Anderson Junior College
Preliminary Examination 2017
H2 Mathematics Paper 2 (9758/02)

1(i)	$\frac{dx}{dt} = 30 - kx, \quad k > 0$ $\Rightarrow \int \frac{1}{30-kx}\,dx = \int dt$ $\Rightarrow -\frac{1}{k}\ln\lvert 30-kx\rvert = t + C$ $\Rightarrow \ln\lvert 30-kx\rvert = -kt - kC$ $\Rightarrow \lvert 30-kx\rvert = e^{-kt-kC}$ $\Rightarrow 30 - kx = Ae^{-kt}, \quad$ where $A = \pm e^{-kC}$ $\Rightarrow x = \frac{1}{k}\left(30 - Ae^{-kt}\right)$ At $t = 0$, $x = 0 \Rightarrow 0 = \frac{1}{k}\left(30 - Ae^{0}\right) \Rightarrow A = 30$ $\Rightarrow x = \frac{1}{k}\left(30 - 30e^{-kt}\right) = \frac{30}{k}\left(1 - e^{-kt}\right)$
1(ii)	For patient to have overdose, $x = \frac{30}{k}\left(1-e^{-kt}\right) > 1000$ Since for $t > 0$, $0 < e^{-kt} < 1$, so $0 < 1 - e^{-kt} < 1$ $\frac{30}{k} > \frac{30}{k}\left(1-e^{-kt}\right) > 1000$ $0 < k < \frac{30}{1000} = 0.03$
(iii)	At $x = 200, \quad 200 = 30(50)\left(1 - e^{-\frac{t}{50}}\right)$ $1 - e^{-\frac{t}{50}} = \frac{2}{15}$ $t = 50\ln\left(\frac{15}{13}\right)$ Using GC, t = 7.16h or 7h 9min
2	Second root is $re^{-i\theta}$. Quadratic factor of P(z) is $\left(z - re^{i\theta}\right)\left(z - re^{-i\theta}\right)$ $= z^2 - \left(re^{i\theta} + re^{-i\theta}\right)z + \left(re^{i\theta}\right)\left(re^{-i\theta}\right)$ $= z^2 - r\left(e^{i\theta} + e^{-i\theta}\right)z + r^2$ $= z^2 - r\left(\cos\theta + i\sin\theta + \cos\theta - i\sin\theta\right)z + r^2$ $= z^2 - \left(2r\cos\theta\right)z + r^2$ root of the equation is $3e^{i\left(\frac{2\pi}{3}\right)}$. So $r = 3$ and $\theta = \frac{2\pi}{3}$. Quadratic factor is $z^2 - 2(3)\left(\cos\frac{2\pi}{3}\right)z + 9 = z^2 + 3z + 9$

hence $z^3 + az^2 + 15z + 18 = (z^2 + 3z + 9)(z + 2)$

By comparing z^2 term, $a = 5$

The roots of the equation $z^3 + az^2 + 15z + 18 = 0$ are

$3e^{i\left(\frac{2\pi}{3}\right)}$, $3e^{i\left(-\frac{2\pi}{3}\right)}$ and $-2 = 2e^{i(\pi)}$

$18z^3 + 15z^2 + az + 1 = 0$

$$z^3\left(18 + 15\left(\frac{1}{z}\right) + a\left(\frac{1}{z^2}\right) + \left(\frac{1}{z^3}\right)\right) = 0$$

Since $z \neq 0$, and let $w = \dfrac{1}{z}$

We have $w^3 + aw^2 - 2w + 18 = 0$

Hence $w = 3e^{i\left(\frac{2\pi}{3}\right)}, 3e^{i\left(-\frac{2\pi}{3}\right)}, -2$

$$\frac{1}{z} = 3e^{i\left(\frac{2\pi}{3}\right)}, 3e^{i\left(-\frac{2\pi}{3}\right)}, -2$$

Since $\left|\dfrac{1}{z}\right| = \dfrac{1}{|z|}$ and $\arg\left(\dfrac{1}{z}\right) = -\arg(z)$

So $z = \dfrac{1}{3}e^{i\left(-\frac{2\pi}{3}\right)}, \dfrac{1}{3}e^{i\left(\frac{2\pi}{3}\right)}, -\dfrac{1}{2}$ are the roots of $18z^3 + 15z^2 + az + 1 = 0$

