Section A: Pure Mathematics [40 marks]

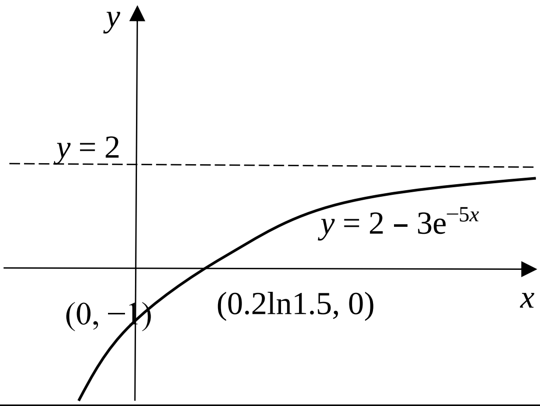
1 A farm sells ginger, spear mint, garlic by price per kilogram.
Cheryl bought 1.2 kg of ginger and 1.5 kg of garlic, while her colleague bought 0.9 kg of spear mint and 0.5 kg of garlic. The amount that Cheryl paid is \$60.60 and her colleague paid \$38.54. Ann, a retailer, bought 36 kg of ginger, 20 kg of spear mint, and 40 kg of garlic. Ann is entitled to have a discount of 25%, and paid \$1783.50 after the discount.
Identify the cost and name of the herb with the highest selling price per kg.
State the cost of 2 kg of garlic, without any discount. [4]

	Let \$x, \$y and \$z be the selling price of a kg of ginger, spear mint and garlic respectively. $1.2x + 1.5z = 60.6$ ----(1) $0.9y + 0.5z = 38.54$ ----(2) $0.75(36x + 20y + 40z) = 1783.50$ $36x + 20y + 40z = 2378$ ----(3)	
	Using GC, $x = 27.50$, $y = 32.60$, $z = 18.40$	
	The most expensive herb is spear mint which cost \$32.60 per kg. Cost of 2 kg of garlic is \$36.80	

2 The curve C has equation $y = 2 - 3e^{-5x}$.

(i) Sketch the graph of C, stating the exact coordinates of any points of intersection with the axes and the equation of the asymptote. [3]

(ii) Without using a calculator, find the exact equation of the tangent to C at the point where $x = 0.2$, expressing y in terms of x. [3]

(i)	When $x = 0$, $y = 2 - 3e^{0} = -1$ When $y = 0$, $2 = 3e^{-5x}$ $-5x = \ln(2/3)$ $x = -0.2\ln(2/3) = 0.2\ln(3/2) = 0.2 \ln 1.5$ 	
(ii)	$y = 2 - 3e^{-5x}$ $\dfrac{dy}{dx} = 15e^{-5x}$	
	At $x = 0.2$, $y = 2 - 3e^{-1}$, $\dfrac{dy}{dx} = 15e^{-1}$	
	Equation of tangent at $(0.2, 2 - 3e^{-1})$ is $y - (2 - 3e^{-1}) = 15e^{-1}(x - 0.2)$ $y = 15e^{-1}x + (2 - 6e^{-1})$	
OR	Substitute $x = 0.2$, $y = 2 - 3e^{-1}$, $m = 15e^{-1}$ into $y = mx + c$. $c = 2 - 3e^{-1} - 15e^{-1}(0.2) = 2 - 6e^{-1}$ $\therefore$ Equation of tangent at $(0.2, 2 - 3e^{-1})$ is $y = 15e^{-1}x + (2 - 6e^{-1})$	

3 A curve C has equation $y = (x - 2)^2 - 5$.

(i) Find the set of values of k such that the line $y = 2x + k$ intersect C twice. [3]

(ii) Find the exact area of the region bounded by C and the line $y = 2x - 6$. [4]

