2017 NYJC JC2 Preliminary Examination 9758/2 Solution

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1(i)	Assume that $\mathbf{a}$ and $\mathbf{b}$ are non-parallel vectors. $\overrightarrow{OC}=5\mathbf{a}$, $\overrightarrow{OD}=2\mathbf{b}$ On the line *BC*, $\overrightarrow{OM}=\lambda(5\mathbf{a})+(1-\lambda)\mathbf{b}$ On the line *AD*, $\overrightarrow{OM}=\mu(2\mathbf{b})+(1-\mu)\mathbf{a}$ Since $\mathbf{a}$ and $\mathbf{b}$ are non-zero, non-parallel vectors, comparing coefficient $\begin{array}{l}5\lambda=1-\mu\\2\mu=1-\lambda\end{array} \Rightarrow \lambda=\frac{1}{9},\mu=\frac{4}{9}$ Thus $\overrightarrow{OM}=\frac{5}{9}\mathbf{a}+\frac{8}{9}\mathbf{b}$
1(ii)	Since $\mathbf{a}$ is a unit vector in the direction of *OC*, shortest distance $=\left\|\overrightarrow{OM}\times\mathbf{a}\right\|$ $=\left\|\left(\frac{5}{9}\mathbf{a}+\frac{8}{9}\mathbf{b}\right)\times\mathbf{a}\right\|$ $=\frac{8}{9}\left\|\mathbf{a}\times\mathbf{b}\right\|$ $k=\frac{8}{9}$
2(a)	For sum to infinity to exist, $\left\|\tan\theta\right\|<1$ $-1<\tan\theta<1$ $-\frac{\pi}{4}<\theta<\frac{\pi}{4}$ $\frac{1}{1-\tan\theta}>2$ $0<1-\tan\theta<\frac{1}{2}$ $\tan\theta>\frac{1}{2}\Rightarrow\theta>\tan^{-1}\frac{1}{2}$ (or $\theta>0.464$) Since $-\frac{\pi}{4}<\theta<\frac{\pi}{4}$, therefore $\{\theta\in\mathbb{R} \mid \tan^{-1}\frac{1}{2}<\theta<\frac{\pi}{4}\}$ (or $\{\theta\in\mathbb{R} \mid 0.464<\theta<0.785\}$ or $\theta:(0.464,0.785)$)

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2(b)	$u_1 = S_1 = 2 \Rightarrow a = 8$ $u_2 = S_2 - S_1 = 10 \Rightarrow d = 8$ $u_{32} = a + (32-1)d = 2 + (32-1)8 = 250$ $\frac{u_{32}}{u_m} = \frac{u_m}{u_2} = \text{constant}$ $\Rightarrow (u_m)^2 = (10)(250) = 2500$ $u_m = 50$ (since it is a positive sequence) $50 = 2 + (m-1)8 \Rightarrow m = 7$ Alternatively, $u_n = S_n - S_{n-1}$ $= 4n^2 - 2n - \left[4(n-1)^2 - 2(n-1)\right]$ $= 8n - 6$ $\frac{u_{32}}{u_m} = \frac{u_m}{u_2}$ $\frac{8(32)-6}{8m-6} = \frac{8m-6}{8(2)-6}$ $(8m-6)^2 = (250)(10) = 2500$ $m = 7$ or $m = -5.5$ (rejected as m is a positive integer)
3(i)	$y = \ln\left(\cos ax - \sin ax\right)$ $e^y = \cos ax - \sin ax$ $e^y \frac{dy}{dx} = -a \sin ax - a \cos ax$ $e^y \frac{d^2 y}{dx^2} + e^y \left(\frac{dy}{dx}\right)^2 = -a^2 \cos ax + a^2 \sin ax$ $e^y \frac{d^2 y}{dx^2} + e^y \left(\frac{dy}{dx}\right)^2 = -a^2 \left(\cos ax - \sin ax\right)$ $e^y \frac{d^2 y}{dx^2} + e^y \left(\frac{dy}{dx}\right)^2 = -a^2 e^y$ $\frac{d^2 y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + a^2 = 0$

