SERANGOON JUNIOR COLLEGE
2017 JC2 PRELIMINARY EXAMINATION
MATHEMATICS (SOLUTIONS)

Higher 1

8865/01

Tuesday **12 Sep 2017**
3 Hours

Additional materials: Writing paper
List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name and class on the cover page and on all the work you hand in.

Write in dark blue or black pen on both sides of the paper.

You may use a soft pencil for any diagrams or graphs.

Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

You are expected to use an approved graphing calculator.

Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.

Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.

You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets **[ ]** at the end of each question or part question.

The total marks for this paper is **100**.

At the end of the examination, fasten all your work securely together.

This document consists of **8** printed pages and **0** blank page.

[**Turn Over**

1	**(a)** Differentiate each of the following functions with respect to x, simplifying your answers: **(i)** $\left(\frac{2}{3}x+1\right)^{-6}$	[1]
	(ii) $5e^{1-2x}+\frac{1}{8x^3}$	[2]
	(b) Find $\int \frac{1}{\sqrt{2-kt}}\,dt$, where k is a constant.	[2]
	Suggested Solution	
	(a) (i) Let $y=\left(\frac{2}{3}x+1\right)^{-6}$ $\frac{dy}{dx}=(-6)\left(\frac{2}{3}x+1\right)^{-7}\left(\frac{2}{3}\right)$ $=-\frac{4}{\left(\frac{2}{3}x+1\right)^{7}}$	
	(ii) Let $y=5e^{1-2x}+\frac{1}{8x^3}$ $\frac{dy}{dx}=5(-2)e^{1-2x}+\frac{1}{8}(-3)x^{-4}$ $=-10e^{1-2x}-\frac{3}{8x^4}$	
	$\int \frac{1}{\sqrt{2-kt}}\,dt=\int(2-kt)^{-\frac{1}{2}}\,dt$ $=\frac{(2-kt)^{\frac{1}{2}}}{\left(\frac{1}{2}\right)(-k)}+c$ $=\frac{-2\sqrt{2-kt}}{k}+c$	

2

(a) Given that the quadratic graph with equation $y = ax^2 + bx + c$ passes through the points with coordinates $(-2,\ 18)$, $(-1, 9)$ and $(1.5,\ 4)$, find the equation of this graph. [3]

(b) Find the exact range of values of k for which the line $y = -x - 2$ does not meet the curve $y = (k-1)x^2 + kx - 3$, where $k \neq 1$. [4]

Suggested Solution

(a)

$y = ax^2 + bx + c$

At $(-2,18)$: $4a - 2b + c = 18$ --------- (1)

At $(-1,9)$: $a - b + c = 9$ --------- (2)

At $(1.5,4)$: $2.25a + 1.5b + c = 4$ --------- (3)

Using GC, we have $a = 2,\ \ b = -3,\ \ c = 4$.

Hence, the equation of this graph is $y = 2x^2 - 3x + 4$.

(b)

