2016 Prelim Paper 2 Solution

1

Let $\mathrm{P}(n)$ be the statement, $\displaystyle\sum_{r=2}^{n}(r-1)\ln\left(\frac{r}{r-1}\right)=\ln\left[\frac{n^{n-1}}{(n-1)!}\right]$, $n\in\mathbb{Z}^{+}$, $n\geq 2$. .

When n = 2,

LHS = $(2-1)\ln\left(\dfrac{2}{2-1}\right)=\ln 2$

RHS = $\ln\left[\dfrac{2^{2-1}}{(2-1)!}\right]=\ln 2$

$\therefore$ LHS = RHS

P(2) is true.

Assume that P(k) is true for some positive integer k, $k\in\mathbb{Z}^{+}$, $k\geq 2$

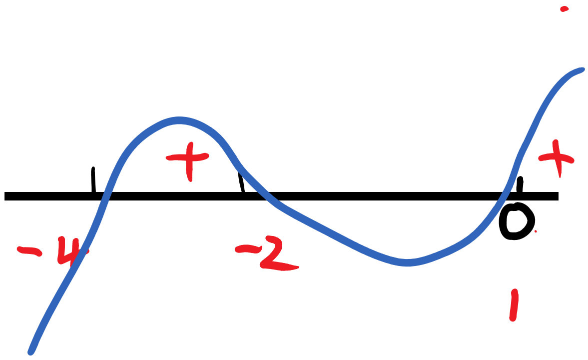
i.e. Assume $\displaystyle\sum_{r=2}^{k}(r-1)\ln\left(\frac{r}{r-1}\right)=\ln\left[\frac{k^{k-1}}{(k-1)!}\right]$.

To prove P(k + 1) is true.

i.e. to prove $\displaystyle\sum_{r=2}^{k+1}(r-1)\ln\left(\frac{r}{r-1}\right)=\ln\left[\frac{(k+1)^{k}}{k!}\right]$

$$\begin{aligned}
\text{LHS} &= \sum_{r=2}^{k+1}(r-1)\ln\left(\frac{r}{r-1}\right)\\
&= \sum_{r=2}^{k}(r-1)\ln\left(\frac{r}{r-1}\right)+k\ln\left(\frac{k+1}{k}\right)\\
&= \ln\left[\frac{k^{k-1}}{(k-1)!}\right]+k\ln\left(\frac{k+1}{k}\right)\\
&= \ln\left[\frac{k^{k-1}}{(k-1)!}\right]+\ln\left(\frac{k+1}{k}\right)^{k}\\
&= \ln\left[\frac{k^{k-1}}{(k-1)!}\times\frac{(k+1)^{k}}{k^{k}}\right]\\
&= \ln\left[\frac{k^{-1}(k+1)^{k}}{(k-1)!}\right]\\
&= \ln\left[\frac{(k+1)^{k}}{k(k-1)!}\right]\\
&= \ln\left[\frac{(k+1)^{k}}{k!}\right]\\
&= \text{RHS}
\end{aligned}$$

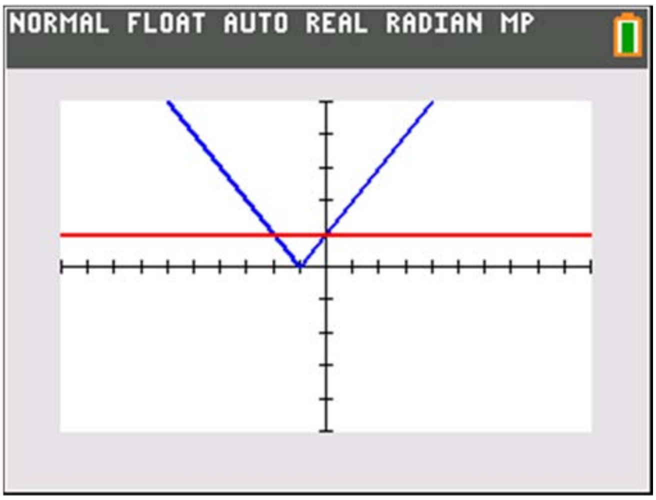
Since P(2) is true and if P(k) is true, it implies that P(k+1) is true. By Mathematical Induction, P(n) is true for all positive integers $n\geq 2$.

