Section A: Pure Mathematics [50 marks]

1. Prove by mathematical induction that $(x+1)^n - nx - 1$ has a factor x^2 for all $n \in \square$, $n \geq 2$. [5]

2. **(i)** Show, with the aid of a sketch graph, that the equation

$$x + k\ln x = 0$$

has exactly one real root, α, in the interval $\frac{1}{2} < x < 1$ if $k > \frac{1}{2\ln 2}$. [3]

(ii) In the case $k = 1$, a student tries to use fixed point iteration in the form of $x = \mathrm{F}(x)$ to find the value of α.

(a) In his first attempt, the student uses $\mathrm{F}(x) = -\ln x$ and $x_0 = 0.5$. Calculate the value of x_1 and x_2, correct to 4 decimal places. Explain why this method will fail to find the value of α. [2]

(b) Suggest a possible $\mathrm{F}(x)$ for the student and using $x_0 = 0.5$, find the value of α, correct to 3 decimal places. [2]

(c) Use a diagram to explain how the iteration in **(b)** converges to α, showing clearly the position of x_0, x_1 and x_2. [2]

3. For each point on the conic *C*, the distance from the focus *F*(0, 0) is half the distance from the directrix $x = 3$.

(i) Find the polar equation of *C*. [2]

(ii) Find the cartesian equation of *C*. [3]

(iii) *R* is the region bounded by the *y*-axis and the part of *C* where $x \leq 0$. Show that the volume formed when *R* is rotated by 2π radians about the *y*-axis is given by $k\int_0^{-3} x\sqrt{4-(x+1)^2}\,\mathrm{d}x$, for some constant *k* to be determined. Using the substitution $x + 1 = 2\sin t$, find the exact volume. [6]

Turn Over

4. **(a)** The point P in an Argand diagram represents the variable complex number z and the point A represents the fixed complex number a, where $0 < \arg a < \dfrac{\pi}{4}$.

Sketch, on a single diagram, the locus of P in each of the following:

(i) $|\mathrm{i}z - a| = |a^*|$, [2]

(ii) $\arg(a - z) = \dfrac{\pi}{4}$. [2]

Hence, shade the region that satisfy the following inequalities:

$$|\mathrm{i}z - a| \geq |a^*| \text{ and } \frac{\pi}{4} \leq \arg\left(a - z\right) \leq \frac{\pi}{2}.$$ [2]

(b) By considering $\mathrm{e}^{\alpha\mathrm{i}} = \cos\alpha + \mathrm{i}\sin\alpha$, show that, provided $\cos\theta \neq 0$,

$$\sum_{k=0}^{11} \left(-1\right)^k \sin\left(2k+1\right)\theta = a\sec\left(b\theta\right)\sin\left(c\theta\right)$$

where a, b and c are constants to be determined. [6]

5. **(a)** Let S be the set of all solutions $y = \mathrm{f}(x)$ of the homogeneous second order linear differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 5\frac{\mathrm{d}y}{\mathrm{d}x} + 6y = 0$$

defined on $[a, b]$.

(i) Given that the set of all functions defined on $[a, b]$ forms a vector space C, show that S is a subspace of C. [4]

(ii) Find a basis of S. [2]

(iii) Determine whether the set of all solutions to the differential equation $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} + 5\dfrac{\mathrm{d}y}{\mathrm{d}x} + 6y = \mathrm{e}^x$ is also a subspace of C. Justify your answer. [1]

(b) The transformation $T : \mathbb{R}^4 \to \mathbb{R}$ is defined in such a way that T maps a column vector to the sum of its entries. For example, $T\left(\begin{pmatrix} 1 \\ -1 \\ 2 \\ 3 \end{pmatrix}\right) = 1 + (-1) + 2 + 3 = 5$.

(i) Show that T is a linear transformation. [3]

(ii) Write down the matrix representing T and find the null space of T. [3]

Section B: Probability and Statistics [50 marks]

6. The mass of a particular brand of mini chocolate bars is advertised as 8 grams. Ten pieces of these bars were removed from their wrappers and weighed. Their masses (in grams) are recorded below.

