**J2 H1 Math Prelim Exam (Solutions)**

**1.** Let the amount of samples types *X*, *Y* and *Z* in 20 grams of sample type *T* be *x*, *y* and *z*.

Fibre:

$$\left(\frac{12}{20}\right)x+\left(\frac{8}{20}\right)y+\left(\frac{6}{20}\right)z=8.8$$

$$0.6x+0.4y+0.3z=8.8 \qquad ---(1)$$

Wheat :

$$\left(\frac{6}{20}\right)x+\left(\frac{6}{20}\right)y+\left(\frac{14}{20}\right)z=7.6$$

$$0.3x+0.3y+0.7z=7.6 \qquad ---(2)$$

Sweetener :

$$\left(\frac{2}{20}\right)x+\left(\frac{6}{20}\right)y+\left(\frac{0}{20}\right)z=3.6$$

$$0.1x+0.3y+0z=3.6 \quad ---(3)$$

From GC

$x=6, \quad y=10, \ z=4$

There are 6 g of sample type *X*, 10 g of sample type *Y* and 4 g of sample type *Z* in 20 g of sample type *T*.

**2.**

$$\frac{a}{x-1}=\frac{a}{x}+5$$

$$\frac{a}{x-1}=\frac{a+5x}{x}$$

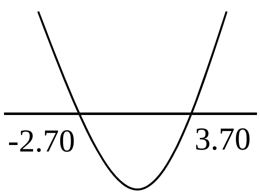
$$ax=ax-a+5x^2-5x$$

$$5x^2-5x-a=0 \ \text{(shown)}$$

To spend not more than \$50 for $x$ kg of cherries $\Rightarrow a\le 50$

$$5x^2-5x\le 50$$

$$x^2-x-10\le 0$$




Note:
The graphical method to solve the inequality must be shown.

$-2.70\le x\le 3.70$

The maximum amount of cherries Mr Woo can buy is 3 kg.

**3(a)** $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{x^2}+\dfrac{1}{px^2+1}\right)=2x\mathrm{e}^{x^2}-\dfrac{2px}{\left(px^2+1\right)^2}$

**(b)** $\displaystyle\int\frac{2x-1}{x+3}\,\mathrm{d}x=\int 2-\frac{7}{x+3}\,\mathrm{d}x$

$=2x-7\ln(x+3)+C$

**4(i)** $\dfrac{\mathrm{d}y}{\mathrm{d}x}=q-\dfrac{4x}{2x^2+1}$

When $x=1$,

$y=q-\ln 3$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x}=q-\dfrac{4}{3}$

Equation of tangent:

$y=\left(q-\dfrac{4}{3}\right)x+c$

$q-\ln 3=\left(q-\dfrac{4}{3}\right)(1)+c\Rightarrow c=\dfrac{4}{3}-\ln 3$

$\therefore y=\left(q-\dfrac{4}{3}\right)x+\dfrac{4}{3}-\ln 3$

Note:

The bracket in $y=\left(q-\dfrac{4}{3}\right)x+\dfrac{4}{3}-\ln 3$ must be clearly shown.

**(ii)** $\dfrac{\mathrm{d}y}{\mathrm{d}x}=q-\dfrac{4x}{2x^2+1}$

For stationary points,

$\dfrac{\mathrm{d}y}{\mathrm{d}x}=q-\dfrac{4x}{2x^2+1}=0$

$2qx^2+q-4x=0$

Since $C$ has 1 stationary point, $b^2-4ac=0$

$(-4)^2-4(2q)(q)=0$

$16-8q^2=0$

$q^2=2$

$q=-\sqrt{2}$ (NA since $q>0$) or $q=\sqrt{2}$

**(iii)** $y=\sqrt{2}x-\ln(2x^2+1)$

$\dfrac{\mathrm{d}y}{\mathrm{d}x}=\sqrt{2}-\dfrac{4x}{2x^2+1}$

Given that the tangent is parallel to the $x$-axis,

$\dfrac{\mathrm{d}y}{\mathrm{d}x}=\sqrt{2}-\dfrac{4x}{2x^2+1}=0$

$2\sqrt{2}x^2+\sqrt{2}-4x=0$

Using GC, $x=0.70711$ or $\dfrac{1}{\sqrt{2}}$.

Hence, the equation of the tangent is $y=1-\ln 2$ or $y=0.307$.

