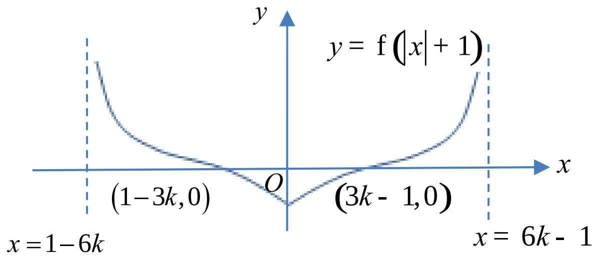
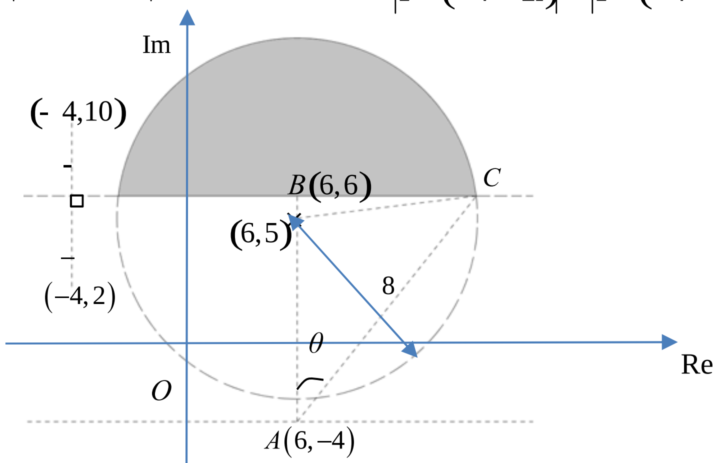
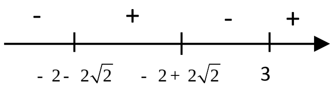
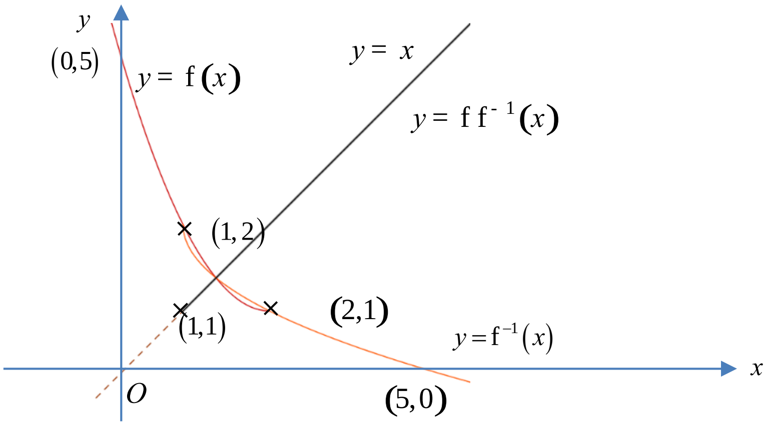
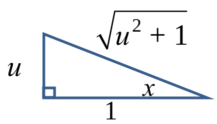
2016 VJC JC2 Prelim Paper 1 Solutions/Comments

