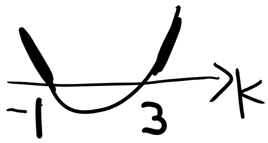
AJC J2 H1 Maths 2017 Prelim

1	$(k-2)x^2-6x+k>0$ for all values of x $3(k-2)>0$ ----(1) and $(-6)^2-4(3)(k-2)(k)<0$ ------(2) From (1): $k>2----(1)$ and From (2): $36-12k(k-2)<0$  $\Rightarrow 36-12k^2+24k<0$ $\Rightarrow -k^2+2k-3<0$ $\Rightarrow k^2-2k+3>0$ $\Rightarrow (k+1)(k-3)>0$ $\Rightarrow k<-1$ or $k>3------(2)$ From (1) and (2) : solution is $k>3$ $y=(k-2)x^3-3x^2+kx+5 \Rightarrow \frac{dy}{dx}=3(k-2)x^2-6x+k$ If function is strictly increasing, $\frac{dy}{dx}>0$ for all values of x So $(k-2)x^2-6x+k>0$ From above, solution is $k>3$
2	(i) $y=\frac{1}{2}e^{1-3x^2} \Rightarrow \frac{dy}{dx}=\frac{1}{2}e^{1-3x^2}\left(-6x\right)=-3xe^{1-3x^2}$ At P, $x=1$, $y=\frac{1}{2}e^{1-3}=\frac{1}{2}e^{-2}$, $\frac{dy}{dx}=-3e^{1-3}=-3e^{-2}$ Equation of tangent is $y-\left(\frac{1}{2}e^{-2}\right)=(-3e^{-2})(x-1)$ $y-\left(\frac{1}{2}e^{-2}\right)=(-3e^{-2})(x-1)$ $y=-3e^{-2}x+3e^{-2}+\frac{1}{2}e^{-2}=-3e^{-2}x+\frac{7}{2}e^{-2}$ (ii) At B, $x=0$, $y=\frac{7}{2}e^{-2}$ At A, $y=0$, $-3e^{-2}x+\frac{7}{2}e^{-2}=0 \Rightarrow x=\frac{\left(-\frac{7}{2}e^{-2}\right)}{-3e^{-2}}=\frac{7}{6}$ $A(\frac{7}{6},0)$ $B(0,\frac{7}{2}e^{-2})$ Midpoint of AB is $\left(\frac{\frac{7}{6}+0}{2},\frac{0+\frac{7}{2}e^{-2}}{2}\right)=\left(\frac{7}{12},\frac{7}{4}e^{-2}\right)$ (iii) $AB=\sqrt{\left(\frac{7}{6}\right)^2+\left(\frac{7}{2}e^{-2}\right)^2}$ =1.26

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a(i)

$$\ln\left(\frac{x^3}{1+x^2}\right)=\ln x^3-\ln(1+x^2)=3\ln x-\ln(1+x^2)$$

$$\frac{d}{dx}\ln\left(\frac{x^3}{1+x^2}\right)=\frac{3}{x}-\frac{2x}{1+x^2}$$

$$=\frac{3(1+x^2)-2x(x)}{x(1+x^2)}$$

$$=\frac{3+3x^2-2x^2}{x(1+x^2)}=\frac{x^2+3}{x(1+x^2)}$$

a(ii)

$$\int_1^2\frac{x^2+3}{2x(1+x^2)}dx=\frac{1}{2}\int_1^2\frac{x^2+3}{x(1+x^2)}dx$$

$$=\frac{1}{2}\left[\ln\frac{x^3}{1+x^2}\right]_1^2=\frac{1}{2}\left[\ln\frac{2^3}{1+2^2}-\ln\frac{1}{2}\right]$$

$$=\frac{1}{2}(\ln\frac{8}{5}-\ln\ \frac{1}{2})=\frac{1}{2}(\ln\frac{\left(\frac{8}{5}\right)}{\left(\frac{1}{2}\right)})$$

$$=\frac{1}{2}\ln\frac{16}{5}$$

3b) $\int_1^2\ln\left(\frac{x^3}{1+x^2}\right)dx=-0.0103$

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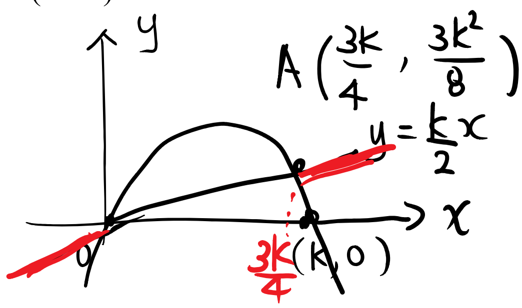
$y=2(k-x)x$



