Solutions to P1 Prelim 2016

1

Let P_n denote the proposition $S_n = \left(\frac{n-1}{n+2}\right)\left(\frac{4^{n+1}}{3}\right)+\frac{2}{3}$ for $n \in \mathbb{Z}^+$.

When $n=1$, $\text{LHS} = S_1 = u_1 = \frac{4^1(1)^2}{(2)(3)} = \frac{2}{3}$.

$$\text{RHS} = \left(\frac{1-1}{1+2}\right)\left(\frac{4^2}{3}\right)+\frac{2}{3} = \frac{2}{3} = \text{LHS}.$$

$\therefore P_1$ is true.

Assume P_k is true for some $k \in \mathbb{Z}^+$, i.e. $S_k = \left(\frac{k-1}{k+2}\right)\left(\frac{4^{k+1}}{3}\right)+\frac{2}{3}$.

To prove P_{k+1} is also true, i.e. $S_{k+1} = \left(\frac{k}{k+3}\right)\left(\frac{4^{k+2}}{3}\right)+\frac{2}{3}$.

$$\begin{aligned}
\text{LHS} &= S_{k+1} \\
&= S_k + u_{k+1} \\
&= \left(\frac{k-1}{k+2}\right)\left(\frac{4^{k+1}}{3}\right)+\frac{2}{3}+\frac{4^{k+1}(k+1)^2}{(k+2)(k+3)} \\
&= \left(\frac{4^{k+1}}{3(k+2)}\right)\left[k-1+\frac{3(k+1)^2}{k+3}\right]+\frac{2}{3} \\
&= \left(\frac{4^{k+1}}{3(k+2)}\right)\left[\frac{(k-1)(k+3)+3(k+1)^2}{k+3}\right]+\frac{2}{3} \\
&= \left(\frac{4^{k+1}}{3(k+2)}\right)\left[\frac{k^2+2k-3+3k^2+6k+3}{k+3}\right]+\frac{2}{3} \\
&= \left(\frac{4^{k+1}}{3(k+2)}\right)\left[\frac{4k^2+8k}{k+3}\right]+\frac{2}{3} \\
&= \left(\frac{4^{k+2}}{3(k+2)}\right)\left[\frac{k(k+2)}{k+3}\right]+\frac{2}{3} = \left(\frac{k}{k+3}\right)\left(\frac{4^{k+2}}{3}\right)+\frac{2}{3} = \text{RHS}.
\end{aligned}$$

Since P_1 is true and P_k is true $\Rightarrow P_{k+1}$ is true, by Mathematical Induction, P_n is true for all $n \in \mathbb{Z}^+$.

2	$\frac{17x^2+23x+12}{(3x+4)(x^2+4)} \equiv \frac{A}{3x+4}+\frac{Bx+C}{x^2+4}$ Solving, A = 2, B = 5 and C = 1 $\int_{-2}^{2}\frac{17x^2+23x+12}{(3x+4)(x^2+4)}\,\mathrm{d}x=\int_{-2}^{2}\frac{2}{3x+4}+\frac{5x+1}{x^2+4}\,\mathrm{d}x$ $=\int_{-2}^{2}\frac{2}{3x+4}+\frac{5}{2}\left(\frac{2x}{x^2+4}\right)+\frac{1}{x^2+4}\,\mathrm{d}x$ $=\frac{2}{3}\Big[\ln\lvert 3x+4\rvert\Big]_{-2}^{2}+\frac{5}{2}\Big[\ln(x^2+4)\Big]_{-2}^{2}+\left[\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\right]_{-2}^{2}$ $=\frac{2}{3}\ln 5+\frac{\pi}{4}$
3	$x=\frac{1}{2}(\sin t\cos t+t)=\frac{1}{2}t+\frac{1}{4}\sin 2t$ and $y=\frac{1}{2}t-\frac{1}{4}\sin 2t$ $\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{1}{2}\cos 2t+\frac{1}{2}$ and $\frac{\mathrm{d}y}{\mathrm{d}t}=\frac{1}{2}-\frac{1}{2}\cos 2t$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1-\cos 2t}{1+\cos 2t}=\frac{1-(1-2\sin^2 t)}{(2\cos^2 t-1)+1}=\tan^2 t$ When $\frac{\mathrm{d}y}{\mathrm{d}x}=1\Rightarrow\tan^2 t=1\Rightarrow t=\pm\frac{\pi}{4}$ Since $t<0$, $t=-\frac{\pi}{4}$, and $x=-\frac{1}{4}-\frac{\pi}{8}$ and $y=\frac{1}{4}-\frac{\pi}{8}$ Equation of normal is $y-\left(\frac{1}{4}-\frac{\pi}{8}\right)=-\left[x-\left(\frac{1}{4}-\frac{\pi}{8}\right)\right]$ $y=-x-\frac{\pi}{4}$ When $y=0$, $x=-\frac{\pi}{4}$. Volume required is $=\frac{1}{3}\pi\left(\frac{1}{4}-\frac{\pi}{8}\right)^2\left(\frac{\pi}{8}-\frac{1}{4}\right)+\pi\int_{-\frac{\pi}{4}}^{0}\left(\frac{1}{2}t-\frac{1}{4}\sin 2t\right)^2\left(\frac{1}{2}\cos 2t+\frac{1}{2}\right)\mathrm{d}t$ $=0.00759$ (5 d.p.) $\frac{\mathrm{d}y}{\mathrm{d}x}=\tan^2 t\approx t^2\to 0$ as t approaches 0. Therefore the tangents are parallel to the x-axis.
4	$\overrightarrow{OM}=\frac{1}{2}\left(\overrightarrow{OA}+\overrightarrow{OC}\right)=\frac{1}{2}(2\mathbf{a}+\mathbf{b})$ $\overrightarrow{ON}=\frac{2}{3}\overrightarrow{OM}=\frac{2}{3}\times\frac{1}{2}(2\mathbf{a}+\mathbf{b})=\frac{1}{3}(2\mathbf{a}+\mathbf{b})$

