MERIDIAN JUNIOR COLLEGE
JC2 Preliminary Examination
Higher 2

H2 Mathematics 9740/01

Paper 1 **13 September 2016**

3 Hours

Additional Materials: Writing paper
List of Formulae (MF 15)

READ THESE INSTRUCTIONS FIRST

Write your name and civics group on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

1 A graphic calculator is not to be used for this question.

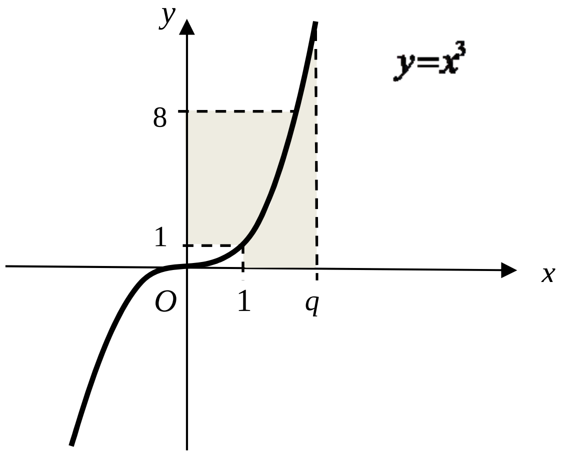
Show algebraically that $x^2 - 2x + 5$ is always positive for $x \in \mathbb{R}$, and solve the inequality

$\dfrac{x}{x^2 - 2x + 5} \le \dfrac{x+2}{x^3 - 2x^2 + 5x}$. [4]

Hence solve the inequality $\dfrac{e^x}{e^{2x} - 2e^x + 5} \ge \dfrac{e^x + 2}{e^{3x} - 2e^{2x} + 5e^x}$. [2]

2 **(a)** Find, in terms of p, $\int_1^p \ln(x)\,dx$, where $p > 1$. [2]

(b)



The diagram shows the curve with the equation $y = x^3$. The area of the region bounded by the curve, the lines $x = 1$, $x = q$ and the x-axis is equal to the area of the region bounded by the curve, $y = 1$, $y = 8$ and the y-axis, where $q > 1$. Find the exact value of q in the form $a^{\frac{1}{b}}$, where a and b are integers. [4]

3 Prove by mathematical induction that

$$1 + \frac{1}{1+2} + \frac{1}{1+2+3} + \ldots + \frac{1}{1+2+3+\ldots+n} = 2\left(1 - \frac{1}{n+1}\right),\ n \in \mathbf{Z}^+. \quad [5]$$

Hence state the value of the infinite series $1 + \dfrac{1}{1+2} + \dfrac{1}{1+2+3} + \ldots + \dfrac{1}{1+2+3+\ldots+n} + \ldots$.

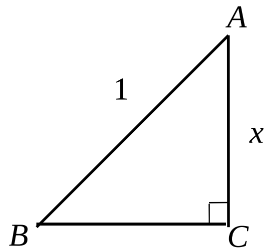
[1]

4 Let $\mathrm{f}(x)=\cos^{-1}x$, where $-1<x<1$ and $0<\mathrm{f}(x)<\pi$. Show that

$$\left(1-x^2\right)\mathrm{f}''(x)=x\,\mathrm{f}'(x). \qquad [2]$$

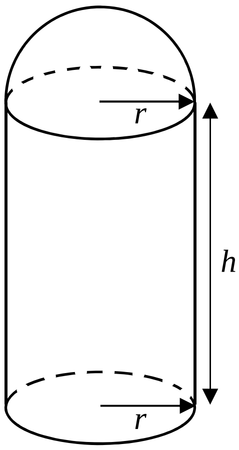
By further differentiation of this result, or otherwise, find the first three non-zero terms in the expansion of $\mathrm{f}(x)$ in ascending powers of x. [3]

The diagram shows a triangle *ABC*. Given that the lengths of *AB* and *AC* are 1 and x units respectively, show that $\sin^{-1}x+\cos^{-1}x=\dfrac{\pi}{2}$. [1]



Hence find the series expansion of $\sin^{-1}x$ in ascending powers of x, up to and including the term in x^3. [1]

