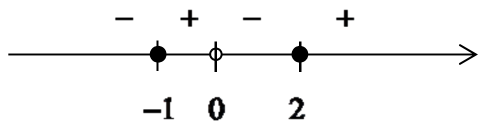
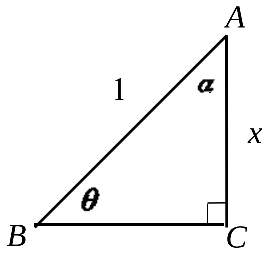
2016 H2 MATH (9740/01) JC 2 PRELIM EXAMINATION – MARKING SCHEME

Qn	Solution
1	**Inequalities**
(i)	$x^2-2x+5=\left(x-1\right)^2-1+5$ $\quad\quad\quad\quad=\left(x-1\right)^2+4\ >0\ \forall x\in\mathbb{R}$
(ii)	$\dfrac{x}{x^2-2x+5}\le\dfrac{x+2}{x^3-2x^2+5x},\ x>0$ $\dfrac{x}{x^2-2x+5}-\dfrac{x+2}{x\left(x^2-2x+5\right)}\le 0$ $\dfrac{x^2-x-2}{x\left(x^2-2x+5\right)}\le 0$ Since $x^2-2x+5>0$ for $\forall x\in\mathbb{R}$, $\dfrac{\left(x-2\right)\left(x+1\right)}{x}\le 0$ $x\le -1$ or $0<x\le 2$ 
(iii)	$\dfrac{\mathrm{e}^x}{\mathrm{e}^{2x}-2\mathrm{e}^x+5}\ge\dfrac{\mathrm{e}^x+2}{\mathrm{e}^{3x}-2\mathrm{e}^{2x}+5\mathrm{e}^x}$ Replace x with e^x. $-1\le \mathrm{e}^x<0$ (rej. $\because \mathrm{e}^x>0$) or $\mathrm{e}^x\ge 2$ $x\ge\ln 2$

Qn	Solution
2	**Definite Integrals**
(a)	$\int_1^p \ln(x)\mathrm{d}x = \left[x\ln x\right]_1^p - \int_1^p \left(\frac{1}{x}\right)x\ \mathrm{d}x$ $= \left[\left(p\ln p - 0\right) - \left(p-1\right)\right]$ $= p\ln p - p + 1$
(b)	$\int_1^q x^3\ \mathrm{d}x = \int_1^8 \sqrt[3]{y}\ \mathrm{d}y$ $\left[\frac{x^4}{4}\right]_1^q = \left[\frac{3y^{\frac{4}{3}}}{4}\right]_1^8$ $\frac{q^4}{4} - \frac{1}{4} = \left(\frac{3}{4}\right)\left(8^{\frac{4}{3}} - 1\right)$ $\frac{q^4}{4} - \frac{1}{4} = \left(\frac{3}{4}\right)(16-1)$ $q^4 = 46$ $q = 46^{\frac{1}{4}}$ $\therefore a = 46,\ b = 4$