$\overrightarrow{AN} = \overrightarrow{ON} - \overrightarrow{OA} = \frac{1}{3}(2\mathbf{a}+\mathbf{b}) - \mathbf{a} = \frac{1}{3}(\mathbf{b}-\mathbf{a}) = \frac{1}{3}\overrightarrow{AB}$

Since $\overrightarrow{AN}$ is parallel to $\overrightarrow{AB}$ and A is the common point, hence A, B and N are collinear. B1

Since P is on AB, $\overrightarrow{OP} = \mathbf{a} + \lambda(\mathbf{b}-\mathbf{a}), where\ \lambda \in \mathbb{R}$

$\overrightarrow{MP} \cdot \overrightarrow{AB} = 0$

$$\left[\mathbf{a} + \lambda(\mathbf{b}-\mathbf{a}) - \frac{1}{2}(2\mathbf{a}+\mathbf{b})\right] \cdot (\mathbf{b}-\mathbf{a}) = 0$$

$$\left[\left(\lambda - \frac{1}{2}\right)\mathbf{b} - \lambda\mathbf{a}\right] \cdot (\mathbf{b}-\mathbf{a}) = 0$$

$$\left(\lambda - \frac{1}{2}\right)|\mathbf{b}|^2 - \left(\lambda - \frac{1}{2}\right)\mathbf{a}\cdot\mathbf{b} - \lambda\mathbf{a}\cdot\mathbf{b} + \lambda|\mathbf{a}|^2 = 0$$

But $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos AOB = 2\times 3\cos 60^o = 3$

Hence, $9\left(\lambda - \frac{1}{2}\right) - 3\left(\lambda - \frac{1}{2}\right) - 3\lambda + 4\lambda = 0$

$$\lambda = \frac{3}{7}$$

$$\overrightarrow{OP} = \mathbf{a} + \frac{3}{7}(\mathbf{b}-\mathbf{a}) = \frac{1}{7}(4\mathbf{a}+3\mathbf{b})$$

<u>Alternative method:</u>

$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos AOB = 2\times 3\cos 60^o = 3$

Using cosine formula $|\mathbf{b}\text{-}\mathbf{a}| = \sqrt{|\mathbf{a}|^2 + |\mathbf{b}|^2 \mathbf{-2}|\mathbf{a}||\mathbf{b}|\cos 60} = \sqrt{7}$

$$\overrightarrow{AP} = \left(\overrightarrow{AM} \cdot \frac{\overrightarrow{AB}}{|\overrightarrow{AB}|}\right)\frac{\overrightarrow{AB}}{|\overrightarrow{AB}|}$$

$$= \left(\frac{1}{2}\mathbf{b} \cdot \frac{(\mathbf{b}-\mathbf{a})}{|\mathbf{b}-\mathbf{a}|}\right)\frac{\mathbf{b}-\mathbf{a}}{|\mathbf{b}-\mathbf{a}|}$$

$$= \frac{1}{2}\left(\frac{\mathbf{b}\cdot\mathbf{b}\text{-}\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}-\mathbf{a}|^2}\right)(\mathbf{b}-\mathbf{a})$$

$$= \frac{1}{2}\cdot\frac{6}{7}(\mathbf{b}-\mathbf{a})$$

$$= \frac{3}{7}(\mathbf{b}-\mathbf{a})$$

$$\overrightarrow{OP} = \mathbf{a} + \frac{3}{7}(\mathbf{b}-\mathbf{a}) = \frac{1}{7}(4\mathbf{a}+3\mathbf{b})$$