2	$\dfrac{x^2+6x+8}{x-1} \ge 0$, $x \ne 1$ $\dfrac{(x+2)(x+4)}{(x-1)} \ge 0$ Multiply by $(x-1)^2$, $(x-1)(x+2)(x+4) \ge 0$,  $-4 \le x \le -2$ or $x > 1$. (Ans)
	$\dfrac{y^2+2y+15}{\lvert y+1\rvert-1} \ge -6$ $\dfrac{y^2+2y+15+6(\lvert y+1\rvert-1)}{\lvert y+1\rvert-1} \ge 0$ $\dfrac{(y+1)^2+14+6\lvert y+1\rvert-6}{\lvert y+1\rvert-1} \ge 0$ $\dfrac{(y+1)^2+6\lvert y+1\rvert+8}{\lvert y+1\rvert-1} \ge 0$ Since $(y+1)^2 = \lvert y+1\rvert^2$ We use the substitution $x = \lvert y+1\rvert$, then we obtain $\dfrac{x^2+6x+8}{x-1} \ge 0$ and from the answer from (i), $-4 < \lvert y+1\rvert < -2$ (no solution since $\lvert y+1\rvert \ge 0$ for all $y \in \mathbb{R}$)

or $|y + 1| > 1$

$\therefore \quad y+1 > 1$ or $y+1 < -1$

(or use of the graphical method)

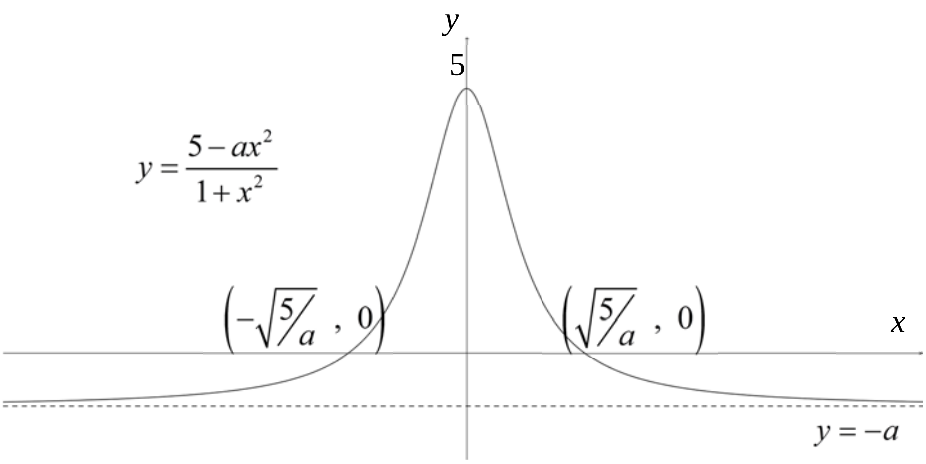


From GC, the intersection points are $(-2,1)$ and $(0,1)$

$y > 0$ or $y < -2$ (Ans)

3(i)

$$\mathrm{f}(x) = \frac{5-ax^2}{1+x^2} = \frac{5-a(1+x^2)+a}{1+x^2} = -a + \frac{5+a}{1+x^2} \qquad , a > 1$$



(ii)

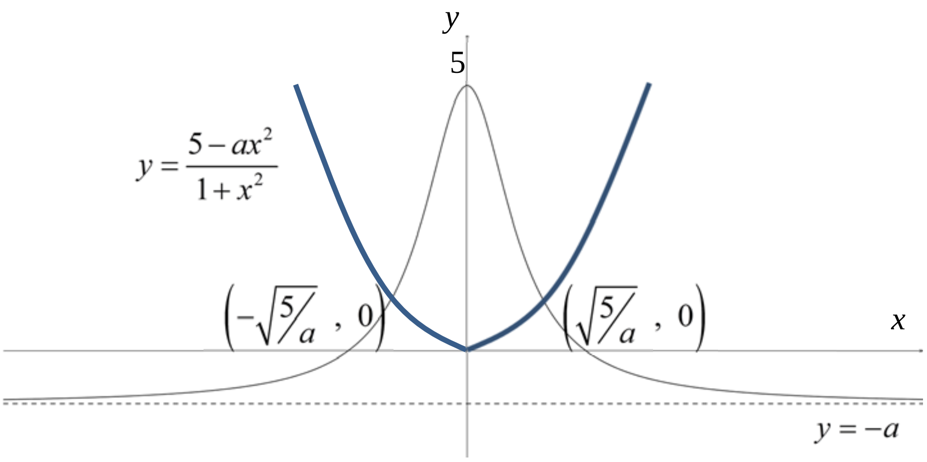
$$x^4+(a+1)x^2-5=0$$

$$x^4+x^2=5-ax^2$$

$$x^2(1+x^2)=5-ax^2$$

$$x^2=\frac{5-ax^2}{1+x^2}$$

Hence we should sketch the curve $y=x^2$.



From the graph we can see that there are only two real roots.

To show: $g(-x)=(-x)^4+(a+1)(-x)^2-5=x^4+(a+1)x^2-5=g(x)$

As there are only two real roots, the other two roots should be complex roots.

