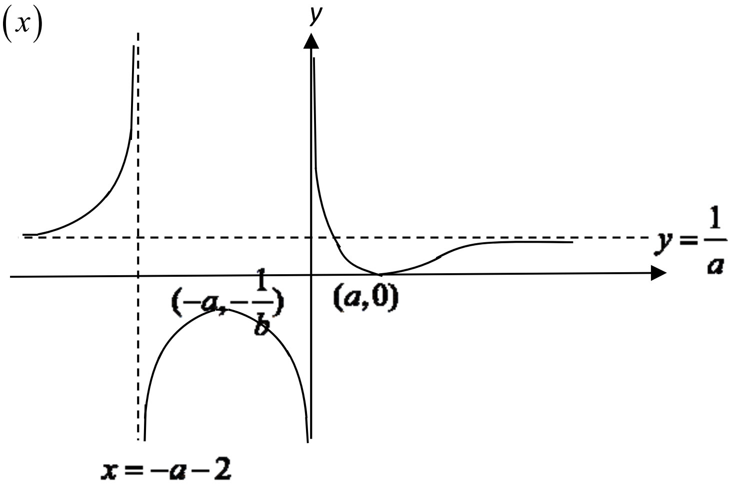
2016 Prelim Paper 1 Solution

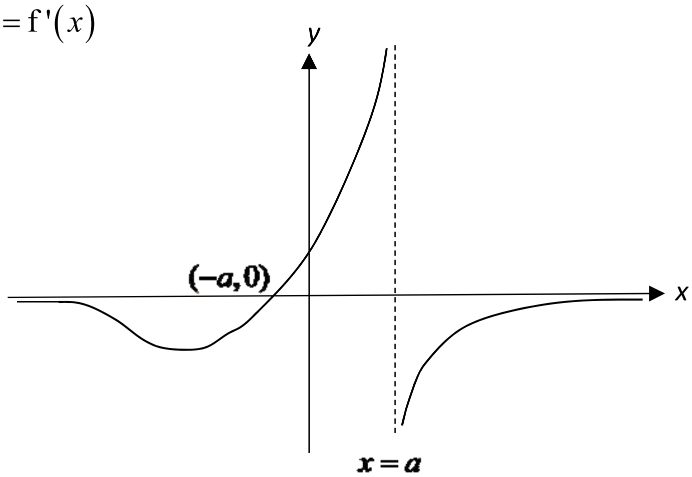
1(i)	$\sum_{r=1}^{n} \frac{1}{(3r-2)(3r+1)}$ $= \frac{1}{3}\sum_{r=1}^{n}\left(\frac{1}{3r-2}-\frac{1}{3r+1}\right)$ $= \frac{1}{3}[\ 1-\cancel{\frac{1}{4}}$ $+\ \cancel{\frac{1}{4}}-\cancel{\frac{1}{7}}$ $+\ \vdots$ $+\ \cancel{\frac{1}{3n-5}}-\cancel{\frac{1}{3n-2}}$ $+\ \cancel{\frac{1}{3n-2}}-\frac{1}{3n+1}]$ $= \frac{1}{3}\left(1-\frac{1}{3n+1}\right)$
1(ii)	As $n \to \infty, S_n \to \frac{1}{3}$ since $\frac{1}{3n+1} \to 0$
(iii)	$\left\|S_n - S\right\| < 2\times10^{-4}$ $\left\|\frac{1}{3}\left(1-\frac{1}{3n+1}\right)-\frac{1}{3}\right\| < 2\times10^{-4}$ $\left\|-\frac{1}{3}\left(\frac{1}{3n+1}\right)\right\| < 2\times10^{-4}$ $\frac{1}{3}\left(\frac{1}{3n+1}\right) < 2\times10^{-4}$ since $n \in \mathbb{Z}^+$ $\frac{1}{3n+1} < \frac{3}{5000}$ $3n+1 > \frac{5000}{3}$ $n > 555.2$ Hence, smallest n = 556.