**5.**

$$\left(\sqrt{x}+\frac{2k}{\sqrt{x}}\right)^2=13k-x$$

$$x+\frac{4k^2}{x}+4k=13k-x$$

$$x^2+4k^2+4kx=13kx-x^2$$

$$2x^2-9kx+4k^2=0$$

$$(2x-k)(x-4k)=0$$

$$x=\frac{1}{2}k \quad \text{or} \quad x=4k \text{ (shown)}$$

Note:
Factorisation
must be shown

Alternative:

$$x=\frac{-(-9k)\pm\sqrt{(-9k)^2-4(2)(4k^2)}}{2(2)}$$

$$=\frac{9k\pm\sqrt{49k^2}}{4}$$

$$=\frac{9k\pm 7k}{4}$$

$$=\frac{16k}{4} \text{ or } \frac{2k}{4}$$

$$=4k \text{ or } \frac{k}{2} \text{ (shown)}$$

$$\text{Required area } =\int_k^{4k}\left(\sqrt{x}+\frac{2k}{\sqrt{x}}\right)^2 \mathrm{d}x+\int_{4k}^{13k}(13k-x)\,\mathrm{d}x$$

$$=\int_k^{4k}\left(x+\frac{4k^2}{x}+4k\right)\mathrm{d}x+\text{Area of triangle}$$

$$=\left[\frac{x^2}{2}+4k^2\ln x+4kx\right]_k^{4k}+\frac{1}{2}(9k)(9k)$$

$$=\left(8k^2+4k^2\ln 4k+16k^2\right)-\left(\frac{1}{2}k^2+4k^2\ln k+4k^2\right)+\frac{81}{2}k^2$$

$$=60k^2+4k^2\ln 4$$

$$=60k^2+8k^2\ln 2$$

**6(i)**

$$7=8(1-\mathrm{e}^{-3k})$$

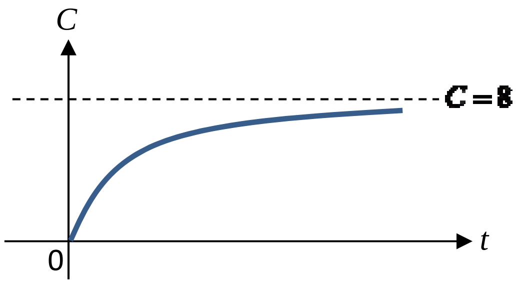
$$\frac{7}{8}=1-\mathrm{e}^{-3k}$$

$$\mathrm{e}^{-3k}=\frac{1}{8}$$

$$-3k=\ln\frac{1}{8}=-3\ln 2$$

$$k=\ln 2$$

**(ii)**




Note:
Curve is in the first
quadrant only

**(iii)** $C = 8(1-\mathrm{e}^{-kt}) = 8 - 8\mathrm{e}^{-t\ln 2}$

$\dfrac{\mathrm{d}C}{\mathrm{d}t} = 8\ln 2\mathrm{e}^{-t\ln 2}$

$\left.\dfrac{\mathrm{d}C}{\mathrm{d}t}\right|_{t=2} = 8\ln 2\mathrm{e}^{-2\ln 2}$

$= 8\ln 2\mathrm{e}^{\ln\frac{1}{4}}$

$= 8\ln 2\left(\dfrac{1}{4}\right)$

$= 2\ln 2$

This value indicates that the number of customers is **increasing** at a rate of 2ln2 **thousands per year at the end of the second year of operation.**

**(iv)** $\dfrac{\mathrm{d}C}{\mathrm{d}t} = -0.1t + 0.7$

$-0.1t + 0.7 = 0$

$t = 7$

$C = -0.05(7)^2 + 0.7(7) + 5.475 = 7.925$

Hence, the maximum number of customers is 7.925 **thousands** customers or 7925 when $t = 7$.

