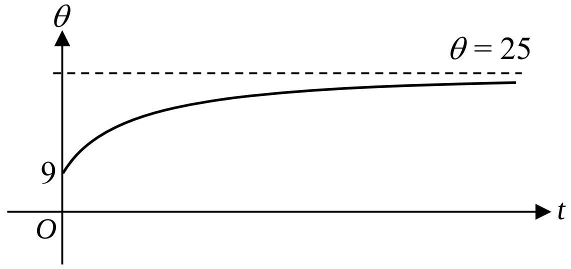
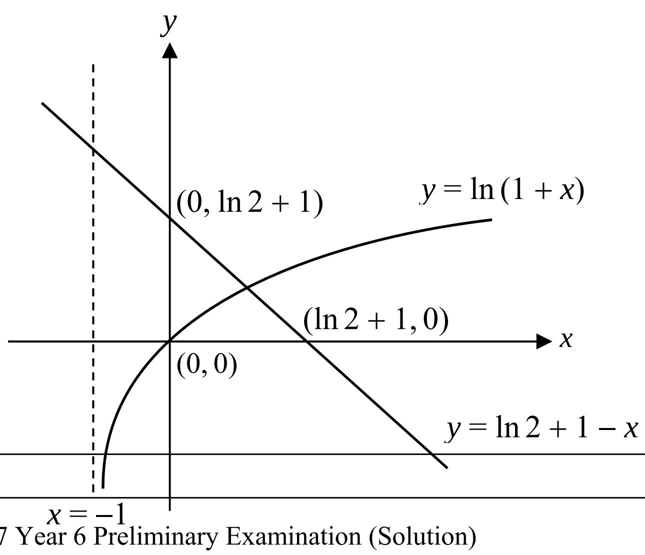
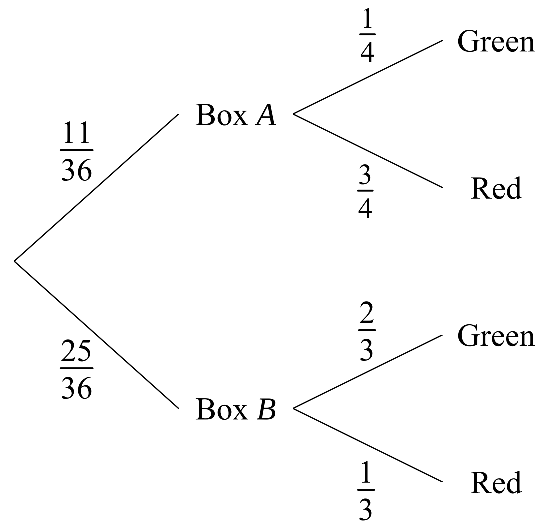
RAFFLES INSTITUTION
H1 Mathematics 8865
2017 Year 6 Preliminary Examination (Solution)