$(k-1)x^2 + kx - 3 = -x - 2$

$(k-1)x^2 + (k+1)x - 1 = 0$

Since the curve does not meet the line,

Discriminant < 0

$(k+1)^2 - 4(k-1)(-1) < 0$

$k^2 + 2k + 1 + 4k - 4 < 0$

$k^2 + 6k - 3 < 0$

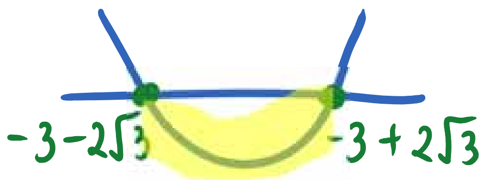
Consider $k^2 + 6k - 3 = 0$

$$k = \frac{-6 \pm \sqrt{6^2 - 4(1)(-3)}}{2(1)}$$

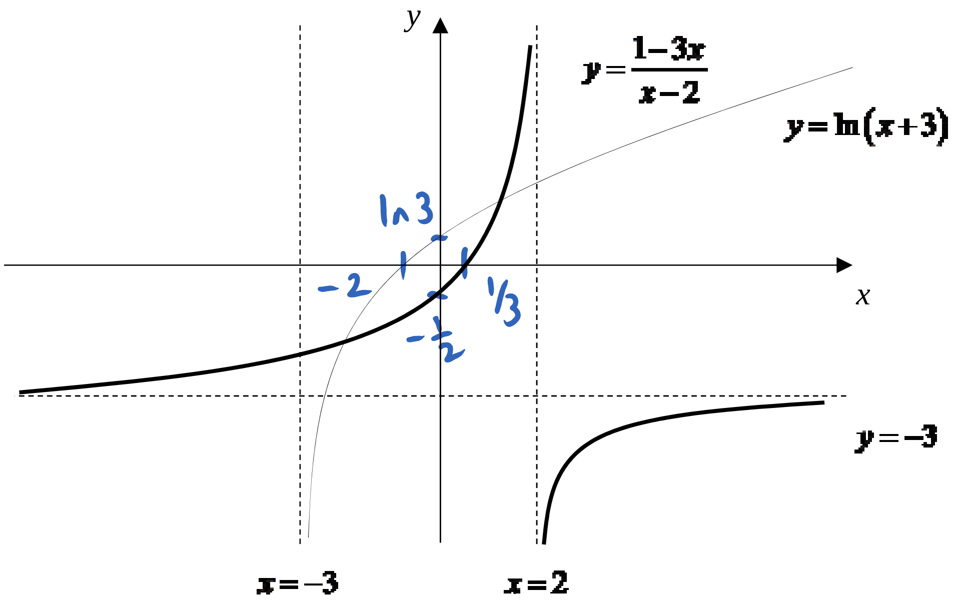
$$= \frac{-6 \pm \sqrt{48}}{2}$$

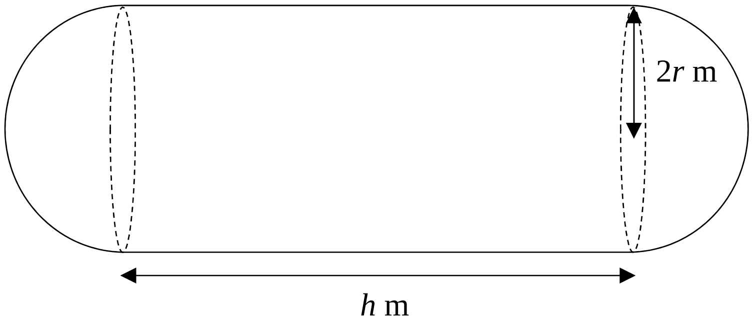
$$= \frac{-6 \pm 4\sqrt{3}}{2}$$

$$= -3 \pm 2\sqrt{3}$$



Hence, $-3 - 2\sqrt{3} < k < -3 + 2\sqrt{3}$

3		
	(i) On a single diagram, sketch the graphs of $y=\dfrac{1-3x}{x-2}$ and $y=\ln(x+3)$, stating clearly the equations of any asymptotes and the axial intercepts.	[5]
	(ii) Find the x-coordinates of the points of intersection, leaving your answer correct to 3 significant figures.	[1]
	(iii) Hence, find the set of values of x for which $\dfrac{1-3x}{x-2}\le\ln(x+3)$.	[1]
	Suggested Solution	
	(i) 	
	(ii) The x-coordinates of the points of intersection are -2.86 and 0.850.	
	(iii) Solution set = $\{x \in \mathbb{R}:\ -2.86 \le x \le 0.850$ or $x>2\}$	

