2017 H2 MATH (9758/01) JC 2 PRELIM – SOLUTIONS

Qn	Solution
1	**Connected Rate of Change**
	$\tan 45^\circ = \frac{r}{h} \Rightarrow r = h$ $V = \frac{1}{3}\pi r^2 h$ $V = \frac{1}{3}\pi h^3$ $\frac{\mathrm{d}V}{\mathrm{d}h} = \pi h^2$ When $h = 0.3$, $\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\mathrm{d}h}{\mathrm{d}V}\frac{\mathrm{d}V}{\mathrm{d}t}$ $= \frac{1}{\pi(0.3)^2}(-2)$ $= -\frac{200}{9\pi} = -7.07$ (3s.f) The depth of water is decreasing at 7.07 cm per minute.

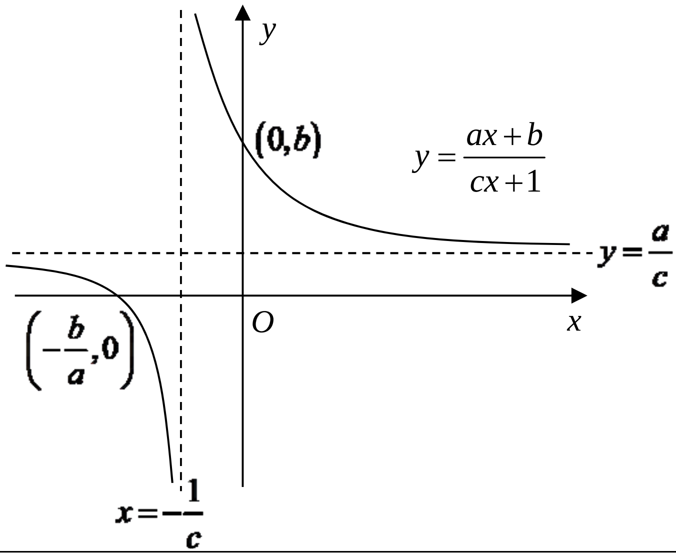
Qn	Solution
2	**Inequalities**
	$\frac{x}{x-1} \leq \frac{4}{x+2}$ $\frac{x}{x-1} - \frac{4}{x+2} \leq 0$ $\frac{x(x+2)-4(x-1)}{(x-1)(x+2)} \leq 0$ $\frac{x^2+2x-4x+4}{(x-1)(x+2)} \leq 0$ $\frac{x^2-2x+4}{(x-1)(x+2)} \leq 0$ $\frac{\left(x-1\right)^2+3}{(x-1)(x+2)} \leq 0$ Since $\left(x-1\right)^2+3>0$ for all $x \in \mathbb{R}$, $(x-1)(x+2)<0$ $-2<x<1$

Qn	Solution
3	**Complex 1**
	$4iz-3w=1+5i$ --------(1) $2z+(1+i)w=2+6i$ --------(2) $(2)\times 2i$ $4iz+2i(1+i)w=2i(2+6i)$ $4iz+2iw-2w=4i-12$ --------(3) $(3)-(1)$: $4iz+2iw-2w-(4iz-3w)=(4i-12)-(1+5i)$ $w+2iw=-13-i$ $(1+2i)w=-13-i$ $w=\left(\frac{-13-i}{1+2i}\right)\left(\frac{1-2i}{1-2i}\right)$ $w=\frac{-13+26i-i-2}{(1)^2-(2i)^2}$ $w=\frac{-15+25i}{5}$ $w=-3+5i$ Substitute $w=-3+5i$ into (2) $2z=2+6i-(1+i)(-3+5i)$ $2z=2+6i-(-3+5i-3i-5)$ $2z=2+6i-(-8+2i)$ $2z=10+4i$ $z=5+2i$ $\therefore w=-3+5i$ and $z=5+2i$.