Time Allowed: 3 hours **Total Marks: 100**

1	(a) $\frac{\mathrm{d}}{\mathrm{d}x}\ln\{7\sqrt{(1+6x^2)}\} = \frac{\mathrm{d}}{\mathrm{d}x}[\ln 7 + \frac{1}{2}\ln(1+6x^2)] = \frac{1}{2}\frac{12x}{1+6x^2} = \frac{6x}{1+6x^2}$ (b) $\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{6(1-7x)^2} = \frac{1}{6}\frac{\mathrm{d}}{\mathrm{d}x}(1-7x)^{-2} = \frac{1}{6}\times[-2(1-7x)^{-3}(-7)] = \frac{7}{3(1-7x)^3}$
2	(i) Firstly, angle DAB is 120°. Next, since triangle AKD is equilateral, angle DAK is 60°, so angle KAB (i.e. angle IAB) is 120° – 60° = 60°. This makes triangle IAB equilateral, so $BI = 4x$ mm. Hence $BF = \frac{BI - FG}{2} = \frac{4x-4y}{2} = 2(x-y)$ mm. (Shown) (ii) Note that by symmetry of the design, angle KGI = angle GIK = 60°. This makes triangle GIK equilateral, so $KI = 2(x-y)$ mm. Hence AD (= AK) = $AI - KI = 4x - 2(x-y) = 2(x+y)$ mm. (Shown) (iii) Perimeter = 566 mm $\Rightarrow 2\times 4x + 4\times 4y + 3\times 2(x-y) + 2\times 2(x+y) = 566$ $\Rightarrow 9x + 7y = 283$ – (1) Note that height of trapezium $ABCD = \sqrt{\{(4x)^2 - (2x)^2\}} = (2\sqrt{3})x$ Note also that area of an equilateral triangle with side $l = \frac{\sqrt{3}}{4}l^2$ Area = $4655\sqrt{3}$ mm^2 $\Rightarrow \frac{1}{2}(8x+4y)\times(2\sqrt{3})x - 2\times\frac{\sqrt{3}}{4}(4y)^2 - \frac{\sqrt{3}}{4}\{2(x+y)\}^2 = 4655\sqrt{3}$ $\Rightarrow 7x^2 + 2xy - 9y^2 = 4655$ – (2) Solving (1) and (2), we have $x = 26$ and $y = 7$.
3	(a) $\frac{(x-4)(x^2-4x+4)}{x+4} \geq 0 \Rightarrow \frac{(x-4)(x-2)^2}{x+4} \geq 0$ $\Rightarrow (x+4)(x-4)(x-2)^2 \geq 0,\ x \neq -4$ $\Rightarrow x = 2$ or $(x+4)(x-4) \geq 0,\ x \neq -4$ Hence $x < -4$ or $x = 2$ or $x \geq 4$

	(b) $(k-1)x^2 - 2x + k + 2 < 0 \Rightarrow a < 0$ and $D < 0$ $\Rightarrow k - 1 < 0$ and $(-2)^2 - 4(k-1)(k+2) < 0$ $\Rightarrow k < 1$ and $-4k^2 - 4k + 12 < 0$ $\Rightarrow k < 1$ and $k^2 + k - 3 > 0$ $\Rightarrow k < 1$ and $k < \dfrac{-1-\sqrt{13}}{2}$ or $k > \dfrac{-1+\sqrt{13}}{2}$ $\Rightarrow k < \dfrac{-1-\sqrt{13}}{2}$
4	(i) When $t = 0$, $\theta = 9 \Rightarrow 9 = 25 - Ae^0 \Rightarrow A = 16$ When $t = 20$, $\theta = 17 \Rightarrow 17 = 25 - 16e^{20k}$ $\Rightarrow e^{20k} = \dfrac{1}{2} \Rightarrow k = \dfrac{1}{20}\ln\dfrac{1}{2}$ (shown) We have $\theta = 25 - 16\left(\dfrac{1}{2}\right)^{\frac{t}{20}}$ (ii) When $t = 25$, $\theta = 25 - 16\left(\dfrac{1}{2}\right)^{\frac{25}{20}} = 18.3\ °\text{C}$ (iii) When $\theta = 23$, $23 = 25 - 16\left(\dfrac{1}{2}\right)^{\frac{t}{20}} \Rightarrow \left(\dfrac{1}{2}\right)^{\frac{t}{20}} = \dfrac{1}{8} \Rightarrow t = 60$ Hence duration = 60 – 20 = 40 minutes. (iv) As $t \to \infty$, $\left(\dfrac{1}{2}\right)^{\frac{t}{20}} \to 0$, so $\theta \to 25$ for large values of t. (v) 
5	(i) 