(i)	$(x-2)^2 - 5 = 2x + k$ $(x-2)^2 - 5 - 2x - k = 0$ $x^2 - 4x + 4 - 5 - 2x - k = 0$ $x^2 - 6x - 1 - k = 0$ As the roots are real and different, $(-6)^2 - 4(-1 - k) > 0$ $36 + 4 + 4k > 0$ $4k > -40$ $k > -10$	
(ii)	To find x–coordinates of points of intersection $(x-2)^2 - 5 = 2x - 6$ $x^2 - 6x + 5 = 0$ $(x-5)(x-1) = 0$ $x = 1$ or 5 $\int_1^5 2x - 6 - \left(\left(x-2\right)^2 - 5\right) \mathrm{d}x$ $= \int_1^5 -5 + 6x - x^2 \mathrm{d}x$ $= \left[-5x + 3x^2 - \frac{x^3}{3}\right]_1^5$ $= \left[-5(5) + 3(5)^2 - \frac{(5)^3}{3}\right] - \left[-5(1) + 3(1)^2 - \frac{(1)^3}{3}\right]$ $= \frac{25}{3} - \left(-\frac{7}{3}\right)$ $= 10\frac{2}{3}$	

4 **(a)** Differentiate the followings with respect to x.

(i) $\ln\left(\dfrac{4}{\sqrt{12+3x^2}}\right)$, [2]

(ii) $\dfrac{1}{\sqrt{2-3x}}$. [2]

(b) Use a non-calculator method to find $\displaystyle\int_0^6 \frac{\sqrt{e^x}+e^2}{e^{3x}}\,dx$. [4]

(a) (i)	Let $y = \ln\left(\dfrac{4}{\sqrt{12+3x^2}}\right) = \ln 4 - \dfrac{1}{2}\ln(12+3x^2)$ $\dfrac{dy}{dx} = -\dfrac{1}{2}\left(\dfrac{6x}{12+3x^2}\right) = -\dfrac{x}{4+x^2}$	
(ii)	$\dfrac{d}{dx}\left(\dfrac{1}{\sqrt{2-3x}}\right) = \dfrac{d}{dx}\left((2-3x)^{-\frac{1}{2}}\right)$ $= -\dfrac{1}{2}(2-3x)^{-\frac{3}{2}}(-3)$ $= \dfrac{3}{2}(2-3x)^{-\frac{3}{2}}$ or $\dfrac{3}{2\left(\sqrt{2-3x}\right)^3}$	
(b)	$\displaystyle\int_0^6 \frac{\sqrt{e^x}+e^2}{e^{3x}}\,dx$ $\displaystyle= \int_0^6 e^{-2.5x}+e^{2-3x}\,dx$ $= \left[\dfrac{e^{-2.5x}}{-2.5}+\dfrac{e^{2-3x}}{-3}\right]_0^6$ $= \left(\dfrac{e^{-15}}{-2.5}+\dfrac{e^{-16}}{-3}\right)-\left(\dfrac{e^0}{-2.5}+\dfrac{e^2}{-3}\right)$ $= \dfrac{2}{5}+\dfrac{e^2}{3}-\dfrac{2}{5e^{15}}-\dfrac{1}{3e^{16}}$	

5 A ship builder manufactures yachts. The rate, C thousand dollars per year, at which the total manufacturing costs change is to be monitored regularly over a period of 5 years. The Chief Financial Officer proposed that the relationship between C and the time, t years, can be modelled by the equation

$$C = 25 - 12t + e^{0.8t}, \text{ for } 0 \le t \le 5.$$

(i) Use differentiation and this model to find the minimum value of C. Justifying that the value obtained is a minimum. [6]

(ii) Sketch the graph of C against t, giving the coordinates of any intersections with the axes. [2]

(iii) Find the area of the region bounded by C, the t-axis and the lines $t = 0$ and $t = 3$ to 3 decimal places. Give an interpretation of the area that you found, in the context of the question. [2]

The annual profit from the sale of these yachts is P thousands dollars per year. The Chief Financial Officer believes that the relationship between P and t is given by

$$P = 200 + 60\ln(t + 2), \text{ for } 0 \le t \le 5.$$

(iv) Find the exact value of t for which $P = 280$. [2]

(v) Sketch the graph of P against t, giving the coordinates of any intersections with the axes. [1]

