2016 H2 MATH (9740/02) JC 2 PRELIM EXAMINATION – MARKING SCHEME

Qn	Solution
1	**Vectors**
	$\dfrac{\lvert \mathbf{a} \cdot \mathbf{b} \rvert}{\lvert \mathbf{b} \rvert} = \dfrac{\lvert \mathbf{b} \cdot \mathbf{a} \rvert}{\lvert \mathbf{a} \rvert}$ $\lvert \mathbf{a} \rvert = \lvert \mathbf{b} \rvert$ as $\lvert \mathbf{a} \cdot \mathbf{b} \rvert \neq 0$ (as **a** and **b** are not perpendicular) $\lvert \mathbf{a} \times \mathbf{b} \rvert$ is the area of rhombus with adjacent sides OA and OB.
(i)	$\lvert \mathbf{a} \rvert = \lvert \mathbf{i} - \mathbf{j} + 3\mathbf{k} \rvert = \sqrt{11}$ $\lvert \mathbf{b} \rvert = \lvert -2\mathbf{i} + \mathbf{j} + p\mathbf{k} \rvert = \sqrt{5 + p^2}$ As $\lvert \mathbf{a} \rvert = \lvert \mathbf{b} \rvert$, $\Rightarrow 11 = 5 + p^2$ $\Rightarrow p^2 = 6$ Since $p < 0, p = -\sqrt{6}$
(ii)	Let θ denote angle AOB. $\cos\theta = \dfrac{\mathbf{a} \cdot \mathbf{b}}{\lvert \mathbf{a} \rvert \lvert \mathbf{b} \rvert}$ $= \dfrac{\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 1 \\ -\sqrt{6} \end{pmatrix}}{\left(\sqrt{11}\right)\left(\sqrt{11}\right)}$ $= \dfrac{-3 - 3\sqrt{6}}{11}$ $\theta = \cos^{-1}\left(\dfrac{-3 - 3\sqrt{6}}{11}\right) = 2.79569 \text{ rad}$ or 160.18126 degrees Area of minor sector OAB $= \dfrac{1}{2} r^2 \theta$ $= \left(\dfrac{\theta}{360}\right)\left(\pi r^2\right)$ $= \dfrac{1}{2}(11)(2.79569)$ OR $= \left(\dfrac{160.18126}{360}\right)(11\pi)$ $= 15.4 \text{ units}^2$ $= 15.4 \text{ units}^2$ $= \left(\dfrac{2.79569}{2\pi}\right)\left(\pi r^2\right)$ OR $= \left(\dfrac{2.79569}{2\pi}\right)(11\pi)$ $= 15.4 \text{ units}^2$