Qn	Solution
4	**Vectors 1**
(a)(i)	Let θ be the angle between $\overrightarrow{OA}$ and $\overrightarrow{OB}$. $\cos\theta = \dfrac{\begin{pmatrix}3\\-1\\3\end{pmatrix}\cdot\begin{pmatrix}-3\\2\\0\end{pmatrix}}{\left\|\begin{pmatrix}3\\-1\\3\end{pmatrix}\right\|\left\|\begin{pmatrix}-3\\2\\0\end{pmatrix}\right\|}$ $\theta = \cos^{-1}\left(\dfrac{-11}{\sqrt{19}\sqrt{13}}\right) = 134.4°$ (1 d.p) = 2.35 radian (3 s.f)
(ii)	Let h be the shortest distance from B to line OA. $\sin 134.42° = \dfrac{h}{\|\mathbf{b}\|}$ $h = \sqrt{13}\sin 134.42°$ $= 2.5752$ $= 2.58$ units (3 s.f) Note: Accept otherwise method
(b)	Let $\mathbf{c}\times\mathbf{d}=\mathbf{s}$. 1) $\mathbf{s}\cdot\mathbf{e}=0 \Rightarrow$ $\mathbf{s}$ is perpendicular to **e**. 2) $\mathbf{c}\times\mathbf{d}=\mathbf{s} \Rightarrow$ $\mathbf{s}$ is perpendicular to both **c** and **d**. Since $\mathbf{s}$ is <u>perpendicular to **c**, **d** and **e**</u> and <u>**c**, **d** and **e** passes through common point *O*</u> $\Rightarrow$ points *O*, *C*, *D* and *E* are coplanar.

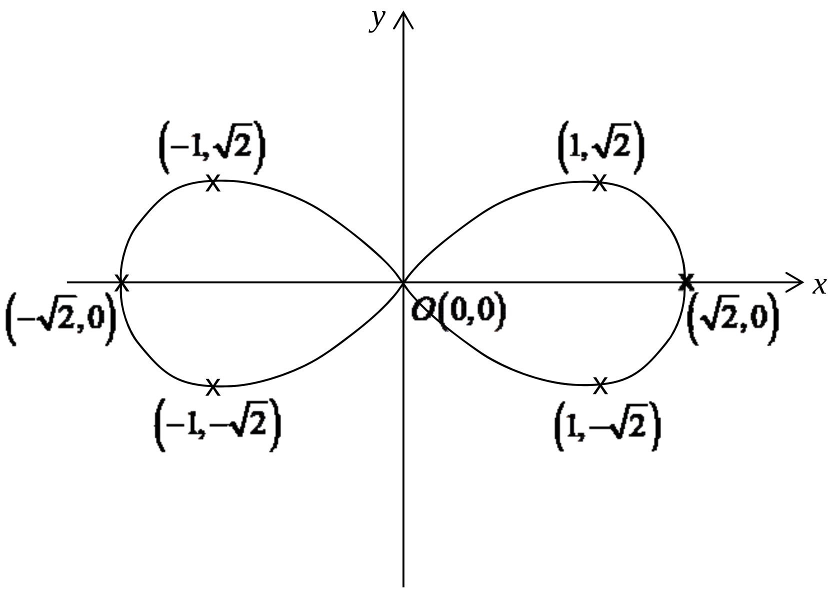
Qn	Solution
5	**Summation and MOD**
(i)	Let $\dfrac{1}{r(r+1)(r+2)}=\dfrac{A}{r}+\dfrac{B}{r+1}+\dfrac{C}{r+2}$ Using 'cover-up' rule, $A=\dfrac{1}{2}, \qquad B=-1, \qquad C=\dfrac{1}{2}$ $\therefore \dfrac{1}{r(r+2)}=\dfrac{1}{2r}-\dfrac{1}{r+1}+\dfrac{1}{2(r+2)}$ $\displaystyle\sum_{r=1}^{n}\frac{1}{r(r+1)(r+2)}=\sum_{r=1}^{n}\left(\frac{1}{2r}-\frac{1}{r+1}+\frac{1}{2(r+2)}\right)$ $= \Big[\dfrac{1}{2}-\dfrac{1}{2}+\dfrac{1}{6}$ $+\dfrac{1}{4}-\dfrac{1}{3}+\dfrac{1}{8}$ $+\dfrac{1}{6}-\dfrac{1}{4}+\dfrac{1}{10}$ $+\dfrac{1}{8}-\dfrac{1}{5}+\dfrac{1}{12}$ $+\ldots$ $+\dfrac{1}{2(n-2)}-\dfrac{1}{n-1}+\dfrac{1}{2n}$ $+\left(\dfrac{1}{2n-1}\right)-\dfrac{1}{n}\quad+\dfrac{1}{2(n+1)}$ $+\dfrac{1}{2n}\qquad-\dfrac{1}{n+1}+\dfrac{1}{2(n+2)}\Big]$ $=\dfrac{1}{2}-\dfrac{1}{2}+\dfrac{1}{4}+\dfrac{1}{2(n+1)}-\dfrac{1}{n+1}+\dfrac{1}{2(n+2)}$ $=\dfrac{1}{4}-\dfrac{1}{2(n+1)}+\dfrac{1}{2(n+2)}$ $=\dfrac{1}{4}-\dfrac{1}{2(n+1)(n+2)}$ (proven) $\therefore k=1$
(ii)	As $n\to\infty, \dfrac{1}{2(n+1)(n+2)}\to 0, \displaystyle\sum_{r=1}^{n}\frac{1}{r(r+1)(r+2)}\to\frac{1}{4}$ $\therefore \displaystyle\sum_{r=1}^{\infty}\frac{1}{r(r+1)(r+2)}$ is a convergent series. $\therefore \displaystyle\sum_{r=1}^{\infty}\frac{1}{r(r+1)(r+2)}=\frac{1}{4}$

