Solutions to P2 Prelim 2016:

1.

$S_n = an^3 + bn^2 + cn + d$

$10 = a + b + c + d$

$16 = 8a + 4b + 2c + d$

$21 = 27a + 9b + 3c + d$

$28 = 64a + 16b + 4c + d$

$a = \frac{1}{2},\ b = -\frac{7}{2},\ c = 13,\ d = 0$

$S_n = \frac{1}{2}n^3 - \frac{7}{2}n^2 + 13n$

$$\begin{aligned} U_n &= S_n - S_{n-1} \\ &= \frac{1}{2}n^3 - \frac{7}{2}n^2 + 13n - \left(\frac{1}{2}(n-1)^3 - \frac{7}{2}(n-1)^2 + 13(n-1)\right) \\ &= \frac{3}{2}n^2 - \frac{17}{2}n + 17 \end{aligned}$$

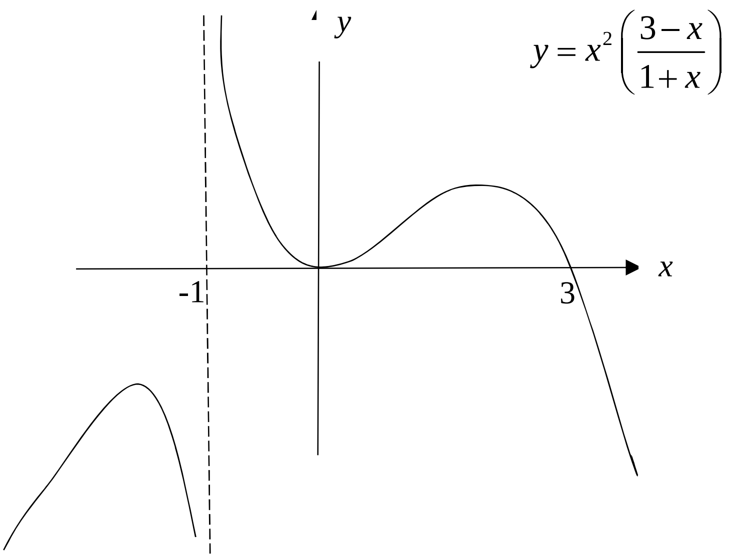
$S_n < 3U_n$

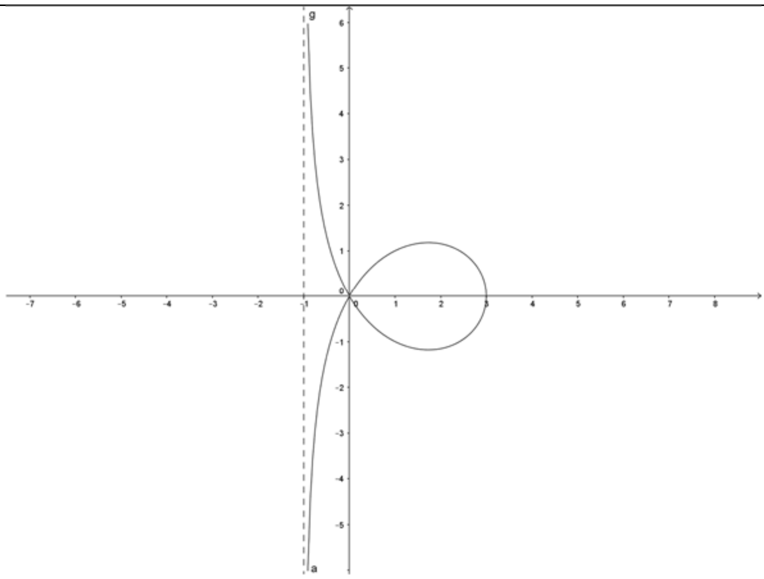
$\frac{1}{2}n^3 - \frac{7}{2}n^2 + 13n < 3\left(\frac{3}{2}n^2 - \frac{17}{2}n + 17\right)$

