2016 Preliminary Examination H2 Mathematics 9740 Paper 1 (Solutions)

1 A fitness assessment walk is conducted where participants walk briskly around a running path. The participants' walking time and heart rate are recorded at the end of the walk.

The formula for calculating the Fitness Index of a participant is as follows:
420 + (Age × 0.2) – (Walking Time × a) – (Body Mass Index × b) – (Heart Rate × c)
where a, b and c are real constants.

Data from 3 participants, Anand, Beng and Charlie are given in the table.

Name	Age	Walking Time	Body Mass Index	Heart Rate	Fitness Index
Anand	32	17.5	25	100	102.4
Beng	19	18.5	19	120	92.6
Charlie	43	17	23	90	121.2

Find the values of a, b and c. [3]

[Solution]

$420+6.4-17.5a-25b-100c=102.4$ $\quad$ $17.5a+25b+100c=324$ ----- (1)

$420+3.8-18.5a-19b-120c=92.6$ $\quad$ or $\quad$ $18.5a+19b+120c=331.2$ ---- (2)

$420+8.6-17a-23b-90c=121.2$ $\quad$ $17a+23b+90c=307.4$ ----- (3)

Using GC, $a=\dfrac{58}{5},\ b=\dfrac{13}{5}, c=\dfrac{14}{25}$

2 Solve the inequality $\dfrac{x^2-2a^2}{x}<a$, giving your answer in terms of a, where a is a positive real constant. [3]

Hence solve $\dfrac{x^2-2a^2}{|x|}<a$. [2]

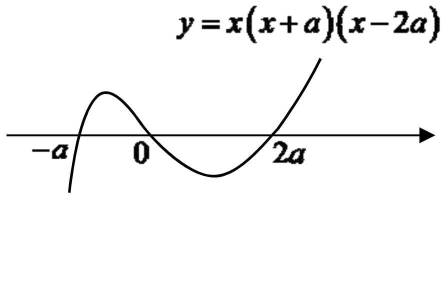
[Solution]

$\dfrac{x^2-2a^2}{x}<a\ ,\quad x\neq 0$

$\dfrac{x^2-ax-2a^2}{x}<0$

$x(x+a)(x-2a)<0$

$x<-a \quad$ or $\quad 0<x<2a$



Replace x by $|x|$,

$|x|<-a \qquad$ or $\qquad 0<|x|<2a$

(no real solution) $\quad -2a<x<2a,\quad x\neq 0$

3 **(i)** Use the substitution $u = x^2$ to find $\int \frac{x}{\sqrt{k^2 - x^2}}\,\mathrm{d}x$ in terms of x and the constant k. [3]

(ii) Find the exact value of $\int_0^2 \mathrm{f}(x)\,\mathrm{d}x$, where

$$\mathrm{f}(x) = \begin{cases} \dfrac{2}{6-x^2}, & 0 \le x < \sqrt{2}, \\ \dfrac{x}{\sqrt{6-x^2}}, & \sqrt{2} \le x < 2. \end{cases}$$

[3]

[Solution]

(i) $\boxed{u = x^2 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 2x}$ or $\boxed{x = \sqrt{u} \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}u} = \frac{1}{2\sqrt{u}}}$

$$\int \frac{x}{\sqrt{k^2 - x^2}}\,\mathrm{d}x = \frac{1}{2}\int \frac{1}{\sqrt{k^2 - u}}\,\mathrm{d}u$$

$$= \frac{1}{2}\,\frac{\sqrt{k^2 - u}}{\left(\frac{1}{2}\right)(-1)} + C$$

$$= -\sqrt{k^2 - x^2} + C$$

(ii) $$\int_0^2 \mathrm{f}(x)\,\mathrm{d}x = \int_0^{\sqrt{2}} \frac{2}{6-x^2}\,\mathrm{d}x + \int_{\sqrt{2}}^2 \frac{x}{\sqrt{6-x^2}}\,\mathrm{d}x$$

$$= \left[\frac{2}{2\sqrt{6}}\ln\left(\frac{\sqrt{6}+x}{\sqrt{6}-x}\right)\right]_0^{\sqrt{2}} + \left[-\sqrt{6-x^2}\,\right]_{\sqrt{2}}^2$$

$$= \frac{1}{\sqrt{6}}\ln\left(\frac{\sqrt{6}+\sqrt{2}}{\sqrt{6}-\sqrt{2}}\right) - \sqrt{2} + 2$$

4 Relative to the origin O, the points A, B, M and N have position vectors **a, b**, **m** and **n** respectively, where **a** and **b** are non-parallel vectors. It is given that $\mathbf{m} = \lambda\mathbf{a} + (1-\lambda)\mathbf{b}$ and $\mathbf{n} = 2(1-\lambda)\mathbf{a} - \lambda\mathbf{b}$ where λ is a real parameter.

