Section A: Pure Mathematics [40 marks]

1 The gradient of a curve C is given by $\frac{\mathrm{d}y}{\mathrm{d}x} = 2k+3-\frac{k+1}{(x-1)^2}$, where $k \in \mathbb{R}$. Find the set of values of k for which C has 2 distinct turning points. [4]

1	$\frac{\mathrm{d}y}{\mathrm{d}x} = 2k+3-\frac{k+1}{(x-1)^2}$ $= \frac{(2k+3)(x-1)^2-k-1}{(x-1)^2}$ $= \frac{(2k+3)x^2-2(2k+3)x+k+2}{(x-1)^2}$ At turning points: $(2k+3)x^2-2(2k+3)x+(k+2)=0$ Since curve has 2 distinct turning points, the equation $(2k+3)x^2-2(2k+3)x+(k+2)=0$ has 2 distinct real roots $\Rightarrow \quad 4(2k+3)^2-4(2k+3)(k+2)>0$ $\Rightarrow \quad 2k^2+5k+3>0$ $\Rightarrow \quad (2k+3)(k+1)>0$ $\therefore k<-\frac{3}{2}$ or $k>-1$	[B1] [M1] correct inequality [A1] correct coefficients [A1]

2 **(a)** Differentiate $\frac{\mathrm{e}^x-1}{\mathrm{e}^{2x}}$. [2]

(b) Find $\int (2x\sqrt{2x+1}+\sqrt{2x+1})\,\mathrm{d}x$. [3]

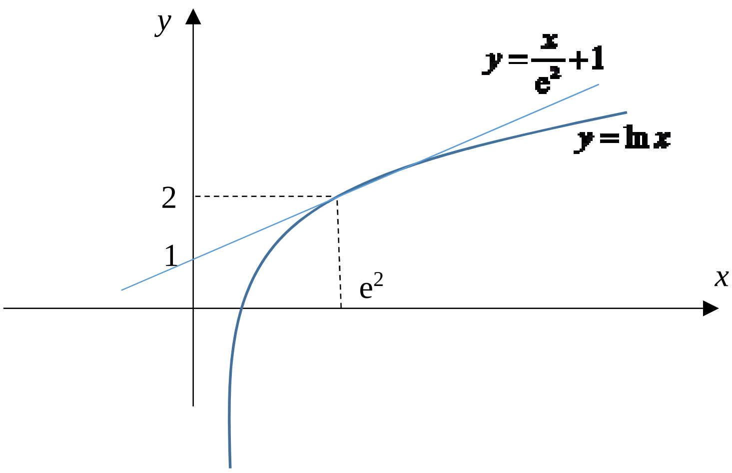
2(a)	$\frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{e}^x-1}{\mathrm{e}^{2x}} = \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\mathrm{e}^x}{\mathrm{e}^{2x}}-\frac{1}{\mathrm{e}^{2x}}\right]$ $= \frac{\mathrm{d}}{\mathrm{d}x}\left[\mathrm{e}^{-x}-\mathrm{e}^{-2x}\right]$ $= -\mathrm{e}^{-x}+2\mathrm{e}^{-2x}$ $= 2\mathrm{e}^{-2x}-\mathrm{e}^{-x}$	[B1] [B1]
2(b)	$\int 2x\sqrt{2x+1}+\sqrt{2x+1}\,\mathrm{d}x = \int (2x+1)\sqrt{2x+1}\,\mathrm{d}x$ $= \int (2x+1)^{\frac{3}{2}}\,\mathrm{d}x$ $= \frac{2}{5}\cdot\frac{(2x+1)^{\frac{5}{2}}}{2}+c$ $= \frac{(2x+1)^{\frac{5}{2}}}{5}+c$	[B1] [M1] [A1]

3 The curve C has equation $y = \ln x,\ x > 0$.

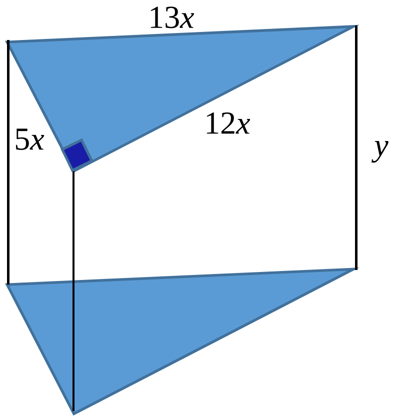
(i) Find the equation of the tangent to C at the point where $x = e^2$, leaving your answer in terms of e. [3]