(i) At point of intersection of $y=\frac{k}{2}x$ and $y=2(k-x)x$

$$\frac{k}{2}x=2(k-x)x\Rightarrow\frac{k}{2}x=2kx-2x^2\Rightarrow 2x^2-2kx+\frac{k}{2}x=0$$

$$\Rightarrow 2x^2-\frac{3k}{2}x=0---------(1)$$

<u>Method 1:</u>

Observe that Discriminant is $D=\left(-\frac{3k}{2}\right)^2-4(2)(0)=\frac{9k^2}{4}>0$ (since

k>0 $\Rightarrow k^2>0\Rightarrow\frac{9}{4}k^2>0$ for all positive values of k.

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Hence, the quadratic equation (1) will have 2 distinct roots.
So the line intersects the curve at two distinct points.

<u>Alternative Method:</u>

From (1) $x\left(2x-\frac{3k}{2}\right)=0 \Rightarrow x=0 \quad \text{or} \quad x=\frac{3k}{4}\neq 0$

Hence, the quadratic equation (1) will have 2 distinct roots.
So the line intersects the curve at two distinct points.

At A, When $x=\frac{3k}{4}, y=\frac{k}{2}\left(\frac{3k}{4}\right)=\frac{3k^2}{8}$ & A $\left(\frac{3k}{4},\frac{3k^2}{8}\right)$

(ii) Area= $\int_0^{\frac{3k}{4}} \left(2(k-x)x-\frac{k}{2}x\right)\ dx$ =

$$\int_0^{\frac{3k}{4}} -2x^2+\frac{3k}{2}x\ dx=\left[-\frac{2x^3}{3}+\frac{3kx^2}{4}\right]_0^{\frac{3k}{4}}$$

$$=\left(-\frac{2}{3}\left(\frac{3k}{4}\right)^3+\frac{3k}{4}\left(\frac{3k}{4}\right)^2\right)-0=-\frac{2}{3}\left(\frac{27k^3}{64}\right)+\frac{3k}{4}\left(\frac{9k^2}{16}\right)$$

$$=-\frac{9k^3}{32}+\frac{27k^3}{64}=\left(-\frac{9}{32}+\frac{27}{64}\right)k^3=\frac{9}{64}k^3$$

<u>Alternative Method</u>:

Area= $\int_0^{\frac{3k}{4}} 2(k-x)x\ dx-\frac{1}{2}\left(\frac{3k}{4}\right)\left(\frac{3k^2}{8}\right)$

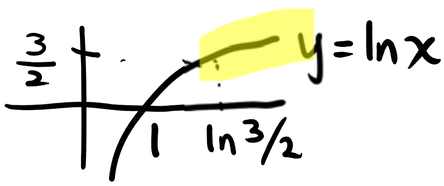
(iii) $2kx-2x^2\leq\frac{k}{2}x$ means $2(k-x)x\leq\frac{k}{2}x\Rightarrow x\leq 0 \quad \text{or} \quad x\geq\frac{3k}{4}$

(iv)Replace x by $\ln x$ and k by 2 in the solution above:

$4\ln x-2(\ln x)^2\leq\ln x$

$\Rightarrow \ln x\leq 0 \quad \text{or} \quad \ln x\geq\frac{3}{2}$

$\Rightarrow 0<x\leq 1 \quad \text{or}\ x\geq e^{\frac{3}{2}}$



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(i) $C=\frac{169}{2x+1}+2x=169(2x+1)^{-1}+2x$

$\frac{dC}{dx}=169(-1)(2x+1)^{-2}(2)+2=\frac{-338}{(2x+1)^2}+2$

Min C: $\frac{dC}{dx}=0\Rightarrow\frac{-338}{(2x+1)^2}+2=0$

$$2=\frac{338}{(2x+1)^2}\Rightarrow(2x+1)^2=\frac{338}{2}=169$$

$$2x+1=13 \quad \text{or} \quad 2x+1=-13$$

$$x=6 \text{ or } x=\text{-}2 \ \ (\text{rejected}, x\geq 0)$$

Method 1: $\dfrac{d^2C}{dx^2} = \dfrac{676}{(2x+1)^3}$

At $x = 6, \dfrac{d^2C}{dx^2} > 0$; so C is minimum when x=6.

