2017 H1 Prelim Solutions

1	Let the time he spend making calls within Canada, the Unites States and Mexico be x, y and z (in min). $0.28x+0.30y+0.84z=\$90$ $2y=z$ $x+z=120$ $x=40.56,\ y=39.71,\ z=79.44$ Total bill by Sunhub $=0.38(x-10)+0.4(y-10)+0.94(z-10)$ $=88.77$ Sunhub is cheaper.
2	$\int_{-1}^{1}\left(e-e^{-x}\right)^2\mathrm{d}x=\int_{-1}^{1}e^2-2e^{1-x}+e^{-2x}\mathrm{d}x$ $=\left[e^2x+2e^{1-x}-\frac{1}{2}e^{-2x}\right]_{-1}^{1}$ $=\left(e^2+2-\frac{1}{2}e^{-2}\right)-\left(-e^2+2e^2-\frac{1}{2}e^2\right)$ $=2+\frac{1}{2}\left(e^2-e^{-2}\right)$
3(i)	$A=7800\left(1+\frac{3.2}{100(12)}\right)^{80}=9652.077\approx 9652.08$ 10 000 – 9652.08 = 347.92
(ii)	$10000=7800(1+\frac{3.2}{100})^t$ $\frac{50}{39}=(\frac{129}{125})^t$ $t=\frac{\ln 1.282}{\ln 1.032}=7.888\approx 8$ It will take 8 years.
4	Let h be height of matchbox. $V=x(1.5x)h=30\Rightarrow h=\frac{30}{1.5x^2}=\frac{20}{x^2}$ $A=3xy+2xh+4yh$ $=3x(1.5x)+2xh+4(1.5x)h$ $=4.5x^2+8xh$ $=4.5x^2+8x(\frac{20}{x^2})$ $=4.5x^2+\frac{160}{x}$  $\frac{\mathrm{d}A}{\mathrm{d}x}=9x-\frac{160}{x^2}=0$ $x^3=\frac{160}{9}\Rightarrow x=\sqrt[3]{\frac{160}{9}}\approx 2.6099$ $y=1.5\times\sqrt[3]{\frac{160}{9}}=\sqrt[3]{\frac{27}{8}}\times\sqrt[3]{\frac{160}{9}}=\sqrt[3]{60}$

	$h=\frac{20}{\left(\sqrt[3]{\frac{160}{9}}\right)^2}=20\left(\frac{9}{160}\right)^{\frac{2}{3}}$ $\frac{d^2A}{dx^2}=9+\frac{320}{x^3}>0$ for all x $\therefore$ Min A

x	2.4	2.61	2.8
$\frac{dA}{dx}$	-6.1778	0	4.7918

5(i)

$$\frac{d}{dx}\left[x-\ln(x^2+1)\right]=1-\frac{2x}{x^2+1}$$

$$y=x-\ln(x^2+1)$$

$$\frac{dy}{dx}=0\Rightarrow 1-\frac{2x}{x^2+1}=0$$

$$\frac{2x}{x^2+1}=1$$

$$x^2-2x+1=0$$

$$(x-1)^2=0$$

$$\therefore x=1,\ y=1-\ln 2$$

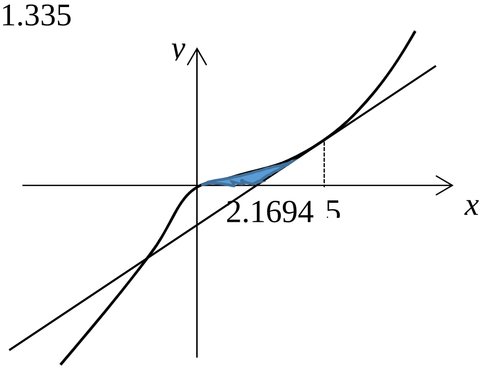
x	1^- 0.5(0.9)	1	1^+ 1.5(1.1)
$\frac{dy}{dx}$	0.2(0.0055)	0	0.0769(0.0045)

$\therefore$(1, 1 – ln2) is a stationary point of inflexion.