As the coefficients of equation are all real, the remaining two roots must be complex roots that form a pair of complex conjugates.

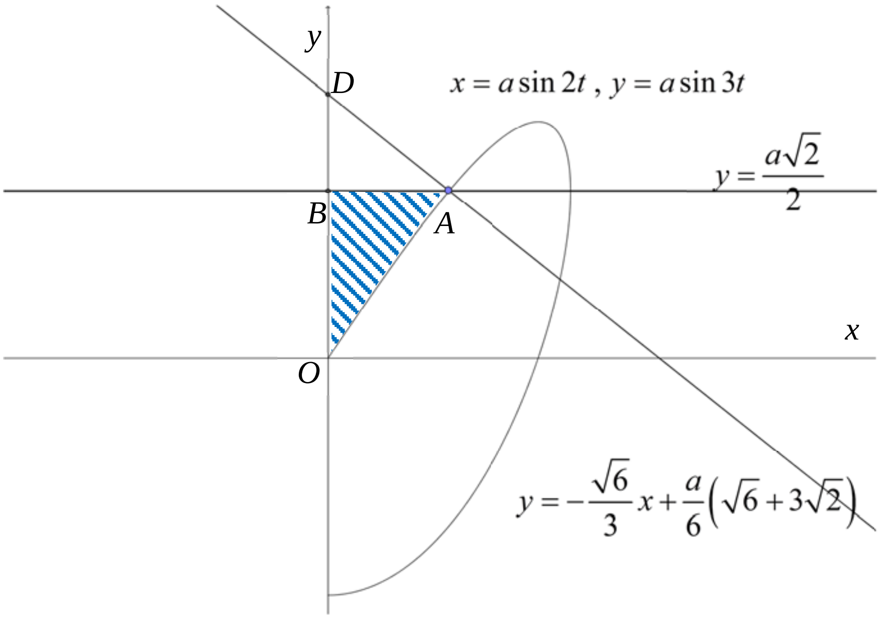
As there is only one pair the complex conjugates, and $g(x)=g(-x)=0$, then the complex conjugates must be purely imaginary.

Explanation:

If $z=x+iy$ is a root where $x,y\in\mathbb{R}$, so $g(x+iy)=0$. Since $g(z)=g(-z)$ for all $z\in\mathbb{C}$ and $g(z)=0$, then $g(-z)=0$ and hence $z=-(x+iy)=-x-iy$ is also a root. As the complex roots need to be in conjugate pairs, then

	$\left(x+\text{i}y\right)^* = -x-\text{i}y$ $x-\text{i}y = -x-\text{i}y$ Comparing the Real Part, $x=-x$ $2x=0$ $x=0$ [Note that the imaginary part is not necessary, as it yields $-y=-y$ which is trivial.] Hence the complex roots must be purely imaginary. Alternative explanation: $x^4+(a+1)x^2-5=0$ $x^2=\dfrac{(a+1)\pm\sqrt{(a+1)^2-4(1)(-5)}}{2}$ $x^2=\dfrac{(a+1)\pm\sqrt{(a+1)^2+20}}{2}$ $x^2=\dfrac{(a+1)+\sqrt{(a+1)^2+20}}{2}$ or $x^2=\dfrac{(a+1)-\sqrt{(a+1)^2+20}}{2}$ $x=\pm\sqrt{\dfrac{(a+1)+\sqrt{(a+1)^2+20}}{2}}$ or $x=\pm\sqrt{\dfrac{(a+1)-\sqrt{(a+1)^2+20}}{2}}$ Since $\dfrac{(a+1)+\sqrt{(a+1)^2+20}}{2}>0$ and $\dfrac{(a+1)-\sqrt{(a+1)^2+20}}{2}<0$ then the complex roots must be purely imaginary.
4(i)	Given $x=a\sin 2t$, $y=a\sin 3t$, $\dfrac{\text{d}x}{\text{d}t}=2a\cos 2t$, $\dfrac{\text{d}y}{\text{d}t}=3a\cos 3t$ Hence, At the point $\left(a\sin 2\theta, a\sin 3\theta\right)$, $\dfrac{\text{d}y}{\text{d}x}=\dfrac{3\cos 3\theta}{2\cos 2\theta}$ As $\theta\to\dfrac{\pi}{4}$, $\cos 2\theta\to\cos\dfrac{\pi}{2}=0$, $\dfrac{\text{d}y}{\text{d}x}\to-\infty$.

