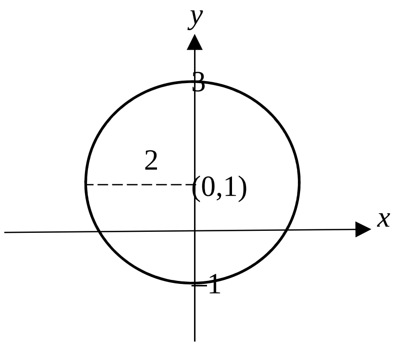
<u>**Jurong Junior College 2017 H1 Mathematics (8865) Preliminary Examination Solutions**</u>

Qn	**Soln**
1	Let \$*x*, \$*y*, \$*z* be the amount invested in stocks, bonds and gold respectively. $x+y+z=100\ 000$ $0.12x=0.08y+0.04z \Rightarrow \quad 0.12x-0.08y-0.04z=0$ $x-2y=4000$ Using GC, $x=28\ 000,\ y=12\ 000,\ z=60\ 000$ Total income $=0.12(28000)+0.08(12000)+0.04(60000)=6720$
2(a)	$x^2+(y-1)^2=4$ Curve *C* is a circle with centre (0,1) and radius 2 units.  $x^2+(k-1)^2=4$ has no real roots $\Rightarrow$ $y=k$ does not intersect the circle $\Rightarrow$ $k<-1$ or $k>3$
(b)	Sub $y=x+k+1$ into $x^2+(y-1)^2=4$, $x^2+(x+k)^2=4$ $x^2+x^2+2kx+k^2=4$ $2x^2+2kx+k^2-4=0$ Since line cuts *C* twice, Discriminant > 0 $(2k)^2-4(2)(k^2-4)>0$ $4k^2-8k^2+32>0$ $-4k^2+32>0$ $k^2-8<0$ $(k+\sqrt{8})(k-\sqrt{8})<0$ $-\sqrt{8}<k<\sqrt{8}$
3(i)	$QR=100-x$ $PR=\sqrt{(100-x)^2-x^2}$ $=\sqrt{10000-200x+x^2-x^2}$ $=\sqrt{10000-200x}$

$$A=\frac{1}{2}x\sqrt{10000-200x}$$

$$=\frac{1}{2}\sqrt{x^2\left(10000-200x\right)}$$

$$=\frac{1}{2}\sqrt{10000x^2-200x^3}$$

$$=\frac{10}{2}\sqrt{100x^2-2x^3}$$

$$=5\sqrt{100x^2-2x^3}\quad\text{(shown)}$$

(ii)

$$\frac{\mathrm{dA}}{\mathrm{d}x}=5\left(\frac{1}{2}\right)\left(100x^2-2x^3\right)^{-\frac{1}{2}}\left(200x-6x^2\right)=\frac{5\left(100x-3x^2\right)}{\sqrt{100x^2-2x^3}}$$

$$\frac{5\left(100x-3x^2\right)}{\sqrt{100x^2-2x^3}}=0$$

$$100x-3x^2=0$$

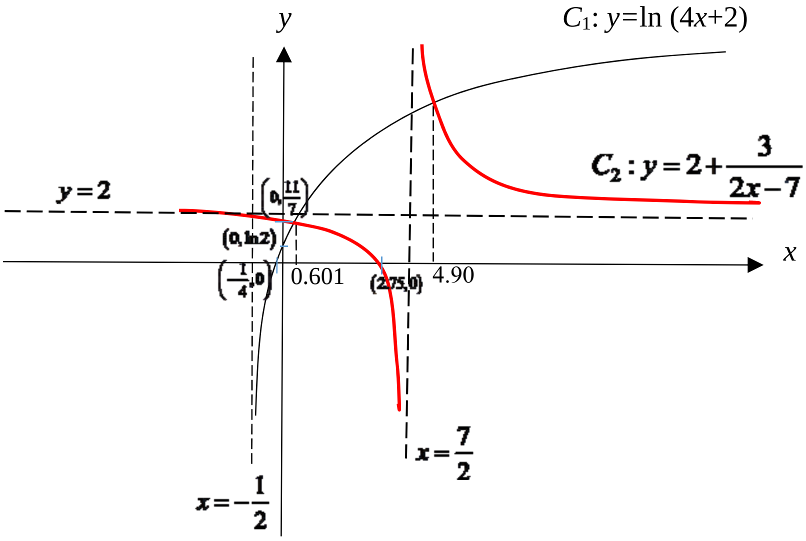
$$x\left(100-3x\right)=0$$

$x=0$ (N.A.) *or* $x=\dfrac{100}{3}$

x	33.2	$\frac{100}{3}=33.3$	33.4
$\frac{\mathrm{dA}}{\mathrm{d}x}$	0.345	0	-0.174
Slope	/	-	\