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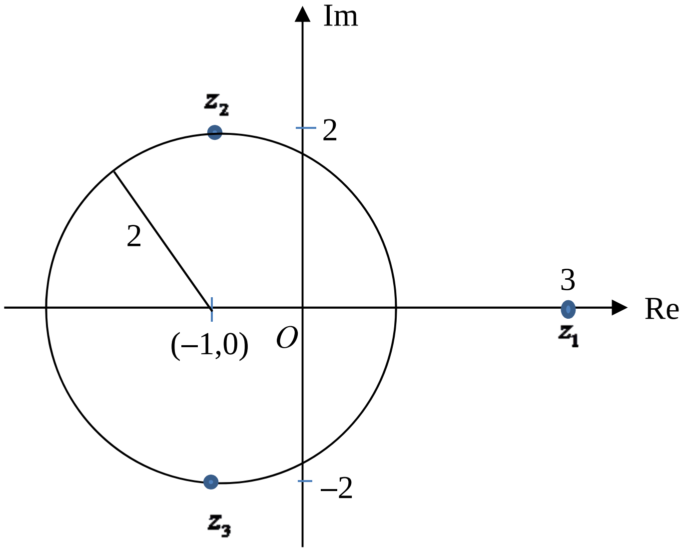
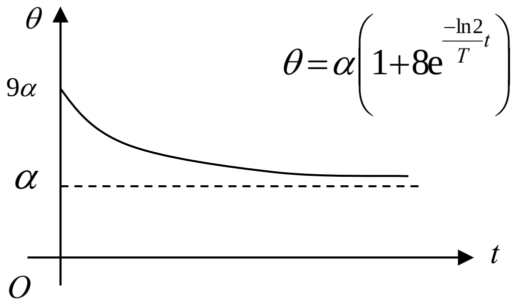
2	**Complex Numbers**
(a)	Using GC, $z_1=3,\ z_2=-1+2\text{i},\ z_3=-1-2\text{i}.$  The locus of points given by $\lvert z+1\rvert=2$ passes through the roots represented by z_2 and z_3 since $\lvert z_2+1\rvert=\lvert -1+2\text{i}+1\rvert=\lvert 2\text{i}\rvert=2$ and $\lvert z_3+1\rvert=\lvert -1-2\text{i}+1\rvert=\lvert -2\text{i}\rvert=2$.
(b) (i)	$1+\text{e}^{\text{i}\theta}=\text{e}^{\text{i}\frac{\theta}{2}}\left(\text{e}^{-\text{i}\frac{\theta}{2}}+\text{e}^{\text{i}\frac{\theta}{2}}\right)$ $=\text{e}^{\text{i}\frac{\theta}{2}}\left(\cos\frac{\theta}{2}-\text{i}\sin\frac{\theta}{2}+\cos\frac{\theta}{2}+\text{i}\sin\frac{\theta}{2}\right)$ $=\text{e}^{\text{i}\frac{\theta}{2}}\left(2\cos\frac{\theta}{2}\right)=2\text{e}^{\text{i}\frac{\theta}{2}}\cos\frac{\theta}{2}$
(ii)	$w=\frac{\text{e}^{\text{i}\theta}}{1+\text{e}^{\text{i}\theta}}$ $=\frac{\text{e}^{\text{i}\theta}}{2\text{e}^{\text{i}\frac{\theta}{2}}\cos\frac{\theta}{2}}=\frac{\text{e}^{\text{i}\frac{\theta}{2}}}{2\cos\frac{\theta}{2}}$ $=\frac{\cos\frac{\theta}{2}+\text{i}\sin\frac{\theta}{2}}{2\cos\frac{\theta}{2}}=\frac{1}{2}+\frac{1}{2}\text{i}\tan\frac{\theta}{2}$ $\therefore\ \text{Im}(w)=\frac{1}{2}\tan\frac{\theta}{2}$

Qn	Solution
3	**Differential Equations**
	$\frac{\mathrm{d}\theta}{\mathrm{d}t} = -k\left(\theta - \alpha\right), k$ is a positive constant
(i)	$\frac{\mathrm{d}\theta}{\mathrm{d}t} = -k\left(\theta - \alpha\right), k > 0$ $\int \frac{1}{\theta - \alpha}\,\mathrm{d}\theta = \int -k\,\mathrm{d}t$ $\ln\left(\theta - \alpha\right) = -kt + c$ since $\theta > \alpha$ $\theta - \alpha = \mathrm{e}^{-kt+c}$ $\theta - \alpha = A\mathrm{e}^{-kt}, A = \mathrm{e}^{c}$ $\theta = \alpha + A\mathrm{e}^{-kt}$ (shown)
(ii)	When $t = 0, \theta = 9\alpha$ $\therefore 9\alpha = \alpha + A \quad \therefore A = 8\alpha$ When $t = T, \theta = 5\alpha$ $5\alpha = \alpha + 8\alpha\mathrm{e}^{-kT}$ $\mathrm{e}^{-kT} = \frac{1}{2}$ $kT = \ln 2$ $k = \frac{\ln 2}{T}$ $\theta = \alpha + 8\alpha\mathrm{e}^{\frac{-\ln 2}{T}t}$ $\theta = \alpha\left(1 + 8\mathrm{e}^{\frac{-\ln 2}{T}t}\right)$ When $\theta = 2\alpha$ $2\alpha = \alpha\left(1 + 8\mathrm{e}^{\frac{-\ln 2}{T}t}\right)$ $\mathrm{e}^{\frac{-\ln 2}{T}t} = \frac{1}{8}$ $-\frac{\ln 2}{T}t = -\ln 8$ $t = \frac{\ln 8}{\ln 2}T = 3T$