n	S_n	$3U_n$
1	10	30
2	16	18
7	91	93

From GC, $\left\{n : n \in \mathbb{Z}^{+}, n = 1, 2, 7\right\}$

2





$$y^2 = x^2\left(\frac{3-x}{1+x}\right)$$

Area enclosed by loop $= 2\int_0^3 x\sqrt{\frac{3-x}{1+x}}\,\mathrm{d}x$

$$= 2\int_0^3 x\sqrt{\frac{(3-x)(3-x)}{(1+x)(3-x)}}\,\mathrm{d}x$$

$$= 2\int_0^3 \frac{x(3-x)}{\sqrt{3+2x-x^2}}\,\mathrm{d}x$$

$$= 2\int_0^3 \frac{3x-x^2}{\sqrt{(4-(x-1)^2)}}\,\mathrm{d}x$$

Using $x-1=2\sin\theta$, we have $\frac{\mathrm{d}x}{\mathrm{d}\theta}=2\cos\theta$

Also, when $x=0$, $\sin\theta=-\frac{1}{2}\Rightarrow\theta=-\frac{\pi}{6}$

And when $x=3, \sin\theta=1\Rightarrow\theta=\frac{\pi}{2}$

Therefore area $= 2\int_0^3 \frac{3x-x^2}{\sqrt{(4-(x-1)^2)}}\,\mathrm{d}x$

$$= 2\int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{3(2\sin\theta+1)-(2\sin\theta+1)^2}{\sqrt{(4-(2\sin\theta)^2)}}(2\cos\theta)\,\mathrm{d}\theta$$

$$= 2\int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} \left(2\sin\theta+2-4\sin^2\theta\right)\mathrm{d}\theta$$

$$=4\int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} \left(\sin\theta+\left(1-2\sin^2\theta\right)\right)\mathrm{d}\theta$$

$$= 4\int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} \left(\sin\theta+\cos 2\theta\right)\mathrm{d}\theta$$

$$= 4\left[-\cos\theta+\frac{\sin 2\theta}{2}\right]_{-\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= 4\left[0-\left(-\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{4}\right)\right]=3\sqrt{3}$$

3(a)	$z^3-2(2-\mathrm{i})z^2+(8-3\mathrm{i})z-5+\mathrm{i}=0$ Let $z=x$ be the real root. $x^3-2(2-\mathrm{i})x^2+(8-3\mathrm{i})x-5+\mathrm{i}=0$ $x^3-4x^2+2\mathrm{i}x^2+8x-3\mathrm{i}x-5+\mathrm{i}=0$ $(x^3-4x^2+8x-5)+(2x^2-3x+1)\mathrm{i}=0$ Since $z=x$ is a root, $x^3-4x^2+8x-5=0$ and $2x^2-3x+1=0$ From GC: $x=1$ Therefore, the real root is $z=1$ $z^3-2(2-\mathrm{i})z^2+(8-3\mathrm{i})z-5+\mathrm{i}=0$ $(z-1)(z^2+Az+(5-\mathrm{i}))=0$ $(z-1)(z^2+(-3+2\mathrm{i})z+(5-\mathrm{i}))=0$ $z=1$ or $z^2+(-3+2\mathrm{i})z+(5-\mathrm{i})=0$ $z=\dfrac{-(-3+2\mathrm{i})\pm\sqrt{(-3+2\mathrm{i})^2-4(5-\mathrm{i})}}{2}$ $=\dfrac{-(-3+2\mathrm{i})\pm(1-4\mathrm{i})}{2}$ $=2-3\mathrm{i}$ or $1+\mathrm{i}$ Roots: 1, $2-3\mathrm{i}$, $1+\mathrm{i}$
3(b)	$1-u^2=1-(\cos\theta+\mathrm{i}\sin\theta)^2$ $=1-\cos^2\theta+\sin^2\theta-2\mathrm{i}\sin\theta\cos\theta$ $=2\sin^2\theta-2\mathrm{i}\sin\theta\cos\theta$ $=2\sin\theta(\sin\theta-2\mathrm{i}\cos\theta)$ $=-2\mathrm{i}\sin\theta(\cos\theta+\mathrm{i}\sin\theta)$ $=-2\mathrm{i}u\sin\theta$ **Alternative** $u=\cos\theta+\mathrm{i}\sin\theta=\mathrm{e}^{\theta\mathrm{i}}$ $1-u^2=1-\mathrm{e}^{2\theta\mathrm{i}}$ $=\mathrm{e}^{\theta\mathrm{i}}(\mathrm{e}^{-\theta\mathrm{i}}-\mathrm{e}^{\theta\mathrm{i}})$ $=u(\cos\theta-\mathrm{i}\sin\theta-\mathrm{i}\sin\theta-\cos\theta)$ $=u(-2\mathrm{i}\sin\theta)$ $=-2\mathrm{i}u\sin\theta$ $\lvert 1-u^2\rvert=\lvert -2\mathrm{i}u\sin\theta\rvert=\lvert -2\sin\theta\rvert\,\lvert \mathrm{i}\rvert\,\lvert u\rvert$ $=2\sin\theta$