5	1. Let $\theta=\sin^{-1}\frac{y}{r}\Rightarrow\sin\theta=\frac{y}{r}\Rightarrow\cos\theta=\sqrt{1-\frac{y^2}{r^2}}$, Diff wrt y : $\cos\theta\frac{\mathrm{d}\theta}{\mathrm{d}y}=\frac{1}{r}\Rightarrow\frac{\mathrm{d}\theta}{\mathrm{d}y}=\frac{1}{r\cos\theta}$ $\therefore m=\tan\theta\Rightarrow\frac{\mathrm{d}m}{\mathrm{d}y}=\sec^2\theta\frac{\mathrm{d}\theta}{\mathrm{d}y}=\frac{1}{r\cos^3\theta}$ ie, $\frac{\mathrm{d}m}{\mathrm{d}y}=\frac{1}{r\left(\frac{r^2-y^2}{r^2}\right)^{\frac{3}{2}}}$ $=\frac{r^2}{\left(r^2-y^2\right)^{\frac{3}{2}}}$ Using $\frac{\mathrm{d}m}{\mathrm{d}t}=\frac{\mathrm{d}m}{\mathrm{d}y}\times\frac{\mathrm{d}y}{\mathrm{d}t}$, we have $\frac{\mathrm{d}m}{\mathrm{d}t}=\frac{r^2}{\left(r^2-y^2\right)^{\frac{3}{2}}}\times\frac{r}{1000}=\frac{r^3}{10^3\left(\sqrt{r^2-y^2}\right)^3}$ $=\left(\frac{r}{10\sqrt{r^2-y^2}}\right)^3$ $\frac{\mathrm{d}m}{\mathrm{d}t}$ is the rate of change of the gradient of the line OP **Alternate method 1:** $m=\tan\left(\sin^{-1}\frac{y}{r}\right)$ $\frac{\mathrm{d}m}{\mathrm{d}y}=\sec^2\left(\sin^{-1}\frac{y}{r}\right)\left[\frac{1}{\sqrt{1-\left(\frac{y}{r}\right)^2}}\left(\frac{1}{r}\right)\right]$ $=\frac{1}{\cos^2\left(\sin^{-1}\frac{y}{r}\right)}\left[\frac{1}{\sqrt{r^2-y^2}}\right]$ $=\frac{1}{\cos^2\theta}\left[\frac{1}{\sqrt{r^2-y^2}}\right]$ where $\theta=\sin^{-1}\frac{y}{r}\Rightarrow\sin\theta=\frac{y}{r}$ $=\frac{1}{\left(x/r\right)^2}\left[\frac{1}{\sqrt{r^2-y^2}}\right]=\frac{r^2}{x^2}\left[\frac{1}{\sqrt{r^2-y^2}}\right]$

$$= \frac{r^2}{r^2 - y^2}\left[\frac{1}{\sqrt{r^2 - y^2}}\right]$$

$$= \frac{r^2}{\left(r^2 - y^2\right)^{3/2}}$$

Using $\frac{\mathrm{d}m}{\mathrm{d}t} = \frac{\mathrm{d}m}{\mathrm{dy}} \times \frac{\mathrm{dy}}{\mathrm{d}t}$

$$= \frac{r^2}{\left(r^2 - y^2\right)^{3/2}} \times \frac{r}{1000}$$

$$= \frac{r^3}{10^3\left(r^2 - y^2\right)^{3/2}}$$

$$= \left(\frac{r}{10\sqrt{r^2 - y^2}}\right)^3$$

$\frac{\mathrm{d}m}{\mathrm{d}t}$ is the rate of change of the gradient of the line *OP*.

Alternate method 2:

$$m = \tan\left(\sin^{-1}\frac{y}{r}\right)$$

$$\tan^{-1} m = \sin^{-1}\frac{y}{r}$$

$$\left(\frac{1}{1+m^2}\right)\frac{\mathrm{d}m}{\mathrm{d}y} = \frac{1}{\sqrt{1-\left(\frac{y}{r}\right)^2}}\left(\frac{1}{r}\right)$$

$$\frac{\mathrm{d}m}{\mathrm{d}y} = \frac{1+m^2}{\sqrt{r^2 - y^2}}$$

$$= \frac{1+\tan^2\theta}{\sqrt{r^2 - y^2}} \qquad \text{where } \theta = \sin^{-1}\frac{y}{r} \quad \Rightarrow \quad \sin\theta = \frac{y}{r}$$

$$= \frac{1+\left(y/x\right)^2}{\sqrt{r^2 - y^2}}$$

$$= \frac{1+\frac{y^2}{x^2}}{\sqrt{r^2 - y^2}}$$

$$= \frac{1+\frac{y^2}{r^2 - y^2}}{\sqrt{r^2 - y^2}}$$

$$= \frac{\left(r^2 - y^2\right) + y^2}{\left(r^2 - y^2\right)\sqrt{r^2 - y^2}}$$

$$= \frac{r^2}{\left(r^2 - y^2\right)^{3/2}}$$

Second part is similar to the above.