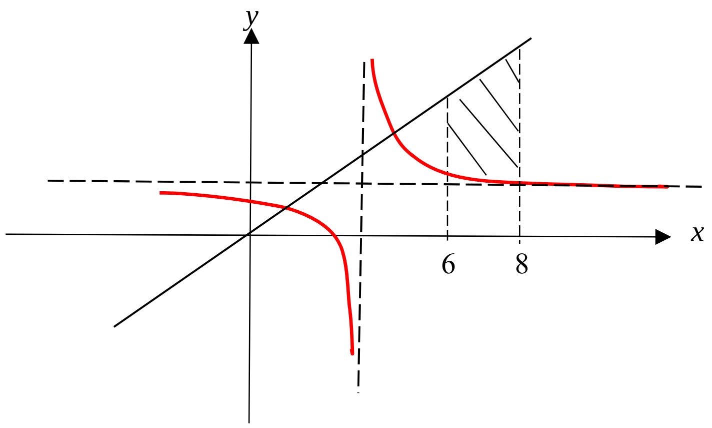
A is maximum when $x=\dfrac{100}{3}$

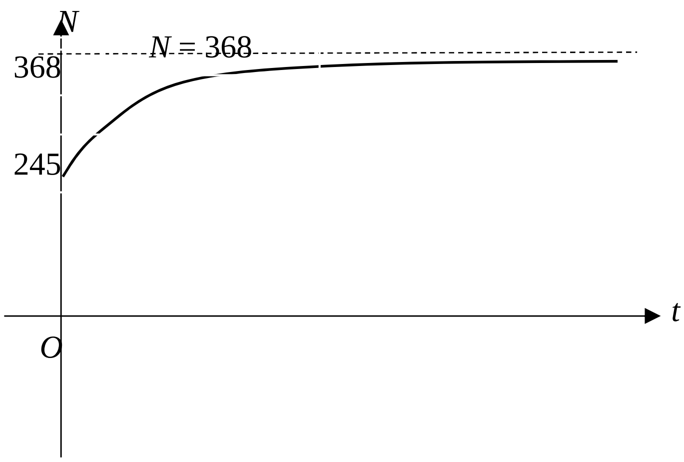
4 (i)

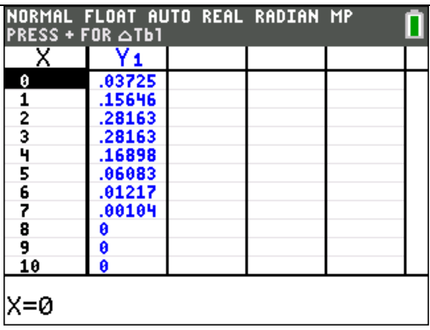


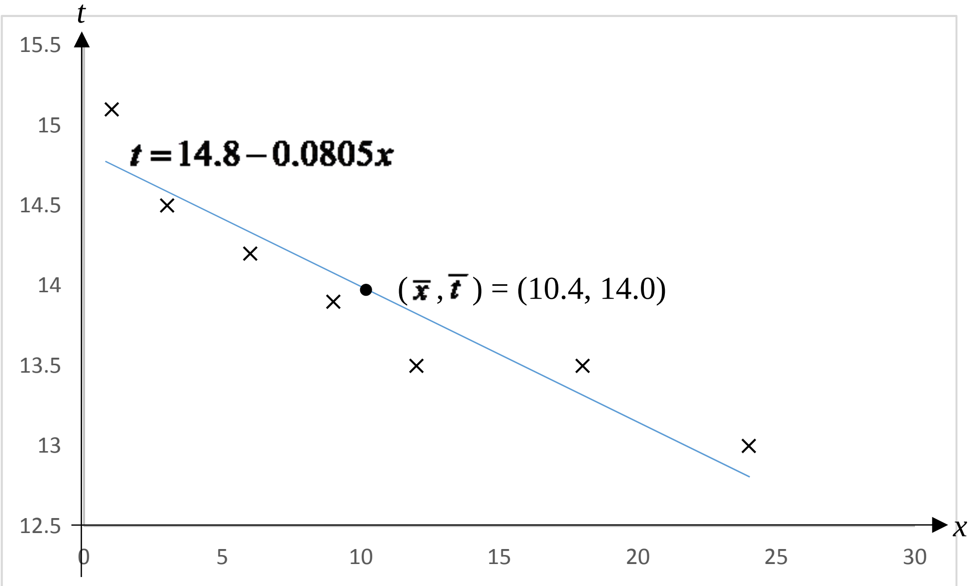
C$_1$: $y=\ln(4x+2)$: $\left(0,\ln 2\right)$ and $\left(-\frac{1}{4},0\right)$. Asymptote: $x=-\frac{1}{2}$

C$_2$: $y=2+\dfrac{3}{2x-7}$: $\left(0,\frac{11}{7}\right)$ and $\left(2.75,0\right)$. Asymptotes: $x=\dfrac{7}{2}$ and $y=2$.