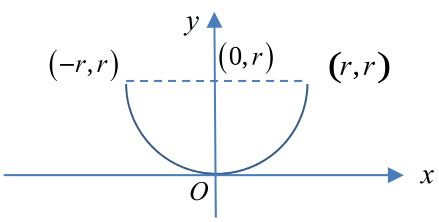
Qn	Solution
1	
2	$\left\lvert \dfrac{z-6-5\mathrm{i}}{2} \right\rvert \le 4$ $\lvert z-(6+5\mathrm{i})\rvert \le 8$ $\lvert 2\mathrm{i}-4-z\rvert \ge \lvert z+4-10\mathrm{i}\rvert$ $\lvert z+4-2\mathrm{i}\rvert \ge \lvert z+4-10\mathrm{i}\rvert$ $\lvert z-(-4+2\mathrm{i})\rvert \ge \lvert z-(-4+10\mathrm{i})\rvert$  By Pythagoras theorem, $BC^2+1^2=8^2 \Rightarrow BC=\sqrt{63}$ $\theta=\tan^{-1}\dfrac{\sqrt{63}}{10}$ Min $\arg(z-6+4\mathrm{i})=\dfrac{\pi}{2}-\tan^{-1}\dfrac{\sqrt{63}}{10}=0.900$ Max $\arg(z-6+4\mathrm{i})=\dfrac{\pi}{2}+\tan^{-1}\dfrac{\sqrt{63}}{10}=2.24$
	Alternative : Equation of circle is $(x-6)^2+(y-5)^2=64 - - (1)$ Equation of perpendicular bisector is $y=6 - - - (2)$ Substituting (2) into (1) $(x-6)^2+(6-5)^2=64 \Rightarrow x=6\pm\sqrt{63}$ Min $\arg(z-6+4\mathrm{i})=\arg\left(6+\sqrt{63}+6\mathrm{i}-6+4\mathrm{i}\right)$ $=\arg\left(\sqrt{63}+10\mathrm{i}\right)=\tan^{-1}\dfrac{10}{\sqrt{63}}=0.900$ Max $\arg(z-6+4\mathrm{i})=\arg\left(6-\sqrt{63}+6\mathrm{i}-6+4\mathrm{i}\right)$ $=\arg\left(-\sqrt{63}+10\mathrm{i}\right)=\pi-\tan^{-1}\dfrac{10}{\sqrt{63}}=2.24$

Qn	Solution
3a	$\dfrac{4-7x}{x-3} \geq x$ $\dfrac{4-7x-x(x-3)}{x-3} \geq 0$ $\dfrac{x^2+4x-4}{x-3} \leq 0$ $\dfrac{(x+2)^2-8}{x-3} \leq 0$ $\dfrac{\left(x+2+2\sqrt{2}\right)\left(x+2-2\sqrt{2}\right)}{x-3} \leq 0$  $\therefore\ x \leq -2-2\sqrt{2}$ or $-2+2\sqrt{2} \leq x < 3$
b	Let e, f and g be the ages of Edwin, his father and his grandfather respectively. $e+f+g = 53\times 3 = 159 \quad ---- \quad (1)$ $\frac{1}{4}e+\frac{1}{3}f+\frac{1}{2}g = 65 \quad ---- \quad (2)$ $g-22 = 2(f-22+e-22)$ $2e+2f-g = 66 \quad ---- \quad (3)$ From GC, $e = 24, f = 51, g = 84$. The ages of Edwin, his father and his grandfather are 24, 51 and 84 respectively.
4ia	When $k=1$, $\mathrm{f}: x \mapsto (x-2)^2+1,\ x \in \mathbb{R},\ x \leq 2$ Let $y = (x-2)^2+1$ $(x-2)^2 = y-1$ $x-2 = \pm\sqrt{y-1}$ $x = 2 \pm \sqrt{y-1}$ Since $x \leq 2$, $x = 2-\sqrt{y-1}$, $\mathrm{f}^{-1}: x \mapsto 2-\sqrt{x-1},\ x \geq 1$
b	
ii	$\{k \in \mathbb{R} : k > 2\}$