(ii) Sketch the graph of C and the tangent to C at the point where $x = e^2$ on the same diagram, stating the coordinates of intersection with the axes and the equation of any asymptote(s). [2]

(iii) By using the result $\int \ln x \, dx = x \ln x - x + c$, where c is an arbitrary constant, find the area of the region bounded by C, the tangent to C at the point where $x = e^2$, the x-axis and the y-axis. Give your answer in terms of e. [3]

3(i)	$y = \ln x \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{x}$ When $x = e^2$, $y = \ln e^2 = 2$, $\frac{dy}{dx} = \frac{1}{e^2}$ Equation of tangent at the point where $x = e^2$: $y - 2 = \frac{1}{e^2}(x - e^2)$ i.e. $y = \frac{x}{e^2} + 1$	[B1] [M1] [A1]
3(ii)	 	[B1] Graph of $y = \ln x$, including x-intercept and vertical asymptote $x = 0$ [B1] Graph of $y = \frac{x}{e^2} + 1$, including y-intercept and intersection with the graph of $y = \ln x$
3(iii)	Required area $= \frac{1}{2}(1+2)e^2 - \int_1^{e^2} \ln x \, dx$ $= \frac{1}{2}(1+2)e^2 - [x \ln x - x]_1^{e^2}$ $= \frac{1}{2}(1+2)e^2 - [e^2 \ln e^2 - e^2 + 1]$ $= 1 + \frac{1}{2}e^2$	[B1] $\int_1^{e^2} \ln x \, dx$ [M1] formulate area [A1]

4 A prism with a cross-section in the shape of a right-angled triangle has dimensions (in cm) as shown in the diagram below.



The volume of the prism is $7200\ \text{cm}^3$. Show that the surface area of the prism is given by

$$S = \frac{7200}{x} + 60x^2 \ \text{cm}^2. \qquad [3]$$

Without using a calculator, find in surd form the value of x that gives a stationary value of S. Hence state, with a reason, whether S is a maximum or minimum. [4]

It is also given that x is decreasing at 0.5 cm/s, find the rate at which the surface area is decreasing when $x = 5$ cm. [2]

<table>
<tr><td>4</td><td>

Given volume of prism = 7200

We have $\frac{1}{2}(5x)(12x)y = 7200 \quad \Rightarrow \quad y = \frac{240}{x^2}$

Surface area of prism

$S = \frac{1}{2}(5x)(12x)\times 2 + 12xy + 13xy + 5xy$

$= 60x^2 + 30xy$

$= 60x^2 + 30x\left(\frac{240}{x^2}\right)$

$= 60x^2 + \frac{7200}{x}$

$\frac{dS}{dx} = \frac{-7200}{x^2} + 120x = \frac{120x^3 - 7200}{x^2}$

At minimum S, $\frac{dS}{dx} = 0 \quad \Rightarrow \quad \frac{120x^3 - 7200}{x^2} = 0$

$\Rightarrow \quad 120x^3 - 7200 = 0$

$\Rightarrow \quad x^3 = 60$

$\therefore \quad x = \sqrt[3]{60}$

Using First Derivative Test:

	$\sqrt[3]{60}^-$	$\sqrt[3]{60}$	$\sqrt[3]{60}^+$
$\frac{dS}{dx}$	-ve	0	+ve

S is minimum at $x = \sqrt[3]{60}$

</td><td>

[B1]

[M1][A1]

[B1]

[M1]

[A1] for surd form only

[B1]

</td></tr>
</table>

	Given that x is decreasing at 0.5 cm/s, when $x=5$ $\frac{\mathrm{d}x}{\mathrm{d}t}=-0.5$ $\frac{\mathrm{d}S}{\mathrm{d}t}=\frac{\mathrm{d}S}{\mathrm{d}x}\times\frac{\mathrm{d}x}{\mathrm{d}x}=\frac{120(5)^3-7200}{5^2}\times -0.5=-156$ Therefore, surface area is decreasing at $156\ \mathrm{cm}^2/\mathrm{s}$	 [M1] [A1]