Qn	Solution
3	**Mathematical Induction + APGP**
(i)	Let P_n be the statement $1+\frac{1}{1+2}+\frac{1}{1+2+3}+...+\frac{1}{1+2+3+...+n}=2\left(1-\frac{1}{n+1}\right), n\in\mathbb{Z}^+$. When $n=1$, LHS = 1. RHS $=2\left(1-\frac{1}{1+1}\right)=1=\text{LHS}$ $\therefore P_1$ is true. Assume that P_k is true for some $k\in\mathbb{Z}^+$ i.e. $1+\frac{1}{1+2}+\frac{1}{1+2+3}+...+\frac{1}{1+2+3+...+k}=2\left(1-\frac{1}{k+1}\right)$ To show that P_{k+1} is also true i.e. $1+\frac{1}{1+2}+\frac{1}{1+2+3}+...+\frac{1}{1+2+3+...+k+1}=2\left(1-\frac{1}{(k+1)+1}\right)$ When $n=k+1$, LHS $=1+\frac{1}{1+2}+\frac{1}{1+2+3}+...+\frac{1}{1+2+3+...+k+1}$ $=2\left(1-\frac{1}{k+1}\right)+\frac{1}{1+2+3+...+k+1}$ $=2\left(1-\frac{1}{k+1}\right)+\frac{1}{\frac{k+1}{2}(1+k+1)}$ $=2\left(1-\frac{1}{k+1}\right)+\frac{2}{(k+1)(k+2)}$ $=2+\frac{-2(k+2)+2}{(k+1)(k+2)}$ $=2+\frac{-2k-2}{(k+1)(k+2)}$ $=2+\frac{-2(k+1)}{(k+1)(k+2)}$ $=2+\frac{-2}{(k+2)}$ $=2\left(1-\frac{1}{((k+1)+1)}\right)$ $\therefore P_k$ is true $\Rightarrow P_{k+1}$ is true Since P_1 is true and P_k is true $\Rightarrow P_{k+1}$ is true, by Mathematical Induction, P_n is true for all $n\in\mathbb{Z}^+$.
	$1+\frac{1}{1+2}+\frac{1}{1+2+3}+...+\frac{1}{1+2+3+...+n}=2\left(1-\frac{1}{n+1}\right)$

	As $n \to \infty, \dfrac{1}{n+1} \to 0 \;\; \therefore 2\left(1-\dfrac{1}{n+1}\right) \to 2$ $\therefore 1+\dfrac{1}{1+2}+\dfrac{1}{1+2+3}+...+\dfrac{1}{1+2+3+...+n}+...=2$

Qn	Solution
4	**Maclaurin Series**
	$\mathrm{f}(x)=\cos^{-1}x$ $\mathrm{f}'(x)=-\dfrac{1}{\sqrt{1-x^2}}$ $\mathrm{f}''(x)=\dfrac{1}{2}\left(1-x^2\right)^{-\frac{3}{2}}\left(-2x\right)$ $\left(1-x^2\right)\mathrm{f}''(x)=-x\left(1-x^2\right)^{-\frac{1}{2}}$ $\left(1-x^2\right)\mathrm{f}''(x)=x\,\mathrm{f}'(x)$ (shown)
	$\left(1-x^2\right)\mathrm{f}''(x)=x\,\mathrm{f}'(x)$ $\left(1-x^2\right)\mathrm{f}'''(x)-2x\,\mathrm{f}''(x)=x\,\mathrm{f}''(x)+\mathrm{f}'(x)$ $\left(1-x^2\right)\mathrm{f}'''(x)=3x\,\mathrm{f}''(x)+\mathrm{f}'(x)$ $\mathrm{f}(0)=\dfrac{\pi}{2},\;\; \mathrm{f}'(0)=-1,\;\; \mathrm{f}''(0)=0,\;\;\; \mathrm{f}'''(0)=-1$ $\cos^{-1}x=\dfrac{\pi}{2}-x-\dfrac{x^3}{6}+...$
	Let $\sin^{-1}x=\theta$ Let $\cos^{-1}x=\alpha$ $\theta+\alpha=\dfrac{\pi}{2}$ $\sin^{-1}x+\cos^{-1}x=\dfrac{\pi}{2}$ (shown) 
	$\sin^{-1}x+\cos^{-1}x=\dfrac{\pi}{2}$ $\sin^{-1}x=\dfrac{\pi}{2}-\cos^{-1}x=\dfrac{\pi}{2}-\left(\dfrac{\pi}{2}-x-\dfrac{x^3}{6}+...\right)$ $\sin^{-1}x=x+\dfrac{x^3}{6}+...$