(iii)	For large values of t, $\mathrm{e}^{\frac{-\ln 2}{T}t} \to 0, \theta \to \alpha$, θ decreases and approaches to α. 

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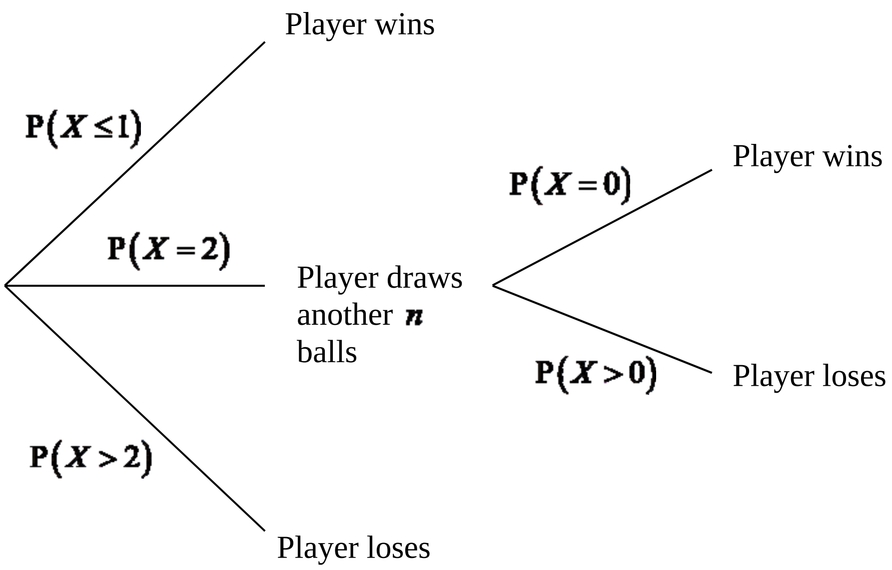
4	**APGP + Summation**
(a)	1^{st} row: number of matches = 3 2^{nd} row: number of matches = 6 3^{rd} row: number of matches = 9 n^{th} row: number of matches = $3+(n-1)(3)=3n$
	1 row: total number of matches = 3 2 rows: total number of matches = 3 + 6 3 rows: total number of matches = 3 + 6 + 9 n rows: total number of matches $=\frac{n}{2}(3+3n)$ OR $=\frac{n}{2}\left[2(3)+(n-1)(3)\right]$ $=\frac{3n(n+1)}{2}$ (shown) $=\frac{3n(n+1)}{2}$ (shown) $\frac{3n(n+1)}{2}\le 2000$ Using GC, When $n=36$, $\frac{3n(n+1)}{2}=1998<2000$ When $n=37$, $\frac{3n(n+1)}{2}=2109>2000$ Maximum number of complete rows $=36$.
(b)	Let $r=\frac{b}{a}$ $\frac{a}{1-r}=a+2b$ $\frac{1}{1-r}=1+2r$ $1=1+r-2r^2$ $2r^2-r=0$ $r(2r-1)=0$ $r=0$ (rejected $\frac{b}{a}\neq 0$) or $r=\frac{1}{2}$ $\therefore$ common ratio = $\frac{1}{2}$ $G_n=\frac{a\left(1-\left(\frac{1}{2}\right)^n\right)}{1-\frac{1}{2}}$ $G_n=2a\left(1-\left(\frac{1}{2}\right)^n\right)$