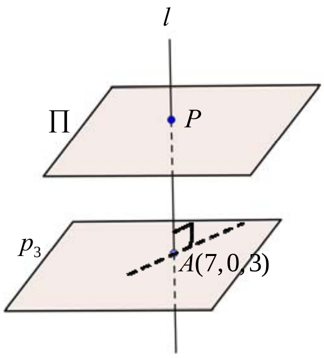
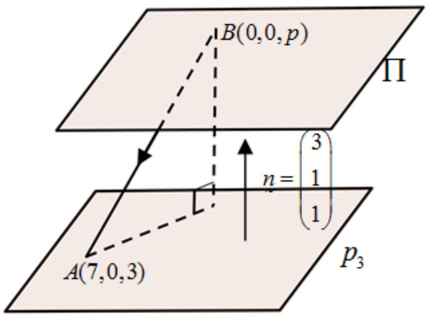
Qn	Solution
5i	$u_2 = u_1 + \dfrac{2-1^2}{(1+2)!} = 0 + \dfrac{1}{6} = \dfrac{1}{6}$ $u_3 = u_2 + \dfrac{2-2^2}{(2+2)!} = \dfrac{1}{6} - \dfrac{2}{24} = \dfrac{1}{12}$ $u_4 = u_3 + \dfrac{2-3^2}{(3+2)!} = \dfrac{1}{12} - \dfrac{7}{120} = \dfrac{1}{40}$
ii	$u_2 = \dfrac{1}{6} = \dfrac{2-1}{(2+1)!}$, $u_3 = \dfrac{1}{12} = \dfrac{2}{24} = \dfrac{3-1}{(3+1)!}$, $u_4 = \dfrac{1}{40} = \dfrac{3}{120} = \dfrac{4-1}{(4+1)!}$ By observation, a conjecture is that $u_n = \dfrac{n-1}{(n+1)!}$
iii	Let P_n be the statement $u_n = \dfrac{n-1}{(n+1)!}$, for all $n \in \mathbb{Z}^+$. Check P_1: LHS = u_1 = 0 RHS = $\dfrac{1-1}{(1+1)!} = 0$ $\therefore$ P_1 is true Assume that P_k is true for some positive integer k i.e. $u_k = \dfrac{k-1}{(k+1)!}$ We want to show that P_{k+1} is true. i.e. $u_{k+1} = \dfrac{k}{(k+2)!}$ LHS $= u_{k+1}$ $= u_k + \dfrac{2-k^2}{(k+2)!}$ $= \dfrac{k-1}{(k+1)!} + \dfrac{2-k^2}{(k+2)!}$ $= \dfrac{(k-1)(k+2)+2-k^2}{(k+2)!}$ $= \dfrac{k^2+k-2+2-k^2}{(k+2)!}$ $= \dfrac{k}{(k+2)!} =$ RHS Since P_1 is true, and P_k is true $\Rightarrow$ P_{k+1} is true, by mathematical induction, P_n is true for all $n \in \mathbb{Z}^+$.
6	If the distance AB is the least, the line segment AB is perpendicular to l.
	$B(b, 2\sqrt{b})$ and $A(a, 2a+1)$ Gradient of $BA = \dfrac{2\sqrt{b}-2a-1}{b-a}$

Qn	Solution
	Since gradient of l is 2, $\dfrac{2\sqrt{b}-2a-1}{b-a}=-\dfrac{1}{2}$ $\Rightarrow 4\sqrt{b}-4a-2=a-b$ $\Rightarrow a=\dfrac{1}{5}\left(b+4\sqrt{b}-2\right)$
	$(AB)^2=\left(2\sqrt{b}-2a-1\right)^2+(b-a)^2$ $=\left(2\sqrt{b}-2a-1\right)^2+\left(4\sqrt{b}-4a-2\right)^2$ $=5\left(2\sqrt{b}-2a-1\right)^2$ $=5\left(2\sqrt{b}-\dfrac{2}{5}\left(b+4\sqrt{b}-2\right)-1\right)^2$ $=5\left(\dfrac{2}{5}\sqrt{b}-\dfrac{2}{5}b-\dfrac{1}{5}\right)^2$ $=\dfrac{1}{5}\left(2\sqrt{b}-2b-1\right)^2$ $=\dfrac{1}{5}\left(2b-2\sqrt{b}+1\right)^2$ $2AB\dfrac{\mathrm{d}AB}{\mathrm{d}b}=\dfrac{2}{5}\left(2b-2\sqrt{b}+1\right)\left(2-\dfrac{1}{\sqrt{b}}\right)$ When $\dfrac{\mathrm{d}AB}{\mathrm{d}b}=0$, $\dfrac{2}{5}\left(2b-2\sqrt{b}+1\right)\left(2-\dfrac{1}{\sqrt{b}}\right)=0$ Consider $\left(2b-2\sqrt{b}+1\right)=0$ Since $(-2)^2-4(2)(1)<0$, $\left(2b-2\sqrt{b}+1\right)=0$ has no real solution. $2-\dfrac{1}{\sqrt{b}}=0\Rightarrow b=\dfrac{1}{4}$ the point on C nearest to l is $\left(b,2\sqrt{b}\right)=\left(\dfrac{1}{4},1\right)$.
	Alternative : $(AB)^2=\dfrac{1}{5}\left(2b-2\sqrt{b}+1\right)^2$ $=\dfrac{4}{5}\left(b-\sqrt{b}+\dfrac{1}{2}\right)^2$ $=\dfrac{4}{5}\left(\left(\sqrt{b}-\dfrac{1}{2}\right)^2-\dfrac{1}{4}+\dfrac{1}{2}\right)^2$ $=\dfrac{4}{5}\left(\left(\sqrt{b}-\dfrac{1}{2}\right)^2+\dfrac{1}{4}\right)^2$

