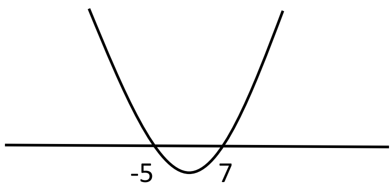
2017 SAJC Preliminary Examination Solutions

1	$kx^2+(k-1)x+\dfrac{9}{k}>0$ Note that $k \neq 0$. For the quadratic expression to be positive, the following 3 conditions must be met: (1) Discriminant <0 and (2) $k>0$ $(k-1)^2-4(k)\left(\dfrac{9}{k}\right)<0$ $k^2-2k+1-36<0$ $k^2-2k-35<0$ $(k+5)(k-7)<0$  $-5<k<7$ Combining (1) and (2), the set of values of k is $\left\{k : k \in \mathbb{R}, 0<k<7\right\}$
2	$5^{x+2}=5^3(5^{y-5})$ $x+2=3+y-5$ $x-y=-4$ $x=y-4$ ---------------------- (1)

	$\log_{\sqrt{3}}(2x-y)-2\log_{\sqrt{3}}2=4$ $\log_{\sqrt{3}}(2x-y)-\log_{\sqrt{3}}4=4$ $\log_{\sqrt{3}}\frac{(2x-y)}{4}=4$ $\frac{(2x-y)}{4}=3^{\frac{4}{2}}$ $2x-y=4\times 9$ $2x-y=36------------(2)$ Sub. (1) into (2) $2(y-4)-y=36$ $y=44$ $x=40$
3	(a) (i) Let $y=\frac{1}{\sqrt{x^2-3x+1}}=\left(x^2-3x+1\right)^{-\frac{1}{2}}$ $\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{1}{2}\left(x^2-3x+1\right)^{-\frac{3}{2}}\left(2x-3\right)$ $=-\frac{2x-3}{2\sqrt{\left(x^2-3x+1\right)^3}}$ (ii) Let $y=\ln\left(\frac{x^2+3x+2}{x^2+4x+3}\right)=\ln\frac{(x+1)(x+2)}{(x+1)(x+3)}=\ln\frac{x+2}{x+3}$ $=\ln\left(x+2\right)-\ln\left(x+3\right)$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{x+2}-\frac{1}{x+3}$ $=\frac{1}{(x+2)(x+3)}$ Alternatively,

$$y=\ln\left(\frac{x^2+3x+2}{x^2+4x+3}\right)=\ln(x^2+3x+2)-\ln(x^2+4x+3)$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2x+3}{x^2+3x+2}-\frac{2x+4}{x^2+4x+3}$$

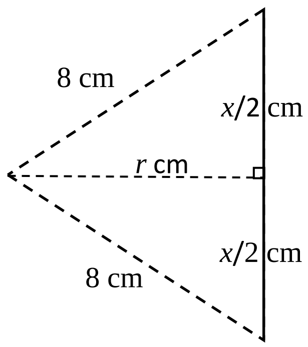
$$=\frac{2x+3}{(x+1)(x+2)}-\frac{2x+4}{(x+1)(x+3)}$$

$$=\frac{(2x+3)(x+3)-(2x+4)(x+2)}{(x+1)(x+2)(x+3)}$$

$$=\frac{x+1}{(x+1)(x+2)(x+3)}$$

$$=\frac{1}{(x+2)(x+3)}$$

(b)(i) By Pythagoras's Theorem,



the radius of the cylinder is $r=\sqrt{8^2-\left(\frac{x}{2}\right)^2}$

$$=\sqrt{64-\frac{x^2}{4}}=\sqrt{\frac{256-x^2}{4}}=\frac{\sqrt{256-x^2}}{2}$$

Volume of cylinder is

$$V=\pi\left(\sqrt{\frac{256-x^2}{4}}\right)^2(x)$$

$$=\frac{\pi x\left(256-x^2\right)}{4}$$

$$=\frac{\pi}{4}\left(256x-x^3\right)$$

(ii) $V=\frac{\pi}{4}\left(256x-x^3\right)$

$$\frac{dV}{dx}=\frac{\pi}{4}\left(256-3x^2\right)$$

When $\dfrac{dV}{dx}=0$, $x^2=\dfrac{256}{3}$

Since the height of the cylinder, x, is positive,

$$x=\sqrt{\frac{256}{3}} \quad (= 9.2376)$$

$$\frac{d^2V}{dx^2}=\frac{\pi}{4}(-6x)= -43.5<0$$ for positive value of x.