Method 2:

x	6^-	6	6^+
$\dfrac{dC}{dx}$	-	0	+
Outline	\	—	/

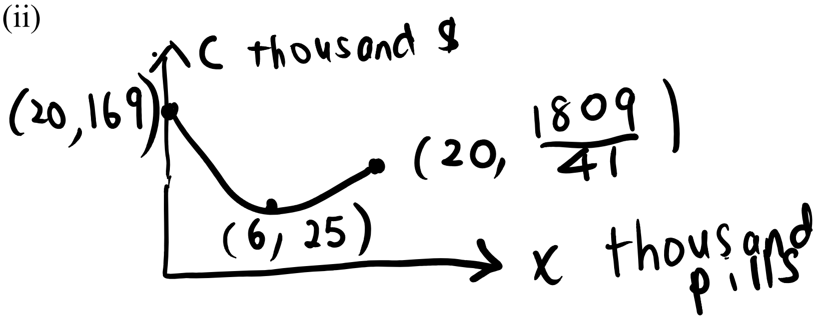
C is minimum.

$C = \dfrac{169}{(2\times6+1)^2} + 2(6) = 25$

6000 pills must be produced.

Minimum production cost is \$25000.

(ii)



(iii)

$\dfrac{dR}{dt} = 3 - e^{-2t}$

$\dfrac{dR}{dt}$ increases and approaches 3 when t is very large.

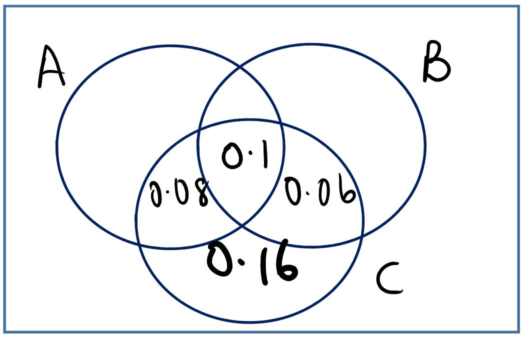
The daily revenue collected increases at a rate of approximately **3 thousand dollars per day** in the **long run**

(iv) $R = \int 3 - e^{-2t} dt = 3t - \dfrac{e^{-2t}}{-2} + C = 3t + \dfrac{e^{-2t}}{2} + C$

t = 0, R=1: $3(0) + \dfrac{e^0}{2} + C = 1 \Rightarrow \dfrac{1}{2} + C = 1 \Rightarrow C = \dfrac{1}{2}$

$R = 3t + \dfrac{e^{-2t}}{2} + \dfrac{1}{2}$

(v)The revenue first reaches \$21500 when $t = 7$

6	(i) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ $= P(A) + P(B) - P(A) \times P(B)$ $\because$ A & B independent $= 0.45 + 0.4 - (0.45)(0.4) = 0.67$ (ii) $P(B\|C) = 0.4 \Rightarrow \dfrac{P(B \cap C)}{P(C)} = 0.4$ $P(B \cap C) = 0.4P(C) = 0.4(0.4) = 0.16$ $P(A' \cap B \cap C) = P(B \cap C) - P(A \cap B \cap C) = 0.16 - 0.1 = 0.06$ (iii) Method 1(Formula) $P(A \cup B \cup C)$ $= P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$ =0.45+0.4+0.4 -0.18-0.16-0.18+0.1=0.83 Alternative method (From Venn diagram) $P(A \cap B' \cap C) = P(A \cap C) - P(A \cap B \cap C) = 0.18 - 0.1 = 0.08$ $P(A' \cap B' \cap C) = P(C) - 0.1 - 0.08 - 0.06 = 0.16$  $P(A \cup B \cup C) = P(A \cup B) + 0.16 = 0.67 + 0.16 = 0.83$ $P(A' \cap B' \cap C') = 1 - P(A \cup B \cup C) = 1 - 0.83 = 0.17$
7	(i) No of ways= ${}^7C_3 \times {}^3C_1 \times {}^6C_2 \times {}^4C_1$ =6300 (ii) No of codes that can be formed = $9 \times 9 \times 9 \times 26 \times 26 = 492804$ (iii) Case 1: one even digit & 2 odd digits, one vowel & one consonant Case 2: one even digit & 2 odd digits, 2 vowels. No of codewords=$3(4 \times 5 \times 5) \times 2(5 \times 21) + 3(4 \times 5 \times 5) \times (5 \times 5)$ =63000+7500=70500 (iv) No of passwords with all different digits, & identical letters $= 9 \times 8 \times 7 \times 26 \times 1 = 13104$ Probability = $\dfrac{1}{13104}$
8	Let X be the number of yellow rose seeds out of 12. $X \sim B(12, 0.3)$