Show that $\mathbf{m}\times\mathbf{n} = (3\lambda^2 - 4\lambda + 2)\ (\mathbf{b}\times\mathbf{a})$. [2]

It is given that $|\mathbf{a}| = 3$, $|\mathbf{b}| = 4$ and the angle between **a** and **b** is $\frac{\pi}{6}$. Find the smallest area of the triangle MON as λ varies. [4]

[Solution]

$$\mathbf{m}\times\mathbf{n} = (\lambda\mathbf{a} + (1-\lambda)\mathbf{b}) \times (2(1-\lambda)\mathbf{a} - \lambda\mathbf{b})$$

$$= 2\lambda(1-\lambda)\ (\mathbf{a}\times\mathbf{a}) - \lambda^2\ (\mathbf{a}\times\mathbf{b}) \quad +2(1-\lambda)^2\ (\mathbf{b}\times\mathbf{a}) - \lambda(1-\lambda)\ (\mathbf{b}\times\mathbf{b})$$

$$= 2(1-\lambda)^2\ (\mathbf{b}\times\mathbf{a}) - \lambda^2\ (\mathbf{a}\times\mathbf{b}) \qquad \text{since } \mathbf{a}\times\mathbf{a} = \underset{\sim}{\mathbf{0}} = \mathbf{b}\times\mathbf{b}$$

$$= (2(1-\lambda)^2 + \lambda^2)\ (\mathbf{b}\times\mathbf{a}) \qquad \text{since } \mathbf{a}\times\mathbf{b} = -\,\mathbf{b}\times\mathbf{a}$$

$$= (3\lambda^2 - 4\lambda + 2)\ (\mathbf{b}\times\mathbf{a})$$

Area of triangle MON $= \frac{1}{2}|\mathbf{m}\times\mathbf{n}| = \frac{1}{2}\left|(3\lambda^2 - 4\lambda + 2)(\mathbf{b}\times\mathbf{a})\right|$

$$= \frac{1}{2}\left|3\lambda^2 - 4\lambda + 2\right|\ \left|\mathbf{b}\times\mathbf{a}\right|$$

$$= \frac{1}{2}\left|3\left(\lambda - \frac{2}{3}\lambda\right)^2 + \frac{2}{3}\right|\ |\mathbf{b}||\mathbf{a}|\sin\frac{\pi}{6}$$

$$= 3\left|3\left(\lambda - \frac{2}{3}\right)^2 + \frac{2}{3}\right|$$

$\therefore$ smallest area is $3\times\frac{2}{3} = 2$ units^2

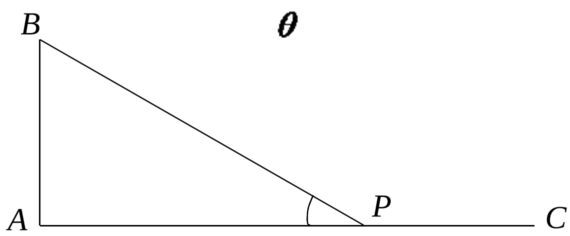
__Alternative solution__

Using GC, the minimum value of $3\lambda^2 - 4\lambda + 2$ occurs when $\lambda = \frac{2}{3}$

$\therefore$ smallest area $= \frac{1}{2}\left[3\left(\frac{2}{3}\right)^2 - 4\left(\frac{2}{3}\right) + 2\right]|\mathbf{b}||\mathbf{a}|\sin\frac{\pi}{6}$

$$= \frac{1}{2}\left[\frac{2}{3}\right]\times 6 = 2 \text{ units}^2$$

5



In the diagram, A and C are fixed points 500 m apart on horizontal ground. Initially, a drone is at point A and an observer is standing at point C. The drone starts to ascend vertically at a steady rate of 3 m s^{-1} as the observer starts to walk towards A with a steady speed of 4 m s^{-1}. At time t, the drone is at point B and the observer is at point P. Given that the angle APB is θ radians, show that $\theta = \tan^{-1}\left(\dfrac{3t}{500-4t}\right)$. [2]

(i) Find $\dfrac{\mathrm{d}\theta}{\mathrm{d}t}$ in terms of t. [2]

(ii) Using differentiation, find the time t when the rate of change of θ is maximum. [4]

[Solution]

At time t, $AB = 3t$, $AP = 500 - 4t$

$$\tan\theta = \frac{AB}{AP} = \frac{3t}{500-4t}$$

$$\theta = \tan^{-1}\left(\frac{3t}{500-4t}\right) \quad \text{(shown)}$$

(i)
$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{1}{1+\left(\dfrac{3t}{500-4t}\right)^2} \times \frac{(500-4t)(3)-3t(-4)}{(500-4t)^2}$$

$$= \frac{(500-4t)^2}{(500-4t)^2+(3t)^2} \times \frac{1500}{(500-4t)^2} = \frac{1500}{9t^2+(500-4t)^2}$$

$$\left(= \frac{1500}{25t^2-4000t+250000} = \frac{60}{t^2-160t+10000} \right)$$

(ii) **<u>Method 1</u>**:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right) = \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = \frac{-1500\left(18t+2(500-4t)(-4)\right)}{\left(9t^2+(500-4t)^2\right)^2} = \frac{-1500(50t-4000)}{\left(9t^2+(500-4t)^2\right)^2}$$