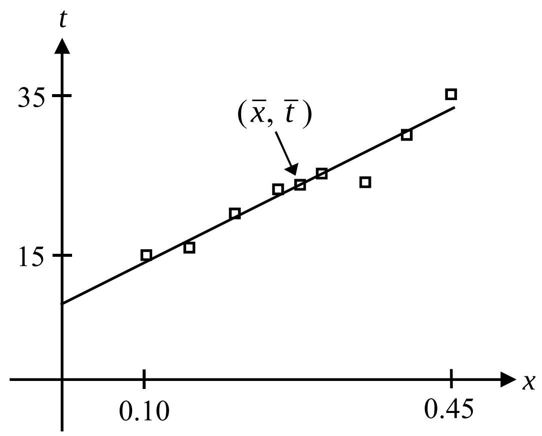
	(ii) For C_1, when $x = 1$, $y = \ln(1+1) = \ln 2$. For C_2, when $x = 1$, $y = \ln 2 + 1 - 1 = \ln 2$. Hence C_1 and C_2 intersect at $x = 1$. (iii) Required area $= \int_0^1 \ln(1+x)\,\mathrm{d}x + \frac{1}{2}(\ln 2)^2 = 0.63$ (2 dp) (iv) Required area $= \int_0^{\ln 2} \mathrm{e}^y - 1\,\mathrm{d}y + \frac{1}{2}(1)^2$ $= \left[\mathrm{e}^y - y\right]_0^{\ln 2} + \frac{1}{2} = (2 - \ln 2) - 1 + \frac{1}{2}$ $= \frac{3}{2} - \ln 2$ where $A = \frac{3}{2}$ and $B = -1$
6	(i) $\mathrm{P}(X < 7) = 0.7 \Rightarrow \mathrm{P}\left(Z < \frac{7-\mu}{\sqrt{7}}\right) = 0.7$ From GC, $\frac{7-\mu}{\sqrt{7}} = 0.52440 \Rightarrow \mu = 5.613$ (3 dp) (ii) $X_1 - X_2 \sim \mathrm{N}(0, 14)$ $\therefore \mathrm{P}(X_1 < X_2 + 1) = \mathrm{P}(X_1 - X_2 < 1) = 0.605$ (3 sf)
7	(i) Let X denote the number of biscuits (out of 8) that contains sweet filings, $X \sim \mathrm{B}(8, p)$ Since mean $= np = 3.2$, $p = 0.4$ Hence $\mathrm{P}(X = 0) = 0.0168$ (ii) $X \sim \mathrm{B}(8, 0.4)$ $\mathrm{P}(X \geq 4) = 1 - \mathrm{P}(X \leq 3) = 0.40591 = 0.406$ (3sf) (Shown) (iii) Let Y denote the number of packs (out of 18) that contains at least four biscuits with sweet fillings, $Y \sim \mathrm{B}(18, 0.406)$ $\mathrm{P}(Y \leq 9) = 0.853$ (or 0.854 if use $p = 0.40591$)
8	P(at least one '6' is shown) $= 1 - \frac{5}{6} \times \frac{5}{6} = \frac{11}{36}$



(i) $\mathrm{P}(\text{Green}) = \frac{11}{36} \times \frac{1}{4} + \frac{25}{36} \times \frac{2}{3} = \frac{233}{432}$

(ii) $\mathrm{P}(\text{Box } A \mid \text{Red}) = \dfrac{\mathrm{P}(\text{Box } A \cap \text{Red})}{\mathrm{P}(\text{Red})} = \dfrac{\frac{11}{36} \times \frac{3}{4}}{1 - \frac{233}{432}} = \frac{99}{199}$

9 (i)



(ii) The regression line of t on x is $t = 8.83 + 53.3x$

(iii) $(\bar{x}, \bar{t}) = (0.275, 23.5)$

(iv) The product moment correlation coefficient, $r = 0.969$

Since $r = 0.969$ is close to 1, there is a strong positive linear correlation between the amount of caffeine consumed and the time for the test subject to fall asleep at night.

(v) When $x = 1.00$ (grams), $t = 62.2$ (minutes).