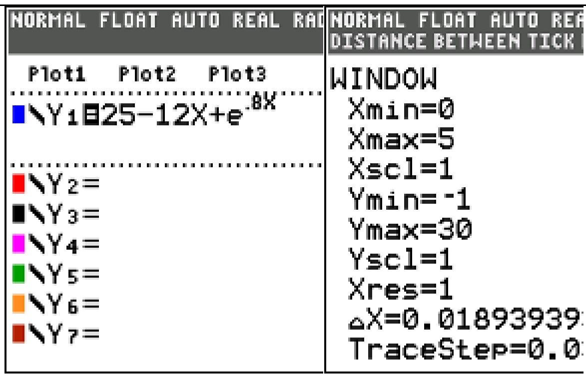
(vi) Find the exact rate at which the annual profit is increasing when $t = 3$. [2]

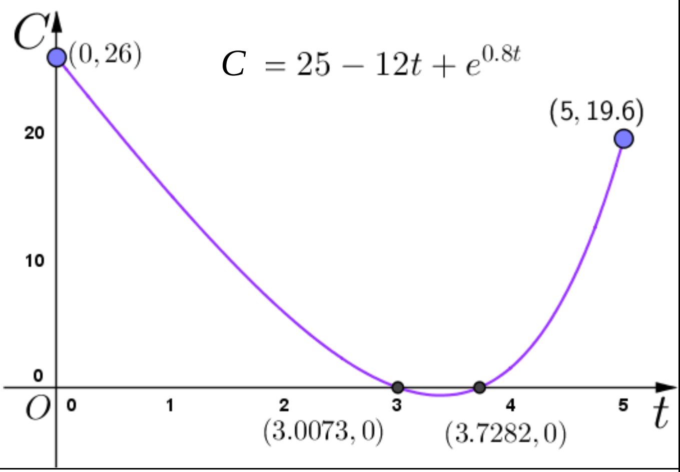
(i)	$C = 25 - 12t + e^{0.8t}$, $\frac{dC}{dt} = -12 + 0.8e^{0.8t}$ For turning points, $\frac{dC}{dt} = 0 \Rightarrow 0.8e^{0.8t} = 12$	
	$0.8e^{0.8t} = 12$ $e^{0.8t} = 15$ $0.8t = \ln 15$ $t = \frac{5}{4}\ln 15$ (or 3.39)	
	The minimum $C = 25 - 15\ln 15 + 15 = 40 - 15\ln 15$ (or -0.621)	
	$\frac{d^2C}{dt^2} = 0.64e^{0.8t}$ When $t = \frac{5}{4}\ln 15$, (ie $e^{0.8t} = 15$) $\frac{d^2C}{dt^2} = 0.64(15) = 9.6 > 0$, which shows C is minimum when $t = \frac{5}{4}\ln 15$	
	Or First derivative test.	

t	3.3	$\frac{5}{4}\ln 15$	3.4
$\frac{dC}{dt}$	-0.789 \	0 _	0.144 /

which shows C is minimum when $t = \frac{5}{4}\ln 15$

(ii)





(iii)

$\int_0^3 25 - 12t + e^{0.8t}\,dt$

$= 33.529$ (3 d.p.)

Represented the total cost during the three years.
or Difference between total cost at the end of the third year and the initial year

(iv)

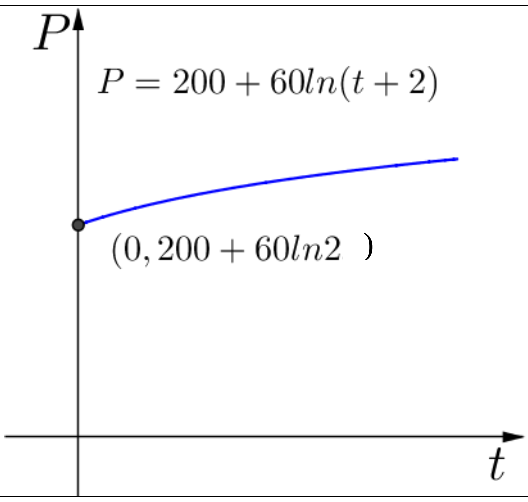
$P = 280$, $280 = 200 + 60\ln(t+2)$

$\ln(t+2) = 8/6$

$t + 2 = e^{4/3}$

$t = e^{4/3} - 2$

(v)



(vi)	$\dfrac{\mathrm{d}P}{\mathrm{d}t} = \dfrac{60}{t+2}$. When $t = 3$, $\dfrac{\mathrm{d}P}{\mathrm{d}t} = \dfrac{60}{3+2}$ = 12 The rate at which the annual profit is increasing when t = 3 is \$12 thousands dollars per year.	