3(i)

Equation of line through point P and perpendicular to π_1 is

$$\mathbf{r} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}, \quad \lambda \in \mathbb{R}$$

Since F lies on plane π_1 ,

$(-2+\lambda) - 2(1-2\lambda) + 2(1+2\lambda) = 7 \quad \Rightarrow \lambda = 1$

$$\overrightarrow{OF} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} + 1\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 3 \end{pmatrix}$$

$$\overrightarrow{PF} = \overrightarrow{OF} - \overrightarrow{OP} = \begin{pmatrix} -1 \\ -1 \\ 3 \end{pmatrix} - \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$$

shortest distance from P to plane Π_1 = $\left|\overrightarrow{PF}\right| = \sqrt{1^2 + (-2)^2 + 2^2} = 3$

3(ii)

Line m is parallel to both planes:

$$\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} \times \begin{pmatrix} a \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 2-6 \\ -(-1-2a) \\ 3+2a \end{pmatrix} = \begin{pmatrix} -4 \\ 1+2a \\ 3+2a \end{pmatrix}$$

Equation of this line $m : r = \begin{pmatrix} -1 \\ -1 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} -4 \\ 1+2a \\ 3+2a \end{pmatrix}$ where $\mu \in \mathbb{R}$

3(iii)

$Q(1, -4, -1)$ lies on line m,

$-1-4\mu = 1$ --- (1)

$-1+(1+2a)\mu = -4$ --- (2)

$3+(3+2a)\mu = -1$ --- (3)

From (1) : $\mu = -\frac{1}{2}$
From (2) : $a = 5/2$
From (3) : $a = 5/2$. Hence the value of a is 5/2

<u>Alternative method</u>

$$\overrightarrow{FQ} = \begin{pmatrix}1\\-4\\-1\end{pmatrix} - \begin{pmatrix}-1\\-1\\3\end{pmatrix} = \begin{pmatrix}2\\-3\\-4\end{pmatrix}$$

Since line m contains F and is parallel to π_1 , line m lies on π_1.

Since line m is on π_1, Q is on π_1. hence $\overrightarrow{FQ}$ is // π_1 and $\perp n_1$

$$\begin{pmatrix}2\\-3\\-4\end{pmatrix} \cdot \begin{pmatrix}a\\3\\-1\end{pmatrix} = 0$$

$2a - 9 + 4 = 0$

$a = 5/2$

(iv) <u>Method 1 (dot product)</u>

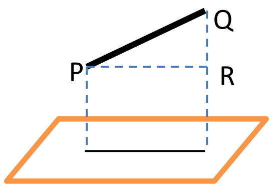
$\overrightarrow{PQ} = \begin{pmatrix}1\\-4\\-1\end{pmatrix} - \begin{pmatrix}-2\\1\\1\end{pmatrix} = \begin{pmatrix}3\\-5\\-2\end{pmatrix}$ and normal to the x-y plane = $\begin{pmatrix}0\\0\\1\end{pmatrix}$

$$QR = \left|\overrightarrow{PQ} \cdot \begin{pmatrix}0\\0\\1\end{pmatrix}\right| = \left|\begin{pmatrix}3\\-5\\-2\end{pmatrix} \cdot \begin{pmatrix}0\\0\\1\end{pmatrix}\right| = 2$$