	$\sum_{n=1}^{N} G_n = 2a\sum_{n=1}^{N}\left(1-\left(\frac{1}{2}\right)^n\right)$ $= 2a\left[N-\sum_{n=1}^{N}\left(\frac{1}{2}\right)^n\right]$ $= 2a\left[N-\frac{\frac{1}{2}\left(1-\left(\frac{1}{2}\right)^N\right)}{1-\frac{1}{2}}\right]$ $= 2a\left(N-\left(1-\left(\frac{1}{2}\right)^N\right)\right)$ $= 2aN-2a\left(1-\left(\frac{1}{2}\right)^N\right)$ $= 2aN-G_N$

Qn	Solution
5	**Sampling Methods**
(i)	Systematic sampling is a sampling method in which the **entire population is listed in some order**. The population is **divided into sampling intervals of *k* members**. After obtaining **a random starting point from the first *k* members, every k^{th} member is chosen** from the list until the required number is achieved.
(ii)	Possible Advantages: • It is easy to conduct the survey as the members of the sample are easily accessible. • It is easy to conduct as the surveyor does not need the list of all the residents in the neighbourhood. Possible Disadvantages: • It is a biased sample as only residents who visit the bakery during the evening rush hour is surveyed. Hence the sample may not be representative. • It is a biased sample as some people may visit the bakery multiple times during the evening rush hours increasing their chances to be selected. • It may not be easy to get residents to visit the bakery in sequence so selection of every k^{th} resident in this case may be difficult.

Qn	Solution
6	**Binomial Distribution**
	Let X be the number of red balls drawn, out of n balls. $X \sim \mathrm{B}(n, p)$  $P = \mathrm{P}(\text{player wins})$ $= \mathrm{P}(X \le 1) + \mathrm{P}(X = 2)\mathrm{P}(X = 0)$ $= \mathrm{P}(X = 0) + \mathrm{P}(X = 1) + \mathrm{P}(X = 2)\mathrm{P}(X = 0)$ $= \binom{n}{0}(0.2)^0(0.8)^n + \binom{n}{1}(0.2)^1(0.8)^{n-1} + \left[\binom{n}{2}(0.2)^2(0.8)^{n-2}\right]\left[\binom{n}{0}(0.2)^0(0.8)^n\right]$ $= (0.8)^n + n(0.2)(0.8)^{n-1} + \left[\binom{n}{2}(0.2)^2(0.8)^{n-2}\right](0.8)^n$ $= (0.8 + 0.2n)(0.8)^{n-1} + \binom{n}{2}(0.2)^2(0.8)^{2n-2}$ (Shown)
(i)	$P = (0.8 + 0.2n)(0.8)^{n-1} + \binom{n}{2}(0.2)^2(0.8)^{2n-2} < 0.1$ Using G.C., When $n = 18,\ P = 0.10218 > 0.1$ When $n = 19,\ P = 0.08509 < 0.1$ $\therefore$ least $n = 19$
(ii)	Let Y be the number of games won, out of 100 games played. $Y \sim \mathrm{B}(100, 0.3)$.