Qn	Solution
	Since $\left(\sqrt{b}-\frac{1}{2}\right)^2 \geq 0$ for all real b, $(AB)^2$ is the least when $\sqrt{b}=\frac{1}{2}$, that is, $b=\frac{1}{4}$. Hence the point on C nearest to l is $\left(b, 2\sqrt{b}\right)=\left(\frac{1}{4},1\right)$
	Alternative : When $(AB)^2$ is the least, tangent to C at B is parallel to l. i.e. gradient of tangent to C = 2 $y^2 = 4x$ $2y\frac{\mathrm{d}y}{\mathrm{d}x} = 4$ $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2}{y}$ At $\left(b, 2\sqrt{b}\right)$, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2}{2\sqrt{b}} = \frac{1}{\sqrt{b}} = 2$ $b = \frac{1}{4}$ $\therefore$ coordinates on C nearest to l is $\left(\frac{1}{4},1\right)$.
7a	$\frac{\mathrm{d}}{\mathrm{d}x}xe^{x^3} = e^{x^3} + x3x^2e^{x^3}$ $= e^{x^3}(1+3x^3)$ $\int x^2\left(1+3x^3\right)e^{x^3}\mathrm{d}x = xe^{x^3}x^2 - \int xe^{x^3}2x\mathrm{d}x$ $= xe^{x^3}x^2 - \frac{2}{3}\int 3x^2e^{x^3}\mathrm{d}x$ $= x^3e^{x^3} - \frac{2}{3}e^{x^3} + C$
b	$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\sec^2 x}{2\sec^2 x + 4\tan x + 7}$ $y = \int \frac{\sec^2 x}{2\sec^2 x + 4\tan x + 7}\mathrm{d}x$ $= \int \frac{1}{2\left(u^2+1\right)+4u+7}\mathrm{d}u$ $= \int \frac{1}{2u^2+4u+9}\mathrm{d}u$ $= \frac{1}{2}\int \frac{1}{u^2+2u+\frac{9}{2}}\mathrm{d}u$ $= \frac{1}{2}\int \frac{1}{(u+1)^2+\frac{7}{2}}\mathrm{d}u$ $u = \tan x$ $\frac{\mathrm{d}u}{\mathrm{d}x} = \sec^2 x$  $\sec x = \sqrt{u^2+1}$ $\sec^2 x = u^2+1$ Or $\sec^2 x = \tan^2 x + 1$ $= u^2+1$

Qn	Solution
	$= \frac{1}{2}\left(\frac{\sqrt{2}}{\sqrt{7}}\right)\tan^{-1}\left(\frac{\sqrt{2}(u+1)}{\sqrt{7}}\right)+C$ $= \frac{1}{\sqrt{14}}\tan^{-1}\left(\frac{\sqrt{2}(\tan x+1)}{\sqrt{7}}\right)+C$
8i	$\sum_{r=2}^{n}\ln\left[\frac{r(r+2)}{(r+1)^2}\right]$ $= \sum_{r=2}^{n}\left(\ln r-2\ln(r+1)+\ln(r+2)\right)$ $\begin{array}{rcccccc} = & \ln 2 & - & 2\ln 3 & + & \cancel{\ln 4} \\ + & \ln 3 & - & \cancel{2\ln 4} & + & \cancel{\ln 5} \\ + & \cancel{\ln 4} & - & \cancel{2\ln 5} & + & \cancel{\ln 6} \\ & & & \vdots & & \\ + & \cancel{\ln(n-2)} & - & \cancel{2\ln(n-1)} & + & \cancel{\ln n} \\ + & \cancel{\ln(n-1)} & - & \cancel{2\ln n} & + & \ln(n+1) \\ + & \cancel{\ln n} & - & 2\ln(n+1) & + & \ln(n+2) \end{array}$ $= \ln 2-\ln 3-\ln(n+1)+\ln(n+2)$ $= \ln\frac{2}{3}+\ln\frac{n+2}{n+1}$
ii	As $n\to\infty$, $\ln\frac{n+2}{n+1}\to\ln 1=0$, $\ln\frac{2}{3}+\ln\frac{n+2}{n+1}\to\ln\frac{2}{3}$. Since the series tends to a constant, it converges. The sum to infinity is $\ln\frac{2}{3}$.
iii	$\sum_{r=2}^{13}\ln\left[\frac{(2r)(2r+4)}{(r+1)^2}\right]=\sum_{r=2}^{13}\left(\ln 4+\ln\left[\frac{r(r+2)}{(r+1)^2}\right]\right)$ $= 12\ln 4+\ln\frac{2}{3}+\ln\frac{15}{14}$ $= \ln\frac{83886080}{7}$
9i	

