2017 SRJC H2FM Prelim P2 Solution

1

Solution:

(i) Let $\mathrm{f}(x) = \cos(x) - \ln(x+1)$. Then

$\mathrm{f}(0) = \cos(0) - \ln(1) = 1 > 0$

$\mathrm{f}(1) = \cos(1) - \ln(2) = -0.15284 < 0$

$$\mathrm{f}\,'(x) = -\left(\sin(x) + \frac{1}{x+1}\right).$$

Since sin (x), $\dfrac{1}{x+1} > 0$ for $x \in [0, 1]$, therefore f '(x) < 0 for all $x \in \mathbb{R}$.

$\Rightarrow$ f is strictly decreasing in the interval [0, 1]

Since f(0)f(1) < 0, f(x) is continuous and strictly decreasing, there is exactly one root to the equation in the interval.

(ii) Let x_n be the n-th approximation using the linear interpolation.

$$x_1 = \frac{0\,|\mathrm{f}(1)| + 1|\mathrm{f}(0)|}{|\mathrm{f}(1)| + |\mathrm{f}(0)|}$$

$= 0.86741$

$= 0.867$ (3 d.p.)

Since f(0.86741) = 0.022251 > 0 , root lies in [0.86741, 1]

Therefore x_1 is an underestimation.

2

<u>Solution</u>

(i) $m^2 - m - 1 = 0 \Rightarrow m = \dfrac{1 \pm \sqrt{1+4}}{2} = \dfrac{1 \pm \sqrt{5}}{2}$. Hence $\rho^* = \dfrac{1-\sqrt{5}}{2}$, $\rho = \dfrac{1+\sqrt{5}}{2}$

(ii) Let P_n be the statement that $F_n = \rho^n + (\rho^*)^n \ \forall\, n \in \mathbb{Z}^+$.

Consider P_1: LHS = $F_1 = \sqrt{5}$

$$\text{RHS} = \frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}$$

$= \sqrt{5}$ = RHS

$\therefore$ P_1 is true.

Consider P_2: LHS = $F_2 = \sqrt{5}$

$$\text{RHS} = \left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^2$$

$$= \frac{1}{4} + \frac{\sqrt{5}}{2} + \frac{5}{4} - \frac{1}{4} + \frac{\sqrt{5}}{2} - \frac{5}{4}$$

$= \sqrt{5}$ = RHS

$\therefore$ P_2 is true.

Assume that P_k and P_{k+1} are true for some $k \in \mathbb{Z}^+$.

i.e. $F_k = \rho^k - (\rho^*)^k$ and $F_{k+1} = \rho^{k+1} - (\rho^*)^{k+1}$

Consider P_{k+2}: LHS = F_{k+2}

$= F_{k+1} + F_k$

$= \rho^{k+1} - (\rho^*)^{k+1} + \rho^k - (\rho^*)^k$

$= \rho^k(\rho+1) - (\rho^*)^k(\rho^*+1)$

$= \rho^k\rho^2 - (\rho^*)^k(\rho^*)^2$ (since ρ and ρ^* satisfies $m^2 = m + 1$)