4	 An engineer has to design an oil tank with a capacity of 120 m^3. The oil tank consists of a cylindrical body of length *h* m and two hemispherical ends of radius 2*r* m each, as shown in the diagram. **(i)** Show that $h=\left(\frac{30}{\pi r^2}-\frac{8}{3}r\right)$ m.	[2]
	(ii) Show that the total surface area of the tank is $\left(\frac{120}{r}+\frac{16}{3}\pi r^2\right)$ m^2.	[2]
	(iii) Use a non-calculator method to find the value of *r* which gives a minimum total surface area of the tank. Hence, find the value of the minimum total surface area of the tank, leaving your answer correct to 2 decimal places. [It is given that a sphere of radius *r* has surface area $4\pi r^2$ and volume $\frac{4}{3}\pi r^3$]	[5]
	Suggested Solution	
	(i) Volume of tank, $\pi(2r)^2 h+\frac{4}{3}\pi(2r)^3=120$ $4\pi r^2 h+\frac{32\pi r^3}{3}=120$ $3\pi r^2 h+8\pi r^3=90$ $h=\frac{1}{3\pi r^2}\left(90-8\pi r^3\right)$ $h=\left(\frac{30}{\pi r^2}-\frac{8}{3}r\right)$ m (Shown)	
	(ii) T.S.A of tank, $A=2\pi(2r)h+4\pi(2r)^2$ $=4\pi rh+16\pi r^2$ $=4\pi r\left(\frac{30}{\pi r^2}-\frac{8}{3}r\right)+16\pi r^2$ $=\frac{120}{r}-\frac{32}{3}\pi r^2+16\pi r^2$ $=\frac{120}{r}+\frac{16}{3}\pi r^2$ (Shown)	

(iii) $A = \dfrac{120}{r} + \dfrac{16}{3}\pi r^2$

$$\frac{dA}{dr} = -\frac{120}{r^2} + \frac{32}{3}\pi r$$

For minimum value of A, $\dfrac{dA}{dr} = 0$

$$-\frac{120}{r^2} + \frac{32}{3}\pi r = 0$$

$$-120 + \frac{32}{3}\pi r^3 = 0$$

$$r = \left(\frac{45}{4\pi}\right)^{\frac{1}{3}} \quad \text{or} \quad \sqrt[3]{\frac{45}{4\pi}}$$

To show A is minimum,

Method I (2nd derivative)

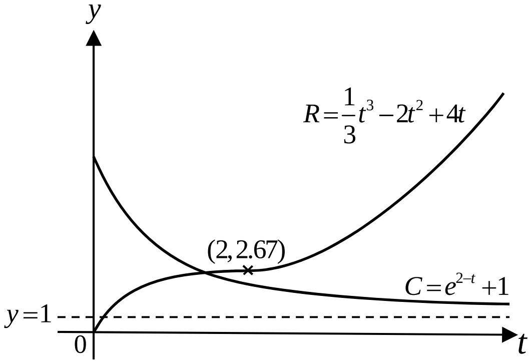
$$\frac{d^2A}{dr^2} = \frac{240}{r^3} + \frac{32}{3}\pi \text{ (positive, hence minimum)}$$

Method II (sign test)

R	$\left(\sqrt[3]{\frac{45}{4\pi}}\right)^-$	$\sqrt[3]{\frac{45}{4\pi}}$	$\left(\sqrt[3]{\frac{45}{4\pi}}\right)^+$
$\frac{dA}{dr}$	–	**0**	+
Slope			

$$A = \frac{120}{\sqrt[3]{\frac{45}{4\pi}}} + \frac{16}{3}\pi\left(\sqrt[3]{\frac{45}{4\pi}}\right)^2$$

$= 117.65 \text{ m}^2$ (2 dec places).