	This prediction is not valid as $x = 1.00$ lies outside the data range of $0.10 \le x \le 0.45$.
10	(i) An unbiased estimate of the population mean, $\bar{x} = \frac{-28}{70} + 10 = 9.6$ An unbiased estimate of the population variance, $s^2 = \frac{1}{69}\left[267 - \frac{(-28)^2}{70}\right] = \frac{1279}{345}$ or 3.71 (3.7072) (ii) Let $\mathrm{E}(X) = \mu$ Null hypothesis, $\mathrm{H}_0 : \mu = 10$ Alternative hypothesis, $\mathrm{H}_1 : \mu < 10$ We perform a one-tail test at the 5 % significance level Under H_0, since n is large, by Central Limit Theorem, $\bar{X} \sim \mathrm{N}\left(10, \frac{1279}{24150}\right)$ (or $\mathrm{N}(10, 0.052961)$) approximately From GC, p-value = 0.0411 Since p-value = $0.0411 < 0.05$, we reject H_0 and conclude that there is sufficient evidence at the 5 % significance level that the farmer's claim is not valid. (iii) $\mathrm{H}_0 : \mu = 10$ $\mathrm{H}_1 : \mu > 10$ Under H_0, since n is large, by Central Limit Theorem, $\bar{X} \sim \mathrm{N}\left(10, \frac{3.31}{70}\right)$ approximately Null hypothesis is rejected $\Rightarrow p < 0.05$ $\Rightarrow \mathrm{P}\left(\bar{X} > \frac{m}{70}\right) < 0.05$ From GC, $\frac{m}{70} > 10.357078 \Rightarrow m > 725.037$ Hence $m \ge 726$ (3sf)
11	(i) Let M (in kg) and F (in kg) denote respectively the masses of the male and the female frogs, $M \sim \mathrm{N}(1.4, 0.28^2)$ and $F \sim \mathrm{N}(0.7, 0.14^2)$ $F - M \sim \mathrm{N}(0.7 - 1.4, 0.14^2 + 0.28^2)$, i.e. $\mathrm{N}(-0.7, 0.098)$ $\mathrm{P}(F > M) = \mathrm{P}(F - M > 0) = 0.0127$ (ii) Let $V = M_1 + M_2 \sim \mathrm{N}(1.4 \times 2, 0.28^2 \times 2)$, i.e. $\mathrm{N}(2.8, 0.1568)$

	Let $W = F_1 + F_2 \sim \mathrm{N}(0.7 \times 2,\ 0.14^2 \times 2)$, i.e. $\mathrm{N}(1.4,\ 0.0392)$ $V + W \sim \mathrm{N}(2.8 + 1.4,\ 0.01568 + 0.0392)$, i.e. $\mathrm{N}(4.2,\ 0.196)$ $\mathrm{P}(V + W \le 4.5) = 0.751$ (iii) Let $S = M_1 + \cdots + M_4 \sim \mathrm{N}(1.4 \times 4,\ 0.28^2 \times 4)$, i.e. $\mathrm{N}(5.6,\ 0.3136)$ $\mathrm{P}(S \le L) \ge 0.95 \Rightarrow \mathrm{P}\left(Z \le \dfrac{L - 5.6}{\sqrt{0.3136}}\right) \ge 0.95$ From GC, $\dfrac{L - 5.6}{\sqrt{0.3136}} \ge 1.6449 \Rightarrow L \ge 6.52$ Hence least $L = 6.6$ kg
12	(i) Number of ways = $7! - 1 = 5039$ (ii) Number of ways = $4! \times 3! = 144$ (iii) Number of ways = $3! \times 5! = 720$ (iv) Number of ways = $4! \times {}^5\mathrm{P}_3 = 1440$ (v) Number of ways = ${}^5\mathrm{P}_2 \times 2! \times 4! = 960$ Note that the letter I must be between the two letters E and O.

Case	Number of consonant between E and O	Number of ways
1	1	${}^4\mathrm{C}_1 \times 2! \times 2! \times 4! = 384$
2	2	${}^4\mathrm{C}_2 \times 3! \times 2! \times 3! = 432$
3	3	${}^4\mathrm{C}_3 \times 4! \times 2! \times 2! = 384$
4	4	$5! \times 2! = 240$

$\therefore$ Probability $= \dfrac{384 + 432 + 384 + 240}{7!} = \dfrac{2}{7}$