Alternative Method to Check for Maximum

x	9.2	$x=\sqrt{\frac{256}{3}}$	9.3
$\dfrac{dV}{dx}$	+1.63	0	–5.67
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Hence the volume V is maximum when $x=\sqrt{\dfrac{256}{3}}$ cm

At Max $V=\dfrac{\pi}{4}\left(256-x^2\right)x$

$$=\frac{\pi}{4}\left(256-\frac{256}{3}\right)\sqrt{\frac{256}{3}}$$

$$=\frac{\pi}{4}\left(\frac{512}{3}\right)\sqrt{\frac{256}{3}}$$

$$=\frac{\pi}{4}\left(\frac{512}{3}\right)\left(\frac{16\sqrt{3}}{3}\right)$$

$$=\frac{2048\sqrt{3}}{9}\pi \text{ cm}^3$$

4	(a)

	$\frac{\mathrm{d}}{\mathrm{d}x}\left(5e^{2x}+1\right)^4$ $=4\left(5e^{2x}+1\right)^3\left(10e^{2x}\right)$ $=40e^{2x}\left(5e^{2x}+1\right)^3$ $\int_0^2 e^{2x}\left(5e^{2x}+1\right)^3 dx=\frac{1}{40}\int_0^2 40e^{2x}\left(5e^{2x}+1\right)^3 dx$ From part (i) $\int_0^2 e^{2x}\left(5e^{2x}+1\right)^3 dx=\frac{1}{40}\left[\left(5e^{2x}+1\right)^4\right]_0^2$ $=\frac{1}{40}\left[\left(5e^4+1\right)^4-\left(5e^0+1\right)^4\right]$ $=\frac{1}{40}\left[\left(5e^4+1\right)^4-\left(5e^0+1\right)^4\right]$ $=\frac{1}{40}\left[\left(5e^4+1\right)^4-1296\right]$ (b) $\int_{-1}^{0}\frac{1}{x+2}\,\mathrm{d}x-(0.5)=\int_{1}^{p}\frac{1}{x+2}\,\mathrm{d}x$ $\left[\ln\|x+2\|\right]_{-1}^{0}-(0.5)=\left[\ln\|x+2\|\right]_{1}^{p}$ $(\ln 2-\ln 1)-0.5=\ln\|p+2\|-\ln 3$ $\ln(p+2)=\ln 2+\ln 3-0.5$ since $p+2>0$ $(p+2)=e^{\ln 6-0.5}=6e^{-0.5}$ $p=6e^{-0.5}-2$
5	(i) $P=500\left(3+\mathrm{e}^{-0.2t}\right)$ When $t=0$, $P=500(3+1)$ $P=2000$ (ii) $P=500\left(3+\mathrm{e}^{-0.2(24)}\right)$ $P=1504$ monkeys

(iii)

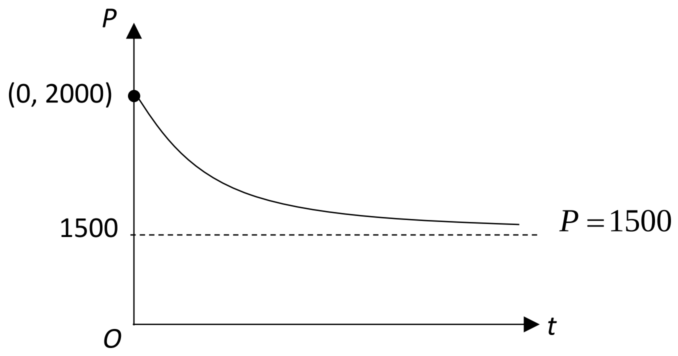
$$500\left(3+e^{-0.2t}\right)<1515$$
$$3+e^{-0.2t}<3.03$$
$$e^{-0.2t}<0.03$$
$$-0.2t<\ln\left(0.03\right)$$
$$t>\frac{\ln\left(0.03\right)}{-0.2}$$
$$t>17.533$$
$$t\approx 18$$

The population of the monkeys will first drop below 1515 after 18 complete months.