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = 0 \Rightarrow -1500(50t-4000) = 0$$

$$\Rightarrow t = 80$$

t	80^-	80	80^+
$\dfrac{d^2\theta}{dt^2}$	+ve	0	−ve
slope	/	—	\

Using first derivative test, rate of change of θ is maximum at $t = 80$

<u>Method 2</u>:

$$\frac{d\theta}{dt} = \frac{60}{t^2 - 160t + 10000} = \frac{60}{(t-80)^2 + 3600}$$

$\dfrac{d\theta}{dt}$ is maximum when $(t-80)^2 = 0$ i.e. when $t = 80$.

6 The functions f and g are defined by

$$f : x \mapsto \ln\left(x^2 - x + 1\right),\ x \in \mathbb{R},\ x \le 1,$$
$$g : x \mapsto e^x,\ x \in \mathbb{R}.$$

Sketch the graph of f and explain why f does not have an inverse. [2]

The function h is defined by

$$h : x \mapsto f(x),\ x \in \mathbb{R},\ x \le k.$$

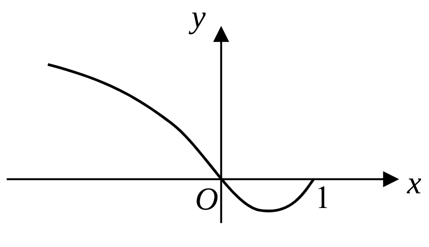
State the maximum value of k such that h^{-1} exists. [1]

Using this maximum value of k,

(i) show that the composite function gh exists, [1]

(ii) find $(gh)^{-1}(x)$, stating the domain of $(gh)^{-1}$. [4]

[Solution]



The line $y = 0$ cuts the graph of f twice, thus f is not one-one and so f does not have an inverse.

Using GC, minimum value of f occurs when $x = \dfrac{1}{2}$

<u>OR</u> $x^2 - x + 1 = \left(x - \dfrac{1}{2}\right)^2 + \dfrac{3}{4} \Rightarrow$ minimum point: $\left(\dfrac{1}{2}, \dfrac{3}{4}\right)$

Hence maximum value of k is $\dfrac{1}{2}$

(i) Since $R_{\text{h}} = \left[\ln\frac{3}{4}, \infty\right) \subseteq \mathbb{R} = D_{\text{g}}$, the function gh exists.

(ii) $\text{gh}(x) = \text{g}\left(\ln\left(x^2 - x + 1\right)\right) = \text{e}^{\ln(x^2 - x + 1)} = x^2 - x + 1, \quad x \le \frac{1}{2}$

Let $y = \text{gh}(x)$

$y = x^2 - x + 1 = \left(x - \frac{1}{2}\right)^2 + \frac{3}{4}$

$\Rightarrow x = \frac{1}{2} - \sqrt{y - \frac{3}{4}} \quad \left(\text{reject } x = \frac{1}{2} + \sqrt{y - \frac{3}{4}} \quad \because x \le \frac{1}{2}\right)$

$\therefore (\text{gh})^{-1}(x) = \frac{1}{2} - \sqrt{x - \frac{3}{4}}$

$\text{D}_{(\text{gh})^{-1}} = \text{R}_{\text{gh}} = \left[\frac{3}{4}, \infty\right)$

7 **(a)** The positive integers are grouped into sets as shown below, so that the number of integers in each set after the first set is three more than that in the previous set.

$\{1\}, \{2, 3, 4, 5\}, \{6, 7, 8, 9, 10, 11, 12\}, \ldots$

Find, in terms of r, the number of integers in the rth set. [1]

Show that the last integer in the rth set is $\frac{r}{2}(3r - 1)$. [2]

Deduce, in terms of r, the first integer in the rth set. [2]

(b) Find $\sum_{r=1}^{n}\left(1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \ldots + \left(\frac{1}{2}\right)^r\right)$ in terms of n. [4]

[Solution]

(a)

	$\{1\}$,	$\{2, 3, 4, 5\}$,	$\{6, 7, 8, 9, 10, 11, 12\}$,	…
Set	1st	2nd	3rd	, …
No. of terms	1	4	7	, …