Alternate method 3:

$$m = \tan\left(\sin^{-1}\frac{y}{r}\right) = \tan\theta, \quad \text{where } \theta = \sin^{-1}\frac{y}{r}$$

$$m = \frac{y}{x} = \frac{y}{\sqrt{r^2 - y^2}}$$

$$\frac{\mathrm{d}m}{\mathrm{d}y} = \frac{\sqrt{r^2 - y^2} - y\left(\frac{1}{2}\right)\left(r^2 - y^2\right)^{-1/2}(-2y)}{\left(r^2 - y^2\right)}$$

$$= \frac{\sqrt{r^2 - y^2} + y^2\left(r^2 - y^2\right)^{-1/2}}{\left(r^2 - y^2\right)}$$

$$= \frac{\left(r^2 - y^2\right) + y^2}{\left(r^2 - y^2\right)^{1/2}\left(r^2 - y^2\right)}$$

$$= \frac{r^2}{\left(r^2 - y^2\right)^{3/2}}$$

Second part is similar to the above.

6(i)

$$\frac{r^2 + r - 1}{(r+2)!} = \frac{A}{r!} + \frac{B}{(r+1)!} + \frac{C}{(r+2)!}$$

$$\frac{r^2 + r - 1}{(r+2)!} = \frac{A(r+1)(r+2) + B(r+2) + C}{(r+2)!}$$

By comparing coefficients,

$A = 1, B = -2, C = 1$

$$\therefore \frac{r^2 + r - 1}{(r+2)!} = \frac{1}{r!} - \frac{2}{(r+1)!} + \frac{1}{(r+2)!}$$

(ii)	$\displaystyle\sum_{r=1}^{n}\frac{r^2+r-1}{(r+2)!}=\sum_{r=1}^{n}\left(\frac{1}{r!}-\frac{2}{(r+1)!}+\frac{1}{(r+2)!}\right)$ $\displaystyle=\Big[\frac{1}{1!}-\frac{2}{2!}+\cancel{\frac{1}{3!}}$ $\displaystyle+\frac{1}{2!}-\cancel{\frac{2}{3!}}+\cancel{\frac{1}{4!}}$ $\displaystyle+\cancel{\frac{1}{3!}}-\cancel{\frac{2}{4!}}+\frac{1}{5!}$ $\displaystyle+\cancel{\frac{1}{4!}}-\frac{2}{5!}+\cancel{\frac{1}{6!}}$ $\vdots$ $\displaystyle+\cancel{\frac{1}{(n-2)!}}-\cancel{\frac{2}{(n-1)!}}+\cancel{\frac{1}{n!}}$ $\displaystyle+\cancel{\frac{1}{(n-1)!}}-\cancel{\frac{2}{n!}}+\frac{1}{(n+1)!}$ $\displaystyle+\cancel{\frac{1}{n!}}-\frac{2}{(n+1)!}+\frac{1}{(n+2)!}\Big]$ $\displaystyle=\frac{1}{2}-\frac{1}{(n+1)!}+\frac{1}{(n+2)!}$
(iii)	Since $r^2-1<r^2+r-1$ for $r>0$, so we have $\displaystyle\sum_{r=1}^{n}\frac{r^2-1}{(r+2)!}<\sum_{r=1}^{n}\frac{r^2+r-1}{(r+2)!}=\frac{1}{2}-\frac{1}{(n+1)!}+\frac{1}{(n+2)!}$ $\displaystyle=\frac{1}{2}-\left[\frac{n+1}{(n+2)!}\right]<\frac{1}{2}$ $\displaystyle\left(\because\frac{n+1}{(n+2)!}>0\ \text{as}\ n\in\mathbb{Z}^{+}\right)$
(iv)	$\displaystyle\sum_{r=4}^{n}\frac{r^2-3r+1}{r!}$ Replace r with $(k+2)$ $\displaystyle=\sum_{k=2}^{n-2}\frac{(k+2)^2-3(k+2)+1}{(k+2)!}$