(iv) As $t\to\infty$, $P\to 500\left(3+0\right)=1500$.

The population of the monkeys will decrease and approach/stabilise at 1500.

(v)



(vi) No. The model may not be accurate in the real world.

In the real world, population of the monkeys may not decrease due to the opening of the factory alone and the monkeys may not stabilise at 1500 due to human interference. (or any possible logical solutions)

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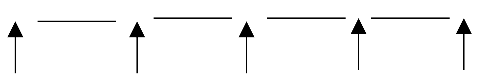
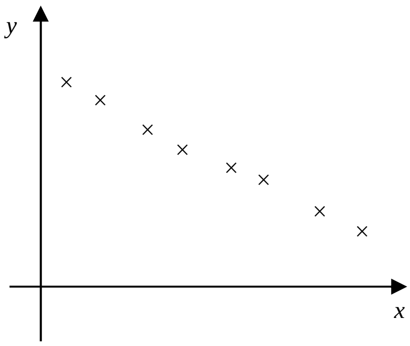
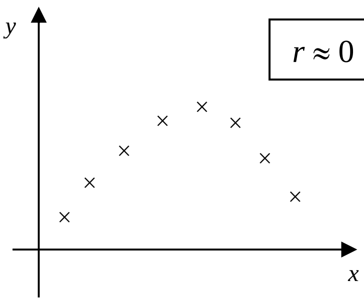
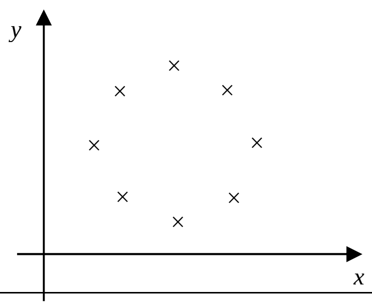
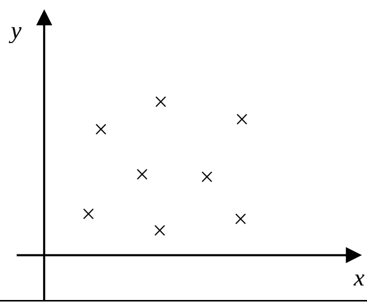
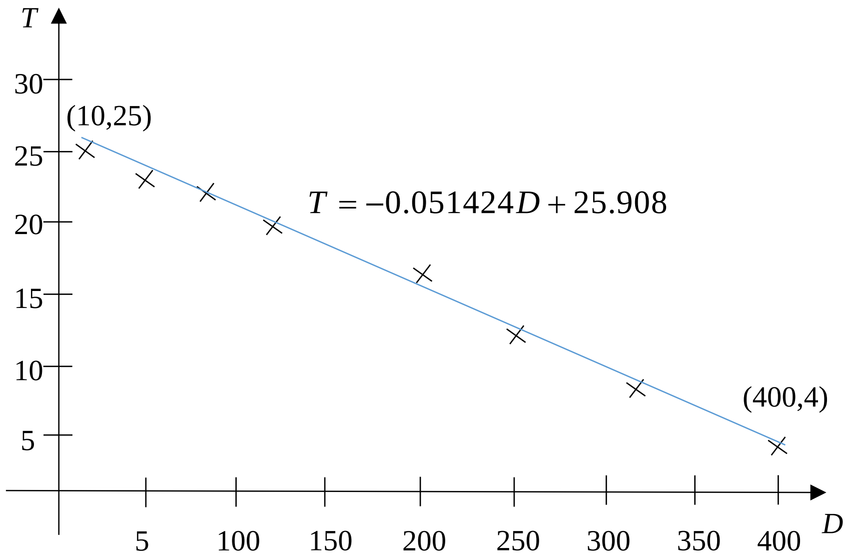
(i) Probability (tiles are arranged in correct order to spell his name) = $\frac{1}{6!}=\frac{1}{720}$

(ii)



4! ways to arrange consonants.