Qn	Solution	
ii	$V=\pi\int_0^h x^2\,\mathrm{d}y$ $=\pi\int_0^h \left(r^2-(y-r)^2\right)\mathrm{d}y$ $=\pi\left[r^2y-\frac{(y-r)^3}{3}\right]_0^h$ $=\pi\left(r^2h-\frac{(h-r)^3}{3}-\frac{r^3}{3}\right)$ $=\pi\left(\begin{array}{l}r^2h-\frac{h^3}{3}+h^2r\\-hr^2+\frac{r^3}{3}-\frac{r^3}{3}\end{array}\right)$ $=\pi\left(h^2r-\frac{h^3}{3}\right)$ $=\frac{\pi h^2}{3}(3r-h)$	Alternative: $=\pi\int_0^h \left(r^2-(y^2-2ry+r^2)\right)\mathrm{d}y$ $=\pi\int_0^h \left(2ry-y^2\right)\mathrm{d}y$ $=\pi\left[ry^2-\frac{1}{3}y^3\right]_0^h$ $=\vdots$
iii	$\frac{\mathrm{d}V}{\mathrm{d}t}=-\frac{\frac{2}{3}\pi r^3}{24}=-\frac{\pi r^3}{36}$ $\frac{\mathrm{d}V}{\mathrm{d}h}=\pi\left(2hr-h^2\right)$ $\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{1}{\pi\left(2hr-h^2\right)}\left(-\frac{\pi r^3}{36}\right)$ $=-\frac{r^3}{36\left(2hr-h^2\right)}$ Rate of decrease is $\frac{r^3}{36\left(2hr-h^2\right)}$ cm^3 s^{-1}.	
iv	The rate of decrease of the depth is the least when the bowl is full, i.e. $h=r$. $\frac{\mathrm{d}h}{\mathrm{d}t}=-\frac{r^3}{36r(2r-r)}=-\frac{r}{36}$ The slowest rate at which the depth of water is decreasing is $\frac{r}{36}$ cm s^{-1}.	
10i	Let θ be the angle between p_1 and p_2. $\cos\theta=\frac{\begin{pmatrix}1\\-5\\2\end{pmatrix}\cdot\begin{pmatrix}-2\\1\\5\end{pmatrix}}{\sqrt{30}\sqrt{30}}=\frac{3}{30}\Rightarrow\theta=84.3^{\circ}$	
ii	$x-5y+2z=13$, $-2x+y+5z=1$, From GC, $x=-2+3\lambda, y=-3+\lambda, z=\lambda$	