length of projection of $\overrightarrow{PQ}$ on the x-y plane

= PR

$= \sqrt{PQ^2 - 2^2} = \sqrt{(3^2 + 5^2 + 2^2 - 2^2} = \sqrt{34}$



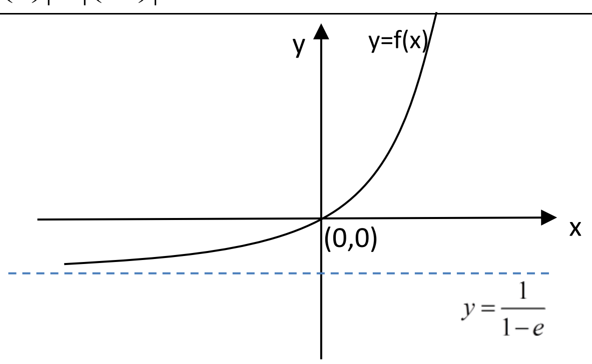
<u>Method 2 (cross product)</u>

length of projection of $\overrightarrow{PQ}$ on the x-y plane

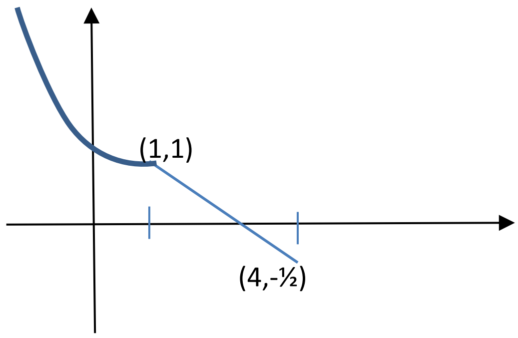
$$= PR = \left|\overrightarrow{PQ} \times \begin{pmatrix}0\\0\\1\end{pmatrix}\right| = \left|\begin{pmatrix}3\\-5\\-2\end{pmatrix} \times \begin{pmatrix}0\\0\\1\end{pmatrix}\right| = \left|\begin{pmatrix}-5\\-3\\0\end{pmatrix}\right| = \sqrt{5^2 + 3^2} = \sqrt{34}$$

4 Soln:

$$R_f = \left(\frac{1}{1-e}, \infty\right)$$



4



$$R_h = \left[-\frac{1}{2}, \infty\right)$$

$$D_{f^{-1}} = R_f = \left(\frac{1}{1-e}, \infty\right) = (-0.582, \infty)$$

Hence $R_h \subseteq D_{f^{-1}}$, so $f^{-1}h$ exists .

Let $\left(f^{-1}h\right)^{-1}(3) = k$

$\Rightarrow f^{-1}h(k) = 3$

$\Rightarrow h(k) = f(3)$

$\Rightarrow h(k) = \dfrac{e^3-1}{e-1} = e^2+e+1$

Since $e^2+e+1 > 1$,
hence $h(x) = (x-1)^2+1$

$(k-1)^2+1 = e^2+e+1$

$\Rightarrow k = 1 \pm \sqrt{e^2+e}$

Since x <1, hence the exact value of $\left(f^{-1}h\right)^{-1}(3) = 1-\sqrt{e^2+e}$.

Alternative method: use $f^{-1}(x)$

Let $\left(f^{-1}h\right)^{-1}(3) = k$

$\Rightarrow f^{-1}h(k) = 3$

$\Rightarrow \ln\left[1+(e-1)h(k)\right] = 3$

$\Rightarrow 1+(e-1)h(k) = e^3$

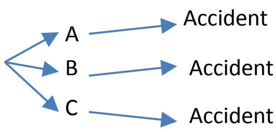
$\Rightarrow h(k) = \dfrac{e^3-1}{e-1} = e^2+e+1$

5(i)

Required probability

$= (0.3\times 0.01)+(0.5\times 0.03)+(0.2\times 0.06)$

= 0.03



5(ii)

$$P(\text{class C/accident}) = \frac{P(\text{accident} \cap \text{class C})}{P(\text{accident})}$$

$$= \frac{0.2\times 0.06}{0.03} = 0.4$$

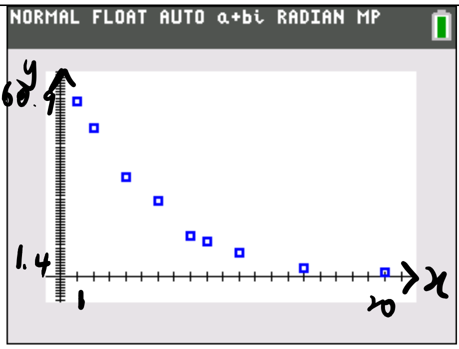
5(iii)

P(all three drivers are of class C and exactly one have accident)

$= (0.2\times 0.94)^2 \times 0.2\times 0.06 \times \dfrac{3!}{2!}$

= 0.00127 (to 3sf)

6(i)