$$=\sum_{k=2}^{n-2}\frac{k^2+k-1}{(k+2)!}$$

$$=\sum_{k=1}^{n-2}\frac{k^2+k-1}{(k+2)!}-\left(\frac{1}{6}\right)$$

$$=\frac{1}{2}-\frac{1}{(n-1)!}+\frac{1}{n!}-\frac{1}{6}$$

$$=\frac{1}{3}-\frac{1}{(n-1)!}+\frac{1}{n!}$$

Alternatively, consider $\sum_{r=1}^{n}\frac{r^2+r-1}{(r+2)!}$ and sub. $r=k-2$. So we have

$$\sum_{r=1}^{n}\frac{r^2+r-1}{(r+2)!}=\sum_{k=3}^{n+2}\frac{k^2-3k+1}{k!}.$$

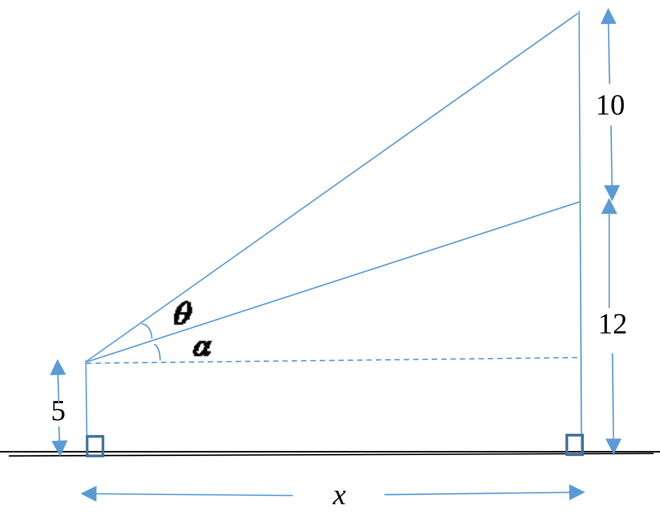
$$\Rightarrow\sum_{k=3}^{n+2}\frac{k^2-3k+1}{k!}=\frac{1}{2}-\frac{1}{(n+1)!}+\frac{1}{(n+2)!}$$

$$\Rightarrow\sum_{k=4}^{n+2}\frac{k^2-3k+1}{k!}=\frac{1}{2}-\frac{1}{(n+1)!}+\frac{1}{(n+2)!}-\left(\frac{3^2-3(3)+1}{3!}\right)$$

$$=\frac{1}{3}-\frac{1}{(n+1)!}+\frac{1}{(n+2)!}$$

$$\therefore\sum_{k=4}^{n}\frac{k^2-3k+1}{k!}=\frac{1}{3}-\frac{1}{(n-1)!}+\frac{1}{n!}$$

7



From diagram, $\tan\alpha = \dfrac{7}{x}$

and $\tan(\theta+\alpha) = \dfrac{17}{x}$

$$\tan\theta = \tan((\theta+\alpha)-\alpha)$$

$$= \frac{\tan(\theta+\alpha)-\tan\alpha}{1-\tan(\theta+\alpha)\tan\alpha}$$

$$= \frac{\frac{17}{x}-\frac{7}{x}}{1-\frac{17}{x}(\frac{7}{x})}$$

$$= \frac{\frac{10}{x}}{1-\frac{119}{x^2}} = \frac{10x}{x^2+119} \quad \text{(shown)}$$

Differentiating wrt x,

$$\sec^2\theta\frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{(119+x^2)(10)-10x(2x)}{(119+x^2)^2}$$

$$= \frac{1190-10x^2}{(119+x^2)^2}$$

For maximum angle,

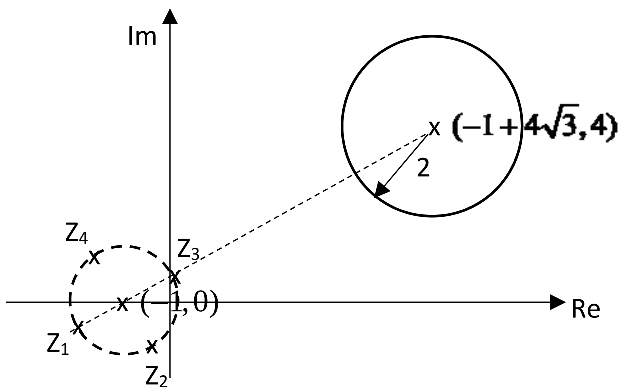
$$\frac{\mathrm{d}\theta}{\mathrm{d}x} = 0 \Rightarrow 10x^2 = 1190$$