No. of integers in rth set $= 1 + (r - 1)3$
$= 3r - 2$

Last integer in rth set $=$ Sum of no. of terms from 1st to rth set

$= 1 + 4 + 7 + \ldots + (3r - 2)$

$= \frac{r}{2}\left[2(1) + (r - 1)(3)\right]$

or $\frac{r}{2}\left[1 + (3r - 1)\right]$

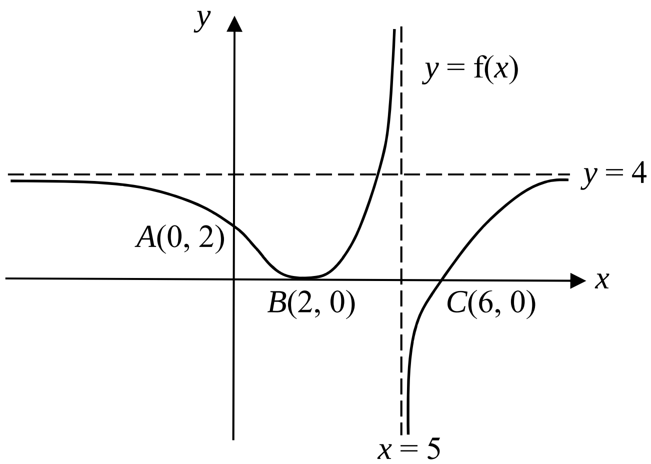
$= \frac{r}{2}(3r - 1)$

Hence first integer in rth set $= \frac{r}{2}(3r - 1) - (3r - 2) + 1$

$$\text{or } \frac{1}{2}(r-1)\left[3(r-1)-1\right]+1$$
$$=\frac{3r^2-7r+6}{2}$$

(b)
$$\sum_{r=1}^{n}\left(1+\frac{1}{2}+\left(\frac{1}{2}\right)^2+\ldots+\left(\frac{1}{2}\right)^r\right)$$
$$=\sum_{r=1}^{n}\left(\frac{1\left(1-\left(\frac{1}{2}\right)^{r+1}\right)}{1-\frac{1}{2}}\right)$$
$$=2\sum_{r=1}^{n}\left(1-\tfrac{1}{2}\left(\tfrac{1}{2}\right)^r\right)$$
$$=2n-\sum_{r=1}^{n}\left(\tfrac{1}{2}\right)^r$$
$$=2n-\frac{\frac{1}{2}\left(1-\left(\frac{1}{2}\right)^n\right)}{1-\frac{1}{2}}$$
$$=2n-1+\left(\tfrac{1}{2}\right)^n$$

8 The graph of $y=\mathrm{f}(x)$ intersects the axes at $A(0, 2)$, $B(2, 0)$ and $C(6, 0)$ as shown below. The lines $y=4$ and $x=5$ are asymptotes to the graph, and $B(2, 0)$ is a minimum point.



On separate diagrams, sketch the graphs of

(i) $y=\mathrm{f}(|x|)$, [2]

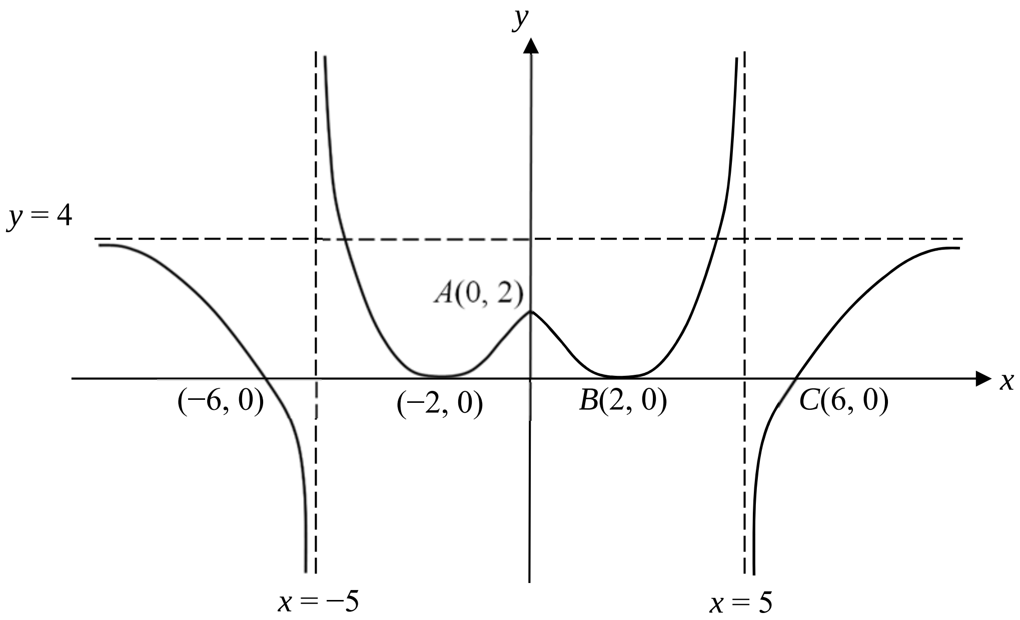
(ii) $y^2=\mathrm{f}(x)$, [3]

(iii) $y=\dfrac{1}{\mathrm{f}(x)}$, [3]