Section B : Probability and Statistics [60 marks]

6 A group of nine people consists of father, mother, their three sons and four daughters. The group arrange themselves in a line for a game. Find the number of different possible arrangements if

(i) the parents are at the two ends and the three sons are all together, [2]

(ii) the four daughters are all separated. [2]

The parents has four tickets for a children ride. If each child is equally likely to be selected, find the probability that there will be more girls selected for the ride. [3]

6(i)	2 ways to arrange the two parents at the ends. Group the sons as one unit before arrange with the 4 daughters. $3!(4+1)!$ Number of ways = $2 \times 3!5! = 1440$	
(ii)	First arrange the parents and sons in 5! ways. $\downarrow X \downarrow X \downarrow X \downarrow X \downarrow X \downarrow$ where X could parents or sons As the daughters are all separated, chose 4 of 6 possible slot made by the parents and sons and arrange the daughters in $\binom{6}{4}$ 4! ways. Number of ways where the four daughters are separated $=5!\binom{6}{4}4! = 43200$	
	Case 1: 4 girls, 0 boys Case 2: 3 girls, 1 boy Numbers of ways with more girls $= 1 + \binom{4}{3}\binom{3}{1} = 13$ P(there will have more girls) = $\dfrac{13}{\binom{7}{4}} = \dfrac{13}{35}$	

7 **(a)** In a travel fair, a survey on two destination packages was conducted on a large number of participants. The participants can select at most one destination package. The survey showed that 20% would select destination package A, 35% would select destination package B and 45% would select neither.

Twelve participants who took part in the survey were randomly selected.

Find the probability that

(i) exactly 4 participants would select destination package A. [1]

(ii) at least 4 participants would select destination package B. [2]

(iii) fewer than 10 participants but more than 2 would not select destination package A. [2]

(iv) Explain the significance of the phrase 'large number' in the first sentence of this question. [1]

(b) A random variable X has a binomial distribution with mean 4 and variance $\frac{4}{3}$. The mean and standard deviation of X are denoted by μ and σ respectively. Find $P(\mu - \sigma < X < \mu + \sigma)$, correct to 4 decimal places. [4]

(a)(i)	Let X be the random variable "number of participants that would select destination package A out of 12." Then $X \sim B(12,0.2)$ $P(X = 4) = 0.132875551 = 0.133$ (3sf)	
(ii)	Let Y be the random variable "number of participants that would select destination package B out of 12." Then $Y \sim B(12,0.35)$ $P(Y \geq 4) = 1 - P(Y \leq 3) = 0.653347304 = 0.6533$ (3sf)	
	~~$P(Y > 4) = 1 - P(Y \leq 4) = 0.087361546 = 0.0874$ (3sf)~~	
(iii)	Let W be the random variable "number of participants that would not select destination package A out of 12." Then $W \sim B(12,0.8)$ $P(2 < W < 10) = P(W \leq 9) - P(W \leq 2)$ $= 0.441649725 = 0.442$ (3sf) OR $1 - P(X \leq 2) - P(X \geq 10)$	
(iv)	'Large number' in the first sentence of this question is included so that changes in the percentages of different groups (select Package A, select Package B, select neither) would be negligible.	
(b)	$np = 4$, $np(1-p) = \frac{4}{3} \Rightarrow (1-p) = \frac{1}{3}$, $p = \frac{2}{3}$ and $n = 6$. So $X \sim B(6, \frac{2}{3})$ $P(\mu - \sigma < X < \mu + \sigma)$ $= P(4 - \sqrt{[4/3]} < X < 4 + \sqrt{[4/3]})$ since $\mu = 4$ and $\sigma^2 = \frac{4}{3}$ $= P(2.845299462 < X < 5.154700538)$ $= P(3 \leq X \leq 5)$ since X takes 0, 1, 2, …, 6. $= P(X = 3) + P(X = 4) + P(X = 5)$ or $P(X \leq 5) - P(X \leq 2)$ $= 0.812071331 = 0.8121$ (4 dp)	