	$\arg\left(1-u^2\right)=\arg\left(-2\mathrm{i}u\sin\theta\right)$ $=\arg\left(-2\mathrm{i}\sin\theta\right)+\arg\left(u\right)$ $=-\dfrac{\pi}{2}+\theta$ $\left(1-u^2\right)^{10}$ is real and negative: $\arg\left(1-u^2\right)^{10}=10\arg\left(1-u^2\right)=\left(2k+1\right)\pi, k\in\mathbb{Z}$ $10\left(-\dfrac{\pi}{2}+\theta\right)=\left(2k+1\right)\pi$ $-5\pi+10\theta=\left(2k+1\right)\pi$ $\theta=\dfrac{\left(2k+6\right)\pi}{10}$ $0<\theta<\dfrac{\pi}{2}$: $\theta=\dfrac{1}{5}\pi,\dfrac{2}{5}\pi$ **Alternative** $\left(1-u^2\right)^{10}=\left(2\sin\theta e^{-\frac{\pi}{2}+\theta}\right)^{10}$ $=\left(2^{10}\sin^{10}\theta\right)\left(\cos\left(-5\pi+10\theta\right)+\mathrm{i}\sin\left(-5\pi+10\theta\right)\right)$ Since $\left(1-u^2\right)^{10}$ is real and negative, and $2^{10}\sin^{10}\theta>0$, $\sin\left(-5\pi+10\theta\right)=0$ and $\cos\left(-5\pi+10\theta\right)<0$ $-5\pi+10\theta=k\pi$, $k\in\mathbb{Z}$ $\theta=\dfrac{\left(k+5\right)\pi}{10}$ $0<\theta<\dfrac{\pi}{2}$: $\theta=\dfrac{1}{10}\pi,\dfrac{1}{5}\pi,\dfrac{3}{10}\pi,\dfrac{2}{5}\pi$ Only when $\theta=\dfrac{1}{5}\pi,\dfrac{2}{5}\pi$ will $\cos\left(-5\pi+10\theta\right)<0$. Therefore, $\theta=\dfrac{1}{5}\pi,\dfrac{2}{5}\pi$.
4(a)(i)	Since $S_{16}=2\pi r$, thus $\dfrac{16}{2}(2r+15d)=2\pi r$ $\Rightarrow 2r+15d=\dfrac{\pi r}{4}$ $\Rightarrow 15d=\left(\dfrac{\pi-8}{4}\right)r$ $\Rightarrow d=\left(\dfrac{\pi-8}{60}\right)r$
(a)(ii)	Since $L_n=r\theta_n$ and $A_n=\dfrac{1}{2}r^2\theta_n$, thus $A_n=\dfrac{1}{2}rL_n$. Hence

	$A_{n+1}-A_n=\frac{1}{2}rL_{n+1}-\frac{1}{2}rL_n$ $=\frac{1}{2}r\left(L_{n+1}-L_n\right)$ $=\frac{1}{2}rd=\text{constant}$ for all $n=2,\ldots,15$ Thus A_n is an arithmetic sequence.
(b)(i)	Since $S_N=\pi a^2$, we have $\frac{a(r^N-1)}{r-1}=\pi a^2$ $\Rightarrow r^N-1=\pi ar-\pi a$ $\Rightarrow r^N-\pi ar+(\pi a-1)=0$
(b)(ii)	Since $0<r<1$, we have $r^N\to 0$ as $N\to\infty$. Thus $-\pi ar+(\pi a-1)=0$ $\Rightarrow r=\frac{\pi a-1}{\pi a}$ Alternative: Since $S_\infty=\pi a^2$, we have $\frac{a}{1-r}=\pi a^2$ $\Rightarrow 1-r=\frac{1}{\pi a}$ $\Rightarrow r=\frac{\pi a-1}{\pi a}$
5	No sampling frame. Station an interviewer at the exits of a local supermarket store during peak hours and he is free to choose 25 male and 25 female customers who buy the products. P(a particular consumer is the first to be selected) = $\frac{1}{2000}$ P(a particular consumer is the third to be selected) $=\frac{1999}{2000}\frac{1998}{1999}\frac{1}{1998}=\frac{1}{2000}$ (shown)
6	(i) $\mathrm{P}(\bar{X}<35.0)=0.97725$ $\mathrm{P}(Z<\frac{35-\mu}{\sigma/\sqrt{n}})=0.97725$ $\frac{35-\mu}{\sigma/\sqrt{n}}=2-----[1]$ $\mathrm{P}(\bar{X}<20.0)=0.15866$