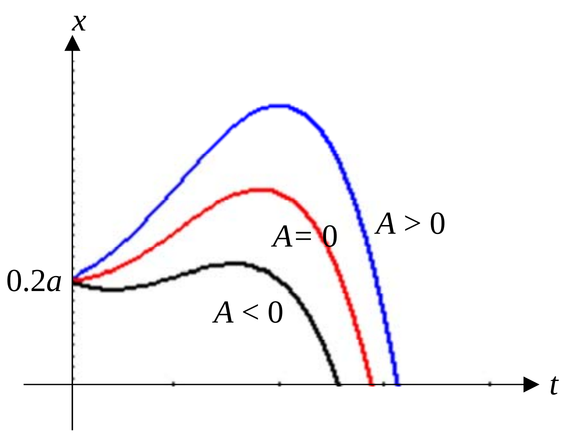
Since $x>0$, $x=\sqrt{119}$

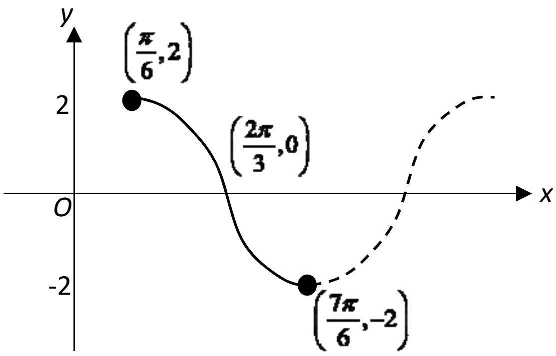
Since $\sec^2\theta > 0$,

	$\left(\sqrt{119}\right)^-$	$\left(\sqrt{119}\right)$	$\left(\sqrt{119}\right)^+$
Sign of $\frac{\mathrm{d}\theta}{\mathrm{d}x}$	+	0	-
slope	/	—	\

Using sign test of derivatives at vicinity of $x=\sqrt{119}$, it can be shown that the angle is maximum.

	Therefore, $\tan\theta=\dfrac{10\sqrt{119}}{119+119}=\dfrac{5\sqrt{119}}{119}\Rightarrow\theta\approx 25^o$
8	*(i)* $w^4=-1+\sqrt{3}\,\mathrm{i}$ $w^4=2\mathrm{e}^{\mathrm{i}\frac{2\pi}{3}}$ $w=2^{\frac{1}{4}}\mathrm{e}^{\mathrm{i}\frac{1}{4}\left(\frac{2\pi}{3}+2k\pi\right)}\qquad k=-2,-1,\ 0,\ 1,$ $w=2^{\frac{1}{4}}\mathrm{e}^{\mathrm{i}\left(-\frac{5\pi}{6}\right)},\ 2^{\frac{1}{4}}\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{3}\right)},\ 2^{\frac{1}{4}}\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{6}\right)},\ 2^{\frac{1}{4}}\mathrm{e}^{\mathrm{i}\left(\frac{2\pi}{3}\right)}$ *(ii)* $(1+z)^4+1-\mathrm{i}\sqrt{3}=0\ \Rightarrow\ (1+z)^4=-1+\mathrm{i}\sqrt{3}$ $z=w-1$ $z=2^{\frac{1}{4}}\mathrm{e}^{\mathrm{i}\left(-\frac{5\pi}{6}\right)}-1,\ 2^{\frac{1}{4}}\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{3}\right)}-1,\ 2^{\frac{1}{4}}\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{6}\right)}-1,\ 2^{\frac{1}{4}}\mathrm{e}^{\mathrm{i}\left(\frac{2\pi}{3}\right)}-1$ $z_1=2^{\frac{1}{4}}\mathrm{e}^{\mathrm{i}\left(-\frac{5\pi}{6}\right)}-1,\ z_2=2^{\frac{1}{4}}\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{3}\right)}-1, z_3=\ 2^{\frac{1}{4}}\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{6}\right)}-1,\ z_4=2^{\frac{1}{4}}\mathrm{e}^{\mathrm{i}\left(\frac{2\pi}{3}\right)}-1$  *(iii)* *Least possible Z_iQ =Z_3Q*$=8-2-2^{1/4}=6-2^{1/4}$ *Greatest possible Z_iQ =Z_1Q*$=8+2+2^{1/4}=10-2^{1/4}$
9	(i) $\dfrac{\mathrm{d}x}{\mathrm{d}t}=x(a-x)$ $\displaystyle\int\frac{\mathrm{d}x}{x(a-x)}=\int\mathrm{d}t$ $\displaystyle\int\frac{\frac{1}{a}}{x}+\frac{\frac{1}{a}}{a-x}\,\mathrm{d}x=\int\mathrm{d}t$ $\dfrac{1}{a}\ln\lvert x\rvert-\dfrac{1}{a}\ln\lvert a-x\rvert=t+C$ $\dfrac{1}{a}\ln\left\lvert\dfrac{x}{a-x}\right\rvert=t+C$ $\ln\left\lvert\dfrac{x}{a-x}\right\rvert=at+aC$ $\dfrac{x}{a-x}=A\mathrm{e}^{at}$, where $A=\pm e^{aC}$