8 For a camping task, the Task score, t %, and the mean amount of sleep during the camp, s minutes were recorded for a random sample of 9 students. The results are given in the following table.

t	20	30	28	52	76	44	48	60	64
s	450	525	193	530	640	483	554	505	554

(i) Draw a scatter diagram for the data. [2]

(ii) Identify the pair of data which should be regarded as an outlier. Give a possible reason for the occurrence of this pair of data. [2]

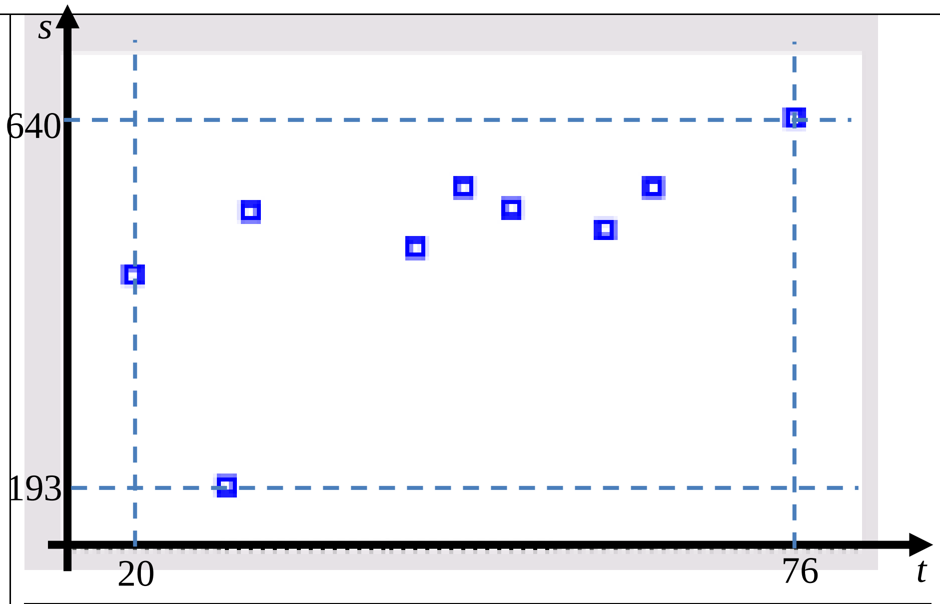
Remove the outlier you have identified in part (ii).

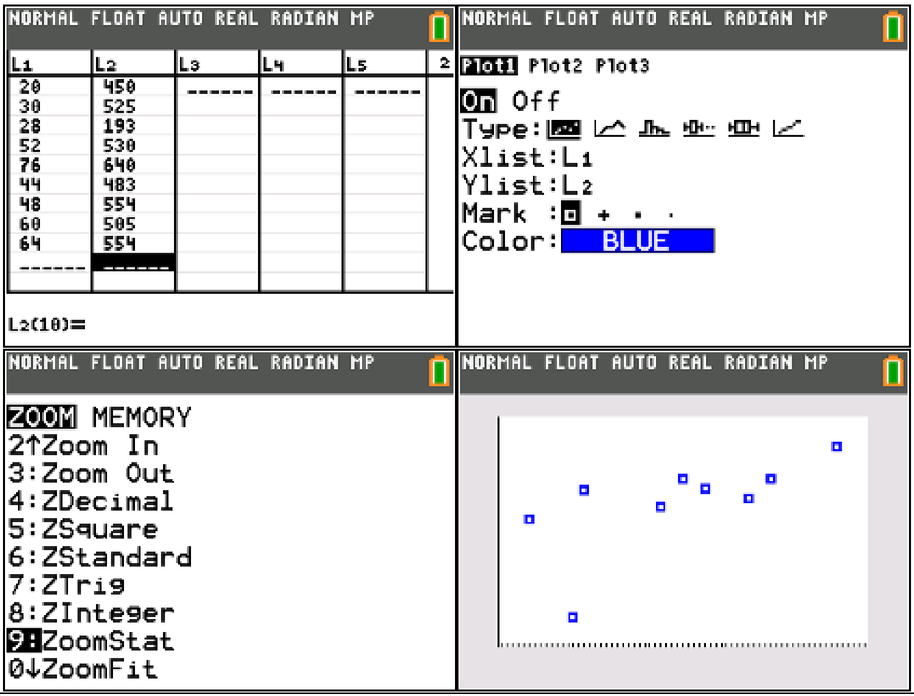
(iii) Calculate the new product moment correlation coefficient and find the regression line of s on t for the revised set of data. [2]