6(ii)	Regression line of y on x is $y = 49.7 - 3.09x$ When x = 17, y = -2.8466…= - 2.85 The linear model is not suitable since 1) the negative value of y is impossible or 2) the scatter diagram shows a curved relationship between the two variables.
6(iii)	Product moment correlation coefficient between W and x = -0.997837…= - 0.998
6(iv)	Since x is the controlled variable, we use the regression line of $\ln y$ on x : $\ln y = 4.3549 - 0.20532x$ [from GC] When $y = \frac{1}{2}(75)$, we have $\ln \frac{75}{2} = 4.3549 - 0.20532x$ $\Rightarrow x = 3.5581... = 3.6$ The weight will drop to half its original value in 3.6 weeks. The estimate is reliable since 1) The product moment correlation coefficient between $\ln y$ and x is -0.998 which is very close to -1, showing a strong negative linear correlation between $\ln y$ and x. 2) The estimate is an interpolation, because $y = \frac{1}{2}(75)$ is in the data range $1.4 \le y \le 60.9$.
7a(i)	Only working adults travelling by train will have a chance of being selected. Those who do not travel by train will have no chance of being chosen. Hence not every working adult in the country has an equal chance to be selected – therefore the sample is not a random sample.
7a(ii)	Let X hours be the number of hours of sleep of a randomly chosen adult and μ be the mean of X. To test $H_o : \mu = 6$ vs $H_1 : \mu > 6$ Since sample size is large, by CLT, $\overline{X} \sim N\left(6, \frac{\sigma^2}{50}\right)$ Since population variance σ^2 is unknown, it is replaced by s^2 Under H_o, test statistic $Z = \frac{\overline{X} - 6}{\frac{s}{\sqrt{50}}} \sim N(0,1)$ We use a one-tailed test at 5% level of significance, that is, reject H_o if p-value < 0.05 Sample readings: $\overline{x} = \frac{320}{50} = 6.4$, $s^2 = \frac{1}{49}\left(2308.5 - \frac{(320)^2}{50}\right) = 5.31633$ From GC, p-value = 0.109967 = 0.110 > 0.05 $\Rightarrow$ we **do not reject** H_o. Hence we conclude that there is **insufficient** evidence at the 5% level of significance to doubt the Health Promotion Board's claim. It is not necessary to assume that the number of hours of sleep follow a normal distribution because since the sample size is large, by the Central Limit Theorem, the sample mean follows a normal distribution approximately.

7b

To test $H_o : \mu = \mu_o$ vs $H_1 : \mu \neq \mu_o$

$s^2 = \frac{50}{49}(2.1)^2$, $\bar{x} = 6.14$,

Since H_o is not rejected at the 5% level,

$$-1.95996 < \frac{\bar{x} - \mu_o}{\frac{s}{\sqrt{n}}} < 1.95996$$

$$\Rightarrow -1.95996 < \frac{6.14 - \mu_o}{\frac{2.1}{\sqrt{49}}} < 1.95996$$

$$\Rightarrow 6.14 - 1.95996\frac{2.1}{\sqrt{49}} < \mu_o < 6.14 + 1.95996\frac{2.1}{\sqrt{49}}$$

$$\Rightarrow 5.552012 < \mu_o < 6.727988$$

$$\Rightarrow 5.55 < \mu_o < 6.73$$

8 (a)

Let X be the number of cream biscuits per box. $X \sim B(10, p)$

$P(X = 1 \text{ or } 2) = 0.15$

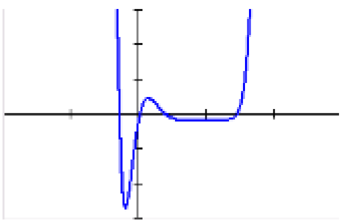
$P(X = 1) + P(X = 2) = 0.15$

$^{10}C_1 p^1(1-p)^9 + {}^{10}C_2 p^2(1-p)^8 = 0.15$

$10p(1-p)^9 + 45p^2(1-p)^8 = 0.15$

$5p(1-p)^8\left[2(1-p) + 9p\right] = 0.15$

$5p(1-p)^8\left(2 + 7p\right) = 0.15$



Draw $y = 5p(1-p)^8(2+7p) - 0.15$

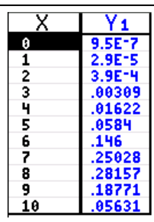
From G.C.,

$p = 0.0162$ or $p = 0.408$

(other values are 1.45 or -0.288 need to be rejected)

8 (b) (i)

$X \sim B(10, \frac{3}{4})$. Let $Y_1 = P(X = x)$.