**7(i)** Case 1: First and last seats occupied by males

Number of ways = $4 \times 3 \times 8!$ = 483 840 ways

OR Number of ways = ${}^4C_2 \times 2! \times 8!$ = 483 840 ways

Case 2: First and last seats occupied by females

Number of ways = $6 \times 5 \times 8!$ = 1209 600 ways

OR Number of ways = ${}^6C_2 \times 2! \times 8!$ = 1209 600 ways

$\therefore$ Total number of ways $= 483\ 840 + 1209\ 600 = 1693\ 440$ ways

**(ii)** Number of ways = $10! =$ 36 288 00 ways

**8(i)** Given $X$ denotes the number of prizes being won out of 80 games on a particular day.

Then $X \sim \mathrm{B}(80, 0.1)$

$\mathrm{P}(X > 5) = 1 - \mathrm{P}(X \le 5) \approx 0.82308 \approx 0.823$

**(ii)** $\mathrm{P}(X \le 5) \approx 0.17692$

Let $Y$ denotes the number of days with at most 5 prizes being won each day out of 10 days. Then $Y \sim \mathrm{B}(10,\ 0.17692)$

Note: For $p$ to be in 5 sf

$\mathrm{P}(Y = 4) \approx 0.063970 \approx 0.0640$

**(iii)** E($X$) = $80 \times 0.1 = 8$
Var($X$) = $80 \times 0.1 \times 0.9 = 7.2$

Since $n$ is large, by Central Limit Theorem, $\overline{X} \sim N(8, \frac{7.2}{n})$ approximately.

$P(\overline{X} > 8.5) < 0.1$

$$\Rightarrow P\left(Z > \frac{8.5-8}{\sqrt{\frac{7.2}{n}}}\right) < 0.1$$

$$\Rightarrow 1 - P\left(Z \le \frac{0.5}{\sqrt{\frac{7.2}{n}}}\right) < 0.1$$

$$\Rightarrow P\left(Z \le \frac{0.5}{\sqrt{\frac{7.2}{n}}}\right) > 0.9$$

Take note on the change in the inequality

$$\Rightarrow \frac{0.5}{\sqrt{\frac{7.2}{n}}} > 1.2816$$

Solving, $n > 47.300$ (5 s.f.)

The minimum value of $n$ is 48.

<u>Alternative (Using table)</u>
Using GC,

When $n$ = 47, $P(\overline{X} > 8.5) \approx 0.10072 > 0.1$

When $n$ = 48, $P(\overline{X} > 8.5) \approx 0.09835 < 0.1$

When $n$ = 49, $P(\overline{X} > 8.5) \approx 0.09605 < 0.1$

The minimum value of $n$ is 48.

**9(i)** The phrase 'large number' in the first sentence is required in order to assume that the probability of a rotten apple is approximately constant at 0.15.

**(ii)** Let $X$ denote the number of rotten apples in a random sample of 20 apples.
Then $X \sim B(20, 0.15)$
$P(1 \le X \le 3) = P(X \le 3) - P(X = 0) \approx 0.60897 \approx 0.609$
Or
$P(1 \le X \le 3) = P(X = 1) + P(X = 2) + P(X = 3) \approx 0.60897 \approx 0.609$

**(iii)** Let $Y$ denote the number of rotten apples in a random sample of 10 apples.
Then $Y \sim \mathrm{B}(10, 0.15)$
P(A randomly chosen box is chosen for export)
$= \mathrm{P}(X = 0) + \mathrm{P}(1 \le X \le 3) \times \mathrm{P}(Y = 0)$
$\approx 0.15865 \approx 0.159$

**(iv)** Required probability =
$$\frac{\text{P(First box is chosen for export)} \times \text{P(One out of the 3 remaining boxes chosen for export)} \times 3}{\text{P(First box chosen for export)}}$$
$\approx\ 0.15865 \times (1-0.15865)^2 \times 3$
$\approx 0.33691 \approx 0.337$

Alternative solution:
Let $W$ denote the number of boxes chosen for export out of 3.
Then $W \sim \mathrm{B}(3,\ 0.15865)$
$\mathrm{P}(W = 1) \approx 0.337$

**10(i)** Unbiased estimate of the population mean, $\bar{x} = \dfrac{3.9}{50} + 14 = 14.078$

Unbiased estimate of the population variance,
$$s^2 = \frac{1}{49}\left[2.7 - \frac{(3.9)^2}{50}\right] \approx 0.048894 \approx 0.0489$$

**(ii)** Possible reasons:
- Keep the recorded values small since they are around 14 hours.
- Give an indication of the variations around the hypothesised mean of 14 hours.

**(iii)** $\mathrm{H}_0 :\ \mu = 14$ vs $\mathrm{H}_1 :\ \mu > 14$

Since $n$ = 50 is large, by Central Limit Theorem, $\overline{X} \sim \mathrm{N}\left(14, \dfrac{0.048894}{50}\right)$ approximately.