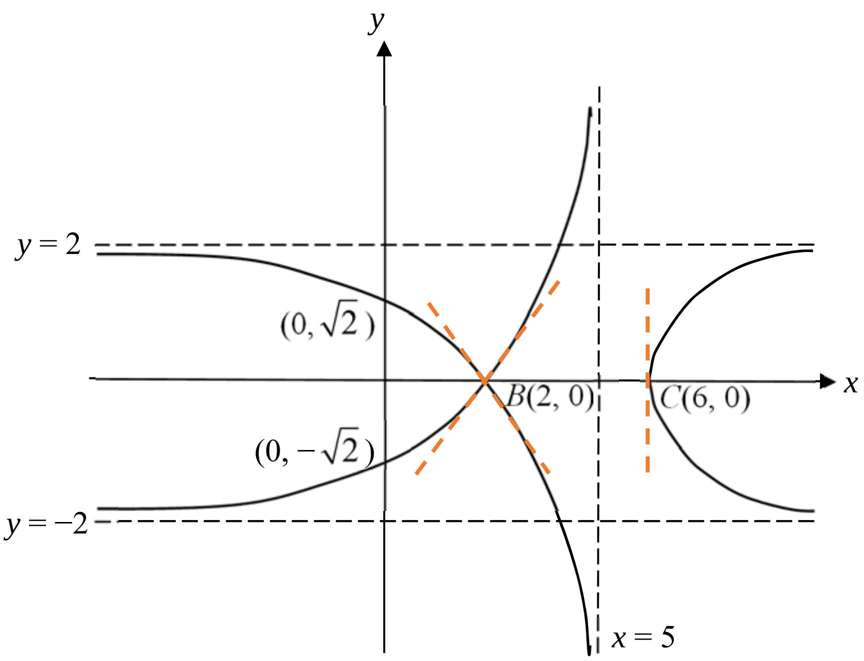
stating the equations of any asymptotes, coordinates of any stationary points and points of intersection with the axes.

[Solution]

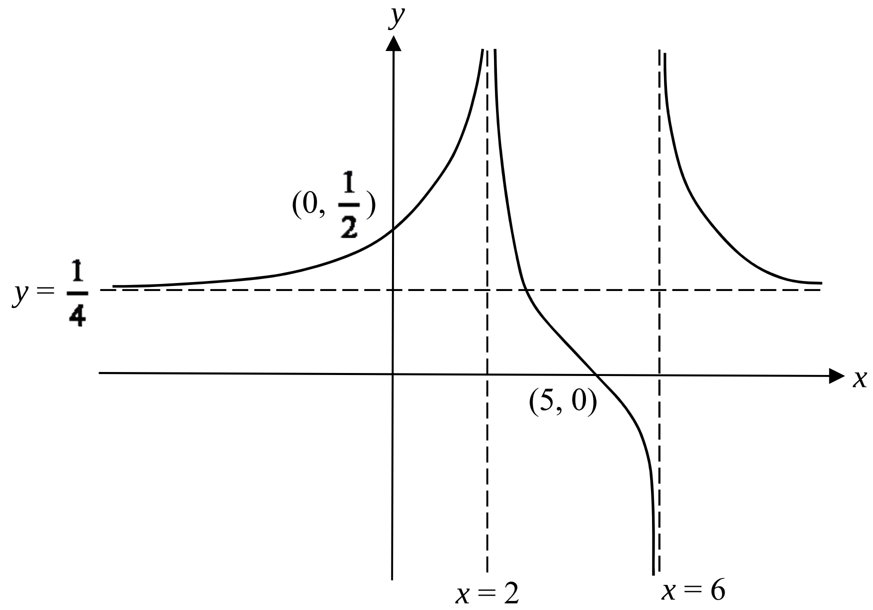
(i) $y = \mathrm{f}(|x|)$



(ii) $y^2 = \mathrm{f}(x)$



(iii) $y = \dfrac{1}{f(x)}$



9 **(a)** Given that x is small such that x^3 and higher powers of x can be neglected, show that

$$\frac{\sqrt{2}\sin\left(\frac{\pi}{4}+x\right)}{\sqrt{2-\cos x}} \approx a + bx + cx^2,$$

for constants a, b and c to be determined. [4]

(b) The curve $y = f(x)$ passes through the point $(0, -1)$ and satisfies the differential equation

$$(1+x^2)\frac{dy}{dx} = e^{-y}.$$

(i) Find the Maclaurin series for y, up to and including the term in x^2. [3]

(ii) By using an appropriate expansion from the List of Formulae (MF15), obtain the Maclaurin series for $\ln(2+y)$, up to and including the term in x^2. [3]

[Solution]