	i.e. $\frac{\mathrm{d}y}{\mathrm{d}x}$ will be undefined. Hence, the tangent to C as $\theta \to \frac{\pi}{4}$ will become a <u>vertical line</u>.
(ii)	At $t = \frac{\pi}{12}$, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3\cos\left(3\left(\frac{\pi}{12}\right)\right)}{2\cos\left(2\left(\frac{\pi}{12}\right)\right)}$ $= \frac{3\left(\frac{1}{\sqrt{2}}\right)}{2\left(\frac{\sqrt{3}}{2}\right)}$ $= \frac{\sqrt{6}}{2}$ Gradient of normal$= -\frac{2}{\sqrt{6}}$ Coordinates of point at $t = \frac{\pi}{12}$: $x = a\sin\left(2\left(\frac{\pi}{12}\right)\right)$ $= \frac{a}{2}$ $y = a\sin\left(3\left(\frac{\pi}{12}\right)\right)$ $= \frac{a\sqrt{2}}{2}$ Equation of normal: $y - \frac{a\sqrt{2}}{2} = -\frac{2}{\sqrt{6}}\left(x - \frac{a}{2}\right)$ $= -\frac{\sqrt{6}}{3}\left(x - \frac{a}{2}\right)$

	$y=-\frac{\sqrt{6}}{3}x+\frac{a\sqrt{6}}{6}+\frac{a\sqrt{2}}{2}$ $=-\frac{\sqrt{6}}{3}x+\frac{a}{6}\left(\sqrt{6}+3\sqrt{2}\right)$
(iii)	 $y=0, t=0$ $y=\frac{a\sqrt{2}}{2}, t=\frac{\pi}{12}$ (from (ii)) Area bounded by curve, y-axis and the line $y=\frac{\sqrt{2}a}{2}$, region OAB $=\int_0^{\frac{\sqrt{2}a}{2}} x\,\mathrm{d}y$ $=\int_0^{\frac{\pi}{12}}\left(a\sin 2t\right)\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)\mathrm{d}t$ $=\int_0^{\frac{\pi}{12}}\left(a\sin 2t\right)\left(3a\cos 3t\right)\mathrm{d}t$ $=3a^2\int_0^{\frac{\pi}{12}}\left(\cos 3t\sin 2t\right)\mathrm{d}t$ (Shown) y-intercept of normal is: $y=\frac{a}{6}\left(\sqrt{6}+3\sqrt{2}\right)$

Hence,

Required area, region *OADB*

$$= 3a^2\int_0^{\frac{\pi}{12}}\left(\cos 3t\sin 2t\right)\mathrm{d}t+\frac{1}{2}\left(\frac{a}{2}\right)\left[\frac{a}{6}\left(\sqrt{6}+3\sqrt{2}\right)-\frac{a\sqrt{2}}{2}\right]$$

$$= \frac{3a^2}{2}\int_0^{\frac{\pi}{12}}\left(\sin 5t-\sin t\right)\mathrm{d}t+\frac{a^2}{4}\left[\frac{\left(\sqrt{6}+3\sqrt{2}\right)}{6}-\frac{\sqrt{2}}{2}\right]$$

$$= \frac{3a^2}{2}\left[-\frac{1}{5}\cos 5t+\cos t\right]_0^{\frac{\pi}{12}}+\frac{a^2}{4}\left(\frac{\sqrt{6}}{6}\right)$$

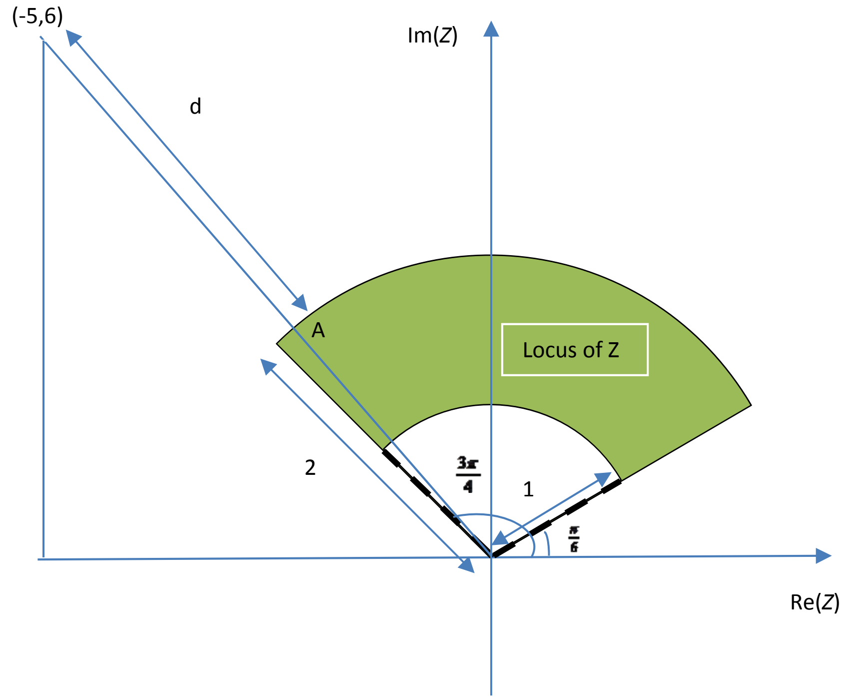
$$= \frac{3a^2}{2}\left[-\frac{1}{5}\cos\frac{5\pi}{12}+\cos\frac{\pi}{12}+\frac{1}{5}\cos 0-\cos 0\right]+\frac{a^2\sqrt{6}}{24}$$