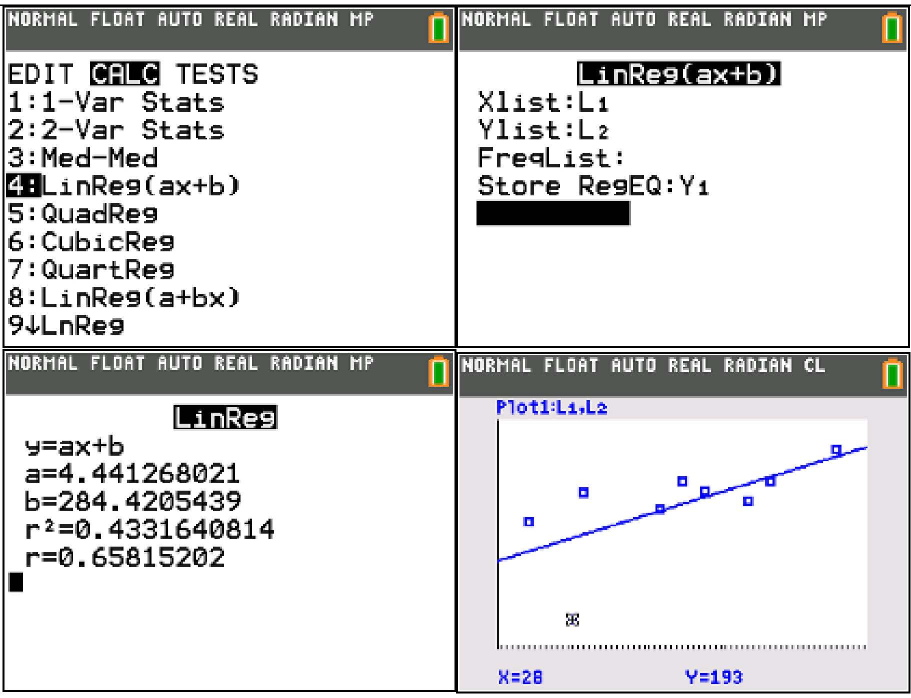
(iv) Estimate the mean amount of sleep of the student if his Task score is 80. Comment on the reliability of this estimate. [2]

(v) Using the appropriate regression line, estimate a student's Task score if his mean amount of sleep is 8.8 hours. [2]

(i)





		
(ii)	(28, 193) is an outlier, inconsistent with the trend of the other points. Possibly Error recording the amount of sleep or task score.	
(iii)	r = 0.78753 = 0.788 (3sf). $s = 2.4576t + 409.09$	
(iv)	Could t depend on s?	
(iv)	$s = 2.4576t + 409.09$ $s = 2.4576(80) + 409.09 = 606$(3sf) Unreliable, as $x = 80$ is outside the data range [20, 76], ~~through r value and scatter diagram show strong linear correlation~~.	
(v)	Use the line of regression of t on s. $t = 0.252355806s - 84.53012155$ First convert the 8.8 hrs to 528 mins $t = 0.252355806\,(528) - 84.53012155 = 48.71374402 = 48.7$ (3sf)	

9 A researcher is testing the mileage of a particular type of electric car which the manufacturer claims to have a range of 380 km. A random sample of 50 cars of this type is tested and the range is measured. The results are summarised by

$$\sum(x-380) = -15 \text{ and } \sum(x-380)^2 = 81.$$

(i) Find the exact unbiased estimates of the population mean and variance. [2]

(ii) The researcher suspects that the manufacturer has overstated the mean range of the cars. Test at the 5% level of significance whether there is sufficient evidence to support the researcher's suspicion. [4]

(iii) Explain, whether it is necessary to assume that the range of the cars follow a normal distribution in order for the test to be valid. [1]