	$\mathrm{P}(Z < \frac{20-\mu}{\sigma/\sqrt{n}}) = 0.15866$ $\frac{20-\mu}{\sigma/\sqrt{n}} = -1 ----[2]$ Eqn [1] –[2]: $\frac{3\sigma}{\sqrt{n}} = 15$ $\sigma = 5\sqrt{n}$ (ii) $\mu = 25$, $\bar{X} \sim N(25, 5^2)$ since $\frac{\sigma}{\sqrt{n}} = 5$ $\mathrm{P}(\bar{X} > 32) = 0.0808$ let M be the mass of a randomly chosen discharge of 15 ice cubes. $M \sim N(375, 750)$ (iii) $P(M > a) = 0.1$ $P(M \le a) = 0.9$ $a = 410.1$ (iv) $M_1 - M_2 \sim N(0, 1500)$ $P(M_1 > M_2) = P(M_1 - M_2 > 0) = 0.5$
7	5M, [3W, A, L]: No. of selections = $\left({}^{8}\mathrm{C}_5\right)\left({}^{7}\mathrm{C}_3\right)$ = 1960 5M, 5W (exclude A and L): No. of selections = $\left({}^{8}\mathrm{C}_5\right)\left({}^{7}\mathrm{C}_5\right)$ = 1176 Total number of selections = $\left({}^{8}\mathrm{C}_5\right)\left({}^{7}\mathrm{C}_3\right) + \left({}^{8}\mathrm{C}_5\right)\left({}^{7}\mathrm{C}_5\right)$ = 1960 + 1176 = 3136
(i)	M_1 A M_2 M_3 L M_4 M_5 W W W Arrange 5M: No. of ways = 5! = 120 Arrange A and L: No. of ways = 2! = 2 Arrange 3W: No. of ways = 1(3!) = 6 Total no. of arrangements = $(5!)\left[(2!)\times(3!)\right]$ = 1440
(ii)	No of arrangements $(8-1)!(2)(10)$ =100800

8(i)

P(Strollers will win in 2018)

$$= (0.7)(0.7)(0.3) + (0.7)(0.3)(0.5)$$
$$+ (0.3)(0.5)(0.3) + (0.3)(0.5)(0.5)$$
$$= 0.372$$

(ii)

Let event A denotes "Strollers first win for at least three years" and event B denotes "Strollers win in 2018"

$$\mathrm{P}(A \mid B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$$
$$= \frac{(0.7)(0.7)(0.3)}{0.372}$$
$$= 0.39516$$
$$= 0.395 \quad (3\text{ s.f.})$$

(iii)

$$(0.5)(0.7)^{n-1} < 0.05$$
$$(0.7)^{n-1} < 0.1$$
$$n - 1 > \frac{\ln(0.1)}{\ln(0.7)} = 6.4557$$
$$n > 7.4557$$

Hence, the smallest value is $n = 8$.