Since $E(X) = 12p = 3.6 \Rightarrow p = \dfrac{3.6}{12} = 0.3$

$P(X \le 3) = 0.4925158 \approx 0.4925$

(ii) Let Y be the number of seeds that are either red or yellow rose seeds

$Y \sim B(12, 0.55)$ Since P(yellow or red)=0.3+0.25=0.55

$P(Y > 6) = P(Y \ge 7) = 1 - P(Y \le 6) = 0.527$

(iii) Let W be the number of packs that contain at most three yellow rose seeds, out of 200 packs . $W \sim B(200, 0.4925)$

$$P(30\% \text{ of } 200 \le W < 60\% \text{ of } 200) = P(60 \le W < 120)$$

$$= P(60 \le W \le 119) = P(W \le 119) - P(W \le 59)$$

$$= 0.998545 \approx 0.999$$

$$\text{(iv)P(at least 2 pink)} = \left(\frac{5}{12} \times \frac{4}{11} \times \frac{3}{10}\right) + 3\left(\frac{5}{12} \times \frac{4}{11} \times \frac{7}{10}\right) = \frac{4}{11}$$

$$\text{(v)P(third seed is pink|at least 2 pink)} = \frac{P(PPP \text{ or } P\bar{P}P \text{ or } \bar{P}PP)}{P(\text{at least 2 pink})}$$

$$= \frac{\left(\frac{5}{12} \times \frac{4}{11} \times \frac{3}{10}\right) + 2\left(\frac{5}{12} \times \frac{4}{11} \times \frac{7}{10}\right)}{\frac{4}{11}} = \frac{\left(\frac{17}{66}\right)}{\left(\frac{4}{11}\right)} = \frac{17}{24}$$

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$X \sim N(50, 8^2)$

(i)Prob $= \left(P(X > 40)\right)^3$ $=(0.894351)^3= 0.715$(ii)

$X_1 + X_2 - 2X_3 \sim N(50 + 50 - 2(50), 8^2 + 8^2 + 4(8^2))$ i.e $N(0, 384)$

$P(X_1 + X_2 - 2X_3 < -15 \text{ or } X_1 + X_2 - 2X_3 > 15)$

$= P(X_1 + X_2 - 2X_3 < -15) + P(X_1 + X_2 - 2X_3 > 15)$

$= 0.221997 + 0.221997 = 0.444$

<u>Alternative:</u> $1 - P(-15 < X_1 + X_2 - 2X_3 < 15) = 1 - 0.556006 = 0.444$

(iii) Let $Y \sim N(\mu, \sigma^2)$

$$P(Y < 42) = P(Y > 78) \Rightarrow E(Y) = \frac{42 + 78}{2} = 60 \text{ (by symmetry)}$$

$$P(Y < 42) = 0.0204 \Rightarrow P(Z < \frac{42 - 60}{\sigma}) = 0.0204 \Rightarrow P(Z < \frac{-18}{\sigma}) = 0.0204$$

$$\frac{-18}{\sigma} = -2.0455567 \Rightarrow \sigma = \frac{-18}{-2.0455567} = 8.79956$$

Var(Y)=8.79956^2=77.4322655=77.432

$Y = aX + b$

$E(Y) = aE(X) + b = a(50) + b = 50a + b$

$50a + b = 60$ --------(1)

$Var(Y) = a^2 Var(X) = 64a^2$

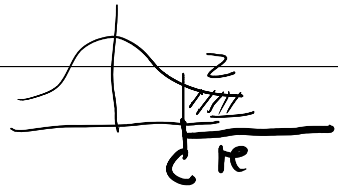
$64a^2$ =77.4333---------(2)