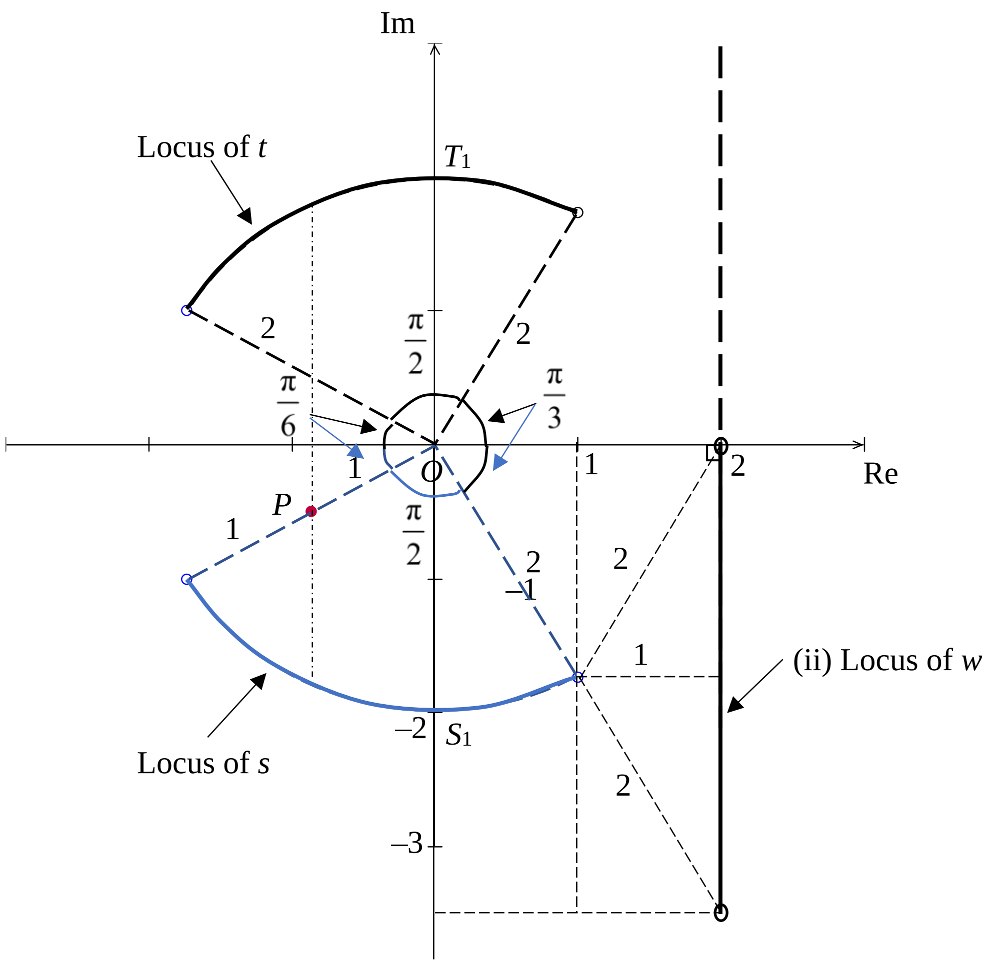
$= \rho^{k+2} - (\rho^*)^{k+2}$ = RHS (*)

$\therefore$ P_{k+2} is true.

Since P_1 and P_2 are true and P_k and P_{k+1} are true $\Rightarrow$ P_{k+2} is true, by mathematical induction, P_n is true $\forall\ n \in \mathbb{Z}^+$.

3 Solution

(i)



For s, O, and t to be collinear, $\theta = \pi/3$

(ii) See diagram above.

4 (i) Suppose $px + q(x^2 - 1) + r(x^3 - 3x) = 0$ is the zero function.

Then $rx^3 + qx^2 + (p - 3r)x - q = 0$

By linear independence of $\{1, x, x^2, x^3\}$, $r = q = p - 3r = 0$

$\Rightarrow p - 3(0) = 0$

$\Rightarrow p = 0$

$\therefore$ $\{x, x^2 - 1, x^3 - 3x\}$ forms a linearly independent set of vectors.

Since B spans Q and is a linearly independent set of vectors, B forms a basis of Q. Since there are 3 vectors in B, the dimension of Q is 3.

(ii) $\mathbf{T}(ax^3+bx^2+cx+d) = (a+d)(x^3-3x)+(b+d)(x^2-1)+(c+d)x$

$= a(x^3-3x)+b(x^2-1)+cx+d(x^3+x^2-2x-1)$

$\therefore$ $\mathbf{T}(x^3) = x^3 - 3x$ (by taking $a = 1$, $b = 0$, $c = 0$, $d = 0$)

$\mathbf{T}(x^2) = x^2 - 1$ (by taking $a = 0$, $b = 1$, $c = 0$, $d = 0$)