Alternative method:

$$(0.5)(0.7)^{n-1} < 0.05$$

Using GC:

n	probability
7	0.05582
8	0.04418 < 0.05
9	0.02882

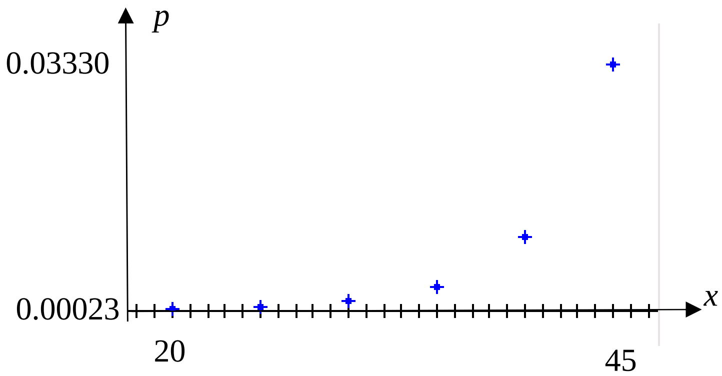
Hence, the smallest value is $n = 8$.

Or

$$(0.7)^{n-1} < 0.1$$

n	probability
7	0.11765
8	0.08235 < 0.01
9	0.05465

Hence, the smallest value is $n = 8$.

9	(i)  (ii)(a) $r = 0.8130$ (b) $r = 0.9960$ (c) $r = 0.8667$ (iii)Since |r| is closest to 1 for model (b), equation would be $\ln p = 0.19409x - 12.322$ (iv)When $x = 32$, $p = 0.0022181$ Expected number = 5000(0.0022181) = 11.1
10	(i) Unbiased estimate of pop mean = $\bar{x} = \dfrac{\sum x}{n} = \dfrac{11872}{250} = 47.488$ Unbiased estimate of population variance, $s^2 = \dfrac{1}{249}(646193 - \dfrac{11872^2}{250}) = 330.986$ (i) To test $H_o : \mu = 49.5$ $H_1 : \mu < 49.5$ At $\alpha\%$ significance level Since n= 250 is large, by Central limit Theorem, $\bar{X} \sim N(49.5, \dfrac{330.986}{250})$ approx. Since H_0 is rejected, p-value $= P(\bar{X} < 47.488) = 0.040179 < \dfrac{\alpha}{100}$ $\alpha > 4.02$ (ii) Since n =250 is large, by Central Limit Theorem, $\bar{X} \sim N(45.292, \dfrac{18.761^2}{250})$ approximately $P(\bar{X} \geq 47.488) \approx 0.0321$
11(i)	The mean number of customers who arrived at the village post office during a random chosen 30 minutes period must be a constant.
(ii)	Let X be the random variable denoting the number of customers who arrive at the village post office between 11.00 a.m. and 11.30 a.m.

	i.e. $X \sim \text{P}_o(9)$ $\text{P}(X \le 4) = 0.054964 = 0.0550 \quad (3\text{ s.f.})$
(iii)	Let Y be the random variable denoting the number of customers who arrive at the village post office in 5 minutes i.e. $Y \sim \text{P}_o(1.5)$ $\text{P}(Y = 0) = 0.22313$ Let W be the random variable denoting the number of periods (of 5 minutes each) out of 6 where $Y = 0$ i.e. $W \sim \text{B}(6, 0.22313)$ $\text{P}(W \le 1) = 0.59867 = 0.599 \quad (3\text{ s.f.})$
(iv)	Let U be the random variable denoting the number of customers who arrive at the village post office in 3.5 hours in the afternoon i.e. $U \sim \text{Po}(21)$ $P(\mu - \sigma < U < \mu + \sigma) = P(16.4 < U < 25.6)$ $= P(U \le 25) - P(U \le 16)$ $= 0.675$
(v)	Let T be the random variable denoting the number of customers who arrive at the village post office in 3.5 hours in the morning $T + U \sim \text{Po}(84)$ $P(T > 38 \mid T + U = 40) = \dfrac{\text{P}(T > 38 \text{ and } T + U = 40)}{\text{P}(T + U = 40)}$ $= \dfrac{\text{P}(T = 39)\text{P}(U = 1) + \text{P}(T = 40)\text{P}(U = 0)}{\text{P}(T + U = 40)}$ $= 1.44 \times 10^{-4}$
(vi)	$T + U \sim \text{Po}(84)$ Since $\lambda = 84 > 10$, hence use the normal distribution for approximation i.e. $T + U \sim \text{N}\left(126, \left(\sqrt{126}\right)^2\right)$ approximately $\text{P}(T + U > 100) \xrightarrow{c.c.} \text{P}(T + U > 100.5) = 0.0359 \quad (3\text{ s.f.})$