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3(ii)	$\frac{\mathrm{d}^3y}{\mathrm{d}x^3}+2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=0$ When $x=0,\quad y=0$ $\frac{\mathrm{d}y}{\mathrm{d}x}=-a\ ,\quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-2a^2,\quad \frac{\mathrm{d}^3y}{\mathrm{d}x^3}=-4a^3$ $y=-ax-a^2x^2-\frac{2}{3}a^3x^3+\cdots$
3(iii)	$\ln\left(\cos 2x-\sin 2x\right)=-2x-4x^2-\frac{16}{3}x^3+\cdots$ $\cos 2x-\sin 2x\approx \mathrm{e}^{-2x-4x^2}$ $\approx 1+\left(-2x-4x^2\right)+\frac{\left(-2x-4x^2\right)^2}{2!}$ (since $\mathrm{e}^x\approx 1+x+\frac{x^2}{2!}$) $\approx 1-2x-4x^2+\frac{\left(-2x\right)^2}{2}$ $=1-2x-2x^2$ where $k=-2$
3(iv)	$\cos 2x-\sin 2x=1-\frac{(2x)^2}{2}-(2x)$ $=1-2x-2x^2$

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4(i)	$$\frac{\mathrm{d}x}{\mathrm{d}t}=k(N-x)x$$ $$\frac{1}{(N-x)x}\frac{\mathrm{d}x}{\mathrm{d}t}=k$$ $$\int\frac{1}{(N-x)x}\,\mathrm{d}x=\int k\,\mathrm{d}t$$ $$\frac{1}{N}\int\frac{1}{N-x}+\frac{1}{x}\,\mathrm{d}x=\int k\,\mathrm{d}t$$ $$\frac{1}{N}\left(-\ln\lvert N-x\rvert+\ln\lvert x\rvert\right)=kt+C$$ $$\frac{1}{N}\ln\left\lvert\frac{x}{N-x}\right\rvert=kt+C$$ $$\ln\left\lvert\frac{x}{N-x}\right\rvert=Nkt+NC$$ $$\left\lvert\frac{x}{N-x}\right\rvert=\mathrm{e}^{Nkt+NC}$$ $$\frac{x}{N-x}=A\mathrm{e}^{Nkt}\quad\text{where } A=\pm\mathrm{e}^{NC}$$ When $t=0$, $x=\frac{1}{3}N$, $$A=\frac{1}{2}$$ $$\frac{x}{N-x}=\frac{1}{2}\mathrm{e}^{Nkt}$$ $$2x=(N-x)\mathrm{e}^{Nkt}$$ $$x\left(2+\mathrm{e}^{Nkt}\right)=N\mathrm{e}^{Nkt}$$ $$x=\frac{N\mathrm{e}^{Nkt}}{2+\mathrm{e}^{Nkt}}\qquad\text{equivalently, } x=\frac{N}{2\mathrm{e}^{-Nkt}+1}$$
	<u>Alternative method</u> (not recommended):

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	$\dfrac{1}{(N-x)x}\dfrac{\mathrm{d}x}{\mathrm{d}t}=k$ $-\displaystyle\int \frac{1}{x^2-Nx}\,\mathrm{d}x=\int k\,\mathrm{d}t$ $-\displaystyle\int \frac{1}{\left(x-\frac{N}{2}\right)^2-\left(\frac{N}{2}\right)^2}\,\mathrm{d}x=\int k\,\mathrm{d}t$ $-\left(\dfrac{1}{2\left(\frac{N}{2}\right)}\ln\left\|\dfrac{\left(x-\frac{N}{2}\right)-\frac{N}{2}}{\left(x-\frac{N}{2}\right)+\frac{N}{2}}\right\|\right)=kt+C$ $-\left(\dfrac{1}{N}\ln\left\|\dfrac{x-N}{x}\right\|\right)=kt+C$ $\dfrac{1}{N}\ln\left\|\dfrac{x}{N-x}\right\|=kt+C$ $\ln\left(\dfrac{x}{N-x}\right)=Nkt+NC$ since $0<x<N$ When $t=0$, $x=\dfrac{1}{3}N$, $\ln\dfrac{1}{2}=NC\Rightarrow C=-\dfrac{1}{N}\ln 2$ $\ln\left(\dfrac{x}{N-x}\right)=Nkt-\ln 2$ $\ln\left(\dfrac{2x}{N-x}\right)=Nkt$ $\dfrac{2x}{N-x}=\mathrm{e}^{Nkt}$ $x\left(2+\mathrm{e}^{Nkt}\right)=N\,\mathrm{e}^{Nkt}$ $x=\dfrac{N\,\mathrm{e}^{Nkt}}{2+\mathrm{e}^{Nkt}}$ equivalently, $x=\dfrac{N}{2\,\mathrm{e}^{-Nkt}+1}$ As $t\to\infty$, $x=\dfrac{N}{2\,\mathrm{e}^{-Nkt}+1}\to N$