$\mathbf{T}(x) = x$ (by taking $a = 0$, $b = 0$, $c = 1$, $d = 0$)

$\mathbf{T}(1) = x^3 + x^2 - 2x - 1$ (by taking $a = 0$, $b = 0$, $c = 0$, $d = 1$)

$$\therefore\ \mathbf{A} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ -3 & 0 & 1 & -2 \\ 0 & -1 & 0 & -1 \end{pmatrix}$$

(iii) From GC, $\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ -3 & 0 & 1 & -2 \\ 0 & -1 & 0 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Since there are 3 non-zero rows in the rref of $\mathbf{A}$, the rank of $\mathbf{A}$ is 3.

Solving,

$w + z = 0 \Rightarrow w = -z$

$x + z = 0 \Rightarrow x = -z$

$y + z = 0 \Rightarrow y = -z$

$$\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -z \\ -z \\ -z \\ z \end{pmatrix} = z\begin{pmatrix} -1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$$

$\therefore$ basis of null space of $\mathbf{A}$ is $\left\{ \begin{pmatrix} -1 \\ -1 \\ -1 \\ 1 \end{pmatrix} \right\}$

(iv) For any vector $px + q(x^2 - 1) + r(x^3 - 3x) \in Q$,

$\mathbf{T}(px^3+qx^2+rx) = px + q\left(x^2-1\right) + r\left(x^3-3x\right)$

Conversely, by definition of $\mathbf{T}$, any vector in the range of $\mathbf{T}$ is a linear combination of vectors in B, therefore the range of $\mathbf{T}$ is Q.

5 <u>Solution</u>

(i) $t = e^x \quad \Rightarrow \quad \dfrac{dt}{dx} = e^x = t$

$$\therefore\ \frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx}$$

$$= t\,\frac{dy}{dt} \qquad \text{(Shown)}$$

	and $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(t\dfrac{dy}{dt}\right) = \dfrac{dt}{dx}\dfrac{d}{dt}\left(t\dfrac{dy}{dt}\right)$ $= t\left(\dfrac{dy}{dt}+t\dfrac{d^2y}{dt^2}\right)$ (Shown) (ii) $\dfrac{d^2y}{dx^2}+\left(2e^x-1\right)\dfrac{dy}{dx}+2e^{2x}y=e^{2x}\sin 3e^x$ $t\left(\dfrac{dy}{dt}+t\dfrac{d^2y}{dt^2}\right)+\left(2t-1\right)t\dfrac{dy}{dt}+2t^2y=t^2\sin 3t$ $t^2\dfrac{d^2y}{dt^2}+t\dfrac{dy}{dt}+2t^2\dfrac{dy}{dt}-t\dfrac{dy}{dt}+2t^2y = t^2\sin 3t$ $t^2\dfrac{d^2y}{dt^2}+2t^2\dfrac{dy}{dt}+2t^2y = t^2\sin 3t$ $\dfrac{d^2y}{dt^2}+2\dfrac{dy}{dt}+2y = \sin 3t$ $\therefore$ a = 2 and b = 2 Let the auxiliary equation be $m^2 + 2m + 2 = 0$ $\therefore$ $m = \dfrac{-2\pm\sqrt{4-8}}{2}=-1\pm i$ $\Rightarrow$ the complementary function is $y = e^{-t}\left[A\cos t+B\sin t\right]$ where A and B are constants. Let the particular integral be $y = p\cos 3t+q\sin 3t$ where p and q are constants $\Rightarrow \dfrac{dy}{dt}=-3p\sin 3t+3q\cos 3t$ $\Rightarrow \dfrac{d^2y}{dt^2}=-9p\cos 3t-9q\sin 3t$. $\therefore$ $-9p\cos 3t-9q\sin 3t+2\left(-3p\sin 3t+3q\cos 3t\right)+2\left(p\cos 3t+q\sin 3t\right) = \sin 3t$ $\left(-7p+6q\right)\cos 3t+\left(-7q-6p\right)\sin 3t=\sin 3t$ $\Rightarrow -7p + 6q = 0$ and $-6p - 7q = 1$ From GC, we have $p = -\dfrac{6}{85}$ and $q = -\dfrac{7}{85}$ $\therefore y = e^{-t}\left[A\cos t+B\sin t\right] -\dfrac{6}{85}\cos 3t-\dfrac{7}{85}\sin 3t$ Hence the general solution required is $y = e^{-e^x}\left[A\cos e^x+B\sin e^x\right]-\dfrac{6}{85}\cos 3e^x-\dfrac{7}{85}\sin 3e^x$
6	(a)(i) $u=\sqrt{x+\dfrac{3}{2}} \Rightarrow \dfrac{du}{dx}=\dfrac{1}{2}\left(x+\dfrac{3}{2}\right)^{-\frac{1}{2}}=\dfrac{1}{2u}$ $\displaystyle\int x\sqrt{x+\frac{3}{2}}\ dx=\int u\left(u^2-\frac{3}{2}\right)(2u)\ du$

