4, $\therefore b = n-4$

Given: $(0.39)^{\frac{n-4}{2}} < 0.1$

$\Rightarrow \dfrac{n-4}{2}\lg(0.39) < \lg 0.1$

$\Rightarrow \dfrac{n-4}{2} > 2.445$

$\Rightarrow n > 8.89$

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</table>

	Least $n = 10$	
8(iii)	P(Claire wins the second game on her 6^{th} attempt | Claire wins the 1^{st} 2 games) $= \dfrac{{}^5C_1(0.4)(0.6)^4(0.65)^4(0.4)}{\left(\dfrac{40}{61}\right)^2}$ = 0.0430 (to 3 s.f.)	
9	Let X denotes the random variable for the number of people arriving at the lift in 15-min interval. $\mathrm{E}(X) = \dfrac{15}{60} \times 10 = 2.5$ Then $X \sim \mathrm{Po}(2.5)$. $\mathrm{P}(X \le 4) = 0.891$ (to 3 s.f.)	
9(i)	$\mathrm{P}(X \le 8) = 0.999$ (to 3 s.f.)	
9(ii)	Let Y_1 and Y_2 denote the random variables for the number of people arriving at the queue in the first and second 10-min interval respectively. Then $Y_1 \sim \mathrm{Po}\left(\dfrac{5}{3}\right), Y_2 \sim \mathrm{Po}\left(\dfrac{5}{3}\right)$ Required probability $= \mathrm{P}(Y_1 \le 4, Y_2 \le 4) + \mathrm{P}(Y_1 = 5, Y_2 \le 3) + \mathrm{P}(Y_1 = 6, Y_2 \le 2)$ $+ \mathrm{P}(Y_1 = 7, Y_2 \le 1) + \mathrm{P}(Y_1 = 8, Y_2 = 0)$ = 0.969 (to 3 s.f.)	
	Let A denotes the random variable for the number of people arriving at the queue in t minute interval. $A \sim \mathrm{Po}\left(\dfrac{t}{6}\right)$ $\mathrm{P}(T > t) = \mathrm{P}(A = 0) = \mathrm{e}^{-\frac{t}{6}},\ t \ge 0$ $\mathrm{P}(T \le t) = 1 - \mathrm{e}^{-\frac{t}{6}},\ t \ge 0$ Probability density function of T is $\mathrm{f}(t) = \begin{cases} \dfrac{1}{6}\mathrm{e}^{-\frac{t}{6}}, & t \ge 0, \\ 0, & \text{otherwise.} \end{cases}$ Thus T has exponential distribution with mean 6 mins. Assume that the lift departs only when a passenger arrives and any passenger who needs to wait for the lift leaves the place. It may not be reasonable to assume that any passenger who needs to wait for the lift leaves the place since tourists invariably will wait for the lift to visit the observatory.	

<table>
<tr><td>10</td><td>Assume that the populations of sizes of mussels at site A and B are normally distributed, and have common variances.

Let μ_A and μ_B be the population mean size of mussels at Site A and B respectively.

Test $\quad$ $H_0 : \mu_A = \mu_B$
against $\quad$ $H_1 : \mu_A < \mu_B$ at 10% significance level

Under H_0, Test Statistic, $T = \dfrac{\bar{A} - \bar{B} - (0)}{s\sqrt{\dfrac{1}{15} + \dfrac{1}{15}}} \sim t(28)$
$n_A = 15,\ \bar{x}_A = 48.53333,\ s_A^{\ 2} = 3.83344^2$
$n_B = 15,\ \bar{x}_B = 50.46667,\ s_B^{\ 2} = 2.89918^2$
$s_p^2 = \dfrac{14s_A^2 + 14s_B^2}{28} = 3.39888^2$

Value of test statistic, $t = -1.56$

p-value = 0.0653

Since p-value = 0.0653 > 0.05, do not reject H_0.
Conclude that there is insufficient evidence that the conservation measures are effective at 5% level of significance.

To improve the accuracy of the testing,
(1) increase the sample size so that there is no need to assume normal distribution
(2) obtain the size of mussels from site A instead of site B to minimise any site differences.</td><td></td></tr>
<tr><td>10(i)</td><td>Let μ_1 be the population mean size of mussels at the site after the fishing ban of 3 years.

A 95% confidence interval for the population mean, μ_1 is
$\left(\bar{x} - 2.10982\dfrac{s}{\sqrt{18}},\ \bar{x} + 2.10982\dfrac{s}{\sqrt{18}}\right)$
Unbiased estimate for population mean, $\bar{x} = \dfrac{45.7 + 49.3}{2} = 47.5$

and $\bar{x} - 2.10982\dfrac{s}{\sqrt{18}} = 45.7 \Rightarrow s = 3.61962$
Unbiased estimate for population variance, $s^2 = 13.1$ (to 3 s.f._)</td><td></td></tr>
<tr><td>10(ii)</td><td>A 95% confidence interval for the population mean size of mussels means that if repeated samples of similar sample size are taken and 95% confidence interval is constructed for each sample, then in the</td><td></td></tr>
</table>

	long run, 95% of the confidence intervals will contain the actual population mean size of mussels. Since 40mm is outside the confidence interval, there is sufficient evidence to reject the hypothesis $\mu_1 = 40$, and conclude that $\mu_1 \neq 40$ at 5% level of significance. Thus there is sufficient evidence to conclude that $\mu_1 > 40$ at 5% level of significance. However, as 51mm is outside the confidence interval, we conclude that $\mu_1 > 51$ at 5% level of significance and thus the site is not ready for fishing ban to be lifted.	
10(iii)	Let μ_2 be the population mean size of mussels at the site after a further 3-year period. Test H₀ : $\mu_2 = 51$ against H₁ : $\mu_2 > 51$ at 10% significance level Under H₀, Test Statistic, $T = \dfrac{\bar{X} - 51}{\dfrac{s}{\sqrt{12}}} \sim t(11)$ Now $\dfrac{s^2}{12} = \dfrac{\frac{12}{11}(k)}{12} = \dfrac{k}{11}$, thus $T = \dfrac{\bar{X} - 51}{\sqrt{\dfrac{k}{11}}}$ Rejection Region: $t > 1.36343$ When H₀ is rejected at 10% significance level, $\Rightarrow \dfrac{52.4 - 51}{\sqrt{\dfrac{k}{11}}} > 1.36343$ $\Rightarrow \sqrt{k} < 3.40558$ $\Rightarrow k < 11.598$ $\Rightarrow k < 11.6$ (to 3 s.f.) Conclude that the conservation measures are working if $k < 11.6$.	