(ii)

Equation of tangent at x = 5 is

$y=0.61538x-1.335$



when y = 0, x = 2.1694

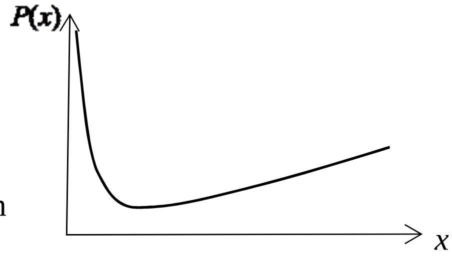
when x = 5, y = 1.7419

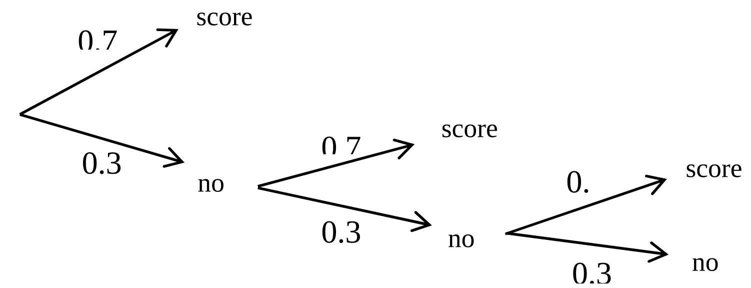
$\int_0^5 y\ dx-$ Area of triangle

$=\int_0^5[x-\ln(x^2+1)]\ dx-0.5(5-2.1694)(1.7419)$

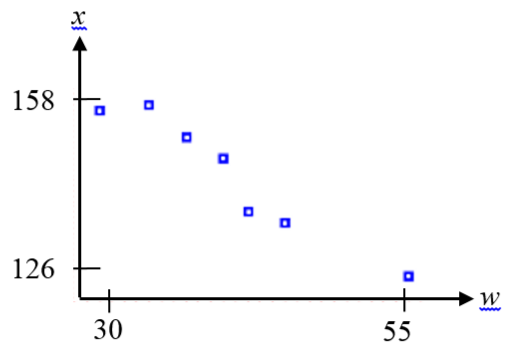
$=3.4627-2.4653$

$=0.99739\approx 0.997$ (3 s.f.)

6(a)	$P'(x) = -\frac{2500}{x^2} + \frac{2}{3} = 0$ $x^2 = 3750$ $\therefore x = 25\sqrt{6}$ Min point (61.237, 81.65) Optimal speed = 81.7 km/h  When $P(x) \le 90,\ 39.1 \le x \le 95.9$.
6(b)	$\frac{\mathrm{d}V}{\mathrm{d}t} = 0.15\pi\sqrt{\pi t + 1}$ $V = \int 0.15\pi\sqrt{\pi t+1}\ \mathrm{d}t = \frac{0.15\pi}{1.5\pi}\left(\pi t+1\right)^{1.5} + C = 0.1\left(\pi t+1\right)^{1.5} + C$ When t = 0, V = 0 $\therefore C = -0.1$ $\therefore V = 0.1\left(60\pi+1\right)^{1.5} - 0.1 = 260.755$ OR $V = \int_0^{60} 0.15\pi\sqrt{\pi t+1}\ \mathrm{d}t = 260.755$ fr GC $\therefore$ After 1 min, amount of fuel in the tank is 260.755 litres. Vol of fuel = 260.755 $\Rightarrow \pi h^3 = 260.755\ \therefore h = 4.36208$ $V = \pi(h)^2 h = \pi h^3$ $\frac{\mathrm{d}V}{\mathrm{d}h} = 3\pi h^2$ $\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\mathrm{d}h}{\mathrm{d}V} \times \frac{\mathrm{d}V}{\mathrm{d}t}$ $= \frac{1}{3\pi h^2} \times 0.15\pi\sqrt{\pi(60)+1}$ when $t = 60$ $= 0.036173 \approx 0.0362$ cm/s
7	Let M be the mass of a randomly chosen Black Pearl durian, and S be the cost of a randomly chosen Black Pearl durian. $S = 21M$ $M \sim N(\mu,\ \sigma^2)$ $S \sim N\left(21\mu,\ (21\sigma)^2\right)$ $P(M > 2) = 0.322$ $P(M < 2) = 0.678$ $P(Z < \frac{2-\mu}{\sigma}) = 0.678$ $\frac{2-\mu}{\sigma} = 0.46211$