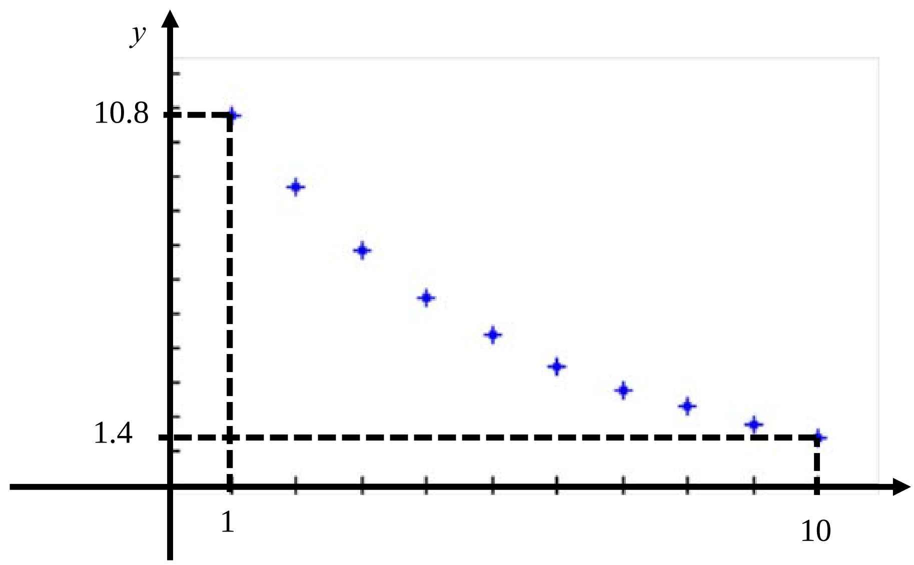
	Required probability $= \mathrm{P}(Y \geq 40)$ $= 1 - \mathrm{P}(\mathrm{Y} \leq 39)$ $= 0.020988$ ≈ 0.0210 (3 s.f.)

Qn	Solution
7	**Hypothesis Testing**
	Let X denote the number of hours of sleep each child gets at night. Let μ denote the population mean hours of sleep each child gets at night. Assumption: $X \sim \mathrm{N}(\mu, \sigma^2)$ $\therefore \overline{X} \sim \mathrm{N}\left(\mu, \dfrac{\sigma^2}{n}\right)$. H0: $\mu = 6.5$ H1: $\mu < 6.5$ Test statistic: $T = \dfrac{\overline{X} - \mu}{S / \sqrt{n}} \sim t_{n-1}$ Level of Significance: 8% Reject $\mathrm{H_0}$ if p-value < 0.08 Under $\mathrm{H_0}$, using GC, p-value = 0.0998 Since p-value = 0.0998 > 0.08, we do not reject $\mathrm{H_0}$ and conclude that there is insufficient evidence at 8% level of significance, that supports Ms Patricia's claim. $s^2 = \dfrac{15}{14}(0.849) = 0.90964$ (5 s.f) Level of Significance: 8% Reject $\mathrm{H_0}$ if t-value < -1.48389 $\dfrac{\overline{x} - 6.5}{\sqrt{0.90964} / \sqrt{15}} < -1.48389$ $\overline{x} < 6.13458$ $\therefore$ set of values of $\overline{x} = \{\overline{x} \in \mathbb{R} : 0 < \overline{x} < 6.13\}$

Qn	Solution
8	**Normal Distribution**
(i)	Let X be the mass of a randomly chosen bar of body soap in grams. Let $\overline{X}$ be the sample mean mass of 20 randomly chosen bars of body soaps in grams. $X \sim \mathrm{N}\left(110, 1.5^2\right)$ $\overline{X} \sim \mathrm{N}\left(110, \frac{1.5^2}{20}\right)$ $\overline{X}_1 - \overline{X}_2 \sim \mathrm{N}\left(0, 2\left(\frac{1.5^2}{20}\right)\right)$ $P\left(\left\|\overline{X}_1 - \overline{X}_2\right\| \le 0.5\right) = P\left(-0.5 \le \overline{X}_1 - \overline{X}_2 \le 0.5\right)$ $= 0.708\,(3\text{ s.f.})$
(ii)	Let W be the mass of a portion of liquefied soap. $W = \frac{X_1 + X_2 + X_3 + X_4 + X_5}{4}$ $W \sim \mathrm{N}\left(\frac{(5)(110)}{4}, \frac{(5)\left(1.5^2\right)}{4^2}\right)$ $W \sim \mathrm{N}\left(\frac{275}{2}, \frac{45}{64}\right)$ $\mathrm{P}(W > 140) = 0.00143\,(3\text{ s.f.})$
(iii)	Unbiased estimate of population mean, $\overline{u} = \frac{\sum u}{n} = \frac{1590}{15} = 106$ Unbiased estimate of population variance, $s^2 = \frac{1}{n-1}\left(\sum u^2 - \frac{\left(\sum u\right)^2}{n}\right)$ $= \frac{1}{15-1}\left(169046 - \frac{(1590)^2}{15}\right)$ $= 36.1\,(3\text{ s.f.})$ OR $\frac{253}{7}$