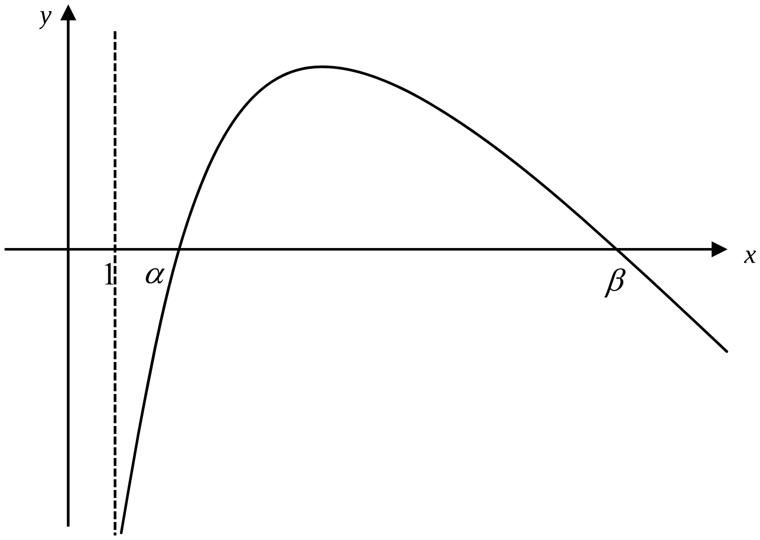
5 [It is given that a sphere of radius r has surface area $4\pi r^2$ and volume $\dfrac{4}{3}\pi r^3$.]



A rice farmer wants to build a new grain silo to store his rice grains. The cylindrical section has height h m and the hemispherical roof has radius r m. After building the grain silo, the farmer will be painting its rooftop and the external curved surface. The time needed to paint the grain silo will be 20 minutes per square metre for the curved surface area of the cylinder and 35 minutes per square metre for the hemispherical roof. Given that a total time of 60 000 minutes is taken to paint the grain silo, find, using differentiation, the value of r which gives a grain silo of maximum volume. [8]

6 The diagram below shows the graph of $y = 2\ln(x-1) + 4 - x$.

The two roots of the equation $2\ln(x-1) + 4 - x = 0$ are denoted by α and β, where $\alpha < \beta$.



(i) Find the values of α and β, correct to 3 decimal places. [2]

A sequence of real numbers $x_1, x_2, x_3, \ldots$ where $x_n > 1$, satisfies the recurrence relation

$$x_{n+1} = \ln(x_n - 1)^2 + 4 \text{ for } n \geq 1.$$

(ii) Prove algebraically that if the sequence converges, it must converge to either α or β. [2]

(iii) Use a calculator to determine the behaviour of the sequence for each of the cases $x_1 = 3,\ x_1 = 12$. [2]

(iv) By considering $x_{n+1} - x_n$ and the graph above, prove that

$$x_{n+1} > x_n \text{ if } \alpha < x_n < \beta,$$

$$x_{n+1} < x_n \text{ if } 1 < x_n < \alpha \text{ or } x_n > \beta.$$ [2]

7 The equations of three planes are

$$x + 2y + z = 60$$
$$4x + 5y + 10z = 180$$
$$2x + 3y + 4z = 100$$

(i) It is given that all three planes meet in the line l. Find a vector equation of l. [2]

(ii) Find a cartesian equation of the plane which contains l and the origin. [3]

A technology company specialises in manufacturing circuit boards that are used for space exploration. It manufactures only 3 types of circuit boards (A, B and C). Each circuit board requires particular amounts of different raw materials for manufacturing. The amounts of raw material (in units) required for each type of circuit board and the total amounts of raw material available to the company are shown in the following table.

	Copper	Lead	Fibreglass
Circuit Board A	1	4	2
Circuit Board B	2	5	3
Circuit Board C	1	10	4
Total amount of material available (in units)	60	180	100

The company is required to use all the materials available to manufacture its circuit boards.

The vector $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is defined such that variables x, y, and z represent the number of circuit boards A, B and C that are manufactured respectively.

(iii) With the aid of your answer in part **(i)**, solve for $\mathbf{r}$. Leave your answer clearly in the form of $\mathbf{a} + \mu\mathbf{b}$ and state the possible values for μ. [2]

(iv) Explain, in context, why your vector equation in part **(i)** is not an appropriate answer for part **(iii)**. [2]

8 Functions f and g are defined by

$$\mathrm{f} : x \mapsto \left(\frac{1}{x+1}\right)^2, \quad x > -1,$$

$$\mathrm{g} : x \mapsto \ln x, \quad x > 0 .$$

(i) Show that gf exists and express gf in a similar form. [3]

(ii) Sketch, in a single diagram, the graphs of g and gf, labelling each graph clearly. Write down the range of gf. [3]

(iii) Describe a sequence of transformations which maps the graph of g onto the graph of gf. [4]