$$= \frac{3a^2}{2}\left[-\frac{1}{5}\cos\frac{5\pi}{12}+\cos\frac{\pi}{12}-\frac{4}{5}\right]+\frac{a^2\sqrt{6}}{24}$$

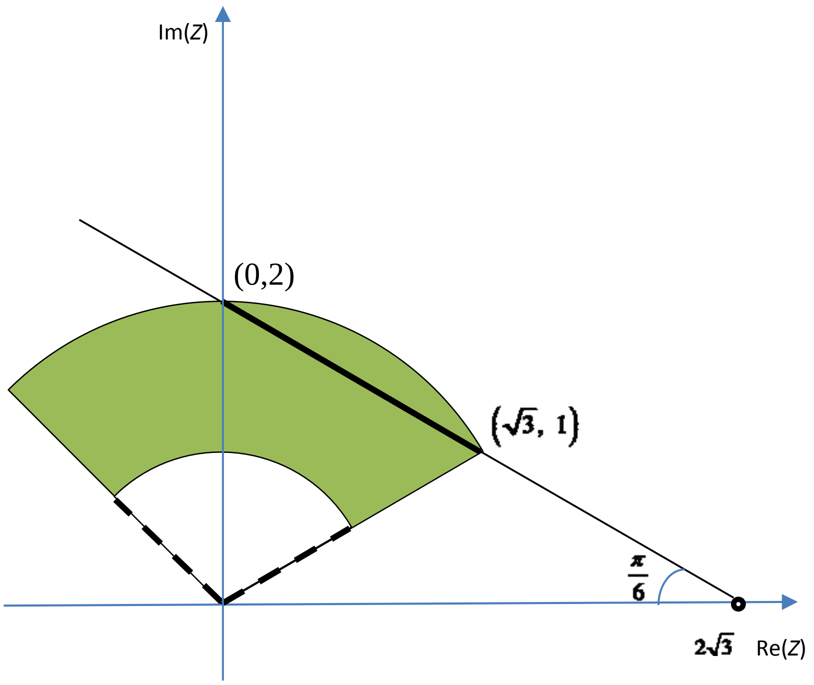
$$= \frac{3a^2}{2}\left[\cos\frac{\pi}{12}-\frac{1}{5}\cos\left(\frac{\pi}{2}-\frac{\pi}{12}\right)\right]+a^2\left(\frac{\sqrt{6}}{24}-\frac{6}{5}\right)$$

$$= \frac{3a^2}{2}\left[\cos\frac{\pi}{12}-\frac{1}{5}\sin\frac{\pi}{12}\right]+a^2\left(\frac{\sqrt{6}}{24}-\frac{6}{5}\right)\text{ units}^2$$

where $k=\frac{3}{2}, b=1, c=-\frac{1}{5}, d=\frac{\sqrt{6}}{24}-\frac{6}{5}$

5(i), (ii)	$\arg(z)=\theta$ $\lvert z\rvert = r$ 
(ii)	The minimum distance, d $=\sqrt{(-5)^2+6^2}-2=\sqrt{61}-2$ Let θ be the basic angle of the point $(-5,6)$. From the diagram, $\tan\theta=\dfrac{6}{5}$ so $\sin\theta=\dfrac{6}{\sqrt{61}}$ and $\cos\theta=\dfrac{5}{\sqrt{61}}$. Hence complex number that corresponds to point A, $z=2\left(-\cos\theta+\mathrm{i}\sin\theta\right)=-2\left(\dfrac{5}{\sqrt{61}}\right)+2\mathrm{i}\left(\dfrac{6}{\sqrt{61}}\right)=\dfrac{1}{\sqrt{61}}\left(-10+12\mathrm{i}\right)$ $=-1.28+1.54\mathrm{i}$

(iii)

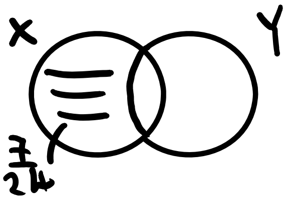


6(i) The people who visits the fitness centre on a particular weekday may not be representative of the members of the club as some members may be working or attending school and only go to club later in the day. Thus the sample collected would contain mainly women who are not working/do not attend school thereby under representing women of some age groups.