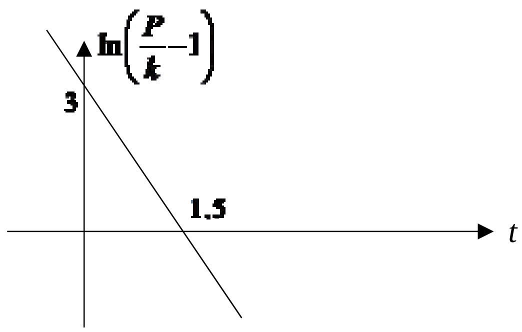
5 The profit P (in thousands dollars) of a company after the start of a promotion "Clearance Sale" can be modelled by the equation

$$P=k(1+b\mathrm{e}^{-rt}),$$

where t is the number of days elapsed since the start of the promotion and b, r and k are positive constants.

(i) Express $\ln\left(\frac{P}{k}-1\right)$ in terms of b, r and t [1]

(ii) The graph of $\ln\left(\frac{P}{k}-1\right)$ against t is given below.



By using the graph above, show that $b=\mathrm{e}^3$ and $r=2$. [3]

(iii) Using differentiation, show that $\frac{\mathrm{d}P}{\mathrm{d}t}<0$ for $t\geq 0$. Hence explain why the maximum profit occurs at t = 0. Given that the maximum profit is $42171, find the value of k, correct to nearest integer value. [5]

The "Clearance Sale" ends after a week and another promotion "Happy Sale" takes place immediately after. The "Happy Sale" lasts for 3 weeks and the profit during "Happy Sale" can be modelled by the equation

$$P=a-\frac{1}{6}(t-14)^2,\quad \text{for } 7\leq t\leq 28.$$

(iv) Given that the company's maximum profit during "Happy Sale" is $18000, find the value of a. [2]

(v) Find the total profit of the company in 4 weeks, correct to the nearest dollars. [3]

5(i)	$P=k(1+b\mathrm{e}^{-rt}) \quad\Rightarrow\quad \frac{P}{k}-1=be^{-rt}$	

	Then, $\ln\left(\frac{P}{k}-1\right)=\ln(be^{-rt})=\ln b-rt$	[B1]
5(ii)	From graph, y-intercept = 3 $\Rightarrow \ln b=3 \therefore b=\mathrm{e}^3$ Gradient of line $=\frac{-3}{1.5}=-2 \Rightarrow -r=-2 \therefore r=2$	[M1][A1] [M1][A1]
5(iii)	$\frac{\mathrm{d}}{\mathrm{d}t}P=\frac{\mathrm{d}}{\mathrm{d}t}k(1+\mathrm{e}^3\mathrm{e}^{-2t})=\frac{\mathrm{d}}{\mathrm{d}t}k(1+\mathrm{e}^3\mathrm{e}^{-2t})=k(-2\mathrm{e}^{3-2t})$ Since $\mathrm{e}^{3-2t}>0$ for $t\ge 0$, then $-2k(\mathrm{e}^{3-2t})<0$ where k is a constant i.e. $\frac{\mathrm{d}P}{\mathrm{d}t}<0$ for $t\ge 0$ P is decreasing for $t\ge 0$ Hence, P is maximum at $t=0$. Maximum profit occurs at $t=0$. Maximum profit = 42171 $\Rightarrow$ $P=\frac{42171}{1000}$ $k(1+\mathrm{e}^{3-2(0)})=42.171 \Rightarrow k\approx 1.999996498$ $\therefore k=2$	[B1] [B1] for $\frac{\mathrm{d}P}{\mathrm{d}t}<0$ [M1] [A1]
5(iv)	Maximum profit occurs at $t=14$, $\frac{18000}{1000}=a-\frac{1}{6}(14-14)^2$ $a=18$	[M1] [A1]
5(v)	Total profit $=\int_0^7 2(1+\mathrm{e}^{3-2t})\,\mathrm{d}t+\int_7^{28}\left(18-\frac{1}{6}(t-14)^2\right)\mathrm{d}t$ ≈ 240.5855202 Total profit of company in 4 weeks is \$240586	[B1] [B1] [B1]

Section B: Statistics [60 marks]

6 Find the number of 6-letter passwords that can be formed using the letters from the word SINGAPORE if

(i) repetitions of letters are not allowed. [1]

(ii) at least two vowels must be chosen and repetition of letters are not allowed. [2]

(iii) three distinct vowels and three distinct consonants are chosen, and vowels and consonants must alternate? [3]

6(i)	No. of ways $=\binom{9}{6}\times 6!=60480$	[B1] for answer
6(ii)	No. of ways $=60480-\binom{4}{1}\times\binom{5}{5}\times 6!=57600$	[M1] for complement method [A1]
6(iii)	No. of ways $=\binom{4}{3}\times\binom{5}{3}\times 3!\times 3!\times 2=2880$	[B1] for choosing 3 vowels and 3 consonants [B1] for 2 cases of alternating [B1] for answer

7 A fixed number, n, of taxis entering VICOM is observed and the number of those taxis that fails inspection is denoted by X.