8.3	8.2	7.9	8.2	8.1
8.4	8.0	8.1	8.1	8.2

(i) Test, at 1% level of significance, whether the mean mass of the mini chocolate bars exceeds 8 grams. [5]

(ii) Find a 98% confidence interval for the mean mass of the bars to 4 decimal places. [1]

(iii) Explain why the confidence interval found in part **(ii)** supports your conclusion in part **(i)**. [1]

7. **(i)** A game played by n players, where $n > 2$, has the following rule. Each player tosses a fair coin. If any of the players' toss is different from that of all the other players' tosses, then that player wins the game. Otherwise all the players toss again until one player wins.

Given that X is the number of tosses each player makes, up to and including the one on which the game is won, show that

$$\mathrm{P}(X = r) = n\left(\frac{1}{2}\right)^{n-1}\left[1 - n\left(\frac{1}{2}\right)^{n-1}\right]^{r-1}$$

for any positive integer r. [3]

(ii) Given that $n = 8$, find

(a) $\mathrm{E}(X)$, [1]

(b) $\mathrm{P}(X = 10 \mid X > 6)$, [2]

(c) the largest integer value of m such that $\mathrm{P}(X \le m) < 0.6$. [2]

8. The random variable X denotes the number of hours (in hundreds) that daffodils will last in a vase of water with a new additive. The function f defined below is a proposed probability density function for X.

$$\mathrm{f}(x)=\begin{cases} k(5-x^2) & 0\le x\le 2 \\ 0 & \text{otherwise.}\end{cases}$$

(a) Show that the value of the constant k is $\dfrac{3}{22}$. [2]

(b) Find the median, m, of X, giving your answers in hours. [3]

(c) 100 independent observations of X are taken. Let N denotes the number of these observations such that X is greater than 1.8. Find $\mathrm{P}(5 < N < 12)$. [3]

From historical data, it is known that several daffodils last more than 250 hours in a vase of water with the new additive.

(d) Explain if the function f is a suitable model for the probability density function of X. [1]

9. **(a)** The table below shows the distribution of the number of hits by bombs in 450 equally sized areas in a certain part of London during the World War II.

Number of hits (x)	0	1	2	3	4	5	6 or more
Frequency (f)	182	171	67	22	6	2	0

Find the expected frequencies of hits given by a Poisson distribution having the same mean and total as the observed distribution. Using χ^2 distribution and a 10% level of significance, test the adequacy of the Poisson distribution as a model for these data. [5]

(b) Based on bomb relics and historical recordings kept during the war, it was found that three types of bombs (i.e. Types A, B and C bombs) were deployed in the hits. The number of the various types of bombs dropped out of the 450 bombs over different zones of London are summarised in the table below.

	Type A bomb	Type B bomb	Type C bomb
North Zone	58	60	44
East Zone	50	64	36
South Zone	50	35	41
West Zone	3	3	6

Test, at 5% significance level, the hypothesis that the zone is independent of the type of bomb deployed in the hit. [5]

Find the smallest integer α such that at $\alpha\%$ significance level, the hypothesis that the zone is independent of the type of bomb deployed is rejected. [2]

10. An exercise is claimed to improve concentration. Two tasks A and B of similar difficulty were designed such that people who concentrate better should complete the tasks in less time. An experiment was conducted on 15 participants to study the effect of the exercise on concentration. Each participant completed one of the two tasks, did the exercise, and then completed the other task. The times taken to complete the tasks are shown in the table below.

Participant	1	2	3	4	5	6	7	8
Before	25	27	24	25	32	19	21	27
After	22	20	25	25	25	17	24	27

Participant	9	10	11	12	13	14	15
Before	25	23	24	29	26	26	23
After	23	22	24	23	29	22	25

(i) Explain why it is not advisable to have all participants do task A before the exercise and task B after. How should the tasks be assigned? [2]

(ii) Explain the difference in the hypotheses for a paired sample *t*-test and a Wilcoxon signed rank test. Which test requires an assumption on the distribution of the times taken and what distribution is that? [4]

Carry out, at 5% level of significance,

(a) a paired sample *t*-test, [3]

(b) a Wilcoxon signed rank test. [5]

End of Paper