	5 possible slots to insert A & O P(the vowels are separated) = $\dfrac{4!\,{}^5P_2}{6!} = \dfrac{2}{3}$ (iii) 4! Ways to arrange consonants. 2! Ways to arrange A&O $\boxed{\text{A}}\ \boxed{\text{T, H, M, S}}\ \boxed{\text{O}}$ P(the vowels are at the two ends) = $\dfrac{4!2!}{6!} = \dfrac{1}{15}$
7	 $\boxed{r \approx -0.9}$  $\boxed{r \approx 0}$  

	Using $\overline{T} = -0.051424\overline{D} + 25.908$, substitute $\overline{D} = \frac{1450}{8} = 181.25, \quad \overline{T} = \frac{k+113}{8}$ $\Rightarrow \frac{k+113}{8} = -0.051424(181.25) + 25.908$ $\Rightarrow k = 19.6992 = 19.7$ (shown)  $r = -0.992$. It indicates a strong negative linear correlation between the depth of water and the temperature of water. As the depth of water increases, the temperature of water decreases. Regression line T on D: $T = -0.051424D + 25.908$ $T = -0.0514D + 25.9$ (3 s.f.) When D = 550, $T = -0.051424(550) + 25.908$ $T = -2.38$ Although $r = -0.992$ (3 s.f.) is close to -1, but the estimate may not be reliable as D = 550 falls outside the data range [10, 400]. Hence, the estimate is unreliable. Regression line T on D where height is given in kilometres: $T = -51.424D + 25.908$

$T = -51.4D + 25.9$ (3 s.f.)

The value of the regression coefficient is -0.000051424.

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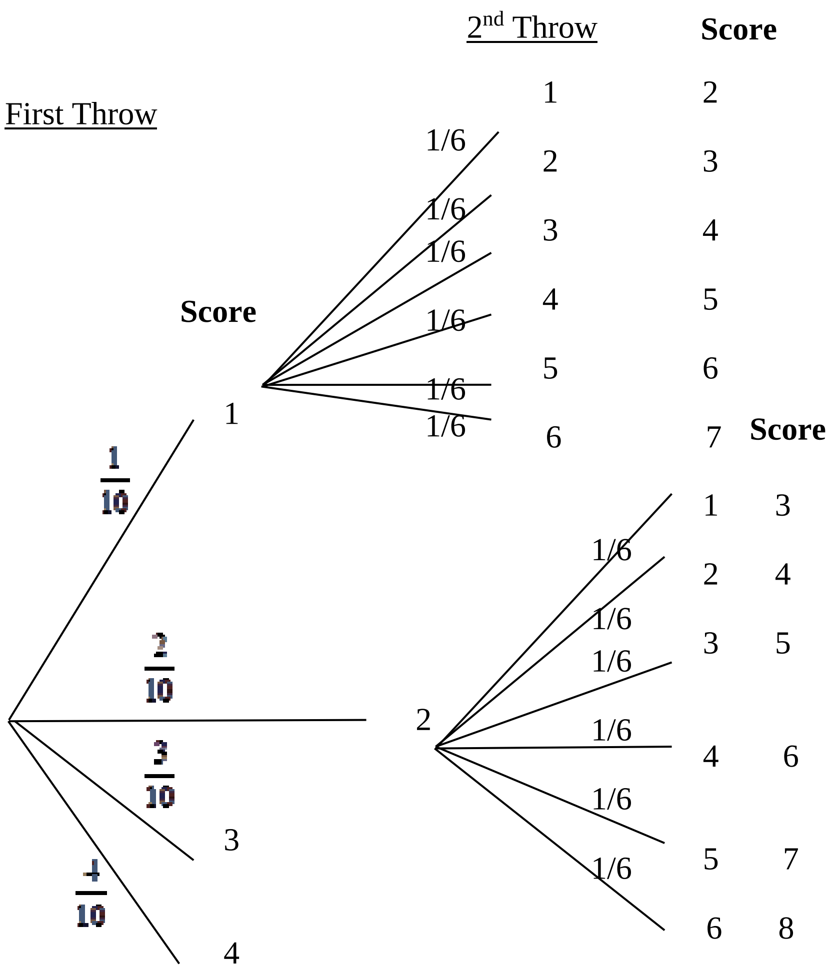