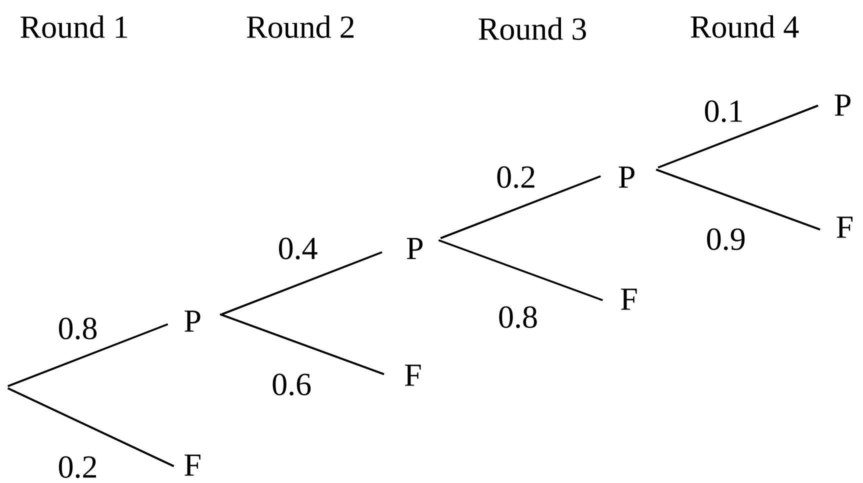
5	The management of HoLi, a chain of bubble tea outlets, is selling a 5-year franchise to operate its newest outlet in Kovan. Past experience in similar franchises suggests that the revenue (R) and the operating costs (C), in hundred thousand dollars per year, at time t years, can be modelled by the graphs of R: $y=\frac{1}{3}t^3-2t^2+4t$ and C: $y=e^{2-t}+1$ respectively, for $0\le t\le 5$.	
	(i) Using the axes of y (hundred thousands of dollars per year) against t, sketch, on the same diagram, the graphs of R and C, indicating the coordinates of any stationary point(s) and equation of asymptote(s), showing any necessary working clearly.	[5]
	(ii) State the coordinates of the point of intersection between the two graphs.	[1]
	The area under the curve of R and C, from t = 0 to t = T, gives the total revenue and total operating costs at t = T respectively.	
	(iii) Find the value of t for which the franchise is expected to break even i.e. where the total revenue just covers the total operating costs for the period of t years.	[3]
	(iv) Compute the approximate total profit, in dollars, expected to be generated over the 5-year period, correcting your answer to 3 significant figures.	[2]
	(v) Explain, in context, a possible meaning of the horizontal asymptote of curve C.	[1]
	Suggested Solution	
	(i) 	

	$\frac{dR}{dt}=0: \quad t^2-4t+4=0$ $(t-2)^2=0 \quad \Rightarrow \; t=2$ When $t=2$, $y=\frac{8}{3}$ (or 2.67, rounded off to 3 s.f.)	
	(ii) Coordinates of point of intersection is: $(1.51, 2.63)$	
	(iii) Let T be the no. of years for the franchise to break even. $\int_0^T \frac{1}{3}t^3-2t^2+4t \, dt = \int_0^T e^{2-t}+1 \, dt$ $\int_0^T \frac{1}{3}t^3-2t^2+4t-e^{2-t}-1 \, dt = 0$ $\left[\frac{1}{12}t^4-\frac{2}{3}t^3+2t^2+e^{2-t}-t\right]_0^T=0$ $\frac{1}{12}T^4-\frac{2}{3}T^3+2T^2+e^{2-T}-T-e^2=0$ Using GC, $T \approx 4.13$	
	(iv) $\int_0^5 \frac{1}{3}t^3-2t^2+4t \, dt - \int_0^5 e^{2-t}+1 \, dt$ $=18.75-12.339$ (using GC) ≈ 6.41 $\therefore$ Total profit over 5 years is \$641,000.	
	(v) **Possible meaning:** The horizontal asymptote $(y=1)$ of curve C means that the operating costs will stabilize at \$100,000 per year in the long run.	