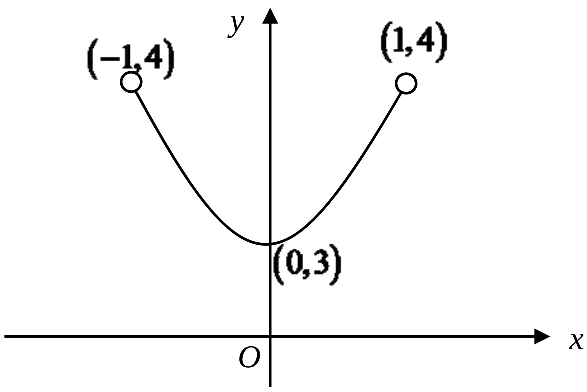
9 It is given that

$$\mathrm{f}(x) = \frac{x}{\sqrt{(1-x^2)}}, \text{ where } -1 < x < 1.$$

(i) Show by differentiation that f is strictly increasing. [3]

(ii) Sketch the graph of $y = \mathrm{f}(x)$, stating the equations of any asymptotes and the coordinates of any points of intersection with the axes. [3]

The diagram below shows the graph of $y = \mathrm{g}(x)$, which is continuous and differentiable on $(-1, 1)$. It has a minimum turning point at $(0, 3)$.



(iii) It is given that $\mathrm{w}(x) = \mathrm{g}(x)\mathrm{f}(x)$, where $-1 < x < 1$. By finding $\mathrm{w}'(x)$ and using your earlier results in **(i)** and **(ii)**, determine the number of stationary points on the graph of w. [4]

10 **(a)** Solve the simultaneous equations

$$z = w + 2\mathrm{i} - 1 \text{ and } z^2 - \mathrm{i}w + \frac{5}{2} = 0,$$

giving z and w in the form $x + y\mathrm{i}$ where x and y are real. [5]

(b) **(i)** Given that $z = w - \frac{1}{w}$ where $w = 2(\cos\theta + \mathrm{i}\sin\theta)$, $-\pi < \theta \leq \pi$, express the real and imaginary parts of z in terms of θ. [3]

(ii) Hence show that locus of z on an Argand diagram lies on the curve with cartesian equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where a and b are constants. [3]

(iii) Sketch this locus on an Argand diagram, indicating clearly the points of intersection with the axes. [2]

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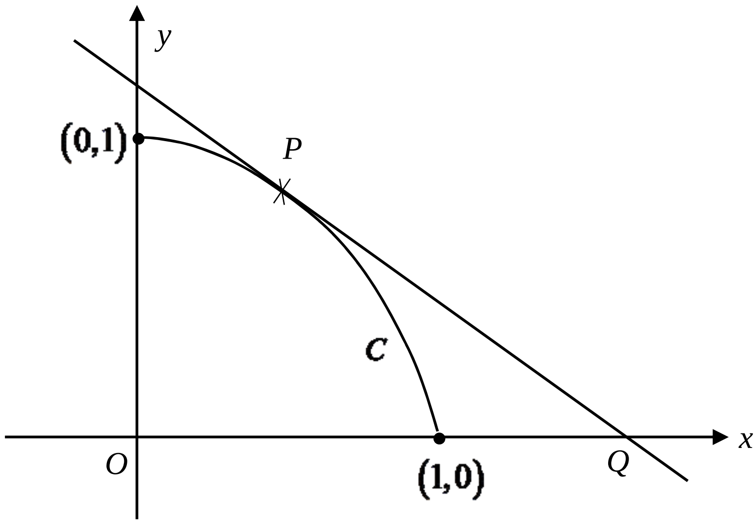

It is given that curve C has parametric equations

$$x = t^3, \quad y = \sqrt{\left(1-t^2\right)} \quad \text{for } 0 \le t \le 1.$$

The diagram shows the curve C and the tangent to C at P. The tangent at P meets the x-axis at Q.

(i) The point P on the curve has parameter p. Show that the equation of the tangent at P is $3p\left(1-p^2\right) - 3py\sqrt{\left(1-p^2\right)} = x - p^3$. [3]

(ii) Given further that the line $y = \left(4\sqrt{3}\right)x$ meets the curve at point P, find the exact coordinates of P. [3]

(iii) Hence find the exact coordinates of Q. [2]

(iv) Show that the area of the region bounded by C, the tangent to C at P, and the x-axis is given by $\frac{9\sqrt{3}}{32} - \int_{\frac{1}{2}}^{1} 3t^2\sqrt{\left(1-t^2\right)}\,\mathrm{d}t$. [3]

Show that the substitution $t = \sin u$ transforms the above integral to $\frac{9\sqrt{3}}{32} - \frac{3}{8}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 1 - \cos 4u \,\mathrm{d}u$. Hence, evaluate this area exactly. [6]

END OF PAPER