(iii)	For all $r \geq 1$, $(r+2)^3 > r(r+1)(r+2)$ $\dfrac{1}{(r+2)^3} < \dfrac{1}{r(r+1)(r+2)}$ $\displaystyle\sum_{r=1}^{n} \frac{1}{(r+2)^3} < \sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)}$ $\displaystyle\sum_{r=1}^{n} \frac{1}{(r+2)^3} < \frac{1}{4} - \frac{1}{2(n+1)(n+2)}$ $\displaystyle\sum_{r=1}^{n} \frac{1}{(r+2)^3} < \frac{1}{4} \qquad \left(\because \frac{1}{2(n+1)(n+2)} > 0 \text{ for all } n \geq 1 \right)$

Qn	Solution
6	**System of Linear Equations and Graphing Techniques**
(i)	
(ii)	Equation of new curve: $y=\frac{1}{2}\left[\frac{a(x-2)+b}{c(x-2)+1}\right]$
(iii)	Since the new curve $y=\mathrm{f}(x)$ passes through the points with coordinates $\left(3,\frac{3}{2}\right)$ and $(6,1)$: $\frac{3}{2}=\frac{1}{2}\left[\frac{a(3-2)+b}{c(3-2)+1}\right]$ $3=\frac{a+b}{c+1}$ $a+b=3c+3$ $a+b-3c=3$ --------(1) $1=\frac{1}{2}\left[\frac{a(6-2)+b}{c(6-2)+1}\right]$ $2=\frac{4a+b}{4c+1}$ $4a+b=8c+2$ $4a+b-8c=2$ ---------(2) Since $y=\frac{3}{4}$ is one of the asymptotes of $y=\mathrm{f}(x)$, $\frac{3}{4}=\frac{1}{2}\left(\frac{a}{c}\right)$ $\frac{a}{c}=\frac{3}{2}$ $2a-3c=0$ --------(3) Solving equations (1), (2) and (3) using GC, $a=3,\ b=6$ and $c=2$. **Note:** If part (ii) is wrong, give max 2 (method for subst 2 points plus attempt to solve) out of 5 marks in part (iii).

Qn	Solution
7	**Maclaurin's Series**
(i)	$\mathrm{f}(x)=\tan\left(\frac{1}{2}x+\frac{1}{4}\pi\right)$ $\mathrm{f}'(x)=\frac{1}{2}\sec^2\left(\frac{1}{2}x+\frac{1}{4}\pi\right)$ $=\frac{1}{2}\left[1+\tan^2\left(\frac{1}{2}x+\frac{1}{4}\pi\right)\right]$ $=\frac{1}{2}\left(1+\left(\mathrm{f}(x)\right)^2\right)$ (shown) $\mathrm{f}''(x)=\mathrm{f}(x)\mathrm{f}'(x)$ $\mathrm{f}'''(x)=\mathrm{f}(x)\mathrm{f}''(x)+\left(\mathrm{f}'(x)\right)^2$ $\mathrm{f}(0)=1$, $\mathrm{f}'(0)=1$, $\mathrm{f}''(0)=1$, $\mathrm{f}'''(0)=2$. $\therefore\ \mathrm{f}(x)=1+x+\frac{1}{2}x^2+\frac{1}{3}x^3\cdots$
(ii)	$\frac{\cos(ax)}{1+bx}=\left(1-\frac{(ax)^2}{2!}+...\right)\left(1+(-1)(bx)+\frac{(-1)(-2)}{2!}(bx)^2+...\right)$ $=\left(1-\frac{a^2x^2}{2}+...\right)\left(1-bx+b^2x^2+...\right)$ $\approx 1-bx+b^2x^2-\frac{a^2x^2}{2}$ $=1-bx+\left(b^2-\frac{a^2}{2}\right)x^2$ Comparing coefficients, $x:b=-1$ $x^2:b^2-\frac{a^2}{2}=\frac{1}{2}\Rightarrow\frac{a^2}{2}=\frac{1}{2}\Rightarrow a=\pm 1$

