H

TEMASEK JUNIOR COLLEGE, SINGAPORE
JC 2
Preliminary Examination 2017
Higher 2

TEMASEK JUNIOR COLLEGE

FURTHER MATHEMATICS

9649/02

Paper 2

12 September 2017

Additional Materials: Answer paper
List of Formulae (MF26)

3 hours

READ THESE INSTRUCTIONS FIRST

Write your Civics group and name on all the work that you hand in.

Write in dark blue or black pen on both sides of the paper.

You may use a soft pencil for any diagrams or graphs.

Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

You are expected to use a graphic calculator.

Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.

Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.

You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

At the end of the examination, fasten all your work securely together.

This document consists of **7** printed pages and **1** blank page.

[Turn over

Section A: Pure Mathematics (50 marks)

1 The outer layer of a contact lens is obtained by rotating the curve with parametric equations $x = 2t^2$, $y = 4t$, $-\sqrt{2} \le t \le \sqrt{2}$ through π radians about the x-axis. Find the exact outer surface area of the contact lens. [4]

2 Show that the equation $\tan^{-1}\left(\mathrm{e}^x\right) - \dfrac{1}{x} = 0$ has exactly one positive root α in the interval $(0,1)$. [2]

To find the root α, two methods are proposed.

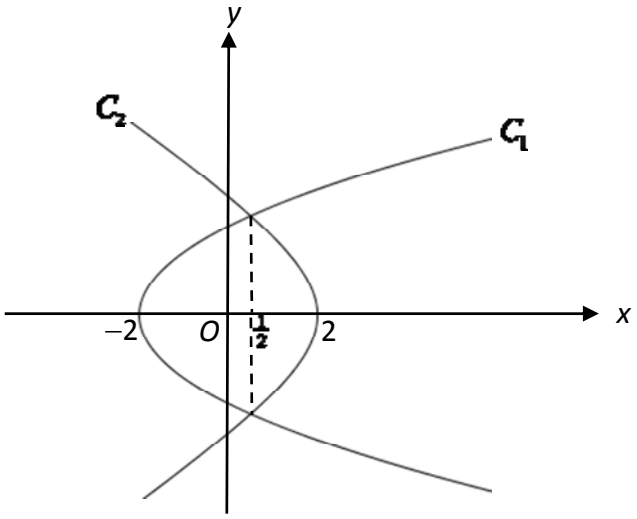
In method 1, Newton-Raphson method is used to estimate this positive root using 1 as an initial value, giving your answer correct to 4 decimal places. [2]

Explain, by illustrating the first two iterations of Newton-Raphson method on the graph of $y = \tan^{-1}\left(\mathrm{e}^x\right) - \dfrac{1}{x}$, why Newton-Raphson method fails when an initial approximation of 2 is being used. [3]

In method 2, the equation $\tan^{-1}\left(\mathrm{e}^x\right) - \dfrac{1}{x} = 0$ is to be written in the form $x = \mathrm{F}(x)$ and the recurrence relation $x_n = \mathrm{F}(x_{n-1})$ where $n \ge 1$ is being used.

Propose a suitable $\mathrm{F}(x)$ that generates a sequence of numbers that converges to α for any initial approximation $x_0 > 0$. Justify your answer. [3]

3



The parabola C_1 and the hyperbola C_2 share the same focus at the origin O. C_1 and C_2 cut the x-axis at (−2, 0) and (2, 0) respectively. Given that C_1 and C_2 intersect at $x = \frac{1}{2}$, find the Cartesian equations of C_1 and C_2. [7]

Hence, show that the distance between the directrix of C_1 and the directrix of C_2 corresponding to the focus at O is $\frac{36}{5}$ units. [2]

4 The matrix **A** is given by $\mathbf{A} = \begin{pmatrix} 3 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{pmatrix}$.

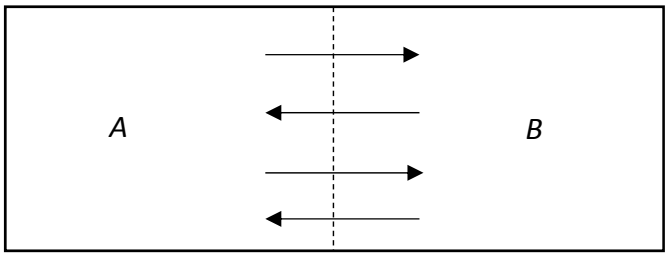
Write down the eigenvalues of **A** and find the corresponding eigenvectors of **A**. Hence find a non-singular matrix **P** and a diagonal matrix **D** such that $\mathbf{A} = \mathbf{PDP}^{-1}$. [7]

The matrix **B** is such that $\mathbf{B} = \mathbf{QAQ}^{-1}$, where $\mathbf{Q} = \begin{pmatrix} -1 & 0 & 1 \\ 1 & -1 & 2 \\ 0 & 1 & -2 \end{pmatrix}$.

By using the expression $\mathbf{PDP}^{-1}$ for **A**, find the eigenvalues and a set of corresponding eigenvectors of **B**. [4]

[Turn over

5 The diagram is a compartmental representation of the exterior and interior of a cell. The concentration of fluid at any time t in compartment A and B are x units and y units respectively. Nutrients are allowed to permeate through the membrane such that the concentration of fluid in compartment A and B increases at a rate of $6(y-kx)$ and $2\left(y-\frac{x}{k}\right)$ respectively where k is a positive constant. Additional nutrients are also constantly supplied so that there is a constant rate of increase of concentration of 13 units to each compartment. The rate of change of x and y are the same at the instant when the concentration of fluid in compartment A is four times that of compartment B.