(ii)	For $\ln(4x+2) > 2+\dfrac{3}{2x-7}$, $0.601 < x < \dfrac{7}{2}$ or $x > 4.90$
(iii)	$y=\ln(4x+2)$ $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{4}{4x+2}=\dfrac{2}{2x+1}$ When $x=2$, $y=\ln 10$, $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{2}{5}$ Equation of the tangent to the curve: $y-\ln 10=\dfrac{2}{5}\left(x-2\right)$ $y=\dfrac{2}{5}x-\dfrac{4}{5}+\ln 10$
(iv)	$\displaystyle\int 2+\frac{3}{2x-7}\,\mathrm{d}x=2x+\frac{3}{2}\ln\left(2x-7\right)+c$
(v)	 Area $\displaystyle=\int_6^8 x\,\mathrm{d}x-\int_6^8 2+\frac{3}{2x-7}\,\mathrm{d}x$ $\displaystyle=\left[\frac{x^2}{2}\right]_6^8-\left[2x+\frac{3}{2}\ln(2x-7)\right]_6^8$ $\displaystyle=\left[\frac{8^2}{2}-\frac{6^2}{2}\right]-\left[\left[2\left(8\right)+\frac{3}{2}\ln(2\left(8\right)-7)\right]-\left[2\left(6\right)+\frac{3}{2}\ln(2\left(6\right)-7)\right]\right]$ $\displaystyle=14-\left[4+\frac{3}{2}\ln 9-\frac{3}{2}\ln 5\right]$ $\displaystyle=10-\frac{3}{2}\ln\frac{9}{5}$
(ii)	<u>Alternative</u> Area = Area of trapezium $\displaystyle-\int_6^8 2+\frac{3}{2x-7}\,\mathrm{d}x$

	$=\frac{1}{2}(6+8)(2)-\int_6^8 2+\frac{3}{2x-7}\,\mathrm{d}x$ $=14-\left[2x+\frac{3}{2}\ln(2x-7)\right]_6^8$ $=14-\left[\left[2(8)+\frac{3}{2}\ln(2(8)-7)\right]-\left[2(6)+\frac{3}{2}\ln(2(6)-7)\right]\right]$ $=14-\left[4+\frac{3}{2}\ln 9-\frac{3}{2}\ln 5\right]$ $=10-\frac{3}{2}\ln\frac{9}{5}$
5 (a) (i)	$N=\frac{K}{1+0.5\mathrm{e}^{-0.6t}}$ When $t = 0$, $N=\frac{K}{1+0.5\mathrm{e}^{0}}=\frac{2}{3}K$
(ii)	As $t\to\infty, \mathrm{e}^{-0.6t}\to 0, N\to K$ The long term population size is K millions.
(iii)	$t = 2$, N=320 $320=\frac{K}{1+0.5\mathrm{e}^{-0.6(2)}}$ $K=320\left(1+0.5\mathrm{e}^{-1.2}\right)=368.191\approx 368$
(iv)	
(v)	$N=\frac{368.19}{1+0.5\mathrm{e}^{-0.6t}}=368.19(1+0.5\mathrm{e}^{-0.6t})^{-1}$ $\frac{\mathrm{d}N}{\mathrm{d}t}=-368.29(1+0.5\mathrm{e}^{-0.6t})^{-2}\left[(-0.6)0.5\mathrm{e}^{-0.6t}\right]$ $=\frac{368.19\times 0.5\times 0.6\mathrm{e}^{-0.6t}}{\left(1+0.5\mathrm{e}^{-0.6t}\right)^2}$ $\approx\frac{110\mathrm{e}^{-0.6t}}{\left(1+0.5\mathrm{e}^{-0.6t}\right)^2}$