Qn	Solution
5	**Maximum/Minimum Problem**

Let volume of silo be V

$$V = \pi r^2 h + \frac{2}{3}\pi r^3$$

Time needed to paint the silo $= 20(2\pi rh) + 35(2\pi r^2)$

$$60000 = 40\pi rh + 70\pi r^2$$

$$h = \frac{60000 - 70\pi r^2}{40\pi r}$$

$$V = \pi r^2\left(\frac{60000 - 70\pi r^2}{40\pi r}\right) + \frac{2}{3}\pi r^3$$

$$= 1500r - \frac{7}{4}\pi r^3 + \frac{2}{3}\pi r^3$$

$$= 1500r - \frac{13}{12}\pi r^3$$

$$\frac{\mathrm{d}V}{\mathrm{d}r} = 1500 - \frac{13}{4}\pi r^2$$

For maximum V, $\frac{\mathrm{d}V}{\mathrm{d}r} = 0$

$$\Rightarrow 1500 - \frac{13}{4}\pi r^2 = 0$$

$$\frac{13}{4}\pi r^2 = 1500$$

$$r^2 = \frac{6000}{13\pi}$$

$$r = \pm\sqrt{\frac{6000}{13\pi}}$$

Since $r > 0$, $\therefore r = \sqrt{\frac{6000}{13\pi}} = 12.1207 = 12.1 \quad (3 \text{ s.f.})$.

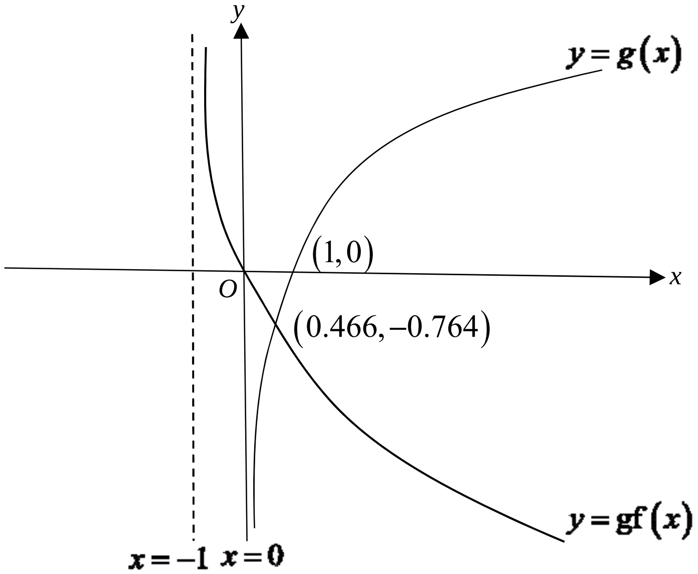
$\frac{\mathrm{d}^2V}{\mathrm{d}r^2} = -\frac{13}{2}\pi r < 0$, for $r = 12.1207$.

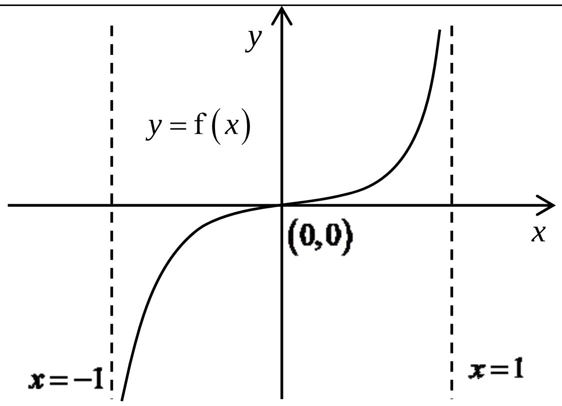
Alternative:

r	$\left(\sqrt{\frac{6000}{13\pi}}\right)^-$	$\sqrt{\frac{6000}{13\pi}}$	$\left(\sqrt{\frac{6000}{13\pi}}\right)^+$
$\frac{\mathrm{d}V}{\mathrm{d}r}$	+ /	0 —	− \