	When $t = 0$, $x = 0.2a$ $\frac{0.2a}{0.8a} = A$ $A = \frac{1}{4}$ When $t = \ln 2$, $x = 0.5a$ $\frac{0.5a}{0.5a} = \frac{1}{4}e^{a\ln 2}$ $4 = e^{a\ln 2}$ $2^a = 4$ $a = 2$ Subst. values of A and a , $\frac{x}{2-x} = \frac{1}{4}e^{2t}$ $4x = 2e^{2t} - xe^{2t}$ $x\left(4+e^{2t}\right) = 2e^{2t}$ $x = \frac{2e^{2t}}{\left(4+e^{2t}\right)} = \frac{2}{4e^{-2t}+1}$ (shown) (ii) $\frac{d^2x}{dt^2} = 10 - 9t^2$ $\frac{dx}{dt} = 10t - 3t^3 + A$ $x = 5t^2 - \frac{3}{4}t^4 + At + B$ When $t = 0$, $x = 0.2a$ $B = 0.2a$ $x = 5t^2 - \frac{3}{4}t^4 + At + 0.2a$ 
10 (a)(i)	$\sin x + \sqrt{3}\cos x = 2\sin\left(x + \frac{\pi}{3}\right)$

(ii)	 For f to have an inverse, f must be one-to-one. Hence largest $k=\frac{7\pi}{6}$. Consider $y=2\sin\left(x+\frac{\pi}{3}\right) \Rightarrow x=\sin^{-1}\left(\frac{y}{2}\right)-\frac{\pi}{3}$ So $\mathrm{f}^{-1}: x \mapsto \sin^{-1}\left(\frac{x}{2}\right)-\frac{\pi}{3}, \quad -2 \le x \le 2.$ For $\mathrm{ff}^{-1}(x)=\mathrm{f}^{-1}\mathrm{f}(x)$, we must have $\mathrm{D}_{\mathrm{f}} \cap \mathrm{D}_{\mathrm{f}^{-1}}$. So the solution set is $x \in \left[\frac{\pi}{6},2\right]$.
(b)(i)	Consider $y=2-\frac{5x}{1+x^2}$ $\Rightarrow 2-y=\frac{5x}{1+x^2}$ $\Rightarrow (2-y)\left(1+x^2\right)=5x$ $\Rightarrow (2-y)x^2-5x+(2-y)=0$ $\mathrm{D}=(-5)^2-4(2-y)(2-y)\ge 0$ $25-4(2-y)^2\ge 0$ $(5-2(2-y))(5+2(2-y))\ge 0$

	$(1+2y)(9-2y)\geq 0$ $\therefore -\dfrac{1}{2}\leq y\leq \dfrac{9}{2}$ So range of g $=\left[-\dfrac{1}{2},\dfrac{9}{2}\right]$
(ii)	$\mathrm{g}(x)=2-\dfrac{5x}{1+x^2}$ $\mathrm{g}\left(-\dfrac{x}{2}\right)=2-\dfrac{5\left(-\dfrac{x}{2}\right)}{1+\left(-\dfrac{x}{2}\right)^2}=2+\dfrac{10x}{4+x^2}$ $\mathrm{g}\left(-\dfrac{x}{2}\right)-2=\dfrac{10x}{4+x^2}$ Scale the graph of g by factor 2 parallel to the *x*-axis followed by a reflection in the *y*-axis followed by a translation of -2 units in the direction of *y*-axis. Or $\mathrm{g}(x)=2-\dfrac{5x}{1+x^2} \quad \to -\mathrm{g}\left(\dfrac{x}{2}\right)=-\left[2-\dfrac{5\left(\dfrac{x}{2}\right)}{1+\left(\dfrac{x}{2}\right)^2}\right]=-2+\dfrac{10x}{4+x^2}$ Scale the graph of g $\to -\mathrm{g}\left(\dfrac{x}{2}\right)+2=\dfrac{10x}{4+x^2}$ by factor 2 parallel to the *x*-axis followed by a reflection in the *x*-axis followed by a translation of 2 units in the direction of *y*-axis.
11	(i) $l_1:\mathbf{r}=\begin{pmatrix}1\\2\\1\end{pmatrix}+\lambda\begin{pmatrix}1\\a\\2\end{pmatrix},\lambda\in\mathbb{R}$ $l_2:\mathbf{r}=\begin{pmatrix}3\\0\\5\end{pmatrix}+\mu\begin{pmatrix}1\\2\\3\end{pmatrix},\mu\in\mathbb{R}$ Equating the equations of the 2 lines, $1+\lambda=3+\mu$ $2+a\lambda=2\mu$ $1+2\lambda=5+3\mu$