6	**(a)** The manager of a bookstore wishes to conduct a survey to seek the customers' opinions on its opening hours. If the manager decides to survey a sample of the first 80 customers who leave the bookstore, give a reason why this sample may not be appropriate.	[1]
	(b) A surveyor decides to obtain a random sample of 20 residents from the apartment block. He randomly selects 20 units from the apartment block and chooses one resident from each unit. **(i)** In the context of the question, explain what is meant by the term 'random sample'. **(ii)** Explain why this method may not be appropriate. **(iii)** Describe an alternative method so that the surveyor will choose a sample of 20 residents at random from the apartment block of 100 residents.	 [1] [1] [1]
	Suggested Solution	
	(a) The first 80 customers may not be representative of all bookstore customers. The mid-day and late night shoppers will be unrepresented. (b)(i) In the context of the question, the term "random sample" means that every resident of the same apartment block has the same probability of being selected. The selection of residents is independent. (b)(ii) Since each unit has different number of people, so the probability of being chosen is not equally likely. (b)(iii) Obtain the name list of the 100 residents living at the apartment block and assign a number from 1 to 100 to all the residents. Use a computer program to generate 20 random numbers from 1 to 100. The person who is assigned the chosen number will be selected.	

7	In a large batch of t-shirts printed, a t-shirt printing company wishes to conduct quality checks for the t-shirts. The random variable X is the number of t-shirts which fail the quality check. A random sample of 10 t-shirts are taken. It is assumed that X follows a binomial distribution with unknown constant p being the probability of a t-shirt failing the quality check and such that $0.2 < p < 1$.	
	(i) Find the number of t-shirts expected to fail the quality check in terms of p.	[1]
	(ii) Given that $\mathrm{P}(X=1)=0.141$, formulate an equation in terms of p. Hence, find the value of p.	[2]
	Taking p = 0.3,	
	(iii) find the largest value of r such that the probability of at least r t-shirts failing the quality check is more than 0.1.	[4]
	Suggested Solution	
	(i) $X \sim \mathrm{B}(10, p)$ $\mathrm{E}(X)=10p$	
	(ii) Given $P(X=1)=0.141$, $\binom{10}{1}p(1-p)^9=0.141$ $10p(1-p)^9=0.141$ Using GC, $p=0.28357=0.284$ (3s.f.) or $p=0.016356$ (rej. since $0.2<p<1$)	
	(iii) $\mathrm{P}(X \ge r)>0.1$ $1-\mathrm{P}(X \le r-1)>0.1$ $\mathrm{P}(X \le r-1)<0.9$ From GC, When $r=4$, $\mathrm{P}(X \le 3)=0.64961(<0.9)$ When $r=5$, $\mathrm{P}(X \le 4)=0.84973(<0.9)$ When $r=6$, $\mathrm{P}(X \le 5)=0.95265(>0.9)$ $\therefore$ The largest value of r is 5.	

8	Find how many different arrangements can be made using all letters of the word *PRELIMS* if	
	(i) there are no restrictions,	[1]
	(ii) the first and last letters must both be vowels,	[2]
	(iii) the letters *R*, *L*, and *M* must be together,	[2]
	(iv) the letters *R*, *L*, and *M* must be separated.	[3]
	Suggested Solution	
	(i) No. of arrangements $=7!=5040$	
	(ii) No. of arrangements $=5!\times 2$ $=240$	
	(iii) No. of arrangements $=5!\times 3!$ (or ${}^5C_1\times 4!\times 3!$) $=720$	
	(iv) No. of arrangements $= {}^5C_3\times 4!\times 3!$ $=1440$ <u>Note</u>: *Award M1 for identifying use of slotting method if student writes* ${}^5C_3\times 4!$ *or* ${}^5C_3\times 3!$	