(iv)	$\sum_{r=0}^{n} \frac{1}{(3r+1)(3r+4)}$ Let $r = k-1$ $3r+1 = 3(k-1)+1 = 3k-2$ $3r+4 = 3(k-1)+4 = 3k+1$ When $r=0, k=1,$ When $r=n, k=n+1$ $\therefore \sum_{r=0}^{n} \frac{1}{(3r+1)(3r+4)}$ $= \sum_{k=1}^{n+1} \frac{1}{(3k-2)(3k+1)}$ $= \frac{1}{3}\left[1-\frac{1}{3(n+1)+1}\right]$ from (i) $= \frac{1}{3}\left(1-\frac{1}{3n+4}\right)$ $(3r+4)^2 > (3r+1)(3r+4)$ for $r \geq 0, r \in \mathbb{Z}$ $\frac{1}{(3r+4)^2} < \frac{1}{(3r+1)(3r+4)}$ $\sum_{r=0}^{n} \frac{1}{(3r+4)^2} < \sum_{r=0}^{n} \frac{1}{(3r+1)(3r+4)} < \frac{1}{3}$ since $1-\frac{1}{3n+4} < 1$ for $n \in \mathbb{Z}^{+} \cup \{0\}$
2 (a)	$y = \ln(4x^2-16x+15)$ $= \ln(4(x^2-4x)+15)$ $= \ln(4(x-2)^2-16+15)$ $= \ln(4(x-2)^2-1)$ Therefore to transform $y = \ln(4(x-2)^2-1)$ to $y = \frac{1}{2}\ln(4x^2-1)$ means: A translation to the left by two units in the direction of the positive x– axis followed by; A scale parallel to y– axis by scale factor of 1/2.

(b)(i) $y=\frac{1}{\mathrm{f}(x)}$



(ii) $y=\mathrm{f}'(x)$



3(a)

Year No	Amount withdrawn at the beginning of N years
1	0
2	$0.05x$
3	$0.05(2x)$
…	
$N+1$	$T_N=0.05Nx$

$S_{N+1} = 0.05x(1+2+3...N)$

$= \dfrac{0.05xN(N+1)}{2}$

$= \dfrac{xN(N+1)}{40}$

(i)

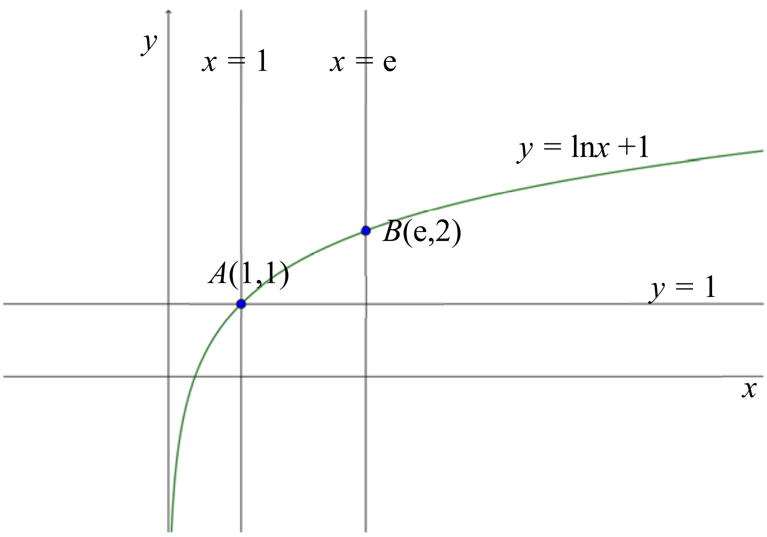
Week No	Amount of vegetables on the farm
1	2000 (0.9) +80
2	$=\left((2000\times 0.9)+80\right)\times 0.9+80$ $=(2000\times 0.9^2)+(80\times 0.9)+80$
3	$=\left((2000\times 0.9^2)+(80\times 0.9)+80\right)\times 0.9)+80$ $=(2000\times 0.9^3)+(80\times 0.9^2)+\left(80\times 0.9\right)+80$
….	
n	$=(2000\times 0.9^n)+(80\times 0.9^{n-1})...+\left(80\times 0.9\right)+80$ $=(2000\times 0.9^n)+80\left[0.9^{n-1}+0.9^{n-2}+...1\right]$ $=(2000\times 0.9^n)+80\left[\dfrac{(1-0.9^n)}{1-0.9}\right]$ $=(2000\times 0.9^n)+800(1-0.9^n)$ $=(1200\times 0.9^n)+800$