Let X be the score of the first throw and Y be the score of the second throw

$$\mathrm{P}(A) = 1 - \mathrm{P}(A') = 1 - \mathrm{P}(\text{James' score is at most a 3})$$
$$= 1 - \mathrm{P}(\text{score is 3 on the 1st throw})$$
$$- \mathrm{P}(X = 1 \cap Y = 1 \text{ or } 2) - \mathrm{P}(X = 2 \cap Y = 1)$$
$$= 1 - \frac{3}{10} - \left(\frac{1}{10} \times \frac{1}{6} \times 2\right) - \left(\frac{2}{10} \times \frac{1}{6}\right) = \frac{19}{30}$$

<u>Alternative Method</u>

$$\mathrm{P}(A) = \mathrm{P}(X{=}4) + \mathrm{P}(X = 1 \cap 3 \le Y \le 6)$$

	$+\ \mathrm{P}(X=2\ \cap\ 2\le Y\ \le 6)$ $=\dfrac{4}{10}+\left(\dfrac{1}{10}\times\dfrac{1}{6}\times 4\right)+\left(\dfrac{2}{10}\times\dfrac{1}{6}\times 5\right)$ $=\dfrac{19}{30}$ $\mathrm{P}(A\mid B')=\dfrac{\mathrm{P}(A\cap B')}{\mathrm{P}(B')}=\dfrac{\text{P(score is at least 4 and even)}}{\text{P(score is even)}}$ $=\dfrac{\mathrm{P}(X=4)+\mathrm{P}(X=1\cap Y=3\text{ or }5)+\mathrm{P}(X=2\cap Y=4\text{ or }6)}{\mathrm{P}(X=4)+\mathrm{P}(X=1\cap Y=1\text{ or }3\text{ or }5)+\mathrm{P}(X=2\cap Y=2\text{ or }4\text{ or }6)}$ $=\dfrac{\dfrac{4}{10}+\left(\dfrac{1}{10}\times\dfrac{1}{6}\times 2\right)+\left(\dfrac{2}{10}\times\dfrac{1}{6}\times 3\right)}{\dfrac{4}{10}+\left(\dfrac{1}{10}\times\dfrac{1}{6}\times 3\right)+\left(\dfrac{2}{10}\times\dfrac{1}{6}\times 3\right)}$ $=\dfrac{8/15}{11/20}$ $=\dfrac{32}{33}$ $\mathrm{P}(A\cap B)$ $\mathrm{P}(X=1\cap Y=4\text{ or }6)+\mathrm{P}(X=2\cap Y=3\text{ or }5)$ $=\left(\dfrac{1}{10}\times\dfrac{1}{6}\times 2\right)+\left(\dfrac{2}{10}\times\dfrac{1}{6}\times 2\right)$ $=\dfrac{1}{10}\neq 0$ Events A and B are not mutually exclusive.
9	(i) Let X and Y be the r.v. denoting the mass of the type A and type B fruits respectively. $X\sim N(1.5, 0.1^2)$