(ii) Stratified sampling method should be used to obtain a representative sample of 200 members.

1. Draw up a list of the members of the fitness centre and calculate the proportion for the different corresponding age group.

18 – 25	26-30	31 – 40	41 – 50	51 & above
500	1000	1500	1500	500
$\frac{500}{5000}\times 200 = 20$	$\frac{1000}{5000}\times 200 = 40$	$\frac{1500}{5000}\times 200 = 60$	$\frac{1500}{5000}\times 200 = 60$	$\frac{500}{5000}\times 200 = 20$

	2. Within each age group, use simple random sampling method to select the required sample size for survey.
7 (a)(i)	$\mathrm{P}(X \cup Y) = \mathrm{P}(Y) + \mathrm{P}(X \cap Y')$ $\frac{5}{8} = \mathrm{P}(Y) + \frac{7}{24}$ $\mathrm{P}(Y) = \frac{8}{24} = \frac{1}{3}$  Given $\mathrm{P}(X' \mid Y) = \frac{9}{16}$, $\frac{\mathrm{P}(X' \cap Y)}{\mathrm{P}(Y)} = \frac{9}{16}$ $\mathrm{P}(X' \cap Y) = \frac{9}{16} . \frac{1}{3} = \frac{3}{16}$
(ii)	$\mathrm{P}(X) = 1 - [\mathrm{P}(X \cup Y)' + \mathrm{P}(X' \cap Y)] = 1 - \frac{3}{8} - \frac{9}{48} = \frac{7}{16}$ $\mathrm{P}(X).\mathrm{P}(Y') = \frac{7}{16} . \frac{2}{3} = \frac{7}{24} = \mathrm{P}(X \cap Y')$ X and Y' are independent.
7(b) **(i)**	Without restriction $= \binom{11}{6}(6-1)!5! = 6652800$ Case 1: Ms Koh sits at round table with two male teachers No. of ways $= \binom{8}{3}\binom{6}{2} 2!(4-1)!5! = 1209600$ Case 2: Ms Koh sits at long table with two male teachers No. of ways $= \binom{8}{2}\binom{6}{2} 2!3!(6-1)! = 604800$ Required Probability

	$=\dfrac{\binom{8}{3}\binom{6}{2}2!(4-1)!5!+\binom{8}{2}\binom{6}{2}2!3!(6-1)!}{\binom{11}{6}(6-1)!5!}$ $=\dfrac{3}{11}$ or 0.273 (to 3.s.f)
(ii)	Required Probability $=\dfrac{P(\text{Ms koh sits between 2 male teachers and male and female teachers alternate})}{P(\text{Ms koh sits between 2 male teachers})}$ $=\dfrac{P(\text{male and female teachers alternate})}{P(\text{ms koh sits between 2 male teachers})}$ $=\dfrac{\dfrac{\binom{6}{3}\binom{5}{3}(3-1)!3!3!2!}{6652800}}{\dfrac{3}{11}}$ OR $\dfrac{\dfrac{6\times5\times4\times5\times4\times2\times3\times2}{6652800}}{\dfrac{3}{11}}$ $=\dfrac{\dfrac{1}{231}}{\dfrac{3}{11}}$ $=\dfrac{1}{63}$ or 0.159 (to 3.s.f)

Alternative Solution:

Required Probability

$= \dfrac{P(\text{ms koh sits between 2 male teachers and male and female teachers alternate})}{P(\text{ms koh sits between 2 male teachers})}$

Case 1: Ms Koh sits at the long table and the male and female teachers alternate

no. of ways = $\binom{6}{3}\binom{4}{1}3!2!3!(3-1)! = 11520$

Case 2: Ms Koh sits at the circular table and the male and female teachers alternate

no. of ways = $\binom{6}{3}\binom{4}{2}(3-1)!3!3!2! = 17280$

Total no. of ways ms koh sits between 2 male teachers and male and female teachers alternate

$= \binom{6}{3}\binom{4}{1}3!2!3!(3-1)! + \binom{6}{3}\binom{4}{2}(3-1)!3!3!2!$

$= 28800$

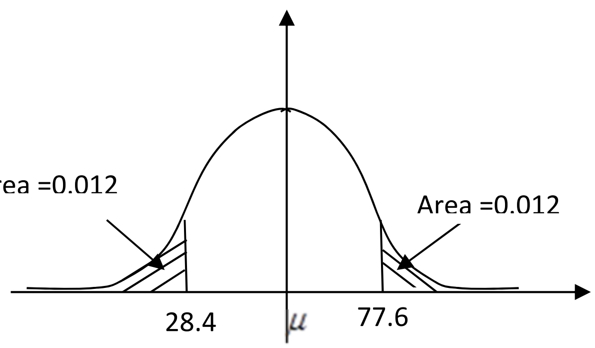
Required Probaility

$= \dfrac{\frac{28800}{6652800}}{\frac{3}{11}}$

$= \dfrac{1}{63}$

[Note that for the males and females to seat on alternate seats, on the round table there must be 3 males and 3 females and the long table there must be 3 males and 2 females]