Another manufacturer claims that the mean range for their similar powered cars is more than 380 km. Another researcher takes a random sample of 80 cars and records their mileage. A test at the 5 % level of significance shows there is sufficient evidence to support the manufacturer's claim. Assuming the population standard deviation range for these cars is 1.8 km, find the set of values of the mean range of the mileage, correct to 2 decimal places. [3]

Solutions*

(i) $\bar{x} = -\frac{15}{50} + 380 = 379.7$ or $379\frac{7}{10}$ (exact)	
$s^2 = \frac{1}{49}\left(81 - \frac{(-15)^2}{50}\right) = \frac{153}{98}$ or $1\frac{55}{98}$ (exact) ~~= 1.56122449 ≈ 1.56~~	
(ii) We perform a one-tail test,	
$H_0 : \mu = 380$ against $H_1 : \mu < 380$ at 5% level of significance. Reject H_0 if p-value < 0.05	
Under H_0, as σ^2 is unknown and $n = 50$ is large, estimate σ^2 with s^2. Then $\bar{X} \sim N\left(380, \frac{153}{98(50)}\right)$ approximately.	
Carry out z test, p–value = 0.0447775078 Since p-value < 0.05, we reject H_0 and	
conclude that there is sufficient evidence at the 5% level of significance to support the researcher's suspicion that the manufacturer has overstated the mean range of the cars.	
(iii) No, as sample size of 50 is large, by Central Limit Theorem, the sample mean of the cars is approximately normally distributed.	
We perform a one-tail test,	
H_0: $\mu = 380$ against H_1: $\mu > 380$ (manufacturer's claim) at 5% level of significance.	
Under H_0, $\bar{X} \sim N\left(380, \frac{1.8^2}{80}\right)$ approximately.	

$z = \dfrac{\bar{x}-380}{\sqrt{\dfrac{1.8^2}{80}}}$ is in the critical region. The critical value is 1.644853627.	
We reject H_o, if $\dfrac{\bar{x}-380}{\sqrt{\dfrac{1.8^2}{80}}} > 1.644853627$	
Thus the set of values of $\bar{x} > 380.3310204$ $\bar{x} > 380.33$ (2dp)	

10 A restaurant sells cooked crabs and lobsters.

The masses, in kg, of the crabs and lobsters have independent normal distributions with means, standard deviations and selling prices as shown in the following table.

	Mean Mass	Standard deviation of mass	Selling price ($ per kg)
Crabs	1.9	0.2	45
Lobsters	1.6	0.15	85

(i) Find the probability that the mass of a randomly chosen crab is within ±0.1 kg of the mean mass of crabs. [2]

(ii) Find the probability that 4 randomly chosen crabs each has a mass of more than 1.8 kg. [2]

Mr Tan goes to the restaurant and randomly chooses 4 crabs and 6 lobsters.

(iii) Find the probability that the total mass of the 6 lobsters is at least 1.5kg more than the total mass of the 4 crabs. [3]

(iv) Find the probability that Mr Tan pays between $1150 and $1170 for the 4 crabs and 6 lobsters. State the mean and variance of the distribution that you use. [4]

(i)	Let X be random variable "the mass of a randomly chosen crab."	
	Then $X \sim N(1.9, 0.2^2)$.	
	$P(1.9 - 0.1 < X < 1.9 + 0.1)$ $= P(1.8 < X < 2)$ $= 0.382924922548026$	
(ii)	$P(X > 1.8) = 0.691462461274013$ P(4 randomly chosen crabs each have a mass of more than 2 kg) $= (0.691462461274013)^4 = 0.22859905507626300 = 0.229$ (3SF)	
(iii)	Let Y be random variable "the mass of a randomly chosen lobster." Then $Y \sim N(1.6, 0.15^2)$.	
	Let $A = (Y_1 + Y_2 + \ldots + Y_6) - (X_1 + X_2 + X_3 + X_4)$ $\sim N(6\times1.6 - 4\times1.9, 6\times0.15^2 + 4\times0.2^2)$	
	$A \sim N(2, 0.295)$	
	$P(X_1 + X_2 + X_3 + X_4 + 1.5 < Y_1 + Y_2 + \ldots + Y_6)$ $P(A > 1.5)$ $= 0.821363720484063 = 0.821$ (3sf)	
(iv)	Let $B = 85(Y_1 + Y_2 + \ldots + Y_6)$ $+ 45(X_1 + X_2 + X_3 + X_4)$ $\sim N(85\times6\times1.6 + 45\times4\times1.9, 85^2\times6\times0.15^2 + 45^2\times4\times0.2^2)$ $B \sim N(1158, 1299.375)$	
	$P(\$1150 < B < \$1170)$	
	$= 0.218212321094298$ $= 0.218$ (3sf)	