(vi)	After 3 days, the rate of increase is: $\dfrac{dN}{dt}=\dfrac{110e^{-0.6(3)}}{\left(1+0.5e^{-0.6(3)}\right)^2}=15.5127\approx 15.5$ millions per day.
(b) (i)	$P=\displaystyle\int_0^5 \frac{60e^{-1.1t}}{\left(1+e^{-1.1t}\right)^2}\,dt=27.051\approx 27.1$
(ii)	P represents the growth of number of Type B bacteria in millions during the first 5 days.
6(i)	$P(R)=\left(\dfrac{2}{6}\times\dfrac{6}{10}\right)+\left(\dfrac{4}{6}\times\dfrac{5}{10}\right)=\dfrac{8}{15}$
(ii)	$P(A'\mid R)=\dfrac{P(A'\cap R)}{P(R)}=\dfrac{\frac{4}{6}\times\frac{5}{10}}{\frac{8}{15}}=\dfrac{5}{8}$
	A and R are not independent Any of the following reasons: (1) $P(R\mid A)=\dfrac{6}{10}\neq P(R)=\dfrac{8}{15}$ OR (2) $P(A)\times P(R)=\dfrac{2}{6}\times\dfrac{8}{15}=\dfrac{8}{45}$ & $P(A\cap R)=\dfrac{2}{6}\times\dfrac{6}{10}=\dfrac{1}{5}$ $P(A\cap R)\neq P(A)\times P(R)$ OR (3) $P(A'\mid R)=\dfrac{5}{8}\neq P(A')=\dfrac{4}{6}$, A' and R are not independent, thus A and R are not independent.
7(i)	(1) The probability that any one appointment will start late remains constant throughout the sample. (2) The punctuality of each appointment is independent of the punctuality of any other appointments. OR Whether an appointment start late is independent of any other appointments that start late.
(ii)	 NORMAL FLOAT AUTO REAL RADIAN MP PRESS + FOR △Tbl X | Y1 0 | .03725 1 | .15646 2 | .28163 3 | .28163 4 | .16898 5 | .06083 6 | .01217 7 | .00104 8 | 0 9 | 0 10 | 0 X=0 Using GC, [GC keystrokes: Y_1= binompdf (7, $\frac{3}{8}$,x)], the most likely numbers of appointments that start late = 2 and 3 [Note: most likely number is referring to the **MODE**, not $E(X)$]
(iii)	$L\sim B\left(7,\dfrac{3}{8}\right)$ $P(L\geq 3.5)=P(L\geq 4)$ $=1-P(L\leq 3)$ $=0.24302$ ≈ 0.243 (3sf)

(iv)	Let X denote the number of days out of 5, with at least half of the appointments starting late. $X \sim B(5, 0.24302)$ $\mathrm{P}(X \le 2) = 0.90371 \approx 0.904$ (3sf)
8(a)(i)	No. of ways $= 12! = 479\,001\,600$
(a)(ii)	No. of ways $= (4! \times 3! \times 3! \times 2!) \times 4! = 41472$
(b)(i)	P(all the tenors are chosen)$= \dfrac{{}^{3}C_{3} \times {}^{9}C_{2}}{{}^{12}C_{5}} = \dfrac{36}{792} = \dfrac{1}{22}$ (or 0.0455 3sf)
(b)(ii)	P(at least one woman is chosen | all the tenors are chosen) $= \dfrac{\text{n(1 woman, 3 tenors, 1 bass) + n (2 woman, 3 tenors)}}{\text{n(all tenors are chosen)}}$ $= \dfrac{\left({}^{7}C_{1} \times {}^{3}C_{3} \times {}^{2}C_{1}\right) + \left({}^{7}C_{2} \times {}^{3}C_{3}\right)}{{}^{3}C_{3}\,{}^{9}C_{2}}$ $= \dfrac{35}{36}$ or $0.972\,(3\text{sf})$
9(i)	
(ii)	$(\bar{x}, \bar{t}) = (10.4, 14.0)$
(iii)	$r = -0.941$
(iv)	$t = 14.796 - 0.080474x$ $t = 14.8 - 0.0805x$ (3 s.f.)
(v)	When $x = 4$ $t = 14.796 - 0.080474(4) = 14.474 \approx 14.5$ hours (3 s.f.) Since $r = -0.941$ is <u>close to – 1</u>, indicating a strong negative linear correlation between the age and average total sleep time of babies and <u>x = 4 is within the data range</u>, this is an interpolation. Hence, the estimate is reliable.
(vi)	x = 32 is outside the data range, we are doing extrapolation. The estimate will not be reliable.