	$P(S<20)=0.215$ $P(Z<\frac{20-21\mu}{21\sigma})=0.215$ $\frac{20-21\mu}{21\sigma}=-0.78919$ Therefore we have two simultaneous equations: $\mu+0.46211\sigma=2$ ----(1) $21\mu-16.573\sigma=20$ -----(2) Solving (1) and (2) $\mu=1.6131=1.61$ (3sf) $\sigma=0.83722=0.837$ (3sf)
8 (a) (b) (c)	 0.3×0.7 $\frac{0.3\times 0.7}{1-\mathrm{P(not\ scoring)}}=\frac{0.3\times 0.7}{1-0.3^3}$ or $\frac{0.3\times 0.7}{0.3+0.3\times 0.7+0.3^2\times 0.7}=\frac{0.21}{0.973}=\frac{30}{139}$ $3!\times 0.7\times 0.21\times(1-0.7-0.21)=0.07938\approx 0.0794$
9	${}^{18}C_7=31\,824$ ${}^{2}C_1\,{}^{16}C_6=2\times 8\ 008=16\ 016$ Probability = P(3 S + 2 B + 2 T) + P(2 S + 3 B + 2 T) + P(2 S + 2 B + 3 T) $=\frac{{}^8C_3\,{}^5C_2\,{}^5C_2+{}^8C_2\,{}^5C_3\,{}^5C_2+{}^8C_2\,{}^5C_2\,{}^5C_3}{{}^{18}C_7}=\frac{11200}{31\,824}=\frac{700}{1989}\approx 0.352$ (3 s.f.)
10(i)	Let L be the mass of a randomly chosen leather jacket steak. Let T be the mass of a randomly chosen tuna steak. Let A be the average mass of 1 randomly chosen leather jacket and 2 randomly chosen tuna steak. $A=\frac{L+T_1+T_2}{3}$ $\mathrm{E}(A)=E\left(\frac{L+T_1+T_2}{3}\right)$ $=\frac{1}{3}[\mathrm{E}(L)+2\mathrm{E}(T)]=\frac{1}{3}(1.6+2\times 1.2)=\frac{4}{3}$

	$\text{Var}(A) = \text{Var}\left(\dfrac{L_1 + T_1 + T_2}{3}\right)$ $= \dfrac{1}{3^2}\left[\text{Var}(L) + 2\text{Var}(T)\right]$ $= \dfrac{1}{9}\left(0.2^2 + 2 \times 0.3^2\right) = \dfrac{11}{450}$ $A \sim N\left(\dfrac{4}{3},\ \dfrac{11}{450}\right)$ $P(A < 1.5) = 0.857$ (3sf)
(ii)	Let X be the cost of a randomly chosen tuna steak, and Y be the cost of a randomly chosen leather jacket steak. $X = 17T$ $Y = 10L$ $X \sim N\left(17 \times 1.2,\ (17 \times 0.3)^2\right) = \left(20.4,\ 26.01\right)$ $Y \sim N\left(10 \times 1.6,\ (10 \times 0.2)^2\right) = \left(16, 4\right)$ $P(X > 22.50) \times P(Y > 16.50)$ = (0.34026)(0.40130) = 0.13654 = 0.137 (3sf)
(iii)	$X + Y \sim N\left(20.4 + 16,\ 26.01 + 4\right) = \left(36.4,\ 30.01\right)$ $P(X + Y > 39) = 0.31753 = 0.318$ (3sf)
(iv)	because the event in (ii) is a subset of the event in (iii), <u>i.e., a tuna steak cost \$24 and a leather jacket steak cost \$16 is not included in (ii) but is included in (iii).</u>
11	Let p be the probability of obtaining a red in a throw. Then $X \sim \text{B}(n, p)$ $\text{E}(X) = np = \dfrac{40}{7}$ and $\text{Var}(X) = np(1-p) = \dfrac{200}{49}$ $\Rightarrow \dfrac{40}{7}(1-p) = \dfrac{200}{49}$ $1 - p = \dfrac{5}{7} \Rightarrow p = \dfrac{2}{7}$ $\therefore np = \dfrac{40}{7}$ $n = \dfrac{40}{7} \times \dfrac{7}{2} = 20$ $P(X > 5) = 1 - P(X \le 5) \approx 0.528$ (3 s.f.)
	Let $\overline{X}$ denotes the mean number of reds obtained per person. $\overline{X} = \dfrac{X_1 + X_2 + \ldots + X_N}{N}$

