1 Prove by Mathematical induction that

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} \cos\frac{n\pi}{2} & \cos\frac{(n+1)\pi}{2} \\ \cos\frac{(n-1)\pi}{2} & \cos\frac{n\pi}{2} \end{pmatrix}$$

for all positive integers n. [5]

2 Biologist calculated that when the concentration of a particular chemical in a sea inlet reaches 7 milligrams per litre (*mg/l*), the level of pollution endangers the life of the fish.

A factory wishes to release waste containing the chemical into the inlet. It is claimed that the discharge will not endanger the life of the fish.

The Local Authority is provided with the following information:

- The inlet contains none of this chemical at present.
- The factory manager has applied for a permit to discharge waste on a weekly basis into the sea inlet. The discharge, which will done at the beginning of each week, will result in an increase in concentration of 2.5 *mg/l* of the chemical in the inlet.
- The tidal streams will remove 7% of the chemical from the inlet every day.

(i) Based on the information, form a recurrence relation for the concentration level of chemical, u_n at the beginning of week n. Hence, find the concentration at the beginning of week n. [4]

(ii) Based on concentration level of the chemical, should the Local Authority allow the factory to go ahead with the discharge? Justify your answer. [1]

[Turn over

3 The equation of a curve is given by

$$y = \ln(2\cos x)\ ,\ -\frac{\pi}{2} < x < \frac{\pi}{2}$$

(i) Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = -\tan x$ [1]

(ii) The portion of the curve that lies above the x-axis is denoted by C. Find the arc length of C, giving your answers in exact form. [3]

(iii) The curved surface area when C is rotated about the x-axis is denoted by S. Evaluate S to three decimal places. [2]

4 **Do not use a calculator in answering this question.**

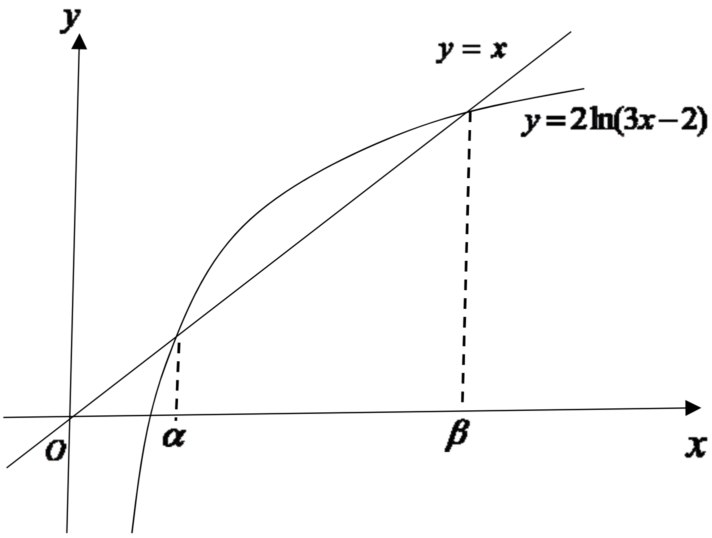
(i) Solve $z^9 = 1$, giving your answers in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \le \pi$. [2]

(ii) Hence, show that the roots of the equation $w^9 - (w + \mathrm{i})^9 = 0$ can be expressed as

$$w = -\frac{1}{2}\left[\cot\left(\frac{k\pi}{9}\right) + i\right]\quad ,\quad k = \pm 1, \pm 2, \pm 3, \pm 4\,.$$ [5]

[Turn over

5 The line $y = x$ intersects the curve $y = 2\ln(3x-2)$ at $x = \alpha$ and $x = \beta$, as shown in the diagram.



(i) Without using a graphing calculator, Show that either α or β lies in the interval $5 < x < 5.5$. [1]

(ii) Use the iteration $x_{n+1} = 2\ln(3x_n - 2)$ to find the value of β, correct to two decimal places. [2]

(iii) With the help of a diagram, illustrate how these iterations converge to β. [1]

(iv) Draw a sketch of the graph $y = 2\ln(3x-2) - x$ showing the two roots α and β. Identify two distinct cases where for some starting values, the Newton-Raphson method will fail to converge to α. [2]

(v) Use the Newton-Raphson method with $x_1 = 1.5$ to find α, correct to 2 decimal places. [2]

6 By using the substitution $u = xy$, find the general solution for the differential equation

$$x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2(x+1)\frac{\mathrm{d}y}{\mathrm{d}x} + (2+x)y = e^{2x}, \; x \neq 0 \; . \qquad [10]$$

[Turn over

7 (i) Given that $z=\cos\theta+i\sin\theta$. Use De Moivre's Theorem to prove that

$$\cos^{2n+1}\theta=2^{-2n}\sum_{r=0}^{n}\left(\binom{2n+1}{r}\cos\left((2n+1-2r)\theta\right)\right),\ n\in\mathbb{Z},\ n\geq 0 \qquad [6]$$