$$= \int 2u^4 - 3u^2 \ \mathrm{d}u$$

$$= \frac{2}{5}u^5 - u^3 + C$$

$$= \frac{2}{5}\left(x+\frac{3}{2}\right)^{\frac{5}{2}} - \left(x+\frac{3}{2}\right)^{\frac{3}{2}} + C$$

(ii) Volume generated $= 2\pi \int_{-\frac{3}{2}}^{-\frac{1}{2}} (-x)\sqrt{x+\frac{3}{2}}\ \mathrm{d}x \ + \ 2\pi \int_{-\frac{1}{2}}^{0} (-x)\left(x-\frac{1}{2}\right)^2 \ \mathrm{d}x$

$$= -2\pi\left[\frac{2}{5}\left(x+\frac{3}{2}\right)^{\frac{5}{2}} - \left(x+\frac{3}{2}\right)^{\frac{3}{2}}\right]_{-\frac{3}{2}}^{-\frac{1}{2}} + 2\pi \int_{-\frac{1}{2}}^{0} \left(-x^3 + x^2 - \frac{1}{4}x\right) \mathrm{d}x$$

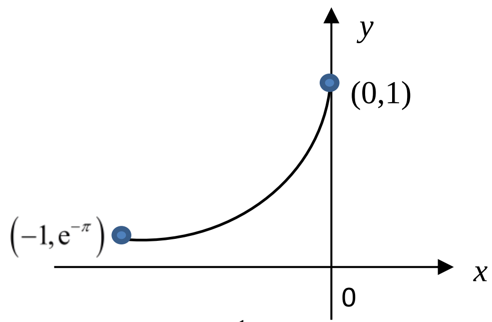
$$= \frac{6}{5}\pi + 2\pi\left[-\frac{x^4}{4} + \frac{x^3}{3} - \frac{x^2}{8}\right]_{-\frac{1}{2}}^{0}$$

$$= \frac{6}{5}\pi + 2\pi\left(+\frac{1}{64} + \frac{1}{24} + \frac{1}{32}\right)$$

$$= \frac{6}{5}\pi + 2\pi\left(\frac{17}{192}\right)$$

$$= \frac{661}{480}\pi$$

(a) (i)



(ii) Volume generated $= 2\pi \int_{e^{-\pi}}^{1} y(-x)\ \mathrm{d}y$

$$= 2\pi \int_{\frac{\pi}{2}}^{0} e^{-2t}(\sin t)\ \left(-2e^{-2t}\right)\mathrm{d}t$$

$$= 4\pi \int_{0}^{\frac{\pi}{2}} e^{-4t} \sin t \ \mathrm{d}t$$

$= 0.734$

7

Solution

(i) The result of each attempt is independent of any previous result.