9	The new private car hire company, Snatch, is expanding into the Southeast Asian market. Snatch is sourcing for suitable candidates to fill up their Marketing Manager positions. A candidate has cleared his interview if he passes the first four rounds of interviews. The probability that a candidate passes the first round of interview is 0.8. If the candidate passes a round of interview, the probability that the candidate will pass the next round of the interview is half the probability of passing the preceding interview. If the candidate fails a round of interview, the candidate will not be allowed to go for the next round of interview. Mr Cheu is shortlisted to go through the interview.	
	(i) Illustrate the possible outcomes for the interview process for a candidate on a tree diagram.	[2]
	(ii) Find the probability that Mr Cheu **(a)** clears his interview, **(b)** fails to clear his interview, given that he passes the second round of interview.	[1] [3]
	(iii) 25 candidates were shortlisted to go through the interview. Find the probability that fewer than 3 candidates clear the interview, leaving your answer correct to 5 significant figures.	[3]
	Suggested Solution	
	(i)  P: Passing the round of interview F: Failing the round of interview	
	(ii)(a) Required probability = $0.8\times0.4\times0.2\times0.1=0.0064$	

	(ii)(b) P(fails to clear interview | passes 2nd round of interview) $= \dfrac{\text{P(fails to clear interview \& passes 2nd round of interview)}}{\text{P(passes 2nd round of interview)}}$ $= \dfrac{0.8\times0.4\times0.8+0.8\times0.4\times0.2\times0.9}{0.8\times0.4}$ = 0.98	
	(iii) Let X be the random variable denoting "the number of candidates, out of 25, who cleared the interview." Then $X \sim \mathrm{B}(25, 0.0064)$ Required probability = $\mathrm{P}(X<3)$ $= \mathrm{P}(X \le 2)$ = 0.99946 (5 significant figures)	

10	A manufacturer claimed that the metal rods produced by their machine has a desired length of 50 cm. 100 metal rods were randomly chosen. The lengths, x cm, are summarised by $$\sum(x-55) = -355 \text{ and } \sum(x-55)^2 = 5622.$$ **(i)** Find the unbiased estimates of the population mean and variance.	[2]
	(ii) Test at 1% level of significance, whether the manufacturer has underestimated the mean length of the metal rods.	[4]
	A new random sample of 100 metal rods is chosen and the mean of this sample is m cm. The population standard deviation is assumed to be 7 cm. A test at 10% level of significance indicates that the manufacturer's claim is valid. **(iii)** Find the range of values of m, giving your answer correct to 2 decimal places.	[4]

Suggested Solution

(i) Unbiased estimate of population mean, $\mu = \frac{-355}{100} + 55 = 51.45$

Unbiased estimate of population variance,

$$s^2 = \frac{1}{99}\left[5622 - \frac{(-355)^2}{100}\right]$$

$$\approx 44.05808081 = 44.1$$

(ii) To test $H_o : \mu = 50$

Against $H_1 : \mu > 50$

Use a right-tailed test at 1% significance level.
Under H_0 and by Central Limit Theorem (since $n = 100$ is large),

$$\bar{X} \sim N\left(50, \frac{44.05808}{100}\right) \text{ approximately}$$

Test statistic, $Z = \dfrac{\bar{X} - \mu_o}{\sqrt{\sigma^2 / n}} \sim N(0, 1)$

$\mu_0 = 50,\ s = \sqrt{44.05808},\ \bar{x} = 51.45,\ n = 100$

Using GC, p-value $= 0.0145$ (to 3s.f.)

Since ***p*-value = 0.0145** $>$ **0.01**, we **do not reject** H_0 and conclude that there is **insufficient** evidence that the manufacturer has underestimated the mean length of the metal rods at the 1% significance level.

(iii) To test H_o: $\mu = 50$
Against H_1: $\mu \neq 50$

Using a two-tailed test at 10% level of significance
Under H_o and by Central Limit Theorem (since n = 100 is large),

$$\bar{X} \sim N\left(50, \frac{7^2}{100}\right) \text{ approximately}$$

Test statistic, $Z = \dfrac{\bar{X} - \mu}{\sigma} \sim N(0, 1)$

Since the manufacturer's claim is valid for this improved experiment at 10% level of significance, the conclusion is not in favour of H_1, we do not reject H_o and z_{test} must lie outside the critical region.