11 A box containing 4 black balls and 6 red balls. Mary draws a ball from the box at random without replacement. One white ball is added to the box, then Jane draws a ball from the box at random.

(i) Draw a probability tree diagram to represent all the possible outcomes. [2]

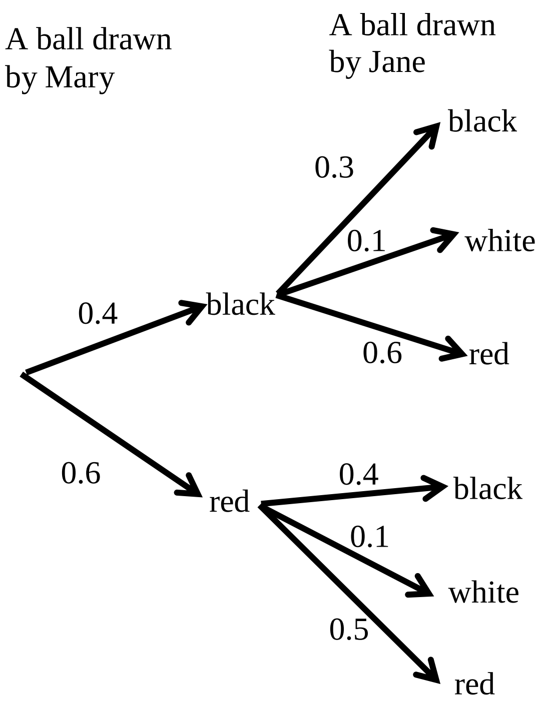
(ii) Find the probability that Jane draws a black ball. [2]

(iii) Find the probability that Mary draws a black ball, given that Jane draws a ball that is not black. [3]

(iv) Find the probability that *either* Mary draws a black ball *or* Jane draws a ball that is not black or *both*. [2]

Suppose Jane wins a prize if she draws the same coloured ball as Mary. Find the probability that Jane wins a prize, given that Jane does not draw a white ball. [3]

Solutions*

(i) 	
Let M be the event "Mary draws a black ball" and J be the event "Jane draws a black ball."	
(ii) P(Jane draws a black ball) = P(J)	
$= 0.4\times0.3 + 0.6\times0.4$	
= 0.36 or 9/25	
(iii) P(Mary draws a black ball given that Jane does not draw a black ball) $= \mathrm{P}(M \mid J')$	
= P(Mary draws a black ball $\cap$ Jane does not draw a black ball) $\div$ P(Jane does not draw a black ball)	
$= 0.4\times(1-0.3)\div(1-0.36)$ = 0.4375 or 7/16	
(iv) P(Mary draws a black ball or Jane does not draw a black ball or both)	
$= \mathrm{P}(M \cup J')$	
$= \mathrm{P}(M) + \mathrm{P}(J') - \mathrm{P}(M \cap J')$	
$= 0.4 + 0.64 - 0.28$	

= 0.76 or 19/25	
P(Jane wins a prize | Jane does not draw a white ball) = P(Jane draws the same colour ball as Mary ∩ Jane does not draw a white ball) ÷ P(Jane does not draw a white ball) = P(Jane draws the same colour ball as Mary) ÷ P(Jane does not draw a white ball) = P(Jane and Mary both draws a black ball) + P(Jane and Mary both draws a red ball) ÷ P(Jane does not draw a white ball) = $(0.4\times0.3 + 0.6\times0.5) \div 0.9$ or $(0.4\times0.3 + 0.6\times0.5) \div (0.4\times0.9 + 0.6\times0.9)$ = 0.467 or 7/15 **Note:** P(Jane does not draw a white ball) = 0.9 since there are 9 non-white out of 10 balls before Jane draws.	