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(ii)	$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}=\frac{-9t}{\left(4+9t^2\right)^2}$ $\int\frac{\mathrm{d}^2x}{\mathrm{d}t^2}\,\mathrm{d}t=\int\frac{-9t}{\left(4+9t^2\right)^2}\,\mathrm{d}t$ $=-\frac{1}{2}\int\frac{18t}{\left(4+9t^2\right)^2}\,\mathrm{d}t$ $\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{1}{2}\left(\frac{1}{4+9t^2}\right)+A$ $\int\frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t=\frac{1}{2}\int\frac{1}{4+9t^2}\,\mathrm{d}t+\int A\,\mathrm{d}t$ $=\frac{1}{18}\int\frac{1}{\left(\frac{2}{3}\right)^2+t^2}\,\mathrm{d}t+\int A\,\mathrm{d}t$ $=\frac{1}{18}\cdot\frac{1}{\frac{2}{3}}\tan^{-1}\left(\frac{t}{\frac{2}{3}}\right)+At+B$ $x=\frac{1}{12}\tan^{-1}\left(\frac{3t}{2}\right)+At+B$ Since the population stabilises in the long run, as $t\to\infty$, $x\to$ finite value, $A=0$ When $t=0$, $x=\frac{1}{3}N$, $B=\frac{N}{3}$ Hence $x=\frac{1}{12}\tan^{-1}\left(\frac{3t}{2}\right)+\frac{N}{3}$ When $t\to\infty$, $\tan^{-1}\left(\frac{3t}{2}\right)\to\frac{\pi}{2}$ Hence $x\to\frac{\pi}{24}+\frac{N}{3}$.
5(i)	Let Y be the payment for an individual. The probability table is as follows: <table><tr><td>y</td><td>100</td><td>200</td><td>300</td><td>325</td><td>350</td></tr><tr><td>$P(Y=y)$</td><td>$\frac{1}{3}$</td><td>$\frac{4}{15}$</td><td>$\frac{1}{5}$</td><td>$\frac{2}{15}$</td><td>$\frac{1}{15}$</td></tr></table> $\mathrm{E}(Y)=100\cdot\frac{1}{3}+200\cdot\frac{4}{15}+300\cdot\frac{1}{5}+325\cdot\frac{2}{15}+350\cdot\frac{1}{15}$ $=213\frac{1}{3}$