Level of significance: 1%
Critical region: $z \ge\ 2.3263$

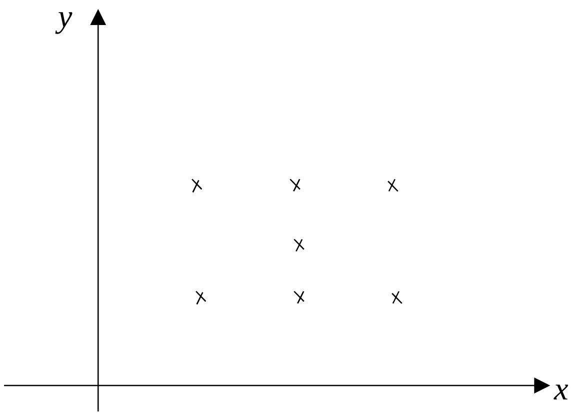
Note: State the distribution of $\overline{X}$

Standardised test statistic $z = \dfrac{14.078-14}{\sqrt{\dfrac{0.048894}{50}}} \approx 2.4943\ > 2.3263$

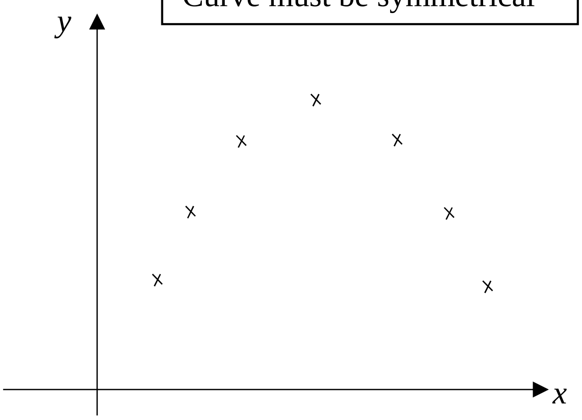
Using GC, $p$-value $\approx 0.0063100\ < 0.01$
Since the $p$-value is smaller than the level of significance, we reject $\mathrm{H}_0$. There is sufficient evidence, at 1% level of significance, to conclude that Henry's claim is invalid.

Alternative conclusion:
Since the standardised test statistic falls inside the critical region, we reject $\mathrm{H}_0$. There is sufficient evidence, at 1% level of significance, to conclude that Henry's claim is invalid.

**11(a)(i)** One possible scatter diagram

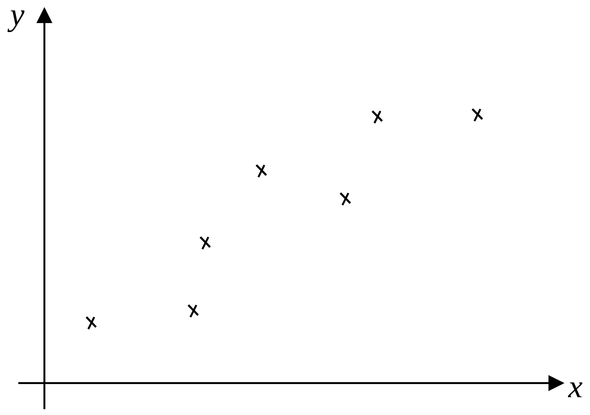



or

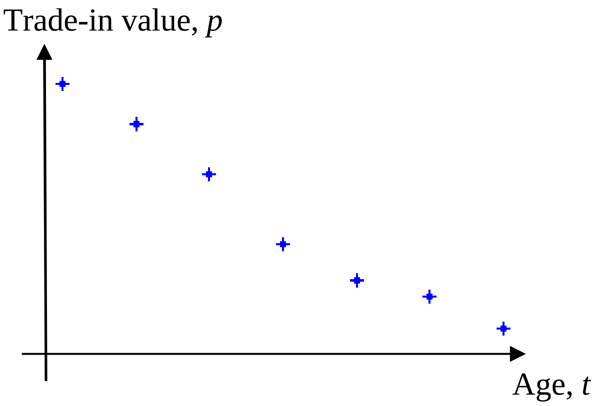



Note:
Curve must be symmetrical

**(ii)** One possible diagram




**(b)(i)**




Note:
Label the axes correctly

**(ii)** $r = -0.986$

There is a strong negative **linear** correlation between the age of the car and the average trade-in value of the car.

**(iii)** $p = -4.0786t + 61.3929$

$p \approx -4.08t + 61.39$ (to 2 d.p.)