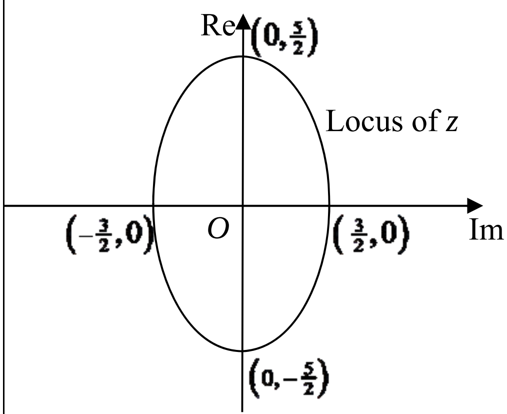
Qn	Solution
6	**Recurrence Relations**
(i)	Using GC, roots of equation are $\alpha = 1.253,\ \beta = 7.848$.
(ii)	As $n \to \infty, \quad x_n \to L, \quad x_{n+1} \to L$ $\therefore L = \ln\left(L-1\right)^2 + 4 \Rightarrow 2\ln\left(L-1\right) + 4 - L = 0$ Since equation is identical to $2\ln\left(x-1\right) + 4 - x = 0$ $\therefore L = 1.253 = \alpha$ or $L = 7.848 = \beta$ Hence the sequence converges to either α or β .
(iii)	Using GC, it can be observed that when $x_1 = 3$, the sequence increases and converges to $7.848 = \beta$. when $x_1 = 12$, the sequence decreases and converges to $7.848 = \beta$.
(iv)	$x_{n+1} - x_n = \ln(x_n - 1)^2 + 4 - x_n$ From graph, if $\alpha < x_n < \beta$, $2\ln(x_n - 1) + 4 - x_n > 0 \Rightarrow \ln(x_n - 1)^2 + 4 > x_n \Rightarrow x_{n+1} > x_n$ if $1 < x_n < \alpha$ or $x_n > \beta$, $2\ln(x_n - 1) + 4 - x_n < 0 \Rightarrow \ln(x_n - 1)^2 + 4 < x_n \Rightarrow x_{n+1} < x_n$.

Qn	Solution
7	**Vectors**
(i)	Using GC, $x = 20 - 5z$ $y = 20 + 2z$ $z = z$ $l : \mathbf{r} = \begin{pmatrix} 20 \\ 20 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -5 \\ 2 \\ 1 \end{pmatrix} \ , \quad \lambda \in \mathbb{R}$
(ii)	Normal Vector $\begin{pmatrix} 20 \\ 20 \\ 0 \end{pmatrix} \times \begin{pmatrix} -5 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 20 \\ -20 \\ 140 \end{pmatrix}$ $\mathbf{r} \cdot \begin{pmatrix} 1 \\ -1 \\ 7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 7 \end{pmatrix} = 0$ Cartesian Equation: $x - y + 7z = 0$ or equivalent
(iii)	$\mathbf{r} = \begin{pmatrix} 20 \\ 20 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -5 \\ 2 \\ 1 \end{pmatrix}$ where $\mu = 0, 1, 2, 3, 4$
(iv)	The vector equation in **(i)** allows for *x*, *y* and *z* to be real numbers. But the circuit boards produced is a physical quantity and must minimally be an integer.

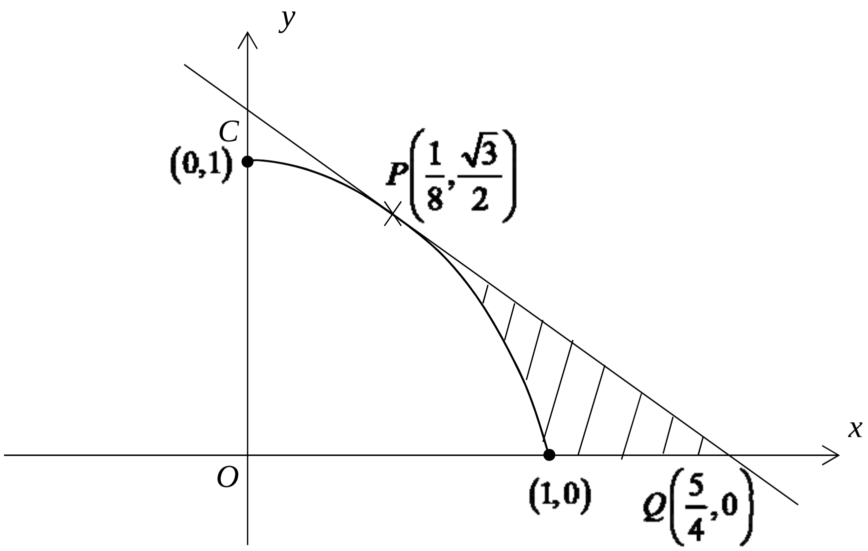
Qn	Solution
8	**Functions & Transformation of Graphs**
(i)	Since $R_f = (0,\infty) \subseteq D_g = (0,\infty)$, gf exists. $\text{gf}: x \mapsto \ln\left(\dfrac{1}{x+1}\right)^2, \quad x > -1.$
(ii)	 $R_{gf} = (-\infty,\infty) = \mathbb{R}$
(iii)	Since $x > -1, x+1 > 0$, $\text{gf}(x) = -2\ln(x+1)$ From $y = \ln x$ to $y = -2\ln(x+1)$: 1. Translation of 1 unit in the negative x-direction 2. Reflection in the x-axis 3. Scaling parallel to y-axis by a factor of 2 [Accept any other possible correct sequence such as 1-3-2]