	Since N is large, $\overline{X} \sim N\left(\frac{40}{7}, \frac{200}{49N}\right)$ approximately by Central limit Theorem. Let $Z = \frac{\overline{X} - \frac{40}{7}}{\sqrt{\frac{200}{49N}}} \sim N(0, 1)$ $\mathrm{P}(\overline{X} < 6) > 0.9$ $\mathrm{P}\left(Z \le \frac{6 - \frac{40}{7}}{\sqrt{\frac{200}{49N}}}\right) > 0.9$ <u>Method 1:</u> $\frac{6 - \frac{40}{7}}{\sqrt{\frac{200}{49N}}} > 1.28155$ $N > \frac{200}{49}\left(\frac{1.28155}{6 - \frac{40}{7}}\right)^2$ $N > 82.11$ The least value of N is 83. **OR**
	Method 2: From GC: $N = 82, \quad \mathrm{P}(\overline{X} < 6) = 0.8998$ $N = 83, \quad \mathrm{P}(\overline{X} < 6) = 0.9012$ The least value of N is 83
12(i)	
(ii)	$r = -0.962$ (3s.f.) There is a <u>strong negative linear relationship</u> between the weight of a boy and the distance of the standing board jump that he can make, i.e., as the weight of a boy increases, the distance of his standing board jump decreases.
(iii)	From G.C., $x = 202.23 - 1.4156w$ (5 s.f.) Meaning of b:

	$b = -1.4156$ means that 1 unit (i.e., kg) increase in the weight of a boy (w) will mean a decrease of 1.4156 units (i.e., cm) in the distance of his standing board jump (x).
(iv)	When $w = 35$ kg, $x = 153$cm (3sf) Since the value of r is close to 1 and w is within the range of data, i.e., $30 \le w \le 55$, the estimate is reliable. When $w = 15$ kg, $x = 181$cm (3sf) Since w is outside the given range of data, i.e., $30 \le w \le 55$, the linear relation may no longer hold, therefore the estimate is unreliable.
(v)	Let the distance of Aaron's standing board jump be d. $\bar{w} = \frac{322}{8} = 40.25$ Since $(\bar{w}, \bar{x})$ lies on the regression line, sub $\bar{w}$ to get $\bar{x}$ $\bar{x} = 202.98 - 1.4249(40.25)$ $\bar{x} = 145.6$ $\bar{x} = \frac{1015 + d}{8}$ $145.6 = \frac{1015 + d}{8}$ $d = 150$ cm (3s.f.)
13 (a) (i)	Unbiased estimate of the population mean, $\bar{x} = 35 + \frac{\sum(x-35)}{80} = 35 + \frac{-40}{80} = 34.5$ Unbiased estimate of the population variance, $s^2 = \frac{1}{80-1}\left[\sum(x-35)^2 - \frac{\left(\sum(x-35)\right)^2}{80}\right]$ $= \frac{1}{79}\left[950 - \frac{(-40)^2}{80}\right] = 11.772 = 11.8$ (3sf)
(ii)	"Keeping the recorded values small since they are around 35 ml" or "Giving an indication of the variations around the hypothesized mean of 35 ml".
(iii)	To test H$_0$: $\mu = 35$ against H$_1$: $\mu < 35$ 1-tail Z-test at the 10% significance level. Since $n = 80$ is large, by Central Limit Theorem, Under Ho, $\bar{X} \sim N\left(35, \frac{11.772}{80}\right)$ approximately. From GC, p-value = 0.096213

	Since p-value < 0.10, reject H_0 We conclude that there is <u>sufficient evidence at the 10% level of significance</u> that average cup of single-shot expresso coffee is less than 35 ml, the manager's claim is not valid.
(b)	To test H_0: $\mu = 35$ against H_1: $\mu \neq 35$ 2-tail Z-test at the 10% significance level. Since n = 80 is large, by Central Limit Theorem, Under Ho, $\bar{X} \sim N\left(35, \frac{10.1}{80}\right)$ approximately. Since the product designer's claim is valid, H_0 is not rejected, $-1.6449 < \dfrac{\bar{x} - 35}{\sqrt{\frac{10.1}{80}}} < 1.6449$ Range of sample mean $34.416 \leq \bar{x} \leq 35.584$ (3 dp)