(a) $$\frac{\sqrt{2}\sin\left(\frac{\pi}{4}+x\right)}{\sqrt{2-\cos x}} = \frac{\sqrt{2}\left(\sin\frac{\pi}{4}\cos x + \cos\frac{\pi}{4}\sin x\right)}{\sqrt{2-\cos x}}$$

$$= \frac{\sin x + \cos x}{\sqrt{2-\cos x}}$$

$$\approx \frac{x+\left(1-\frac{x^2}{2}\right)}{\sqrt{2-\left(1-\frac{x^2}{2}\right)}}$$

$$= \frac{1+x-\frac{x^2}{2}}{\sqrt{1+\frac{x^2}{2}}}$$

$$= \left(1+x-\frac{x^2}{2}\right)\left(1+\frac{1}{2}x^2\right)^{-\frac{1}{2}}$$

$$= \left(1+x-\frac{x^2}{2}\right)\left(1-\frac{1}{4}x^2+\ldots\right)$$

$$\approx 1+x-\frac{3}{4}x^2+\ldots$$

(b)(i) Given $(1+x^2)\frac{\mathrm{d}y}{\mathrm{d}x}=\mathrm{e}^{-y}$

Implicit differentiate w.r.t. x, $(1+x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+2x\frac{\mathrm{d}y}{\mathrm{d}x}=-\mathrm{e}^{-y}\frac{\mathrm{d}y}{\mathrm{d}x}$

When $x=0$, $y=-1$, $\frac{\mathrm{d}y}{\mathrm{d}x}=\mathrm{e}$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-\mathrm{e}^2$

So $y=-1+\mathrm{e}x-\frac{\mathrm{e}^2}{2}x^2+\ldots$

(ii) $\ln(2+y)=\ln\left(1+\mathrm{e}x-\frac{\mathrm{e}^2}{2}x^2+\ldots\right)$

$$= \left(\mathrm{e}x-\frac{\mathrm{e}^2}{2}x^2+\ldots\right)-\frac{1}{2}\left(\mathrm{e}x-\frac{\mathrm{e}^2}{2}x^2+\ldots\right)^2$$

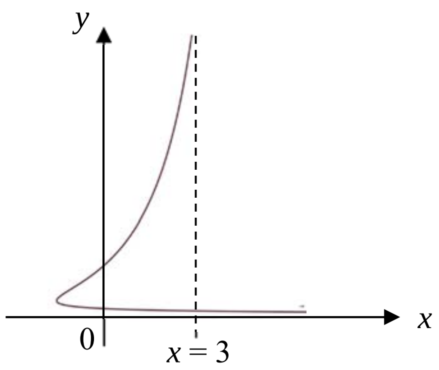
$$\approx \mathrm{e}x-\frac{\mathrm{e}^2}{2}x^2-\frac{1}{2}(\mathrm{e}x)^2$$

$$= \mathrm{e}x-\mathrm{e}^2x^2$$

10

The diagram shows the curve with parametric equations

$$x = 2t + t^2, \quad y = \frac{1}{(1-t)^2}, \quad \text{for } t < 1.$$



The curve has a vertical asymptote $x = 3$.

(i) Find the coordinates of the points where the curve cuts the y-axis. [2]

(ii) Find the equation of the tangent to the curve that is parallel to the y-axis. [4]

(iii) Express the area of the finite region bounded by the curve and the y-axis in the form $\int_a^b \mathrm{f}(t)\,\mathrm{d}t$, where a, b and f are to be determined. Use the substitution $u = 1 - t$ to find this area, leaving your answer in exact form. [5]

[Solution]

(i) When $x = 0$, $t(2+t) = 0 \Rightarrow t = 0$ or $t = -2$

Coordinates are $(0, 1)$ and $(0, \frac{1}{9})$

(ii) $\frac{\mathrm{d}x}{\mathrm{d}t} = 2 + 2t$, $\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{2}{(1-t)^3}$

$\therefore \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{(1+t)(1-t)^3}$

When tangent is parallel to y-axis,

$(1+t)(1-t)^3 = 0 \Rightarrow t = -1$ or $t = 1$ (vertical asymptote)

Equation of tangent is $x = -1$

(iii)
$$
\begin{aligned}
\text{Area} &= -\int_{\frac{1}{9}}^{1} x \quad \mathrm{d}y \\
&= -\int_{-2}^{0} (2t + t^2)\cdot\frac{2}{(1-t)^3}\,\mathrm{d}t \\
&= \int_3^1 (2(1-u) + (1-u)^2)\cdot\frac{2}{u^3}\,\mathrm{d}u \\
&= -2\int_1^3 \frac{u^2 - 4u + 3}{u^3}\mathrm{d}u \\
&= -2\int_1^3 \left(\frac{1}{u} - \frac{4}{u^2} + \frac{3}{u^3}\right)\mathrm{d}u \\
&= -2\left[\ln u + \frac{4}{u} - \frac{3}{2u^2}\right]_1^3 \\
&= -2\left[\left(\ln 3 + \frac{4}{3} - \frac{3}{18}\right) - \left(4 - \frac{3}{2}\right)\right] = \frac{8}{3} - 2\ln 3
\end{aligned}
$$

Let $u = 1 - t$
$\frac{\mathrm{d}u}{\mathrm{d}t} = -1$
When $t = 0$, $u = 1$
When $t = -2$, $u = 3$

11 On the remote island of Squirro, ecologists introduced a non-native species of insects that can feed on weeds that are killing crops. Based on past studies, ecologists have observed that the birth rate of the insects is proportional to the number of insects, and the death rate is proportional to the square of the number of insects. Let x be the number of insects (in hundreds) on the island at time t months after the insects were first introduced.