Qn	Solution
9	**Curve sketching and differentiation**
(i)	$\mathrm{f}(x)=\dfrac{x}{\sqrt{\left(1-x^2\right)}}$ $\mathrm{f}'(x)=\dfrac{\sqrt{\left(1-x^2\right)}\cdot 1-x\cdot\frac{1}{2}\cdot\left(1-x^2\right)^{-\frac{1}{2}}\cdot(-2x)}{\left(1-x^2\right)}$ $=\dfrac{1}{\sqrt{\left(1-x^2\right)}}+\dfrac{x^2}{\left(1-x^2\right)^{\frac{3}{2}}}$ $=\dfrac{1}{\left(1-x^2\right)^{\frac{3}{2}}}>0\left(\because -1<x<1, \therefore \left(1-x^2\right)^{\frac{3}{2}}>0\right)$ Since $\mathrm{f}'(x)>0$ for $-1<x<1$, f is strictly increasing.
(ii)	
(iii)	$\mathrm{w}'(x)=\mathrm{g}(x)\mathrm{f}'(x)+\mathrm{g}'(x)\mathrm{f}(x)$ From **(i)**, $\mathrm{f}'(x)>0$ for $-1<x<1$. From **(ii)**, $\mathrm{f}(x)=\begin{cases}<0 & \text{for } -1<x<0\\ =0 & \text{for } x=0\\ >0 & \text{for } 0<x<1\end{cases}$ From the given graph, $\mathrm{g}(x)>0$ for $-1<x<1$ and $\mathrm{g}'(x)=\begin{cases}<0 & \text{for } -1<x<0\\ =0 & \text{for } x=0\\ >0 & \text{for } 0<x<1\end{cases}$ Therefore, When $-1<x<0$, $\mathrm{w}'(x)>0$. . When $x=0$, $\mathrm{w}'(x)>0$. When $0<x<1$, $\mathrm{w}'(x)>0$. Since $\mathrm{w}'(x)>0\neq 0$ for $-1<x<1$, there are no stationary points in the graph of w.