X	Y1
0	9.5E-7
1	2.9E-5
2	3.9E-4
3	.00309
4	.01622
5	.0584
6	.146
7	.25028
8	.28157
9	.18771
10	.05631

From G.C.,

since P(X = 8) is the highest,

The most likely no. of cream biscuits = 8

(ii)

Let Y denote the random variable: Number of boxes with X = 5.

$Y \sim B(18, p)$ where $p = P(X=5) = 0.058399$

$P(3 \le Y < 7) = P(Y \le 6) - P(Y \le 2)$

$= 0.0843$ (3 s.f.)

(iii)

Let U = no. of boxes sold at Usual price

Let D = no. of boxes sold at Discounted price

Let W: Total income per day.

W = 10U+8D

$E(W) = 10E(U)+8E(D) = 180\times\$10 + 840\times\$8 = \$8520$

$Var(W) = 10^2 Var(U)+8^2 Var(D) = 64\times10^2 + 169\times8^2 = 17216$

Let $T = W_1 + W_2 + \ldots + W_{30}$

Since n = 30 is large, by Central Limit Theorem,

$T \sim N(30\times8520,\ 30\times17216) = N(255600,\ \sqrt{516480}^2)$ approximately

$P(T \ge \$255000) = 0.798$

9(i)

family	A		B		C		D	
	Adult	kids	Adult	kids	Adult	kids	Adult	kids
	2	2	1	2	1	3	1	1

4 family units, No. of ways = $4!\times 4!\times 3!\times 5!\times 2!$ = 829, 440

9(ii)

Case 1: 3,3,2 No. of ways = $\dfrac{{}^8C_3\times{}^5C_3\times{}^2C_2}{2!}$ = 280

Case 2: 2,2,2,2 No. of ways = $\dfrac{{}^8C_2\times{}^6C_2\times{}^4C_2\times{}^2C_2}{4!}$ = 105

Total no. of ways = 280 + 105 = 385

9(iii)

There is only 1 way to divide the 8 children and the adults into 2 circles to satisfy all conditions. Family A and B (3 adults & 4 kids) must be in 1 circle and Family C & D are in another circle.
Arrange the children in 1 circle : (4-1)!
Slot in adults in between children : ${}^4C_3\times 3!$

No. of ways = $[(4-1)!\times{}^4C_3\times 3!]\times[(4-1)!\times{}^4C_3\times 3!]$
= 20736

10(i)

Let M denote the random variable: Height of a male visitor in cm. $M \sim N(165, 12^2)$
Let F denote the random variable: Height of a female visitor in cm. $F \sim N(155, \sigma^2)$

$$P(150 < F < 160) = 0.3829$$
$$P(-\frac{5}{\sigma} < Z < \frac{5}{\sigma}) = 0.3829$$
$$P(Z < -\frac{5}{\sigma}) = \frac{1-0.3829}{2} = 0.30855$$

From G.C. $-\dfrac{5}{\sigma} = -0.4999646$

$\Rightarrow \sigma$ = 10.0 cm (3 sig. fig.)

(ii)

$$\frac{3}{4}M - F \sim N(\frac{3}{4}\times 165 - 155, \frac{9}{16}\times 12^2 + 10^2) = N(-31.25, 181)$$
$$P(\left|\frac{3}{4}M - F\right| \le 20) = P(-20 \le \frac{3}{4}M - F \le 20)$$
$= 0.201$ (3 sig. fig.)

Assumption: The heights of all male and female visitors are independent of one another.

(iii)

Probability Distribution of X:

***x* (in $)**	**P(X = x)**
0	$\frac{1}{2}P(M<120)+\frac{1}{2}P(F<120)=0.00016056$
10	$\frac{1}{2}P(120\le M<150)+\frac{1}{2}P(120\le F<150)=0.20693$
m	$\frac{1}{2}P(M\ge 150)+\frac{1}{2}P(F\ge 150)=0.79291$

Given E(X) = 17.93 = 0 (0.00016056)+10(0.20693)+m(0.79291)
$\Rightarrow m$ = 20.00 (shown)

(iv)

$P(X_1 + X_2 + X_3 > 40)$ = P(20, 20, 20) + 3. P(20, 20, 10)
= 0.889