Initially, 10 insects were released on the island. When the number of insects is 50, it is changing at a rate that is $\frac{3}{4}$ times of the rate when the number of insects is 100. Show that

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \beta x(2-x)$$

where β is a positive real constant. [3]

Solve the differential equation and express x in the form $\frac{p}{1+q\mathrm{e}^{-2\beta t}}$, where p and q are constants to be determined. [6]

Sketch the solution curve and state the number of insects on the island in the long run. [3]

[Solution]

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \text{birth rate} - \text{death rate}$$

$$= \lambda x - \beta x^2 \qquad \text{where } \lambda \text{ and } \beta \text{ are positive real constants}$$

Given $\left.\frac{\mathrm{d}x}{\mathrm{d}t}\right|_{x=\frac{1}{2}} = \frac{3}{4}\times\left.\frac{\mathrm{d}x}{\mathrm{d}t}\right|_{x=1}$

$$\lambda\left(\frac{1}{2}\right) - \beta\left(\frac{1}{2}\right)^2 = \frac{3}{4}(\lambda-\beta) \Rightarrow \lambda = 2\beta$$

Hence $\frac{\mathrm{d}x}{\mathrm{d}t} = \beta x(2-x)$

$$\int \frac{1}{2x-x^2}\,\mathrm{d}x = \beta\int \mathrm{d}t$$

$$\frac{1}{2}\int\left(\frac{1}{x}+\frac{1}{2-x}\right)\mathrm{d}x = \beta\int \mathrm{d}t$$

$$\frac{1}{2}[\ln|x| - \ln|2-x|] = \beta t + c$$

$$\frac{1}{2}\left[\ln\left|\frac{x}{2-x}\right|\right] = \beta t + c$$

$$\frac{x}{2-x} = A\mathrm{e}^{2\beta t} \text{ where } A = \pm\mathrm{e}^{2c}$$

Subst $t=0,\ x=0.1 \Rightarrow \frac{0.1}{1.9} = A \Rightarrow A = \frac{1}{19}$

$$x = \frac{2}{19}\mathrm{e}^{2\beta t} - \frac{1}{19}x\,\mathrm{e}^{2\beta t}$$

$$x=\frac{\frac{2}{19}\mathrm{e}^{2\beta t}}{1+\frac{1}{19}\mathrm{e}^{2\beta t}}$$

$$=\frac{2\mathrm{e}^{2\beta t}}{19+\mathrm{e}^{2\beta t}}=\frac{2}{1+19\mathrm{e}^{-2\beta t}}$$

<u>Alternative solution:</u>

$$\int\frac{1}{2x-x^2}\,\mathrm{d}x=\beta\int\mathrm{d}t$$

$$-\int\frac{1}{(x-1)^2-1}\,\mathrm{d}x=\beta\int\mathrm{d}t$$

$$-\frac{1}{2}\left[\ln\left|\left(\frac{x-1-1}{x-1+1}\right)\right|\right]=\beta t+c$$

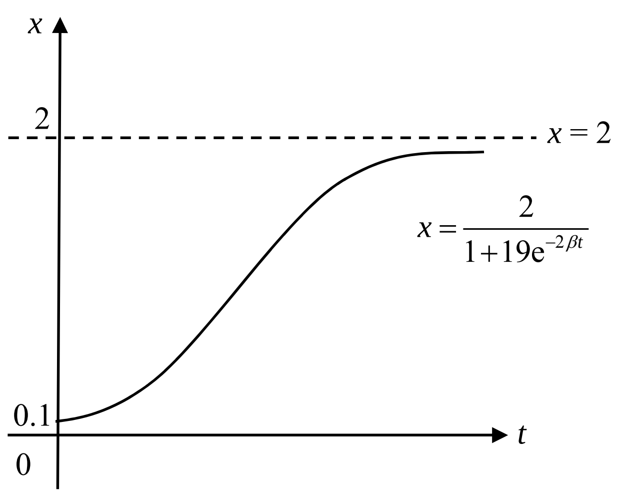
$$-\frac{1}{2}\left[\ln\left|\left(\frac{x-2}{x}\right)\right|\right]=\beta t+c$$

$$\frac{x-2}{x}=A\mathrm{e}^{-2\beta t}\quad\text{where } A=\pm\mathrm{e}^{-2c}$$

Subst $t=0,\ x=0.1\ \Rightarrow\ \frac{-1.9}{0.1}=A\ \Rightarrow\ A=-19$

$$x\left(1+19\mathrm{e}^{-2\beta t}\right)=2$$

$$x=\frac{2}{1+19\mathrm{e}^{-2\beta t}}$$



The number of insects will approach 200 in the long run.