(i) State, in context, two assumptions needed for X to be well modelled by a binomial distribution. [2]

Assume now that X has the distribution $\mathrm{B}(n,\ p)$.

(ii) Given that $n = 20$ and $p = 0.15$, find $\mathrm{P}(X = 2 \text{ or } 3)$. [2]

(iii) Given that $n = 10$ and $p = 0.3$, find $\mathrm{P}(3 \le X < 6)$. [2]

(iv) Given that $n = 3$ and $\mathrm{P}(X \le 1) = 0.5$, show that $4p^3 - 6p^2 + 1 = 0$. Hence find the value of p. [3]

<table>
<tr><td>7(i)</td><td>• Probability that a taxi fails inspection is constant for all the taxis
• Whether a taxi fails inspection or not is independent of another taxi</td><td>[B1]

[B1]</td></tr>
<tr><td>7(ii)</td><td>$X \sim \mathrm{B}(20,\ 0.15)$
$\mathrm{P}(X = 2 \text{ or } 3) = \mathrm{P}(X = 2) + \mathrm{P}(X = 3) = 0.47216 = 0.472$</td><td>[M1][A1]</td></tr>
<tr><td>7(iii)</td><td>$X \sim \mathrm{B}(10,\ 0.3)$
$\mathrm{P}(3 \le X < 6) = \mathrm{P}(X \le 5) - \mathrm{P}(X \le 2) = 0.56986 = 0.570$</td><td>[B1] [B1]</td></tr>
<tr><td>7(iv)</td><td>$X \sim \mathrm{B}(3,\ p)$
$\mathrm{P}(X \le 1) = 0.5$
$\Rightarrow \quad (1-p)^3 + 3p(1-p)^2 = 0.5$
$\Rightarrow \quad 1 - 3p + 3p^2 - p^3 + 3p - 6p^2 + 3p^3 = 0.5$
$\Rightarrow \quad 2p^3 - 3p^2 + 1 = 0.5$
$\Rightarrow \quad 4p^3 - 6p^2 + 1 = 0$ (shown)
Using GC, $p = 0.5$ or 1.366 (rejected $\because p < 1$) or -0.366 (rejected $\because p > 0$)</td><td>[M1] for use of formula
[A1]

[A1]</td></tr>
</table>

8 The number of hours, x, spent daily on revision for mathematics and the marks, y, obtained for the mathematics year-end examination are recorded for 10 randomly selected students. The results are given in the following table.

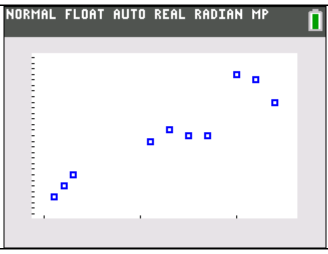
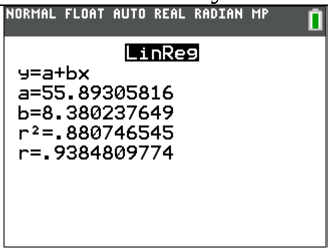
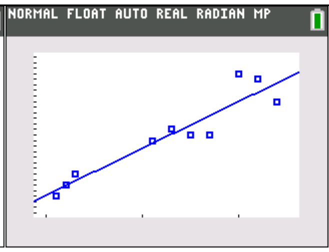
x	1.3	2.1	1.1	2.3	2.7	1.2	3.2	3.4	3.0	2.5
y	68	74	64	76	75	66	85	81	86	75

(i) Give a sketch of the scatter diagram for the data, as shown on your calculator. [2]

(ii) Find the product moment correlation coefficient and comment on its value in the context of the data. [2]

(iii) Find the equation of the regression line of y on x, in the form $y = mx + c$, giving the values of m and c correct to 4 significant figures. Sketch this line on your scatter diagram. [2]