Qn	Solution
8	**APGP**
(a)(i)	Let a be the number of gold coins the most junior pirate will get. $\frac{10}{2}\left[2a+(10-1)(3)\right]=305$ $a=17$ No of gold coins for most senior pirate $=17+(10-1)(3)$ $=44$
(a)(ii)	Least no of gold coins $=\frac{10}{2}\left[2(1)+(10-1)(3)\right]$ $=145$
(b)(i)	Let b be the length of shift for the most junior pirate $\frac{b}{1-0.9}\left(1-0.9^{10}\right)=24$ $b=3.6848 \text{ hr} \left(\text{to 4 d.p.}\right)$ $\qquad$ (shown)
(b)(ii)	Length of shift for 6th most junior pirate $=3.6848(0.9)^5$ $=2.18 \text{ hr}$ Length of 1st 5 shifts $=\frac{3.6848}{1-0.9}\left(1-0.9^5\right)$ $=15.090$ $=15 \text{ hrs } 5 \text{ mins}$ Start time of shift $=1.05\text{pm}$

Qn	Solution
9	**Tangent/Normal and Integration**
(i)	$x=\sqrt{2}\cos\frac{t}{2} \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t}=-\frac{\sqrt{2}}{2}\sin\frac{t}{2}$ $y=\sqrt{2}\sin t \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}t}=\sqrt{2}\cos t$ $\therefore \frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{2\cos t}{\sin\frac{t}{2}}$ At $t=\frac{\pi}{2}$, $\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{2\cos\frac{\pi}{2}}{\sin\frac{\pi}{4}}=0$ (verified) When $t=\frac{\pi}{2}$, $x=\sqrt{2}\cos\left(\frac{\pi}{4}\right)=1$ Equation of normal: $x=1$
(ii)	

(iii)	$\text{Area} = 4\int_0^{\sqrt{2}} y \ \mathrm{d}x$ $= 4\int_{\pi}^{0} \sqrt{2}\sin t \cdot \left(-\frac{\sqrt{2}}{2}\sin\frac{t}{2}\right) \mathrm{d}t$ $= 4\int_0^{\pi} \sin t \cdot \sin\frac{t}{2} \ \mathrm{d}t$ $= 8\int_0^{\pi} \sin^2\frac{t}{2}\cos\frac{t}{2} \ \mathrm{d}t$ $= 8\left[\frac{2}{3}\sin^3\frac{t}{2}\right]_0^{\pi}$ $= \frac{16}{3} \text{ units}^2$ **Alternative Method** $\text{Area} = 4\int_0^{\sqrt{2}} y \ \mathrm{d}x$ $= 4\int_{\pi}^{0} \sqrt{2}\sin t \cdot \left(-\frac{\sqrt{2}}{2}\sin\frac{t}{2}\right) \mathrm{d}t$ $= 4\int_0^{\pi} \sin t \cdot \sin\frac{t}{2} \ \mathrm{d}t$ $= -2\int_0^{\pi} \cos\frac{3t}{2} - \cos\frac{t}{2} \ \mathrm{d}t$ $= -2\left[\frac{2}{3}\sin\frac{3t}{2} - 2\sin\frac{t}{2}\right]_0^{\pi}$ $= \frac{16}{3} \text{ units}^2$

Qn	Solution
10	**Vectors**
(i)	$\begin{pmatrix}3\\-1\\2\end{pmatrix}\times\begin{pmatrix}2\\1\\1\end{pmatrix}=\begin{pmatrix}-3\\1\\5\end{pmatrix}$ $\mathbf{r}\cdot\begin{pmatrix}-3\\1\\5\end{pmatrix}=\begin{pmatrix}-1\\1\\16\end{pmatrix}\cdot\begin{pmatrix}-3\\1\\5\end{pmatrix}=84$ Cartesian equation of p_1 is $-3x+y+5z=84$.
(ii)	$\overrightarrow{OA}=\begin{pmatrix}5\\-6\\7\end{pmatrix}$ Let the foot of perpendicular from *A* to p_1 be *F*. $\overrightarrow{OF}=\begin{pmatrix}5\\-6\\7\end{pmatrix}+\beta\begin{pmatrix}-3\\1\\5\end{pmatrix}=\begin{pmatrix}5-3\beta\\-6+\beta\\7+5\beta\end{pmatrix}$ for some $\beta\in\mathbb{R}$ Since *F* lies on p_1, $\begin{pmatrix}5-3\beta\\-6+\beta\\7+5\beta\end{pmatrix}\cdot\begin{pmatrix}-3\\1\\5\end{pmatrix}=84$ $35\beta+14=84$ $\beta=2$ $\therefore\overrightarrow{OF}=\begin{pmatrix}5-6\\-6+2\\7+10\end{pmatrix}=\begin{pmatrix}-1\\-4\\17\end{pmatrix}$
	Note that *A* lies on p_2 since $\begin{pmatrix}5\\-6\\7\end{pmatrix}\cdot\begin{pmatrix}1\\-2\\5\end{pmatrix}=52$. Let A' be the point of reflection of *A* about p_1. Note that A' lies on p_3. $\overrightarrow{OF}=\dfrac{\overrightarrow{OA}+\overrightarrow{OA'}}{2}$ $\overrightarrow{OA'}=2\overrightarrow{OF}-\overrightarrow{OA}=2\begin{pmatrix}-1\\-4\\17\end{pmatrix}-\begin{pmatrix}5\\-6\\7\end{pmatrix}=\begin{pmatrix}-7\\-2\\27\end{pmatrix}$ $p_1:\ -3x+y+5z=84$. $p_2:\ \mathbf{r}\cdot\begin{pmatrix}1\\-2\\5\end{pmatrix}=52\ \Rightarrow\ x-2y+5z=52$