(ii) The region A is bounded by the curve $y=\dfrac{\cos^5 x}{x}$, the lines $y=0$, $x=\dfrac{7}{4}\pi$ and $x=\dfrac{13}{6}\pi$. Use the result in (i) to find the volume of revolution when the region A is rotated four right angles about the y-axis, giving your answer in the form $\dfrac{\pi}{240}\left(a+b\sqrt{2}\right)$, where a and b are integers to be determined. [5]

8 A comet travels along a parabolic path with the Sun at the focus. The position of the comet is measured with respect to a fixed polar axis with the Sun at the pole. The polar axis does not coincide with the axis of the parabola.
The equation of the trajectory is given by

$$r=\frac{d}{1+\cos\left(\theta-\theta_0\right)}$$

where θ_0 is acute and d is a constant.
The distances are measured in Astronomical Units $(1\,A.U. \approx 150 \text{ million kilometres})$.
At a certain time, it is reported that the comet is at the position P whose polar coordinates are $\left(2\sqrt{2},\dfrac{\pi}{2}\right)$. Thirty days later, the comet is reported to be seen at the position $Q\left(3,\dfrac{\pi}{4}\right)$.

(i) Show that $3\cos\theta_0-\sin\theta_0+\left(3\sqrt{2}-4\right)=0$ [3]

(ii) Find θ_0, correct to 5 decimal places. [1]

(iii) Draw a sketch of the trajectory of the comet relative to a horizontal polar axis. Your sketch should show the position of the Sun, the axis of the parabola and also the positions P and Q. [2]

(iv) Find the shortest distance from the comet to the Sun. [2]

(v) Find the average speed of the comet as it travels along arc PQ. Express your answer in kilometres per second. [4]

[Turn over

9 The matrix **A** is given by $\begin{pmatrix} \alpha & -2 & 0 \\ 0 & -\alpha & 0 \\ -1 & \beta & 2 \end{pmatrix}$, $\alpha, \beta \in \mathbb{R}$, $\alpha \neq 0$. The set S_λ is given by

$$\left\{ \mathbf{x} \in \mathbb{R}^3 : \mathbf{Ax} = \lambda \mathbf{x}, \lambda \in \mathbb{R} \right\}.$$

(i) Show that S_λ is a vector space for all $\lambda \in \mathbb{R}$.

(ii) Find the values of λ for which $\dim(S_\lambda) \neq 0$, giving your answers in terms of α and β where necessary. [5]

It is given that $\dim(S_\lambda) \leq 1$ for all $\lambda \in \mathbb{R}$. Find, in any order,

(iii) the set of conditions to be satisfied by α and β,

(iv) a basis for S_λ for each value of λ found in (ii), giving your answers in terms of α and β where necessary. [8]

Given further that $\alpha = 1$, find

(v) a matrix **P** and a diagonal matrix **D** such that $\mathbf{A} = \mathbf{PDP}^{-1}$,

(vi) the matrix $\mathbf{A}^9$,

giving your answers in terms of β where necessary. [5]

[Turn over

10 Initially, a tank is fully filled with 100 litres of pure water. There exists a tap at the top of the tank. This tap supplies brine, containing 1g of salt per litre, into the tank at a rate of 1 litre per minute. There also exists another tap at the bottom of the tank which allows the mixture to flow out at a constant rate of 2 litres per minute. At time T (in minutes), the amount of salt and the volume of the mixture in the tank are denoted by S (in grams) and V (in litres) respectively. Both taps are turned on simultaneously at time $T = 0$. The tap at the bottom of the tank is turned off at time $T = 75$. The mixture in the tank is assumed to be well-stirred and homogenous at all times.

(i) Show that $\dfrac{\mathrm{d}S}{\mathrm{d}T} = \dfrac{100 - T - 2S}{100 - T}$, $0 < T < 75$. [1]

(ii) By solving the differential equation, show that the amount of salt in the tank after 75 minutes is 18.75 grams. [5]

At the instance when the tap at the bottom is turned off, a crack is accidentally created at the bottom of the tank. According to Torricelli law, the mixture flow out from the crack at a rate proportional to the square-root of its volume. It can be assumed that mixture flow obeys Torricelli law, regardless of its viscosity. Let the amount of salt and the volume of the mixture in the tank be denoted by s (in grams) and v (in litres) respectively, t minutes after the crack has been accidentally created. It has been observed that the volume of the mixture in the tank stays constant at 36 litres after a long period of time.

(iii) Show that $\dfrac{\mathrm{d}v}{\mathrm{d}t} = \dfrac{6 - \sqrt{v}}{6}$. [2]

Estimate the time taken for the mixture in the tank to rise to 26 litres after the crack has been created, by using

(a) Euler's Method with two iterations, [2]

(b) Simpson's Rule with two strips. [3]

(iv) Show that $\dfrac{\mathrm{d}s}{\mathrm{d}v} = \dfrac{6\sqrt{v} - s}{6\sqrt{v} - v}$. Use the Improved Euler's Method with one iteration to estimate the amount of salt in the tank at the instance when the mixture in the tank rises to 26 litres after the crack has been created. Give your answer to 4 decimal places. [5]

End of Paper