(iv) Use the equation of your regression line to estimate the marks spent by a student who spends 1.5 hours a day on revision for mathematics. Comment on the reliability of your estimate. [3]

8(i)		[B1] for shape [B1] for axes correctly labelled
8(ii)	$r = 0.938$ (3 s.f.) This indicates a strong positive linear correlation between revision hours and the marks obtained, i.e. as the hours for revision increases, the marks obtained increases linearly.	[B1] [B1]
8(iii)	  $y = 8.3802x + 55.893$ $m = 8.380$ (4 s.f.) and $c = 55.89$ (4 s.f.)	[B1] for regression line [B1] for m and c
8(iv)	When $x = 1.5$, $y = 68.5$ (3 s.f.) The estimate is reliable since r-value is close to 1 and $x = 1.5$ is within the data range	[B1] [B1] [B1]

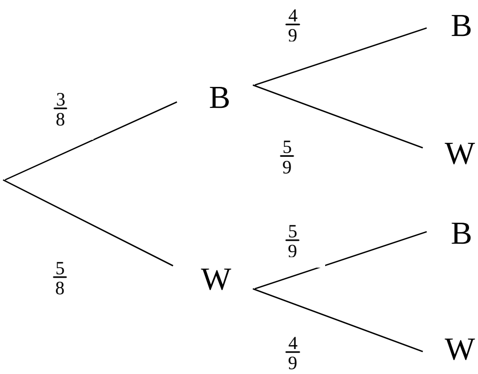
9 A bag contains 3 black balls and 5 white balls. Paul draws a ball at random from the bag and notes the colour. If a black ball is selected, Paul replaces it in the bag and adds an additional black ball into the bag. If a white ball is selected, Paul does not replace it in the bag but adds 2 black balls into the bag. Paul then draws a ball at random from the bag again and notes the colour.

Draw a tree diagram to represent the information of the two draws. [3]

(i) Show that the probability that Paul selects a black ball on both draws is $\frac{1}{6}$. [1]

(ii) Find the probability that Paul selects a white ball either on his first or second draw, or both. [2]

(iii) Find the probability that Paul selects a white ball on his first draw, given that he selects a black ball on his second draw. [3]

9		

9(i)	P(selects B on both draws) $=\frac{3}{8}\times\frac{4}{9}=\frac{1}{6}$ (shown)	[B1]
9(ii)	P(selects W on first or second draw) $=\frac{5}{8}+\left(\frac{3}{8}\times\frac{5}{9}\right)=\frac{5}{6}$ or $1-\frac{3}{8}\times\frac{4}{9}=\frac{5}{6}$	[M1][A1]
9(iii)	P(selects W on first draw | selects B on second draw) $=\dfrac{\frac{5}{8}\times\frac{5}{9}}{\frac{3}{8}\times\frac{4}{9}+\frac{5}{8}\times\frac{5}{9}}$ $=\frac{25}{37}$	[M1][A1] [A1]

10 A company claims that their electric-powered V1 cars is designed to travel a mean distance of 500 km on one full charge. To test this claim, a random sample of 80 V1 cars is taken and the distance, x km, travelled on one full charge are summarised by

$$\sum(x-500)=46 \quad \text{and} \quad \sum(x-500)^2=460.$$

(i) Find the unbiased estimates of the population mean and variance. [3]

(ii) Suggest a reason why, in this context, the given data is summarised in terms of $(x-500)$ instead of x. [1]

(iii) Test, at the 5% level of significance, whether the company's claim is valid. [4]

(iv) State, with a reason, whether it is necessary to assume a normal distribution for the test to be valid. [1]

The company introduces a new electric-powered V2 car and claims that the V2 can travel more than the mean distance of the V1 on one full charge. The new population variance of the V2 cars is known to be 13 km^2. A random sample of 50 V2 cars is taken.