Qn	Solution
9	**Probability**
(ai)	Since $\mathrm{P}(A'\mid B)=\frac{3}{4}=\mathrm{P}(A')$, A' and B are independent events $\Rightarrow A$ and B are independent events $\mathrm{P}(A\cup B)=\mathrm{P}(A)+\mathrm{P}(B)-\mathrm{P}(A\cap B)$ $=\mathrm{P}(A)+\mathrm{P}(B)-\mathrm{P}(A)\,\mathrm{P}(B)$ (Since A and B are independent) $=\frac{1}{4}+\frac{1}{2}-\frac{1}{4}\cdot\frac{1}{2}$ $=\frac{5}{8}$
(ii)	$\mathrm{P}(C\mid A)=\frac{\mathrm{P}(C\cap A)}{\mathrm{P}(A)}=\frac{2}{3}$ $\Rightarrow \mathrm{P}(A\cap C)=\frac{2}{3}\mathrm{P}(A)$ $=\frac{2}{3}\cdot\frac{1}{4}=\frac{1}{6}$
(bi)	Required number of ways $=4!\times\frac{3!}{2!}\times{}^{5}C_{2}$ $=720$ Choose 2 out of 5 slots to put 'Y's Arrange vowels. AEE V R D Group vowels together, and arrange these 4 groups.
(ii)	Required number of ways $={}^{4}C_{2}\times 2!\times 4!$ $=288$ 3. The 4 remaining letters can permute themselves within the 4 remaining blanks. _ E Y _ _ Y E _ 1. Choose 2 out of the 1st 4 blanks to put E & Y. $\times 2!$ Because E & Y can switch positions. 2. Correspondingly, the remaining E & Y would take their respective positions in the next 4 blanks.

Qn	Solution
10	**Poisson Distribution**
(i)	The average number of tins sold per week is constant. The sale of one tin is independent of another throughout the week.
(ii)	Let X be the number of tins for chocolate cookies sold in a week. $X \sim \text{Po}(2.4)$ Let Y be the number of tins for raisin cookies sold in a week. $Y \sim \text{Po}(1.8)$ $X+Y \sim \text{Po}(4.2)$ $P(X+Y>9)=1-P(X+Y\le 9)=0.0111$ (3 s.f.)
(iii)	Let W be the total number of tins sold in 4 weeks. $W \sim \text{Po}(16.8)$ Since $\lambda = 16.8 > 10$, $\therefore W \sim \text{N}(16.8, 16.8)$ approximately. $\text{P}(15 \le W \le 25) = \text{P}(14.5 < W < 25.5)$ after continuity correction $= 0.69576 = 0.696$ (3 s.f.)
(iv)	The mean number of tins sold per week might not be constant from one week to another because of seasonal fluctuations such as sales and holidays.

Qn	Solution
11	**Correlation and Regression**
(i)	Using GC, $y = 10.30667 - 0.99939x$ $y = 10.307 - 0.999x$
(ii)	Using GC, $\sum(y-Y)^2 = 6.3689 = 6.37$ (to 3 s.f.)
(iii)	$\sum(y-Y')^2 \ge 6.37$
	

(v)	For x and y, $r = -0.9635$ For x and $\ln y$, $r = -0.9999$
(vi)	Since the scatter diagram shows that the population decreases at a decreasing rate as the years pass and the r value for the model $\ln y = c + dx$ is closer to –1 than that of $y = a + bx$, $\ln y = c + dx$ is the better model.
(vii)	Using GC, when $x = 20, \ln y = -1.89587 \Rightarrow y = 0.150187$. The population in 20^{th} year will be 150.