	$Y \sim N(1.8, 0.2^2)$ $X_1 + X_2 + X_3 \sim N(4.5, 0.03)$ $P\left(X_1 + X_2 + X_3 > 5\right) = 0.00195$ (to 3 sf) (ii) Let $T = X_1 + X_2 + X_3 - 2Y$ $E(T) = 4.5 - 3.6 = 0.9$ $Var(T) = 0.03 + 4 \times 0.2^2 = 0.19$ $T \sim N(0.9, 0.19)$ $P\left(\lvert T \rvert \geq 0.5\right)$ $= P(T \leq -0.5) + P(T \geq 0.5)$ $= 0.000659548 + 0.8206023$ = 0.821 (3 sf) (iii) Let $C = 30(X_1 + X_2) + 35(Y)$ $E(C) = 30(1.5 + 1.5) + 35(1.8) = 153$ $Var(C) = 30^2(0.01 + 0.01) + 35^2(0.04) = 67$ $C \sim N(153, 67)$ $P(C > k) = 0.1$ $P(C \leq k) = 0.9$ $k = 163.49 \approx 163$ (3 sig. fig.) (iv) Required Probability $= P(X > 1.3)P(X \leq 1.3) \times 2!$ $= 0.97725 \times 0.02275 \times 2$ = 0.0445 (3 sf)

10	(i) The probability of making a faulty teapot is a constant. The event that a teapot is found faulty is independent of other another teapot. (ii) Let X be the number of faulty teapots in a batch of 30. $X \sim \mathrm{B}(30.0.08)$ $P(X>2)=1-P(X\le 2)=0.435$ (iii) Let Y be the number of batches which contain exactly one faulty teapot each. $P(X=1)=0.21382$ $Y \sim \mathrm{B}(240, 0.21382)$ E(Y) = $240\times 0.21382=51.3$ (iv) Let $\overline{W}$ be the mean number of faulty teapots per batch. Since n = 50 is large, by Central Limit Theorem $\overline{W} \sim \mathrm{N}\left(30\times 0.08, \dfrac{30\times 0.08\times 0.92}{50}\right)$ approximately $\overline{W} \sim \mathrm{N}\left(2.4, 0.04416\right)$ $P(\overline{W}>2.5)=0.317$

11	Unbiased estimate of the population mean, $\bar{x} = \frac{970}{100} = 9.7$ Unbiased estimate of the population variance, $s^2 = \frac{1}{99}\left[9800 - \frac{970^2}{100}\right] = \frac{391}{99} = 3.9495$ (ii) It is not necessary to assume a normal distribution for the test to be valid. Since *n* is large, by Central Limit Theorem, the mean mass of salt is normally distributed approximately. (iii) Let X be the mass of one packet of salt in grams and μ be the population mean of the mass of one packet of salt in grams. $H_o : \mu = 10$ $H_1 : \mu < 10$ Under H_o, Since n = 100 is large, by Central Limit Theorem, $\overline{X} \sim N\left(10, \frac{391}{9900}\right)$ approximately, Use a left tailed z-test at 10 % level of significance, Test Statistic: $Z = \frac{\overline{X} - 10}{s/10} \sim \mathrm{N}(0,1)$ Using GC, the p-value = $0.0656 \leq 0.10$ Reject the null hypothesis and conclude that there is sufficient evidence at 10% significance level to reject the company's claim that the mass is at least 10g. (iv) There is a probability of 0.10 that we concluded that the mean mass of the packets of salt is less than 10 grams when it is actually at least 10 grams. The p-value is the lowest significance level at which the sample mean mass of salt is at least 10 grams. (v) Let Y be the random variable of the mass of each packet of salt in the new packaging system.

	$H_o : \mu = 10$ $H_1 : \mu < 10$ Since n = 30 is large, by Central Limit Theorem Under H_o, $Z = \dfrac{\overline{Y} - 10}{\frac{0.9}{\sqrt{30}}} \sim N(0,1)$ approximately Since company's claim is valid, H_o is not rejected at 5% level of significance, $\dfrac{m - 10}{\frac{0.9}{\sqrt{30}}} > -1.6449$ $m > 9.7297$ Least possible value of m is 9.73.