(v) Find the set of values within which the mean distance of this sample must lie, such that there is not enough evidence from the sample to support the company's claim at the 1% level of significance. [4]

10(i)	$\bar{x}=\frac{46}{80}+500=500.575$ $s^2=\frac{1}{79}\left[460-\frac{46^2}{80}\right]=5.4879$	[B1] [B2]
10(ii)	Values may be too large	[B1]
10(iii)	$\text{H}_0: \mu=500$ $\text{H}_1: \mu\neq 500$ Level of significance: 5% Test statistic: $Z=\dfrac{\bar{X}-\mu}{\frac{s}{\sqrt{n}}} \sim \text{N}(0,1)$ approx. by CLT Under H_0 and using GC, we have $p=0.0281$ (3 s.f.)	[B1] for hypotheses [B1] for application of CLT and s instead of σ [B1] for p-value

	Since $p < 0.05$, we reject H_0. There is sufficient evidence at the 5% level of significance to conclude that the company's claim is invalid.	[B1] for conclusion
10(iv)	Not necessary since n is large and $\overline{X}$ is still approximately normal by CLT	[B1]
10(v)	$H_0: \mu = 500$ $H_1: \mu > 500$ Level of significance: 1% Test statistic: $Z = \dfrac{\overline{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1)$ approx. by CLT Since company's claim is not accepted. H_0 is not rejected $\Rightarrow \dfrac{\bar{x} - 500}{\sqrt{\frac{13}{50}}} < 2.3263$ $\Rightarrow \bar{x} < 501.18$	[B1] for hypotheses [B1] for test statistic with σ and CLT seen or implied [M1] [A1]

11 Three friends Anand, Beng and Charlie goes racing regularly at the Temasek Circuit, which offers a standard route on Track 1 or a more challenging route on Track 2. The time taken, in minutes, taken by them to complete a round on Tracks 1 and 2 have independent normal distributions with means and standard deviations as shown in the following table.

	Track	Mean	Standard deviation
Anand	1	3.15	0.21
Beng	1	3.18	0.10
Charlie	2	4.22	0.15

(i) Find the probability that Beng takes less than 3.15 minutes to complete a randomly chosen round on Track 1. [1]

(ii) Find the probability that Anand and Beng each takes less than 3.15 minutes to complete a randomly chosen round on Track 1. [2]

(iii) Find the probability that Beng is faster than Anand in completing a randomly chosen round on Track 1. [3]

(iv) Find the probability that out of 8 complete rounds on Track 2, there are more than 4 rounds in which Charlie takes less than 4.20 minutes to complete. [3]

Temasek Circuit charges customers $18 per minute on Track 1 and $22 per minute on Track 2. On a particular day, Beng completes 10 rounds on Track 1 and Charlie completes 8 rounds on Track 2.

(v) Find the probability that Beng and Charlie pay a total of less than $1300. [5]

11(i)	Let X and Y be the time taken by Anand and Beng to complete a round on Track 1 $X \sim N(3.15,\ 0.21^2)$ and $Y \sim N(3.18,\ 0.10^2)$ $P(Y < 3.15) = 0.38208 = 0.382$ (3 s.f.)	[B1]
11(ii)	Required probability $= 0.5 \times 0.38208 = 0.19104 = 0.191$	[M1][A1]

11(iii)	$Y - X \sim N(3.18 - 3.15,\ 0.10^2 + 0.21^2)$ i.e. $Y - X \sim N(0.03,\ 0.0541)$ $P(Y - X < 0) = 0.44868 = 0.449$ (3 s.f.)	[B2] for $E(Y - X)$ and $Var(Y - X)$ or equivalent [B1]
11(iv)	Let W be the time taken by Charlie to complete a round on Track 2 $W \sim N(4.22,\ 0.15^2)$ $P(W < 4.2) = 0.44696$ Let S be the number of rounds which Charlie takes less than 4.20 minutes to complete $S \sim B(8,\ 0.44696)$ $P(S > 4) = 1 - P(S \le 4) = 0.25460 = 0.255$ (3 s.f.)	[B1] for 5 s.f. [M1][A1]
11(v)	Let C be the total cost $C = 18(Y_1 + Y_2 + \ldots + Y_{10}) + 22(W_1 + W_2 + \ldots + W_8)$ $E(C) = (18 \times 10 \times 3.18) + (22 \times 8 \times 4.22) = 1315.12$ $Var(C) = (18^2 \times 10 \times 0.10^2) + (22^2 \times 8 \times 0.15^2) = 119.52$ $\therefore \quad C \sim N(1315.12,\ 119.52)$ $P(C < 1300) = 0.083327 = 0.0833$ (3 s.f.)	[B1] for mean and variance of time [B1] for $a^2 Var(Y)$ [B2] for $E(C)$ and $Var(C)$ [B1]

End of Paper