A =1200, B = 0.9 , C = 800

(ii)

$(1200\times 0.9^n)+800<835$

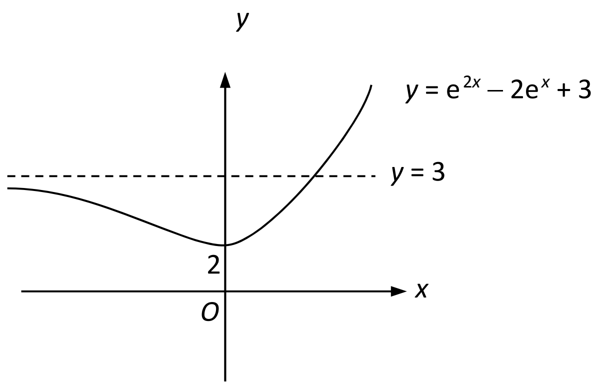
Solving,

	$(1200 \times 0.9^n) < 35$ $0.9^n < \dfrac{35}{1200}$ $n > 33.549$ Least number is 34 weeks
4(i)	$U = u^2 \quad \dfrac{dV}{du} = e^u$ $\dfrac{dU}{du} = 2u \quad V = e^u$ $\int u^2 e^u \ du$ $= \left[u^2 e^u\right] - 2\int ue^u \ du$ $U = u \quad \dfrac{dV}{du} = e^u$ $\dfrac{dU}{du} = 1 \quad V = e^u$ $u^2 e^u - 2\int ue^u \ du$ $= u^2 e^u - 2\left\{\left[ue^u\right] - \int e^u \ du\right\}$ $= u^2 e^u - 2ue^u + 2e^u + C$, where C was an arbitrary constant
4	At $x = 1$, $y = 1$ At $x = \mathrm{e}$, $y = 2$ 

	After translations, the graph is: $y_1 = \ln x$, with A' (1 , 0) and B' (e , 1) $u = \ln x$ $x = e^u$ $\frac{\mathrm{d}x}{\mathrm{du}} = e^u$ $x = 1, u = 0$ $x = e, u = 1$ $V = \pi \int_1^e \left[(\ln x)^2 \right] \mathrm{d}x$ $= \pi \int_0^1 u^2 \frac{\mathrm{d}x}{\mathrm{du}} \mathrm{du}$ $= \pi \int_0^1 u^2 e^u \ \mathrm{d}u$ $= \pi \left[u^2 e^u - 2ue^u + 2e^u \right]_0^1$ $= \pi \left[e - 2e + 2e - 2 \right]$ $= \pi (e-2) \text{ units}^3$

5

(i)



As the horizontal line $y = k$, $k \in (2,3)$ cuts the graph $y = \mathrm{f}(x)$ more than once, f is not one - one. Hence f^{-1} does not exist.

(ii) Max $a = 0$.

(iii) Let $y = \mathrm{e}^{2x} - 2\mathrm{e}^{x} + 3$

$y = (\mathrm{e}^{x})^2 - 2\mathrm{e}^{x} + 3$

$y = (\mathrm{e}^{x} - 1)^2 + 2$

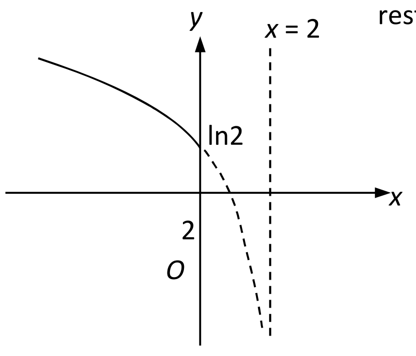
$x = \ln(1+\sqrt{y-2})$ (rejected, since $x \le 0$) .

or $\ln(1-\sqrt{y-2})$

$\therefore\ \mathrm{f}^{-1} : x \mapsto \ln(1-\sqrt{x-2})$, $x \in \mathbb{R}$, $2 \le x < 3$.