By GC, the line of intersection between p_1 and p_2 is

$$\mathbf{r}=\begin{pmatrix}-44\\-48\\0\end{pmatrix}+\alpha\begin{pmatrix}3\\4\\1\end{pmatrix},\quad \alpha\in\mathbb{R}$$

A vector parallel to p_3 is $\overrightarrow{OA'}-\begin{pmatrix}-44\\-48\\0\end{pmatrix}=\begin{pmatrix}-7\\-2\\27\end{pmatrix}-\begin{pmatrix}-44\\-48\\0\end{pmatrix}=\begin{pmatrix}37\\46\\27\end{pmatrix}$

$$\begin{pmatrix}37\\46\\27\end{pmatrix}\times\begin{pmatrix}3\\4\\1\end{pmatrix}=\begin{pmatrix}-62\\44\\10\end{pmatrix}=2\begin{pmatrix}-31\\22\\5\end{pmatrix}$$

$$\mathbf{r}\cdot\begin{pmatrix}-31\\22\\5\end{pmatrix}=\begin{pmatrix}-7\\-2\\27\end{pmatrix}\cdot\begin{pmatrix}-31\\22\\5\end{pmatrix}=308$$

A cartesian equation of p_3 is $-31x+22y+5z=308$

Qn	Solution
11	**Maxima/Minima**

(i)

$V = \pi h r^2$

$h = \dfrac{V}{\pi r^2}$ -------- $(*)$

$$\begin{aligned}
C &= \pi(h+k)(r+2k)^2 - \pi h(r+k)^2 \\
&= \pi\left(h\left((r+2k)^2-(r+k)^2\right)+k(r+2k)^2\right) \\
&= \pi\left(h\left((r^2+4rk+4k^2)-(r^2+2rk+k^2)\right)+k(r+2k)^2\right) \\
&= \pi\left(\frac{V}{\pi r^2}(2rk+3k^2)+k(r+2k)^2\right) \qquad (\text{from } (*)) \\
&= k\left(V\frac{(2r+3k)}{r^2}+\pi(r+2k)^2\right) \\
&= k\left(\frac{2V}{r}+\frac{3kV}{r^2}+\pi(r+2k)^2\right) \qquad (\text{shown})
\end{aligned}$$

(ii)

$$\frac{\mathrm{d}C}{\mathrm{d}r} = k\left(\frac{-2V}{r^2}-\frac{6kV}{r^3}+2\pi(r+2k)\right)$$

When $\dfrac{\mathrm{d}C}{\mathrm{d}r}=0$,

$$k\left(\frac{-2V}{r^2}-\frac{6kV}{r^3}+2\pi(r+2k)\right)=0$$

$$-Vr-3kV+\pi r^3(r+2k)=0$$

$$\pi r^4+2k\pi r^3-Vr-3kV=0 \qquad (\text{Shown})$$

(iii)

From GC,

$r_1 = 4.3736 \quad (\text{since } r>0)$

r	r_1^-	r_1	r_1^+
$\dfrac{\mathrm{d}C}{\mathrm{d}r}$	\	—	/

$\dfrac{\mathrm{d}^2C}{\mathrm{d}r^2} = 5.33 > 0 \Rightarrow C$ is a minimum

$C = 49.7 \quad (3 \text{ s.f.})$

Minimum volume of ceramic casing is 49.7 cm^3.

Note: cannot use part (iv) graph to solve this part.

(iv)