Qn	Solution
	$\therefore$ equation of l is $\underset{\sim}{r} = \begin{pmatrix}-2\\-3\\0\end{pmatrix} + \lambda\begin{pmatrix}3\\1\\1\end{pmatrix}, \lambda \in \mathbb{R}$
iii	The point of intersection of π_1, π_2 and π_3 is the point of intersection of l and π_3. $(a,0,b)$ is a point on l. $\begin{pmatrix}a\\0\\b\end{pmatrix} = \begin{pmatrix}-2\\-3\\0\end{pmatrix} + \lambda\begin{pmatrix}3\\1\\1\end{pmatrix} \Rightarrow$ $a = -2 + 3\lambda$, $\lambda = 3$, $b = \lambda$ $\therefore\ a = -2 + 3\times 3 = 7$ and $b = 3$ Alternatively, subst $x = a, y = 0, z = b$ into equation of planes $a + 2b = 13$ --- (1) $-2a + 5b = 1$ --- (2) $(1)\times 2 \quad 2a + 4b = 26$ --- (3) $(2)+(3) \quad 9b = 27 \Rightarrow b = 3$ $a = 7$
iv	l is perpendicular to π_3 and intersect π_3 at $A(7,0,3)$. Let P be a point on l such that $AP = 4\sqrt{11}$, then P lies in Π. $\overrightarrow{AP} = \pm 4\sqrt{11}\times\frac{1}{\sqrt{11}}\begin{pmatrix}3\\1\\1\end{pmatrix} = \pm 4\begin{pmatrix}3\\1\\1\end{pmatrix}$ $\overrightarrow{OP} = \overrightarrow{OP} + \overrightarrow{AP}$ $= \begin{pmatrix}7\\0\\3\end{pmatrix} \pm 4\begin{pmatrix}3\\1\\1\end{pmatrix} = \begin{pmatrix}-5\\-4\\-1\end{pmatrix}$ or $\begin{pmatrix}19\\4\\7\end{pmatrix}$  $\underset{\sim}{r}\cdot\begin{pmatrix}3\\1\\1\end{pmatrix} = \begin{pmatrix}-5\\-4\\-1\end{pmatrix}\cdot\begin{pmatrix}3\\1\\1\end{pmatrix} = -20$ or $\underset{\sim}{r}\cdot\begin{pmatrix}3\\1\\1\end{pmatrix} = \begin{pmatrix}19\\4\\7\end{pmatrix}\cdot\begin{pmatrix}3\\1\\1\end{pmatrix} = 68$ two possible cartesian equations of Π are $3x + y + z = -20$ and $3x + y + z = 68$. **Alternatively,** The cartesian equation of Π is of the form $3x + y + z = p$. $x = 0$, $y = 0$ and $z = p$ satisfy $3x + y + z = p$, $B(0,0,p)$ is a point in Π.  Distance between Π and π_3 is $4\sqrt{11}$.

<table>
<tr><th>Qn</th><th>Solution</th></tr>
<tr><td></td><td>

$\left|\overrightarrow{BA}\cdot\tilde{n}\right| = 4\sqrt{11}$

$\left|\begin{pmatrix}7\\0\\3-p\end{pmatrix}\cdot\frac{1}{\sqrt{11}}\begin{pmatrix}3\\1\\1\end{pmatrix}\right| = 4\sqrt{11}$

$\frac{1}{\sqrt{11}}\left|24-p\right| = 4\sqrt{11}$

$\left|p-24\right| = 44$

$p-24 = -44$ or 44

$p = -20$ or 68

two possible cartesian equations of π are

$3x+y+z=-20$ and $3x+y+z=68$.

</td></tr>
<tr><td>11a</td><td>

$\frac{a+4d}{a+2d} = \frac{a+11d}{a+4d}$

$a^2+8ad+16d^2 = a^2+13ad+22d^2$

$5ad = -6d^2$

Since the terms are distinct, $d \neq 0$, $d = -\frac{5}{6}a$

Required sum $= \frac{n}{2}\left((a+d)+(a+(2n-1)d)\right)$

$= \frac{n}{2}(2a+2nd) = n(a+nd)$

</td></tr>
<tr><td>bi</td><td>

<table>
<tr><th>n</th><th>Beginning</th><th>End</th></tr>
<tr><td>1</td><td>40000</td><td>40000(1.005)</td></tr>
<tr><td>2</td><td>40000(1.005)</td><td>$40000(1.005)^2$</td></tr>
<tr><td>3</td><td>$40000(1.005)^2 - x$</td><td>$40000(1.005)^3 - 1.005x$</td></tr>
<tr><td>4</td><td>$40000(1.005)^3 - 1.005x - x$</td><td>$40000(1.005)^3 - 1.005^2x - 1.005x$</td></tr>
<tr><td>5</td><td>$40000(1.005)^4 - 1.005^2x - 1.005x - x$</td><td></td></tr>
</table>