(iv) $[2,3) \xrightarrow{\mathrm{f}^{-1}} (-\infty,0] \xrightarrow{\mathrm{g}} [\ln 2,\infty)$

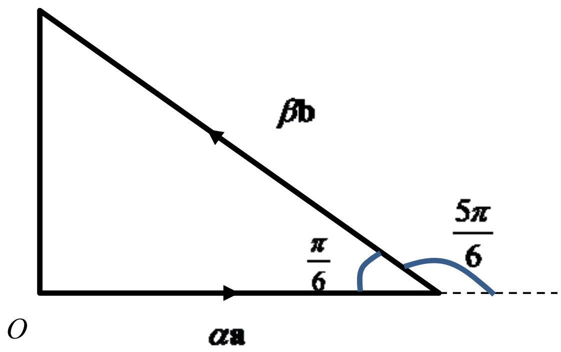
$\mathrm{R}_{\mathrm{gf}^{-1}} = [\ln 2,\infty)$



Graph of g when D_g is restricted to $(-\infty,0]$

6	$y=\sqrt{4+\sin 2x}$ $y^2=4+\sin 2x$ Differentiating implicitly with respect to x, $2y\frac{\mathrm{d}y}{\mathrm{d}x}=2\cos 2x$ $y\frac{\mathrm{d}y}{\mathrm{d}x}=\cos 2x$ …. (1)
(i)	Differentiating (1) implicitly with respect to x, $y\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2=-2\sin 2x$ …. (2) When $x = 0$, $y=\sqrt{4+0}=2$ From (1) $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{2}$ From (2) $2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\left(\frac{1}{2}\right)^2=0 \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-\frac{1}{8}$ The Maclaurin's Series of y is $y=2+\frac{1}{2}x-\frac{1}{8}\times\frac{x^2}{2!}+\cdots$ $y\approx 2+\frac{1}{2}x-\frac{x^2}{16}$, up to and including the term in x^2
(ii)	By using the standard series of $\sin x$, $\sin 2x\approx 2x$ $\sqrt{4+\sin 2x}\approx\sqrt{4+2x}$ $=(4+2x)^{\frac{1}{2}}$ $=2\left(1+\frac{x}{2}\right)^{\frac{1}{2}}$

	$\sqrt{4+\sin 2x}=2\left[1+\frac{1}{2}\left(\frac{x}{2}\right)+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}\left(\frac{x}{2}\right)^2+\cdots\right]$ $=2\left(1+\frac{x}{4}-\frac{x^2}{32}+\cdots\right)$ $=2+\frac{x}{2}-\frac{x^2}{16}+\cdots$
(iii)	$\int_0^{0.1}\sqrt{4-\sin 2x}\ \mathrm{d}x=\int_0^{0.1}\sqrt{4+\sin(-2x)}\ \mathrm{d}x$ $=\int_0^{0.1}\left(2-\frac{x}{2}-\frac{x^2}{16}\right)\mathrm{d}x$ (Replace x with $-x$) ≈ 0.19748 (to 5 s.f.)
7 (i)	$\mathbf{a}\cdot(\alpha\mathbf{a}+\beta\mathbf{b})=0$ $\alpha\lvert\mathbf{a}\rvert^2+\beta\mathbf{a}\cdot\mathbf{b}=0$ $\alpha+\beta\mathbf{a}\cdot\mathbf{b}=0$ $\mathbf{a}\cdot\mathbf{b}=-\frac{\alpha}{\beta}$ Since angle between **a** and **b** is $\frac{5\pi}{6}$, $\cos\left(\frac{5\pi}{6}\right)=\frac{\mathbf{a}\cdot\mathbf{b}}{\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert}$ $-\frac{\sqrt{3}}{2}=\frac{-\frac{\alpha}{\beta}}{\lvert\mathbf{b}\rvert}$ $\lvert\mathbf{b}\rvert=\frac{2\sqrt{3}}{3}\left(\frac{\alpha}{\beta}\right)$ (shown) Or