Solving, $\mu = 0, \lambda = 2$

$\therefore a = -1$

(ii) Coordinates of B is (3, 0, 5)

Shortest distance from B to p_1 $= \dfrac{\left|\overrightarrow{AB} \cdot \begin{pmatrix}1\\2\\3\end{pmatrix}\right|}{\sqrt{14}}$

$$= \frac{\left|\left[\begin{pmatrix}3\\0\\5\end{pmatrix} - \begin{pmatrix}1\\2\\1\end{pmatrix}\right] \cdot \begin{pmatrix}1\\2\\3\end{pmatrix}\right|}{\sqrt{14}} = \frac{\left|\begin{pmatrix}2\\-2\\4\end{pmatrix} \cdot \begin{pmatrix}1\\2\\3\end{pmatrix}\right|}{\sqrt{14}} = \frac{10}{\sqrt{14}}$$

Alternative solution 1:

Equation of plane p_1 : $\mathbf{r} \cdot \dfrac{\begin{pmatrix}1\\2\\3\end{pmatrix}}{\sqrt{14}} = \dfrac{\begin{pmatrix}1\\2\\2\end{pmatrix} \cdot \begin{pmatrix}1\\2\\3\end{pmatrix}}{\sqrt{14}} = \dfrac{8}{\sqrt{14}}$

Equation of plane parallel to p_1 and containing point B is

$$\mathbf{r} \cdot \frac{\begin{pmatrix}1\\2\\3\end{pmatrix}}{\sqrt{14}} = \frac{\begin{pmatrix}3\\0\\5\end{pmatrix} \cdot \begin{pmatrix}1\\2\\3\end{pmatrix}}{\sqrt{14}} = \frac{18}{\sqrt{14}}$$

Shortest distance from B to p_1 is $\dfrac{18}{\sqrt{14}} - \dfrac{8}{\sqrt{14}} = \dfrac{10}{\sqrt{14}}$

Alternative solution 2:

Let F be the foot of perpendicular from point B to plane p_1

Line l_2 meets plane p_1 at point F

$$\overrightarrow{OF} = \begin{pmatrix}3\\0\\5\end{pmatrix} + \lambda \begin{pmatrix}1\\2\\3\end{pmatrix} \text{ for some } \lambda \in \mathbb{R}$$

$$\left[\begin{pmatrix}3\\0\\5\end{pmatrix} + \mu \begin{pmatrix}1\\2\\3\end{pmatrix}\right] \cdot \begin{pmatrix}1\\2\\3\end{pmatrix} = 8$$

$$3 + 15 + \mu(1+4+9) = 8$$

$$\mu = -\frac{5}{7}$$

Shortest distance from B to p_1 is $\left|\overrightarrow{BF}\right| = \left|-\dfrac{5}{7}\begin{pmatrix}1\\2\\3\end{pmatrix}\right| = \dfrac{5}{7}\sqrt{1+4+9} = \dfrac{5}{7}\sqrt{14}$

$\left|\overrightarrow{AB}\right| = \sqrt{2^2 + (-2)^2 + 4^2} = \sqrt{24}$

$\sin\theta = \dfrac{10}{\sqrt{14}}\dfrac{1}{\sqrt{24}}$

$\theta \approx 33.1^0$

(iii)A normal vector to p_2 is $\begin{pmatrix}1\\2\\3\end{pmatrix} \times \begin{pmatrix}1\\-1\\2\end{pmatrix} = \begin{pmatrix}7\\1\\-3\end{pmatrix}$

Equation of p_2 is $r \cdot \begin{pmatrix}7\\1\\-3\end{pmatrix} = \begin{pmatrix}1\\2\\1\end{pmatrix} \cdot \begin{pmatrix}7\\1\\-3\end{pmatrix} = 6$

Cartesian equation of p_2 is $7x + y - 3z = 6$

(iv) Equation of x-y plane is $z = 0$

Let α be the acute angle between p_2 and x-y plane.

$\cos\alpha = \dfrac{\left|\begin{pmatrix}7\\1\\-3\end{pmatrix} \cdot \begin{pmatrix}0\\0\\1\end{pmatrix}\right|}{\sqrt{59}} = \dfrac{3}{\sqrt{59}}$

$\alpha = 67.0^o$