8 (i)



Method 1 (recognise μ is the midpoint)

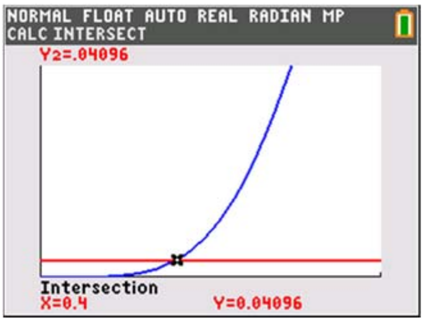
By symmetry, $\mu = \dfrac{28.4+77.6}{2} = 53$

$P(X < 28.4) = 0.012$

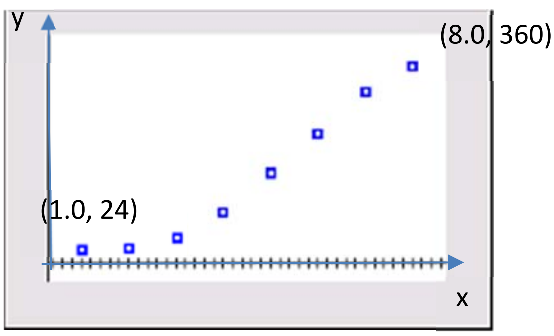
$P(Z < \dfrac{28.4-53}{\sigma}) = 0.012$

Using GC,

	$\frac{-24.6}{\sigma} = -2.25712924$ $\sigma = 10.8988$ $\sigma = 10.9$ (3 s.f.) <u>Method 2 (simultaneous equations- not recommended)</u> $P(X < 28.4) = 0.012$ $\qquad$ $P(X < 77.6) = 0.988$ $P(Z < \frac{28.4-\mu}{\sigma}) = 0.012$ $\qquad$ $P(Z < \frac{77.6-\mu}{\sigma}) = 0.012$ $\frac{28.4-\mu}{\sigma} = -2.25712924$ $\qquad$ $\frac{77.6-\mu}{\sigma} = 2.25712924$ $28.4-\mu = -2.25712924\sigma$ $\qquad$ $77.6-\mu = 2.25712924\sigma$ Solve simultaneously, $\mu = 53$, $\quad \sigma = 10.8988$
(ii)	Let X be the weight of a packet of sweets in grams. $X \sim N(53, 10.8988^2)$ <u>Method 1 (expression in terms of mass)</u> $X_1 + X_2 + \ldots X_4 \sim N(212, 475.13536)$ $\$1.20 \to 100g$ $\$2.60 \to \frac{2.6\times 100}{1.2} = 216.667g$ $P(X_1 + X_2 + \ldots X_4 \le 216.667) = 0.58476$ $= 0.585$ (3 s.f.) $P(X_1 + X_2 + \ldots X_4 \le \frac{2.6\times 100}{1.2}) = 0.58476$ $= 0.585$ (3 s.f.) <u>Method 2 (expression in terms of cost)</u> Let C denote the cost of 4 packets of sweets. $C = \frac{1.20}{100}(X_1 + X_2 + \ldots X_4) = 0.012(X_1 + X_2 + \ldots X_4)$ Then $E(C) = 0.012(4\times 53) = 2.544$ $Var(C) = 0.012^2(4\times 10.8988^2) = 0.0684195$

	$P(C \le 2.60) = 0.5847618 = 0.585$ (3 s.f.)
9 (i)	The probability of picking a brown egg from a box is constant. The colour of an egg is independent of other eggs.
(ii)	Let X be the r.v. "number of brown eggs in a box of 6 eggs" $X \sim B(6, p)$ $P(X \ge 5) = 0.04096$ $P(X = 5) + P(X = 6) = 0.04096$ $\binom{6}{5} p^5 (1-p) + \binom{6}{6} p^6 (1-p)^0 = 0.04096$  Using GC, $p = 0.4$
(iii)	Let A be the r.v. "number of boxes that contain at most 4 brown eggs in a box out of 100 boxes" $A \sim B(100, p_1)$, where $p_1 = 1 - 0.04096 = 0.95904$. Let Y be the r.v. "number of boxes that contain at least 5 brown eggs in a box out of 100 boxes" $Y \sim B(100, 0.04096)$ Note that $A + Y = 100$ Since $n = 100$ is large, $np = 100(0.04096) = 4.096 < 5$, $Y \sim P_0(4.096)$ approximately $P(A \ge 90) = P(100 - Y \ge 90) = P(Y \le 10) = 0.997$
(iv)	$Y \sim B(100, 0.04096)$

