INNOVA JUNIOR COLLEGE
JC 2 PRELIMINARY EXAMINATION
in preparation for General Certificate of Education Advanced Level
Higher 2

CANDIDATE NAME

CLASS | INDEX NUMBER

FURTHER MATHEMATICS 9649/01

Paper 1

24 August 2017
3 hours

Additional Materials: Answer Paper
Cover Page
MF26

READ THESE INSTRUCTIONS FIRST

Do not open this booklet until you are told to do so.

Write your name, class and index number on all the work you hand in.

Write in dark blue or black pen on both sides of the paper. You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

You are expected to use a graphic calculator.

Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.

You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.

The number of marks is given in brackets [ ] at the end of each question or part question.

1 A and B are two arbitrary points on the parabola $y^2 = 4cx$ such that chord AB passes through the focus of the parabola. Prove that the locus of the mid-point of AB is a parabola with focus $\left(\frac{3c}{2}, 0\right)$ and directrix $x = \frac{c}{2}$. [4]

2 Prove by induction that, for all positive integers n,

$$\binom{2n}{n} \le 2^{2n-1}.$$ [5]

3 The equation $\frac{1}{x} - 2 + \ln x = 0$ has exactly two positive real roots α and β, where $\alpha < \beta$. Using the graph of $y = \frac{1}{x} - 2 + \ln x$, explain why neither $x_1 = 0.9$ nor $x_1 = 1$ is a suitable initial value for the use of the Newton-Raphson method to estimate α. [3]

Taking $x_1 = 0.3$ as the initial value, use the Newton-Raphson method to find an approximation to the root α, giving your answer correct to four decimal places. [4]

4 Obtain the general solution of the differential equation $x\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = \frac{2\sin x}{x\cos^3 x}$ in the form $y = \frac{1}{x^2}\left(\tan^2 x + C\right)$, where C is an arbitrary constant. (No marks will be given for merely verifying this solution.)

Hence sketch the family of solution curves for $-\frac{3\pi}{2} < x < \frac{3\pi}{2}$. [8]

5 Given that $z = \mathrm{e}^{\mathrm{i}\frac{\pi}{4}}$, prove that

$$z + z^3 + z^5 + \cdots + z^{2n-1} = \sqrt{2}\left(\sin\frac{n\pi}{4}\right)\left(\cos\frac{n\pi}{4} + \mathrm{i}\sin\frac{n\pi}{4}\right).$$ [4]

Hence find the sum of the series

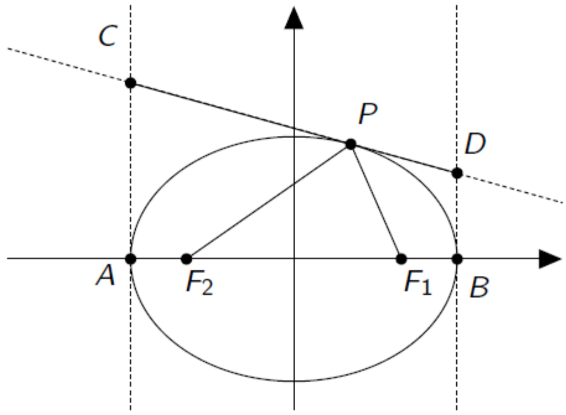
$$\cos\left(\theta + \frac{\pi}{4}\right) + \cos\left(\theta + \frac{3\pi}{4}\right) + \cos\left(\theta + \frac{5\pi}{4}\right) + \cdots + \cos\left(\theta + \frac{(2n-1)\pi}{4}\right).$$ [3]

6 The point $P(a\cos\theta, b\sin\theta)$ lies on the ellipse with equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. The foci of the ellipse are $F_1(ae, 0)$ and $F_2(-ae, 0)$, where e is the eccentricity.

(i) Find $|PF_1|$ in terms of a, e and θ. [2]

Hence show that $|PF_1||PF_2| = a^2\left(1-e^2\cos^2\theta\right)$. [2]

The tangent to the ellipse at P intersects the tangents to vertices A and B at C and D respectively, as shown in the diagram below. The tangent to the ellipse at P makes an acute angle ϕ with the x-axis.



(ii) Show that $|PC| = \frac{a\cos\theta + a}{\cos\phi}$.