10(i)	$\bar{x}=\frac{\sum(x-500)}{60}+500=\frac{318}{60}+500=505.3$ $s^2=\frac{1}{n-1}\left[\sum(x-500)^2-\frac{\left(\sum(x-500)\right)^2}{n}\right]$ $=\frac{1}{59}\left[25548.4-\frac{(318)^2}{60}\right]$ $=\frac{23863}{59}$ or $404.46\approx 404$ (3sf)
(ii)	$\text{H}_0: \mu=500$ $\text{H}_1: \mu\neq 500$ Under H_0, $\bar{X}\sim \text{N}\left(500,\frac{23863/59}{60}\right)$ approximately by CLT since n = 60 is large. Test statistic $Z=\frac{\bar{X}-500}{\sqrt{\frac{23863/59}{60}}}\sim \text{N}(0,\ 1)$ approximately. α = 0.03 From GC, p-value = 0.0412 Since p-value = 0.0412 > α = 0.03, we do not reject H_0 at the 3% level of significance and conclude there is insufficient evidence that the population mean weight of bags of cashew nuts is not 500g. ie, there is insufficient evidence at the 3% level of significance that the claim is invalid.
(iii)	It is not necessary to assume a normal distribution as the sample size, n = 60, is large, by Central Limit Theorem, $\bar{X}$ has a normal distribution approximately.
(iv)	For H_0 to be rejected, $\alpha\geq 0.0412$. If the test is conducted at the 5% level of significance, since $\alpha=0.05>0.0412=p$-value, we reject H_0 at the 5% level of significance and conclude there is sufficient evidence that the population mean weight of bags of cashew nuts is not 500g. ie, there is sufficient evidence at the 5% level of significance that the claim is invalid.
(v)	$s^2=\frac{70}{69}(23.4^2)=\frac{63882}{115}=555.50\approx 556$ $\text{H}_0: \mu=500$ $\text{H}_1: \mu<500$ Under H_0, $\bar{X}\sim \text{N}\left(500,\frac{555.50}{70}\right)$ Test statistic $Z=\frac{\bar{X}-500}{\sqrt{\frac{555.50}{70}}}\sim \text{N}(0,\ 1)$

	$\alpha = 0.05$ mean weight of bags of cashew nuts is not at least 500 g $\Rightarrow$ H_0 is rejected at 5% level of significance $\Rightarrow \dfrac{\bar{x}-500}{\sqrt{\frac{555.50}{70}}} \le -1.6449$ $\Rightarrow \bar{x} \le 500-1.6449\sqrt{\frac{555.50}{70}}$ $\Rightarrow \bar{x} \le 495.37(2\text{dp})$
11	Let X and Y denote the mass of a cheese tart and an empty box respectively. $X \sim \mathrm{N}(60, 3.5^2)$ and $Y \sim \mathrm{N}(52, 0.8^2)$
(i)	P(X<58) = 0.28385 $\approx$ 0.284 (3sf)
(ii)	Let N be the number of tarts in a box with mass less than 58g. $N \sim \mathrm{B}(6, 0.28385)$ $\mathrm{P}(N=2) = 0.31790 \approx 0.318$ (3sf)
	Alternatively, Probability $= {}^6C_2(0.28385)^2(1-0.28385)^4$ $= 0.31790 \approx 0.318$ (3sf)
(iii)	Let $T = X_1 + X_2 + \ldots + X_6 + Y$ $\mathrm{E}(T) = 6\mathrm{E}(X) + \mathrm{E}(Y) = 6(60) + 52 = 412$ $\mathrm{Var}(T) = 6\mathrm{Var}(X) + \mathrm{Var}(Y) = 6(3.5^2) + 0.8^2 = 74.14$ $T \sim \mathrm{N}(412, 74.14)$
	P(T > 415) = 0.36376 $\approx$ 0.364 (3sf)
(iv)	$C = 2.1(X_1 + X_2 + \ldots + X_6) + 0.3Y$ $\mathrm{E}(C) = 2.1 \times 6\mathrm{E}(X) + 0.3\mathrm{E}(Y)$ $= (2.1)(6)(60) + 0.3(52)$ $= 771.6$ $\mathrm{Var}(C) = 2.1^2 \times 6\mathrm{Var}(X) + 0.3^2\mathrm{Var}(Y)$ $= 2.1^2(6)(3.5^2) + 0.3^2(0.8^2)$ $= 324.1926$ $C \sim \mathrm{N}(771.6\ ,\ 324.1926)$
	P(747 <C < 774) = 0.467 (3sf)
(v)	Let M denote the mass of a mini cheese tart. $\mathrm{E}(M) = 35$, $\mathrm{Var}(M) = 3.5^2$ $\bar{M} \sim \mathrm{N}(35, \frac{3.5^2}{30})$ approximately by CLT since n = 30 is large. $\mathrm{P}(\bar{M} > a) = 0.2$ $\mathrm{P}(\bar{M} \le a) = 1 - 0.2 = 0.8$ $a = 35.5$ (3sf)