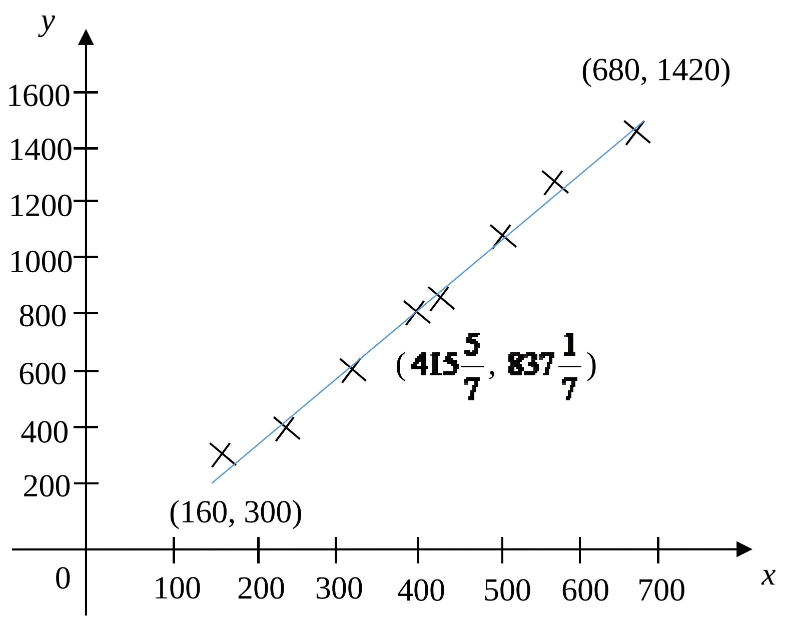
$$-1.6449 < z_{test} < 1.6449$$

$$-1.6449 < \frac{m - 50}{\frac{7}{\sqrt{100}}} < 1.6449$$

48.85 < m < 51.15 (2 decimal places)

11	Mandy sells her homemade matcha-flavoured macarons. The numbers, x, sold in the first seven months of the year 2017, together with the profits, y dollars, on the sale of these macarons are given in the following table.	
	(i) Give a sketch of the scatter diagram for the data as shown on your calculator.	[2]
	(ii) Find $\bar{x}$ and $\bar{y}$ and mark the point $(\bar{x}, \bar{y})$ on your scatter diagram.	[2]
	(iii) Calculate the equation of the regression line of y on x, and draw this line on your scatter diagram.	[2]
	(iv) Calculate the product moment correlation coefficient, and comment on its value in relation to your scatter diagram.	[2]
	(v) Mandy expects to sell 600 matcha-flavoured macarons for the month of August in 2017. Using an appropriate regression line, estimate her profit for this month.	[1]
	(vi) Due to an error in her document, Mandy recorded her profits incorrectly. There is a shortfall of 100 dollars for all the profits recorded from January 2017 to July 2017 (as shown in the table above). Comment on whether this recording error will affect the value of the product moment correlation coefficient found in **(iv)**.	[1]
	Suggested Solution	
	(i)	

	Jan	Feb	Mar	Apr	May	Jun	Jul
x	430	580	320	240	680	160	500
y	850	1240	600	400	1420	300	1050



	(i) **G1 –** Axes labelled with evenly-spaced scale **G1 –** Correct data points plotted (end points labelled) (ii) **G1 –** Correct ($\bar{x}$, $\bar{y}$) plotted (iii) **G1 –** Correct line sketched	
	(ii) $\bar{x} = 415\frac{5}{7}$ (exact) (or 416) $\bar{y} = 837\frac{1}{7}$ (exact) (or 837)	
	(iii) $y = 2.2676x - 105.51$ $\approx 2.27x - 106$ (to 3 sig. fig.)	
	(iv) $r = 0.997$. There is a strong positive linearly correlation between the number of macarons sold and the profit earned from the sale of macarons. As the number of macarons sold increases, the profit from the sale of macarons increases. This explains why the data points show an upward trend and the data points are close to the regression line in the scatter diagram.	
	(v) $y = 2.2676(600) - 105.51$ ≈ 1255.05 ≈ 1260 Mandy's estimated profit for the month of August is \$1260.	
	(vi) As *r* measures the degree of scatter of the data points, an increment for all the values of the profits (values of *y*) will not change the scatter of the data. Hence, there will be no change in the value of *r*.	