Hence show that $|PF_1||PF_2| = |PC||PD|$. [4]

7 In an Argand diagram, the point P represents the complex number z such that

$$|z-2+4\text{i}| \le 4 \quad \text{and} \quad -\frac{\pi}{4} \le \arg(z+2\text{i}) < 0.$$

(i) Sketch the locus of P. [4]

(ii) Find the exact range of values of $|z+4\text{i}|$. [3]

(iii) Find the range of values of $\arg(z+4\text{i})$. [4]

8 The change in the charge, Q coulombs, of a capacitor with time, t seconds, in an electrical circuit is governed by *Kirchoff's Second Law*, which gives the differential equation

$$L\frac{\mathrm{d}^2Q}{\mathrm{d}t^2}+R\frac{\mathrm{d}Q}{\mathrm{d}t}+\frac{Q}{C}=E(t),$$

where L is the inductance (in henries), R the resistance (in ohms), C the capacitance (in farads) and $E(t)$ the electromotive force (in volts).

The initial charge Q and initial current $\frac{\mathrm{d}Q}{\mathrm{d}t}$ in a circuit are both zero. Given that L = 1 henries, R = 4 ohms, $C = \frac{1}{13}$ farads and $E(t)=145\cos 2t$, solve for the charge Q at time t. [11]

9 Suppose that the national income (in million dollars) of a small country in year n is given by

$$I_n = S_n + P_n + G_n,$$

where S_n, P_n and G_n represent national spending (in million dollars) by the population, private investment (in million dollars) and government spending (in million dollars) respectively. Assume that the population will spend a proportion of the previous year's income, such that $S_{n+1}=\frac{1}{6}I_n$, and that the national income in the first year is b million dollars.

(a) In one study, we assume that the private investment is one-third the national spending by the population that year and that the government spending every year is fixed at b million dollars. Show that the national income is given by the recurrence relation

$$I_n = aI_{n-1}+b,$$

where $n \geq 2$ and a is a constant to be determined. [1]

(i) Solve for I_n, giving your answer in terms of b. [3]

(ii) Determine the long term state of the economy. [1]

(b) In another study, we assume that the private investment is equal to the difference between the spending by the population that year and the previous year. The government spending is assumed to be half of the previous year's national income. Show that the national income is given by the recurrence relation

$$I_n = \frac{5}{6} I_{n-1} - \frac{1}{6} I_{n-2}, \qquad \text{where } n \geq 3.$$

Given that the national income in the second year is $\frac{3}{8}b$ million dollars, solve the recurrence relation. [6]

10 The linear transformations $\mathrm{T}_1 : \mathbb{R}^4 \to \mathbb{R}^4$ and $\mathrm{T}_2 : \mathbb{R}^4 \to \mathbb{R}^4$ are represented by the matrices $\mathbf{M}$ and $\mathbf{M}^2$ respectively, where $\mathbf{M} = \begin{pmatrix} 4 & 2 & 1 & -2 \\ 0 & -1 & 2 & 4 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$.

Let R_1 and K_1 be respectively, the range and kernel of T_1.

(i) Find a basis for R_1 [2]

(ii) Find a basis for K_1. [2]

(iii) Show that $R_1 \cap K_1 = \{\mathbf{0}\}$. [3]

(iv) Deduce a basis for the kernel of T_2. [3]

(v) Determine, with reasons, if $R_1 \cup K_1$ is a vector space under standard addition and scalar multiplication. [4]

11 **(a)** Given that $I_n = \int \sec^n x \, dx$, prove that

$$(n+1)I_{n+2} = \tan x \sec^n x + nI_n .$$ [3]

(b) The curve C has the parametric equations

$$x = \frac{2a\cos\theta}{1+\cos\theta}, \qquad y = 2a\tan\left(\frac{\theta}{2}\right),$$

where $-\pi < \theta < \pi$ and $a > 0$.

(i) Show that $\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2} = a\sec^3\left(\frac{\theta}{2}\right)$. [4]

(ii) Two points A and B on C have coordinates (a, 0) and (0, 2a) respectively. Using the result in **(a)**, find, in terms of a, the length of arc AB. [4]

(iii) Find, in terms of a, the exact area of the surface formed by rotating the arc AB through 2π radians about the x-axis. [3]

END OF PAPER