Qn	Solution
10	**Complex Numbers**
(a)	$z = w + 2\mathrm{i} - 1$ --- (1) $z^2 - \mathrm{i}w + \frac{5}{2} = 0$ --- (2) **<u>Method 1</u>** From (1): $w = z - 2\mathrm{i} + 1$ --- (3) Substitute (3) into (2): $z^2 - \mathrm{i}\left(z - 2\mathrm{i} + 1\right) + \frac{5}{2} = 0$ $z^2 - \mathrm{i}z - \mathrm{i} + \frac{1}{2} = 0$ $z = \dfrac{-(-\mathrm{i}) \pm \sqrt{(-\mathrm{i})^2 - 4(1)\left(-\mathrm{i} + \frac{1}{2}\right)}}{2(1)}$ $= \dfrac{\mathrm{i} \pm \sqrt{-3 + 4\mathrm{i}}}{2}$ $= \dfrac{\mathrm{i} \pm \left(1 + 2\mathrm{i}\right)}{2}$ $z = \frac{1}{2} + \frac{3}{2}\mathrm{i},\ w = \frac{3}{2} - \frac{1}{2}\mathrm{i}$, or $z = -\frac{1}{2} - \frac{1}{2}\mathrm{i}, w = \frac{1}{2} - \frac{5}{2}\mathrm{i}$, ---- **<u>Method 2</u>** Substitute (1) into (2): $\left(w + 2\mathrm{i} - 1\right)^2 - \mathrm{i}w + \frac{5}{2} = 0$ $w^2 + \left(2\mathrm{i} - 1\right)^2 + 2\left(2\mathrm{i} - 1\right)w - \mathrm{i}w + \frac{5}{2} = 0$ $w^2 + w\left(3\mathrm{i} - 2\right) - \frac{1}{2} - 4\mathrm{i} = 0$ $w = \dfrac{-\left(3\mathrm{i} - 2\right) \pm \sqrt{\left(3\mathrm{i} - 2\right)^2 - 4(1)\left(-\frac{1}{2} - 4\mathrm{i}\right)}}{2(1)}$ $w = \dfrac{-\left(3\mathrm{i} - 2\right) \pm \left(1 + 2\mathrm{i}\right)}{2}$ $w = \frac{3}{2} - \frac{1}{2}\mathrm{i},\ z = \frac{1}{2} + \frac{3}{2}\mathrm{i}$ or $w = \frac{1}{2} - \frac{5}{2}\mathrm{i},\ z = -\frac{1}{2} - \frac{1}{2}\mathrm{i}$
(i)	$z = w - \frac{1}{w} = 2\cos\theta + 2\mathrm{i}\sin\theta - \left(\frac{1}{2}\cos\theta - \frac{1}{2}\mathrm{i}\sin\theta\right) = \frac{3}{2}\cos\theta + \frac{5}{2}\mathrm{i}\sin\theta$

(ii)	$\text{Re}(z)=\frac{3}{2}\cos\theta,\quad \text{Im}(z)=\frac{5}{2}\sin\theta$ $x=\frac{3}{2}\cos\theta,\quad y=\frac{5}{2}\sin\theta$ $\therefore \cos\theta=\frac{2}{3}x,\quad \sin\theta=\frac{2}{5}y$ Since $\cos^2\theta+\sin^2\theta=1$ $\left(\frac{2}{3}x\right)^2+\left(\frac{2}{5}y\right)^2=1$ $\frac{x^2}{\left(\frac{3}{2}\right)^2}+\frac{y^2}{\left(\frac{5}{2}\right)^2}=1$ $\therefore a=\frac{3}{2},\quad b=\frac{5}{2}$ 

Qn	Solution
11	**Parametric Equations + Applications of Differentiation and Integration**
(i)	$\frac{\mathrm{d}x}{\mathrm{d}t}=3t^2$, $\quad \frac{\mathrm{d}y}{\mathrm{d}t}=-\frac{t}{\sqrt{1-t^2}}$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}t}\div\frac{\mathrm{d}x}{\mathrm{d}t}$ $=-\frac{t}{\sqrt{1-t^2}}\times\frac{1}{3t^2}$ $=-\frac{1}{3t\sqrt{1-t^2}}$ At point P, $x=p^3$, $y=\sqrt{1-p^3}$. Equation of tangent at P: $\quad y-\sqrt{1-p^2}=-\frac{1}{3p\sqrt{1-p^2}}\left(x-p^3\right)$ $3p\left(1-p^2\right)-3py\sqrt{}\left(1-p^2\right)=x-p^3$ (Shown)
(ii)	Substitute $x=p^3$, $y=\sqrt{1-p^2}$ into $y=4\sqrt{3}\ x$ $\sqrt{1-p^2}=4\sqrt{3}\ p^3$ $1-p^2=48p^6$ $48p^6+p^2-1=0$ Using GC, $p=-\frac{1}{2}$ or $p=\frac{1}{2}$ (N.A. since $0\le p\le 1$) $\therefore$ Exact coordinates of p are $\left(\frac{1}{8},\frac{\sqrt{3}}{2}\right)$.