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5(ii)	$\mathrm{E}(Y^2)=100^2\cdot\frac{1}{3}+200^2\cdot\frac{4}{15}+300^2\cdot\frac{1}{5}+325^2\cdot\frac{2}{15}+350^2\cdot\frac{1}{15}$ $=54250$ $\mathrm{Var}(Y)=\mathrm{E}(Y^2)-\left[\mathrm{E}(Y)\right]^2=8738\frac{8}{9}$ Since $n=100$ is large, by Central Limit Theorem, $T=\sum_{i=1}^{100}Y_i \sim \mathrm{N}\left(21333.33, 873888.89\right)$ approx. $\text{Prob. Req'd} = P(T > 24000)$ ≈ 0.00217
6	Probability required = $\frac{2\times {}^3C_2\times {}^5C_2}{{}^8C_4}=\frac{6}{7}$
6(i)	Required probability = $\frac{(8-1)!\,{}^8C_2\,2!}{(10-1)!}=\frac{7}{9}$ OR $\quad 1-\frac{(9-1)!\,2!}{(10-1)!}=\frac{7}{9}$
6(ii)	Let X be the event that the remaining 3 girls are separated. Let Y be the event that Ann and Alice are not seated together. $\mathrm{P}\left(X\mid Y\right)=\frac{\mathrm{P}\left(X\cap Y\right)}{P\left(Y\right)}$ $=\frac{\mathrm{P}\left(X\right)-\mathrm{P}\left(X\cap Y'\right)}{P\left(Y\right)}$ $=\frac{\frac{\left(7-1\right)!\,{}^7C_3\,3!-\left(6-1\right)!2!\,{}^6C_3\,3!}{(10-1)!}}{7/9}$ $=\frac{85/252}{7/9}$ $=\frac{85}{196}$

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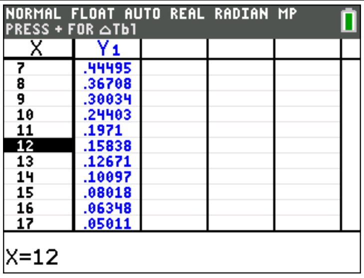
7(i)	$\mathrm{P}(R=6)=2\mathrm{P}(R=4)$ ${}^{10}C_6 p^6(1-p)^4 = 2\,{}^{10}C_4 p^4(1-p)^6$ $p^2 = 2(1-p)^2$ $p^2-4p+2=0$ $p = 0.586$
7(ii)	$R \sim \mathrm{B}(n, 0.25)$ $\mathrm{P}(R<2)>0.15$ $\mathrm{P}(R\le 1)>0.15$ $n = 12$ 
7(iii)	$R \sim \mathrm{B}(11, 0.7)$ $\mathrm{P}(R\ge 5 \mid R\le 8)=\dfrac{\mathrm{P}(R\ge 5 \text{ and } R\le 8)}{\mathrm{P}(R\le 8)}$ $=\dfrac{\mathrm{P}(5\le R\le 8)}{\mathrm{P}(R\le 8)}$ $=\dfrac{\mathrm{P}(R\le 8)-\mathrm{P}(R\le 4)}{\mathrm{P}(R\le 8)}$ $= 0.969$
8(a)(i)	The value of 0.073 indicates that there is a weak linear correlation between petal size and the amount of water but there could be some non-linear relation.
8(a)(ii)	The approximate value of -1 indicates that there is a strong negative linear correlation between the risk of heart disease and amount of red wine intake. It does not mean that red wine intake decreases the risk of heart disease.

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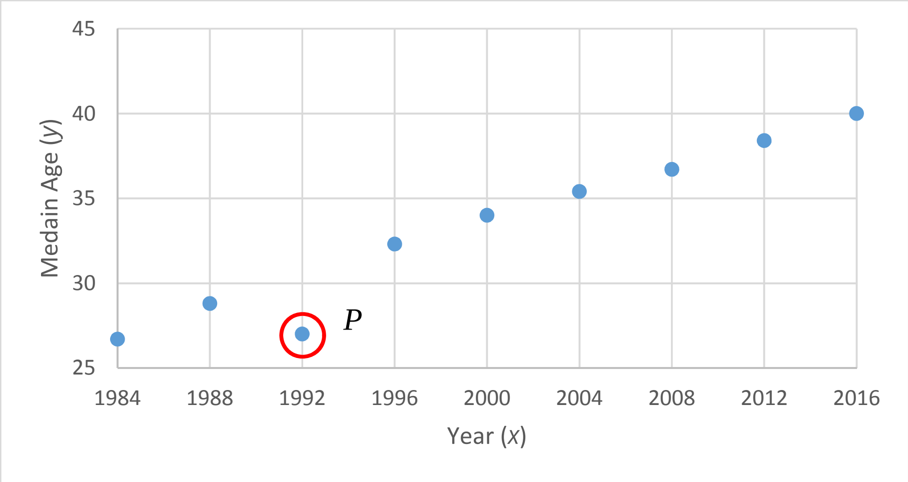
8(b)(i)	
8(b)(ii)	For Model A, $r = -0.9985438 = 0.99854$ For Model B, $r = 0.9984431 = 0.99844$
8(b)(iii)	Model A as the $\lvert r \rvert$ value is closer to 1. The suitable regression line is $y = 848.24 - \dfrac{1629165.57}{x}$
8(b)(iv)	This is because age (x) is the controlled variable
8(b)(v)	The rise in the median age is due to the drop in the growth of the population.
9(i)	To test $H_0 : \mu = 0.5$ $H_1 : \mu \neq 0.5$ Level of significance: 5% Under H_0, $Z = \dfrac{\overline{X} - 0.5}{0.04/\sqrt{25}} \sim N(0,1)$ Reject H_0 if p -value ≤ 0.05 Calculation: $\overline{x} = 0.51$, p -value = 0.211 Since p -value > 0.05, we do not reject H_0. Thus there is insufficient evidence at 5% level of significance that the manufacturing process is producing ball bearings of different diameters. Distribution of the diameter of the ball bearings is normal.