(ii) $X \sim \text{Geo}(0.4)$

Required probability $= \mathrm{P}(X \geq 4)$

$= 1 - \mathrm{P}(X \leq 3)$

$= 1 - 0.784$

$= 0.216$

	(iii) Let n be the number of attempts made by Jeremy for him to have at least 95% chance of achieving his first success. $\mathrm{P}(X \le n) > 0.95$ From GC, When $n = 5$, $\mathrm{P}(X \le 5) = 0.92224$ When $n = 6$, $\mathrm{P}(X \le 6) = 0.95334$ Least $n = 6$ Alternatively, Let Y be the random variable number of times a goal is scored when n attempts are made by Jeremy. $Y \sim \mathrm{B}(n, 0.4)$ $\mathrm{P}(Y \ge 1) > 0.95 \Rightarrow \mathrm{P}(Y = 0) < 0.05$ From GC, For $n = 5$, $\mathrm{P}(Y = 0) = 0.07776$ For $n = 6$, $\mathrm{P}(Y = 0) = 0.04666$ Least $n = 6$
8	Solution: (i) $\int_2^4 a(x-2)(4-x)\,\mathrm{d}x = 1$ $\int_2^4 a(-x^2+6x-8)\,\mathrm{d}x = 1$ $a\left[-\frac{x^3}{3}+3x^2-8x\right]_2^4 = 1$ $a\left[\left(-\frac{64}{3}+48-32\right)-\left(-\frac{8}{3}+12-16\right)\right] = 1$ $\frac{4}{3}a = 1$ $a = \frac{3}{4}$ (ii) By symmetry, $\mathrm{E}(X) = \frac{2+4}{2} = 3$ (iii) $\mathrm{P}(X > 3.5) = \int_{3.5}^4 \frac{3}{4}(-x^2+6x-8)\,\mathrm{d}x$ $= 0.1563$ (iv) For Process (I): Expected profit = $-190\mathrm{P}(X > 3.5) + 80\mathrm{P}(X \le 3.5)$ $= -190(0.1563) + 80(1-0.1563)$ $= 37.799$ cents For Process (II): Expected profit = $\mathrm{E}(190 - 50X)$ $= 190 - 50\mathrm{E}(X)$ = 40 cents

A1 - for correct expected profit for both processes.
Since Process (II) yields a higher profit, the butcher would choose Process (II).

9 Solution

(i)

Agent	A	B	C	D	E	F	G	H	I
Diff.	+3	+4	+2	−4	+11	−2	+6	0	+5
Rank	3	4.5	1.5	4.5	8	1.5	7	NA	6

(ii) For a sign test, test statistic M = number of agencies which ranked Harvey University with a larger (numerical) ranking compared to its previous ranking.
Since agency J produced a non-zero change in university ranking, $n = 9$.

$M \sim B(9,0.5)$

Using GC,
$P(M \geqslant 7) = 1 - P(M \leqslant 6) = 0.08984 < 0.1$, and
$P(M \geqslant 6) = 1 - P(M \leqslant 5) = 0.2539 > 0.1$,
$\Rightarrow M_{\text{test}} = 6$
⇒ The change in university ranking by agency J is negative
⇒ Harvey university ranking by agency J is better than before.

(iii) There is no assumption on the distribution of the university ranking by different agencies following a normal distribution, which is unlikely to be the case.

(iv) Null hypothesis: University rankings of Asian universities remain unchanged
Alternative hypothesis: University rankings of Asian universities have improved.

One-tail Wilcoxon matched-pair signed rank test at 5% level of significance.
Test statistics, T^+ = sum of ranks of positive changes in Asian university rankings.

Since $n = 350$ is large, $Z = \dfrac{T^+ - \frac{n}{4}(n+1)}{\sqrt{\frac{1}{24}n(n+1)(2n+1)}} \sim N(0, 1)$ approximately.

$$z_{\text{test}} = \frac{27150 - \frac{350}{4}(351)}{\sqrt{\frac{350}{24}(351)(701)}} = -1.8806 < -1.6448$$

OR

p-value $= P(Z \leq -1.8806) = 0.03000 < 0.05$
Since p-value < 0.05, we reject the null hypothesis and conclude that there is sufficient evidence at 5% level of significance that the university rankings of Asian universities have improved.