12 **(a)** The complex numbers z_1 and z_2 satisfy the following simultaneous equations

$$2z_1 + \mathrm{i}z_2^* = 7 - 6\mathrm{i},$$
$$z_1 - \mathrm{i}z_2 = 6 - 6\mathrm{i}.$$

Find z_1 and z_2 in the form $x + y\mathrm{i}$, where x and y are real. [4]

(b) It is given that $w = \frac{1}{2} - \frac{1}{2}\mathrm{i}$. Find the modulus and argument of w, leaving your answers in exact form. [2]

It is also given that the modulus and argument of another complex number v is 2 and $\frac{\pi}{6}$ respectively.

(i) Find the exact values of the modulus and argument of $\frac{v}{w^*}$. [3]

(ii) By first expressing v in the form $\sqrt{c} + d\,\mathrm{i}$ where c and d are integers, find the real and imaginary parts of $\frac{v}{w^*}$ in surd form. [3]

(iii) Deduce that $\tan\left(\frac{\pi}{12}\right) = 2 - \sqrt{3}$. [2]

[Solution]

(a) $2z_1 + \mathrm{i}z_2^* = 7 - 6\mathrm{i}$ --- (1)

$z_1 - \mathrm{i}z_2 = 6 - 6\mathrm{i}$ --- (2)

(1) – (2) x 2: $\mathrm{i}z_2^* + 2\mathrm{i}z_2 = 7 - 6\mathrm{i} - 2(6 - 6\mathrm{i}) = -5 + 6\mathrm{i}$

$z_2^* + 2z_2 = 6 + 5\mathrm{i}$

Since $z_2^* + 2z_2 = 3\mathrm{Re}(z_2) + \mathrm{Im}(z_2)\mathrm{i} = 6 + 5\mathrm{i}$, $z_2 = 2 + 5\mathrm{i}$

Sub $z_2 = 2 + 5\mathrm{i}$ into (2): $z_1 = 6 - 6\mathrm{i} + \mathrm{i}(2 + 5\mathrm{i}) = 1 - 4\mathrm{i}$

(b) $|w| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2} = \frac{1}{\sqrt{2}}$

$\arg(w) = \tan^{-1}(-1) = -\frac{\pi}{4}$

(i) $\left|\frac{v}{w^*}\right| = \frac{|v|}{|w^*|} = \frac{|v|}{|w|} = \frac{2}{\left(\frac{1}{\sqrt{2}}\right)} = 2\sqrt{2}$

$\arg\left(\frac{v}{w^*}\right) = \arg(v) - \arg(w^*) = \arg(v) + \arg(w) = \frac{\pi}{6} - \frac{\pi}{4} = -\frac{\pi}{12}$

(ii) $v = 2\left(\cos\frac{\pi}{6} + \mathrm{i}\sin\frac{\pi}{6}\right) = \sqrt{3} + \mathrm{i}$

$$\frac{v}{w^*} = \frac{\sqrt{3}+\mathrm{i}}{\frac{1}{2}+\frac{1}{2}\mathrm{i}} = \frac{2\left(\sqrt{3}+\mathrm{i}\right)}{1+\mathrm{i}} \times \frac{1-\mathrm{i}}{1-\mathrm{i}}$$

$$= \left(\sqrt{3}+1\right) + \left(1-\sqrt{3}\right)\mathrm{i}$$

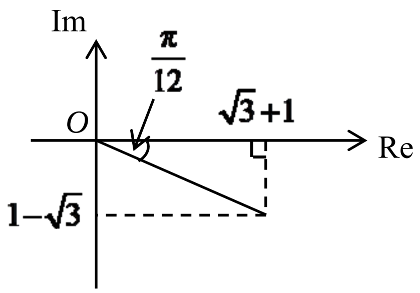
$$\therefore\ \mathrm{Re}\left(\frac{v}{w^*}\right) = \sqrt{3}+1 \quad \text{and} \quad \mathrm{Im}\left(\frac{v}{w^*}\right) = 1-\sqrt{3}$$

<u>**Alternative solution**</u>

$$\frac{1}{w^*} = \sqrt{2}\left[\cos\left(-\frac{\pi}{4}\right) + \mathrm{i}\sin\left(-\frac{\pi}{4}\right)\right] = 1-\mathrm{i}$$

$$\frac{v}{w^*} = \left(\sqrt{3}+\mathrm{i}\right)\left(1-\mathrm{i}\right) = \sqrt{3} - \sqrt{3}\mathrm{i} + \mathrm{i} + 1 = \left(\sqrt{3}+1\right) + \left(1-\sqrt{3}\right)\mathrm{i}$$

(iii) Using results in **(i)** and **(ii)**,



From the Argand diagram, $\tan\left(\frac{\pi}{12}\right) = \frac{\sqrt{3}-1}{\sqrt{3}+1} \times \frac{\sqrt{3}-1}{\sqrt{3}-1} = 2-\sqrt{3}$