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9(ii)	To test $H_0 : \mu = 0.5$ $H_1 : \mu > 0.5$ Level of significance: 5% Under H_0, by Central Limit Theorem, $Z = \dfrac{\bar{X} - 0.5}{s/\sqrt{100}} \sim N(0,1)$ approx Reject H_0 if p-value ≤ 0.05 Calculation: $\bar{x} = 0.506$, $\Sigma(x - 0.5) = 50.6 - 50 = 0.6$ $s^2 = \dfrac{1}{n-1}\left[\Sigma(x-0.5)^2 - \dfrac{\left(\Sigma(x-0.5)\right)^2}{n}\right]$ $= \dfrac{1}{99}\left(0.08345 - \dfrac{0.6^2}{100}\right)$ $= 8.07 \times 10^{-4}$ p-value = 0.0173 Since p-value < 0.05, we reject H_0. Thus there is sufficient evidence at 5% level of significance that the manufacturing process is producing oversized ball bearings.
9(iii)	For the new sample, $\bar{x} = 0.506$. However, $s^2 = \dfrac{1}{199}\left(2(0.08345) - \dfrac{\left(2(0.6)\right)^2}{200}\right)$ $= 8.02513 \times 10^{-4}$ $Z_{calc} = 2.995$, p-value = 0.00137 Since p-value < 0.05, we will still reject H_0. The conclusion remains the same.
10(i)	Let *X* denotes the diameter of bolt from manufacturer A. $X \sim \mathrm{N}(1.56, 0.16^2)$ Let *Y* denotes the diameter of bolt from manufacturer B. $Y \sim \mathrm{N}(\mu, 0.16^2)$ $\mathrm{P}(Y < 1.52) = 0.242$ $\mathrm{P}(Z < \dfrac{1.52 - \mu}{0.16}) = 0.242$ $\dfrac{1.52 - \mu}{0.16} = -0.6998836 \Rightarrow \mu = 1.63198 = 1.632$
10(ii)	$W = X - Y \sim \mathrm{N}(1.56 - 1.632, 0.16^2 + 0.16^2)$ $\mathrm{P}(\lvert W \rvert < 0.1) = \mathrm{P}(-0.1 < W < 0.1) = 0.326$

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10(iii)	$X_1+X_2+X_3+X_4+X_5-5Y \sim \text{N}(5(1.56)\text{-}5(1.632),\ 5(0.16^2)+5^2(0.16)^2)$ $\text{P}(X_1+X_2+X_3+X_4+X_5>5Y)$ $=\text{P}(X_1+X_2+X_3+X_4+X_5-5Y>0)$ $=0.341$
10(iv)	$\text{P}(X<1.52)=0.40129$ $\text{Prob. Req'd}=(0.44)(0.4012)+0.56(0.242)$ $=0.3120876$ $=0.312$