	$E(Y)=100(0.04096)=4.096$ $Var(Y)=100(0.04096)(0.95904)=3.9282$ In 8 weeks, there are 56 days altogether. Mean number of boxes with at least 5 brown eggs is $\overline{Y}=\dfrac{Y_1+Y_2+Y_3+\cdots+Y_{56}}{56}$ Since sample size = 56 is large, by Central Limit Theorem, $\overline{Y} \sim \mathrm{N}\left(4.096, \dfrac{3.9282}{56}\right)$ approximately i.e. $\overline{Y} \sim \mathrm{N}(4.096, 0.0701469)$ $\mathrm{P}\left(4<\overline{Y}<7\right)=0.641$ (correct to 3 sig fig)
10(i)	Let X be the random variable ‘ length of time a patient spent with the doctor’ Given $\sum x=147$, $\sum x^2=1927.91$ The unbiased estimates of population mean μ and population variance σ^2 are $\overline{x}=12.25$, $s^2=\dfrac{1}{n-1}[\sum x^2-\dfrac{(\sum x)^2}{n}]=11.56$ **Assumption:** The length of time, X, a patient spent with the doctor follows a normal distribution. Test H_0: $\mu=10$ H_1: $\mu>10$ Under H_0, $T=\dfrac{\overline{X}-\mu}{S/\sqrt{n}} \sim t(11)$ Use a right tailed t-test at the 5% level of significance. From GC, p-value = 0.0213

	Since p-value $= 0.0213 < 0.05$, we reject H_0. There is sufficient evidence to conclude at the 5% level of significance, the mean time spent with a patient is more than 10 minutes.
ii	Since population variance is given, z-test should be used. Test H_0: $\mu = 10$ H_1: $\mu > 10$ Under H_0, $Z = \dfrac{\overline{X} - \mu}{\sigma / \sqrt{n}} \sim N(0,1)$ Since H_0 is not rejected, $z = \dfrac{12.25 - 10}{15/\sqrt{n}} < 1.64485$ $0.15\sqrt{n} < 1.64485$ $n < 120.25$ $\therefore$ Maximum n = 120. Set of values of n is $\{\ n \in \square : 0 < n \le 120\ \}$
11(i)	$r = 0.974$
(ii)	 Although the value of product moment correlation coefficient indicates a strong positive linear correlation however from the scatter diagram, X and Y follows a non-linear relationship, hence it is advisable to interpret the data using both the scatter diagram and the value of product moment correlation coefficient.

(iii)	Using GC, $Y = 9.67 + 5.81X^2$ $r = 0.993$ Since the value of product moment correlation coefficient for Y on X^2 is closer to 1, compared to the value of product moment correlation coefficient for Y on X, the new proposed model is a better model for the data set.
(iv)	$Y = 9.6714285 + 5.81190X^2$ When $y = 120$, $x = 4.36$ (3 s.f.) The line of Y on X^2 is used because the value of X is fixed precisely and hence X is the independent variable.
12(i)	It must be assumed that the <u>average</u> number of customers who bought the package per week is a constant. The number of customers who bought the package per week varies, and cannot be a constant.
(ii)	Let X and Y be the number of customers who bought the Luxury Cruise Package in a week at Branch A and Branch B respectively. X~Po(3.5) Y~Po(4.5) X+Y~Po(8) $\mathrm{P}(5 < X + Y < 10) = \mathrm{P}(X + Y \le 9) - \mathrm{P}(X + Y \le 5) \approx 0.525$
(iii)	Let U be the number of customers who bought the Luxury Cruise Package in n weeks at Branch A. U~Po(3.5n) Given $\mathrm{P}(U \le 1) < 0.1$ From GC, when $n = 1$, $\mathrm{P}(U \le 1) = 0.136 > 0.1$ when $n = 2$, $\mathrm{P}(U \le 1) = 0.0073 < 0.1$ Least $n = 2$
(iv)	Let S and T be the number of customers who bought the Luxury Cruise Package in one month at Branch A and Branch B respectively. S~Po(14)

	Since $\lambda > 10$, S~N(14, 14) approximately. T~Po(18) Since $\lambda > 10$, T~N(18, 18) approximately. T-S ~ N(4, 32) $\text{P}(0 < T - S \leq 5) \xrightarrow{C.C.} \text{P}(0.5 < T - S \leq 5.5) \approx 0.337$
	The mean number of customers who bought the Luxury Package might not be a constant from one week to another because of fluatuations such as sales, holidays, the economic climate etc. Hence the Poisson distribution may not be a good model for the number of customers who bought the package in a year.