**(iv)** Let the mean trade-in value and mean age of the 6 cars be $a$ and $b$ respectively.

$p = 61.45 - 4.19t \Rightarrow p + 4.19t = 61.45$
$t = 14.39 - 0.23p \Rightarrow 0.23p + t = 14.39$
Solving using GC, $a = 31.843$ and $b = 7.0661$

Mean age for 13 cars
$= \dfrac{2+3+4+5+6+7+8+6(7.0661)}{13}$
$\approx 5.95$ (3 s.f)

Mean age for 13 cars is 5.95

Mean trade-in value for 13 cars
$= \dfrac{54+50.1+45.3+38.6+35.1+33.5+30.4+6(31.843)}{13}$
$= 36.8$
Mean trade-in value for 13 cars is 36.8 thousand dollars (or $36800).

**12(i)** P($T$) = $\dfrac{156+y}{287+x+y}$ , P($V$) = $\dfrac{180}{287+x+y}$ , P($T \cap V$) = $\dfrac{102}{287+x+y}$
Given that $T$ and $V$ are independent, this means that
P($T \cap V$) = P($T$) × P($V$)
$\Rightarrow \dfrac{102}{287+x+y} = \dfrac{156+y}{287+x+y} \times \dfrac{180}{287+x+y}$
$\Rightarrow$ 102 (287 + $x$ + $y$) = 180 (156 + $y$)
$\Rightarrow 29\ 274 + 102x + 102y = 28\ 080 + 180y$
$\Rightarrow 1194 + 102x - 78y = 0$
$\Rightarrow -39y + 51x + 597 = 0$
$\Rightarrow 39y - 51x = 597$
$\therefore 13y - 17x = 199$ (Shown) ----- (1)

**(ii)** P($T \cup B$) = $\dfrac{379}{450}$
$\Rightarrow \dfrac{216+x+y}{287+x+y} = \dfrac{379}{450}$
$\Rightarrow 450(216+x+y) = 379(287+x+y)$
$\Rightarrow 97\ 200 + 450x + 450y = 108\ 773 + 379x + 379y$
$\Rightarrow 71x + 71y = 11\ 573$
$\therefore x + y = 163$ -----(2)

Using GC, solving (1) and (2) simultaneously, $x = 64$ and $y = 99$

**(iii)** P($B \cap (T \cup V)$) = $\dfrac{164}{450} = \dfrac{82}{225}$

**(iv)** $\mathrm{P}(T \mid V) = \dfrac{\mathrm{n}(T \cap V)}{\mathrm{n}(V)} = \dfrac{102}{180} = \dfrac{17}{30}$

**(v)** Required probability = $\dfrac{166}{450} \times \dfrac{99}{449} \times \dfrac{53}{448} \times 3! \approx 0.0577$

**13(i)** Let $C$ denote the mass of cod fish. Then $C \sim \mathrm{N}(a,\ 0.1^2)$.

$\mathrm{P}(C < 0.2 + a) = \mathrm{P}\left(Z < \dfrac{(0.2+a)-a}{0.1}\right) = \mathrm{P}(Z < 2) \approx 0.97725 \approx 0.977$

**(ii)** Given that $\mathrm{P}(C \geq 0.5) = 0.2$

$\Rightarrow \mathrm{P}(C < 0.5) = 0.8$

$\Rightarrow \mathrm{P}\left(Z < \dfrac{0.5-a}{0.1}\right) = 0.8$

$\Rightarrow \dfrac{0.5-a}{0.1} \approx 0.84162$

$\therefore\ a \approx 0.416$

**(iii)** Using $a = 0.4$,

Consider $T = C_1 + C_2 + C_3 + C_4 - 2S \sim \mathrm{N}(0.4,\ 0.13)$

$\mathrm{P}(-0.1 < T < 0.1) \approx 0.11993 \approx 0.120$

**(iv)** Using $a = 0.4$,

Let $68C$ be the selling price of cod fish. Then $68C \sim \mathrm{N}(27.2,\ 46.24)$

Let $30S$ be the selling price of salmon fish. Then $30S \sim \mathrm{N}(18,\ 20.25)$

$\mathrm{P}(68C > 25) \times \mathrm{P}(30S > 15) \approx 0.46858 \approx 0.469$

Alternative

$\mathrm{P}\left(C > \dfrac{25}{68}\right) \times \mathrm{P}\left(S > \dfrac{15}{30}\right) \approx 0.46858 \approx 0.469$

**(v)** The mass of all the fish are independent of one another.