	$a^2 = \frac{77.4322655}{64} = 1.209879149$ $a = 1.099995 \approx 1.10$ 50(1.099995)+b=60 b=5.0025$\approx$5 (v) $\overline{C} = \frac{C_1 + C_2 + \ldots C_{40}}{40}$ Since sample size=40>30 is large, By **CLT**, $\overline{C} \sim N(52, \frac{10^2}{40})$ $P(52-1 < \overline{X} < 52+1) = P(51 < \overline{C} < 53) = 0.473$
10	(i) Let X be the mass of a randomly chosen 'Xtra' loaf of bread, and μ the population mean. *X* has a unknown distribution Test $H_0: \mu = 800$ (baker's claim) vs $H_1: \mu \neq 800$ Test statistic: Under Ho and since sample size n=50≥30is large**, by Central Limit Theorem,** $\overline{X} \sim N\left(800, \frac{10.1^2}{50}\right)$ approximately, $Z = \frac{\overline{X} - 800}{\sqrt{\frac{10.1^2}{50}}} \sim N(0,1)$ Two **tailed** test at the 5% level of significance. From sample, $\overline{x} = 797.7$,z = - 1.61, p=0 .107 Since *p* = 0.107>0.05 , do **not reject H₀**. There is **insufficient evidence** at the **5%** level to conclude that the average mass is not 800 g. We do **not** reject the baker's claim. OR: There is **insufficient evidence** at the **5%** level to conclude that the baker's claim is not valid. (ii) If Test $H_0: \mu = 800$ (baker's claim) vs $H_1: \mu < 800$ **(baker is overstating)** Then p=0.05367 If bakery is overstating , reject Ho at k%, p=0.05367<$\frac{k}{100}$ $\Rightarrow$ k>5.367 smallest k is 5.37 (iii) Let Y be the mass of compound in a randomly chosen healthy loaf and μ the population mean. *Y* has a normal distribution Test $H_0: \mu = 150$ (bakery's claim) vs $H_1: \mu > 150$ (understating **)** Test statistic: Under Ho $\overline{Y} \sim N\left(150, \frac{\sigma^2}{60}\right)$ and $Z = \frac{\overline{Y} - 150}{\frac{\sigma}{\sqrt{60}}} \sim N(0,1)$ One-**tailed** test at the 6% level of significance.

Critical Value:

$P(Z \le C) = 0.94 \Rightarrow C = 1.554774$

Reject Ho if $z > 1.554774$



Since our sample mean $\bar{y} = \dfrac{\sum(y-150)}{60} + 150 = \dfrac{60}{60} + 150 = 151$ Bakery is understating (Reject Ho)

$$\frac{151-150}{\frac{\sigma}{\sqrt{60}}} > 1.554774 \quad \Rightarrow \frac{1}{\left(\frac{\sigma}{\sqrt{60}}\right)} > 1.554774 \quad \Rightarrow \frac{\sqrt{60}}{\sigma} > 1.554774$$

$$\Rightarrow \sqrt{60} > 1.554774\sigma \Rightarrow \sigma < \frac{\sqrt{60}}{1.554774} = 4.98205$$

$$\Rightarrow \sigma < \frac{\sqrt{60}}{1.554774} = 4.98205$$

$0 \le \sigma \le 4.98$

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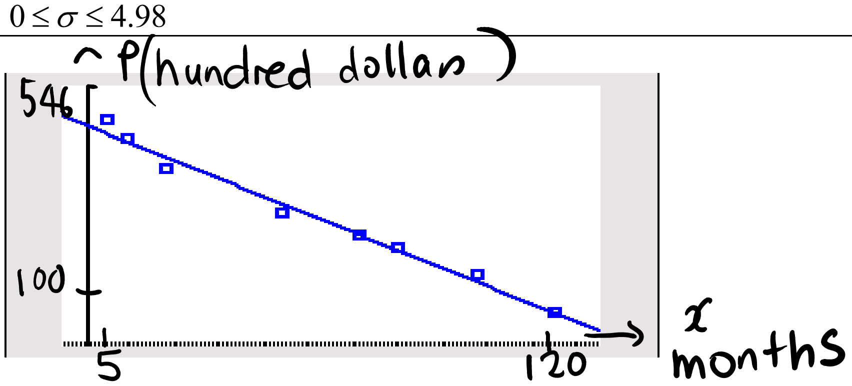
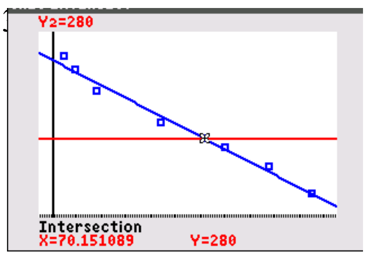


(ii) r= -0.992

Since r is close to -1, there is a strong negative linear correlation between the age of the car (x) and the advertised selling price (P). As the age of the car increases, the advertised selling price tends to decrease.

(iii)Regression line is $P = -3.60x + 5$



(iv)Using P on x:

$280 = -3.59669x + 532.3118$

$x = 70.2$

The estimated age of the car is 64.4 months.

The estimate is reliable because r= -0.992 is close to -1, and $P = 280$ is within the sample data range of $130 < P < 546$. Interpolation for2 strongly linearly correlated variables is reliable.

(v)

$y = 120 - x \Rightarrow x = 120 - y$. Replace x by $120 - y$:

$P = -3.5966918(120 - y) + 532.3118$

$P = 3.60y + 101$