(iii)	Equation of tangent to C at P: $$3\left(\frac{1}{2}\right)\left[1-\left(\frac{1}{2}\right)^2\right]-\frac{3}{2}y\sqrt{1-\left(\frac{1}{2}\right)^2}=x-\left(\frac{1}{2}\right)^3$$ $$\frac{9}{8}-\frac{3y}{2}\sqrt{\frac{3}{4}}=x-\frac{1}{8}$$ When $y=0$, $$\frac{9}{8}=x-\frac{1}{8}$$ $$x=\frac{5}{4}$$ $\therefore$ Exact coordinates of Q are $\left(\frac{5}{4},0\right)$.
(iv)	 When $x=\frac{1}{8}$, $t=\frac{1}{2}$. When $x=1$, $t=1$. Area of shaded region = Area of required region $$=\frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)\left(\frac{5}{4}-\frac{1}{8}\right)-\int_{\frac{1}{8}}^{1} y_C \, \mathrm{d}x$$ $$=\frac{9\sqrt{3}}{32}-\int_{\frac{1}{2}}^{1}\sqrt{1-t^2}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)\mathrm{d}t$$ $$=\frac{9\sqrt{3}}{32}-\int_{\frac{1}{2}}^{1}3t^2\sqrt{1-t^2}\,\mathrm{d}t$$ (Shown)

	Let $t=\sin u$, $\dfrac{\mathrm{d}t}{\mathrm{d}u}=\cos u$. When $t=\dfrac{1}{2}$, $u=\dfrac{\pi}{6}$. When $t=1$, $u=\dfrac{\pi}{2}$. $\dfrac{9\sqrt{3}}{32}-\int_{\frac{1}{2}}^{1} 3t^2\sqrt{1-t^2}\ \mathrm{d}t=\dfrac{9\sqrt{3}}{32}-\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 3\sin^2 u\sqrt{1-\sin^2 u}\ \cos u\ \mathrm{d}u$ $=\dfrac{9\sqrt{3}}{32}-\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 3\sin^2 u\cos^2 u\ \mathrm{d}u$ $=\dfrac{9\sqrt{3}}{32}-\dfrac{3}{4}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(2\sin u\cos u\right)^2\ \mathrm{d}u$ $=\dfrac{9\sqrt{3}}{32}-\dfrac{3}{4}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin^2 2u\ \mathrm{d}u$ $=\dfrac{9\sqrt{3}}{32}-\dfrac{3}{8}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 1-\cos 4u\ \mathrm{d}u$ $\dfrac{9\sqrt{3}}{32}-\dfrac{3}{8}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 1-\cos 4u\ \mathrm{d}u=\dfrac{9\sqrt{3}}{32}-\dfrac{3}{8}\left[u-\dfrac{1}{4}\sin 4u\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}\mathrm{d}u$ $=\dfrac{9\sqrt{3}}{32}-\dfrac{3}{8}\left[\left(\dfrac{\pi}{2}-\dfrac{1}{4}\sin 2\pi\right)-\left(\dfrac{\pi}{6}-\dfrac{1}{4}\sin\dfrac{2\pi}{3}\right)\right]$ $=\dfrac{9\sqrt{3}}{32}-\dfrac{3}{8}\left[\dfrac{\pi}{2}-\dfrac{\pi}{6}+\dfrac{1}{4}\left(\dfrac{\sqrt{3}}{2}\right)\right]$ $=\dfrac{9\sqrt{3}}{32}-\dfrac{3}{8}\left[\dfrac{\pi}{3}+\dfrac{\sqrt{3}}{8}\right]$ $=\dfrac{9\sqrt{3}}{32}-\dfrac{3\sqrt{3}}{64}-\dfrac{\pi}{8}$ $=\dfrac{15\sqrt{3}}{64}-\dfrac{\pi}{8}$