10 Solution:

(i) Let X be the r.v. "number of divorces and annulments per year".

$\bar{x} = 7422.167$

	$s^2 = 147.132^2 = 21647.77$ A 95% confidence interval for μ is $= \left(7422.1667 \pm 2.57058\sqrt{\frac{147.132^2}{6}}\right)$ $= (7267.76, 7576.57)$ Assume that X follows a normal distribution. (ii) Since $\mu = 7420$ **lies in the confidence interval**, we do not reject H_0 and conclude that there is insufficient evidence that the mean number of divorces and annulments per year in Singapore is not 7420 at the 5% level of significance. (iii) There is a probability of 0.05 that the interval will not enclose the actual value of μ (iv) Let B be the r.v. "mean age at first marriage for brides" and G be the r.v. "mean age at first marriage for grooms". Let $D = G - B$ $\bar{d} = 2.1167$ $s_D{}^2 = (0.09832)^2 = 0.0096667$ To test $H_0: \mu_D = 2$ against $H_1: \mu_D > 2$ using a 1-tailed t - test at 5% level. Under H_0, test statistic: $T = \frac{\bar{D} - \mu_D}{\frac{S_D}{\sqrt{n}}} \sim t(n-1)$ where $n = 6$. From GC, p-value = 0.0168. Since p-value = 0.0168 < 0.05, we reject H_0 and conclude that there is sufficient evidence at the 5% level that John's claim is not valid.
11	Solution (i) The arrival of ships at the terminal occur independently of the arrival of other ships. OR The mean number of ships arrival at the terminal is uniform over time or constant over time. (ii) Let X be the r.v. "no. of ships arriving at the terminal in a randomly chosen 45 minutes" Mean number of ships arriving at the port = 0.75 × 9 = 6.75 $X \sim \text{Po}(6.75)$ From GC, $P(X = 5) = 0.1367$ $P(X = 6) = 0.1538$

P(X = 7) = 0.1483

$\therefore$ most probable number of ships arrival in 45 minutes is 6.

Most probable number of ships concurrently anchored at the port terminal is 6.

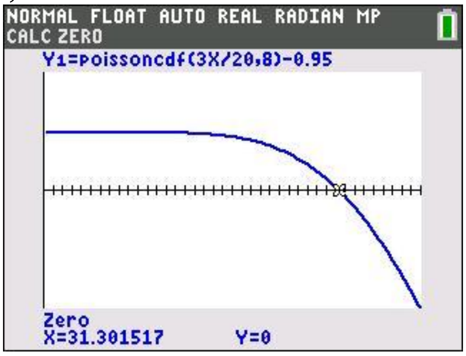
(iii) Let Y be the r.v. "no. of ships arriving at the terminal in a randomly chosen T minutes".

$$Y \sim \text{Po}\left(\frac{3T}{20}\right)$$

For no queuing at the port, there are

P($Y \le 8$) > 0.95

From GC,



T = 31.30 (2 d.p.)

(iv) $C \sim$ Exp (0.5) (in hour) or $C \sim$ Exp (30)

P($C \le 1.5 | C > 1$) = P($C \le 0.5$) = 0.632 (to 3 s.f.)   OR   P($C \le 90 | C \ge 60$) = P($C \le 30$) = 0.632 (to 3 s.f.)

(v) Using observed frequencies,

Observed frequency	Time of Arrival		
	00 00 – 08 00	08 00 – 16 00	16 00 – 00 00
Oil Tanker	35(40)	50(40)	15(20)
Container Ship	45(40)	30(40)	25(20)

Null hypothesis: Time of arrival and type of ships are independent

Alternative hypothesis: Time of arrival and type of ships are dependent

χ^2-test at 5% level of significance.

Degree of freedom = (3 – 1)(2 – 1) = 2.

Test statistics, $\chi^2(2)$

Using GC, p-value = 0.012588 < 0.05

Since p-value < 0.05, we reject the null hypothesis and conclude that there is sufficient evidence at 5% level of significance that time of arrival and types of ships are dependent.