Membrane

(i) Write down two differential equations, one relating x and t and another relating y and t. Hence show that

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+2\frac{\mathrm{d}x}{\mathrm{d}t}+10x=52\,. \qquad [5]$$

(ii) Given that when $t = 0$, $x = y = 0$, find the solution of x in terms of t. [7]

(iii) By sketching the solution curve of x against t, describe what happens to the concentration of fluid in compartment A in the long term. [2]

(iv) Given that as $t \to \infty, y \to \frac{13}{10}$, explain how these long term values of x and y can be found without solving the second order differential equations. [2]

Section B: Statistics (50 marks)

6 In a population, a person is either left-handed or right-handed. On average, a proportion, p of the population is left-handed. A random sample of people is chosen, one by one, until a left-handed person is selected. Find in terms of p, the probability that the number of people in the sample is less than 15. [2]

Another random sample is chosen, one by one, until at least one left-handed and at least one right-handed person have been obtained. The number of people in this sample is denoted by N. Show that $\mathrm{P}(N = n) = pq^{n-1} + qp^{n-1}$, for $n \geq 2$, where $q = 1 - p$. Hence find $\mathrm{E}(N)$ in terms of p and q. [5]

7 A metalworker makes circular discs of various sizes but with the same thickness of 0.1 cm. The length, X cm, of the radius of a randomly chosen disc has a uniform distribution in the interval $\left[\frac{10}{\pi}, \frac{60}{\pi}\right]$. The volume of a randomly chosen disc is denoted by Y cm^3.

Show that $\mathrm{E}(Y) = \frac{430}{3\pi}$ and $\mathrm{Var}(Y) = \frac{c}{9\pi^2}$, where c is a constant to be determined. [6]

A random sample of 200 discs made by the metalworker has mean volume 40 cm^3. Obtain a 90% confidence interval for the mean volume of the discs produced by the metalworker. [1]

Give a possible reason why the interval you have found may not contain the population mean. [1]

8 Given that the random variable X has the Poisson distribution with mean a, show that

$$\sum_{r=0}^{n} r\mathrm{P}(X = r) = a\mathrm{P}(X \leq n-1).$$ [3]

A garage receives delivery of new cars at the beginning of each month and accepts as many new cars as is necessary to bring its stock of new cars to 10. The monthly demand for new cars at the garage has a Poisson distribution with mean 8.

(i) Find the probability that in a particular month there will be insufficient cars to sell. [1]

(ii) Calculate the minimum number of stock of new cars that are to be increased to have at least 90% chance of meeting the demand. [3]

(iii) Find to two significant figures, the mean of the number of new cars sold per month by the garage. [3]

[Turn over

9 **(a)** A sample of 1000 observations of the continuous random variable X was obtained and the results are summarised in the following table, in which s is an unknown integer.

Interval	$0 \le x < 0.2$	$0.2 \le x < 0.4$	$0.4 \le x < 0.6$	$0.6 \le x < 0.8$	$0.8 \le x \le 1.0$
Observed frequency	s	$65 - s$	159	310	466

It is assumed that these results are consistent with a probability distribution having the following probability density function:

$$f(x) = \begin{cases} 3x^k, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$, where k is a constant to be determined.

A χ^2 -test is to be carried out to test the goodness of fit of this distribution. The expected frequencies are given in the following table where a and b are constants.

Interval	$0 \le x < 0.2$	$0.2 \le x < 0.4$	$0.4 \le x < 0.6$	$0.6 \le x < 0.8$	$0.8 \le x \le 1.0$
Expected frequency	8	56	a	b	488

(i) Show that $a = 152$ and find the value of b. [3]

(ii) Find the largest value of s that would result in the null hypothesis not being rejected at the 5% level of significance. [5]

(b) The following table gives the percentage of the different grades obtained by candidates taking Statistics and Pure Mathematics in a school examination. The two groups of candidates are mutually exclusive.

Grade	A, B or C	D or E	S or U
Statistics	41.5	23.8	34.7
Pure Mathematics	29.2	26.2	44.6

(i) What further information is needed to form a 3×2 expected frequency table from which the independence of subjects and grades can be tested? [1]

(ii) When such a table was formed, the calculated value of

$$\sum \frac{\left(O_{ij} - E_{ij}\right)^2}{E_{ij}}$$ was 13.16.

Assume that all E_{ij} are more than 5, carry out the test using a 0.5% significance level. [4]

10 A study was carried out to investigate the effect of a certain drug on blood cholesterol levels. The levels were measured before and after the drug was administered to a random sample of 9 people. The results, in suitable units, are given in the following table.

Person	A	B	C	D	E	F	G	H	I
Before	170	190	200	188	206	247	191	222	263
After	161	199	190	179	195	235	191	225	252

(i) Perform a suitable *t*-test at the 5% significance level whether the drug is effective in reducing the blood cholesterol level, stating any necessary conditions for the validity of the test. [5]

(ii) Find the greatest integer value of a for which it could be claimed at the 10% significance level that the drug is effective in reducing the blood cholesterol level by more than a units. [3]

(iii) If the conditions for the test in **(i)** are not met, test whether the drug is effective in reducing the blood cholesterol level at the 5% significance level using the Wilcoxon matched-pairs signed rank test. [4]

-------- ***End of Paper*** --------