Amount at the beginning of 5th month

$= 40000(1.005)^4 - 1.005^2x - 1.005x - x$

$= 40000(1.005)^4 - \frac{x(1.005^3-1)}{1.005-1}$

$= 40000(1.005)^4 - 200x(1.005^3-1)$

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<tr><td>ii</td><td>

He wishes to repay his in 5 years, $n = 60$

$40000(1.005)^{59} - 200x(1.005^{58}-1) \le 0$

$x \ge \frac{40000(1.005)^{59}}{200(1.005^{58}-1)}$

$x \ge 800.17$

His minimum repayment is \$800.17

</td></tr>
<tr><td>iii</td><td>

Amount interest bank earned = \$(800.17(58) – 40000)

= \$6410.06 = \$6410 (nearest dollar)

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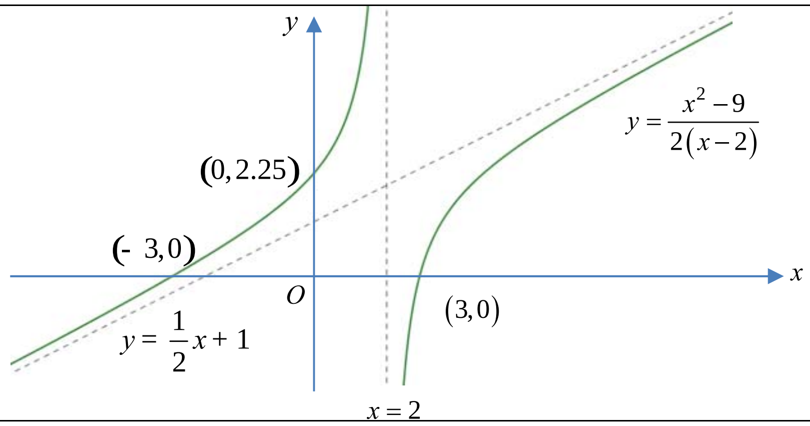
</table>

Qn	Solution
12	Since f(x) is a quadratic expression and $\mathrm{f}(3)=\mathrm{f}(-3)=0$, $\mathrm{f}(x)=k\left(x^2-9\right)$. $\frac{k\left(x^2-9\right)}{x+a}=\frac{1}{2}x+1+\frac{b}{x+a}$ $\frac{kx^2-9k}{x+a}=\frac{\left(\frac{1}{2}x+1\right)(x+a)+b}{x+a}$ $=\frac{\frac{1}{2}x^2+\left(1+\frac{1}{2}a\right)x+a+b}{x+1}$ Comparing coefficients, $k=\frac{1}{2} \Rightarrow \mathrm{f}(x)=\frac{1}{2}\left(x^2-9\right)$ $\therefore 1+\frac{1}{2}a=0 \Rightarrow a=-2$ (shown) $a+b=-\frac{9}{2} \Rightarrow b=-\frac{5}{2}$
	
	$y=\frac{x^2-9}{2(x-2)}=\frac{1}{2}x+1-\frac{5}{2(x-2)}$ $\therefore \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{2}+\frac{5}{2(x-2)^2}$ When $\frac{\mathrm{d}y}{\mathrm{d}x}=1$, $\frac{1}{2}+\frac{5}{2(x-2)^2}=1 \Rightarrow \frac{5}{2(x-2)^2}=\frac{1}{2}$ $(x-2)^2=5$ $x=2\pm\sqrt{5}$
	When $x=2+\sqrt{5}$, $y=\frac{1}{2}(2+\sqrt{5})+1-\frac{5}{2\sqrt{5}}=2$ When $x=2-\sqrt{5}$, $y=\frac{1}{2}(2-\sqrt{5})+1-\frac{5}{2(-\sqrt{5})}=2$ The equations of tangent are $y-2=x-\left(2\pm\sqrt{5}\right)$ $y=x-\sqrt{5}$ or $y=x+\sqrt{5}$