12	The masses, in kilograms, of watermelons and papayas sold by a supermarket have normal distributions with means and standard deviations as shown in the following table.	

	Mean	Standard Deviation
Watermelon	5.42	0.51
Papaya	2.18	0.35

	(i) Find the probability that the mass of a watermelon chosen at random is between 5.2 kg and 6.5 kg.	[1]
	(ii) Find the probability that the total mass of four randomly chosen watermelons is less than 22.8 kg, stating clearly the mean and variance of the distribution that you use.	[3]
	(iii) Find the probability that the total mass of eight randomly chosen papayas is more than the total mass of four randomly chosen watermelons, stating clearly the mean and variance of the distribution that you use.	[3]
	(iv) Watermelons cost \$2.80 per kilogram and papayas cost \$1.80 per kilogram. Find the mean and the variance of the total cost of four randomly chosen watermelons and eight randomly chosen papayas and hence, find the probability that the total cost is between \$80 and \$100.	[4]
	(v) State an assumption for the calculations in parts **(iii)** and **(iv)** to be valid.	[1]
	Suggested Solution	
	(i) Let X denote the random variable of the mass of a randomly chosen watermelon. $X \sim \mathrm{N}\left(5.42, 0.51^2\right)$ $\mathrm{P}(5.2 < X < 6.5) \approx 0.64980 = 0.650$ (3 sig fig.)	
	(ii) Let $T = X_1 + X_2 + X_3 + X_4$ $\mathrm{E}(T) = 4(5.42) = 21.68$ $\mathrm{Var}(T) = 4\left(0.51^2\right) = 1.0404$ $T \sim \mathrm{N}(21.68, 1.0404)$ $\mathrm{P}(T < 22.8) = 0.86391 \approx 0.864$ (3 sig fig)	

	(iii) Let Y denote the random variable of the mass of a randomly chosen papaya. $Y \sim \mathrm{N}\left(2.18, 0.35^2\right)$ Let $S = Y_1 + Y_2 + Y_3 + Y_4 + Y_5 + Y_6 + Y_7 + Y_8$ $S \sim \mathrm{N}\left(8(2.18), 8\left(0.35^2\right)\right)$ $S \sim \mathrm{N}(17.44, 0.98)$ $\mathrm{E}(S-T) = 17.44 - 21.68 = -4.24$ $\mathrm{Var}(S-T) = 0.98 + 1.0404 = 2.0204$ $S - T \sim \mathrm{N}(-4.24, 2.0204)$ $\mathrm{P}(S - T > 0) \approx 0.0014275 = 0.00143$ (3 sig fig)	
	(iv) Let C_1 be $1.8S$ (cost of 6 papayas) $C_1 \sim \mathrm{N}\left(1.8(17.44), 1.8^2(0.98)\right)$ $C_1 \sim \mathrm{N}(31.392, 3.1752)$ Let C_2 be $2.8T$ (Cost of 4 water melons) $C_2 \sim \mathrm{N}\left(2.8(21.68), 2.8^2(1.0404)\right)$ $C_2 \sim \mathrm{N}(60.704,\ 8.156736)$ $C_1 + C_2 \sim$ N(31.392+60.704, 3.1752+8.156736) $C_1 + C_2 \sim$ N(92.096, 11.331936) $\mathrm{P}(80 < C_1 + C_2 < 100) \approx 0.99040 = 0.990$ (3 sig fig).	
	(v) The assumption is that the masses, in kilograms, of watermelons and papayas sold by a supermarket are independently normally distributed.	