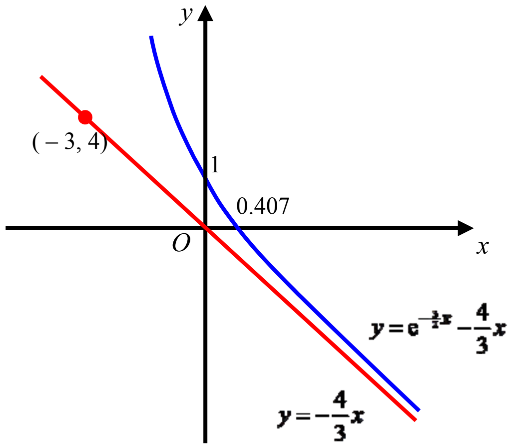
	 $\cos\left(\frac{\pi}{6}\right)=\frac{\alpha\lvert\mathbf{a}\rvert}{\beta\lvert\mathbf{b}\rvert}$ $\lvert\mathbf{b}\rvert=\frac{\alpha\lvert\mathbf{a}\rvert}{\beta\cos\left(\frac{\pi}{6}\right)}$ $\lvert\mathbf{b}\rvert=\frac{\alpha}{\beta\frac{\sqrt{3}}{2}}$ $=\frac{2\alpha}{\sqrt{3}\beta}$ $=\frac{2\sqrt{3}}{3}\left(\frac{\alpha}{\beta}\right)$
(ii)	$\lvert\mathbf{a}\cdot\mathbf{b}\rvert$ is the length of projection of $\mathbf{b}$ onto $\mathbf{a}$ $\lvert\mathbf{a}\cdot\mathbf{b}\rvert=\left\lvert\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\cos\left(\frac{5\pi}{6}\right)\right\rvert$ $=\lvert\mathbf{b}\rvert\frac{\sqrt{3}}{2}$ $=\frac{2\sqrt{3}}{3}\left(\frac{\alpha}{\beta}\right)\frac{\sqrt{3}}{2}$ $=\left(\frac{\alpha}{\beta}\right)$
(iii)	By Ratio theorem, $\overrightarrow{OM}=\lambda\mathbf{b}+(1-\lambda)\mathbf{a}$

	$\overrightarrow{ON} = \overrightarrow{OM} + \overrightarrow{MN}$ $= \left[\lambda\mathbf{b} + (1-\lambda)\mathbf{a}\right] + \overrightarrow{OB}$ $= \left[\lambda\mathbf{b} + (1-\lambda)\mathbf{a}\right] + \mathbf{b}$ $= (\lambda+1)\mathbf{b} + (1-\lambda)\mathbf{a}$ Area of triangle *OAN* $= \frac{1}{2}\left\|\overrightarrow{OA} \times \overrightarrow{ON}\right\|$ $= \frac{1}{2}\left\|\mathbf{a} \times \left[(\lambda+1)\mathbf{b} + (1-\lambda)\mathbf{a}\right]\right\|$ $= \frac{1}{2}\left\| (\lambda+1)\mathbf{a}\times\mathbf{b} + (1-\lambda)\mathbf{a}\times\mathbf{a}\right\|$ $= \frac{1}{2}(\lambda+1)\left\|\mathbf{a}\times\mathbf{b}\right\|$ since $\left\|\lambda+1\right\| = \lambda+1$ as $0<\lambda<1$ $= \frac{1}{2}(\lambda+1)\left\|\mathbf{a}\right\|\left\|\mathbf{b}\right\|\left\|\sin\left(\frac{5\pi}{6}\right)\right\|$ $= \frac{(\lambda+1)}{2}\left(\frac{2\sqrt{3}}{3}\right)\left(\frac{\alpha}{\beta}\right)\left(\frac{1}{2}\right)$ $= \frac{(\lambda+1)\sqrt{3}}{6}\left(\frac{\alpha}{\beta}\right)$

8(i)	$\frac{dw}{dx}=-(\frac{3}{2}w+2)$ $\int \frac{1}{\frac{3}{2}w+2}\ dw=\int -1\ dx$ $\frac{2}{3}\int \frac{\frac{3}{2}}{\frac{3}{2}w+2}\ dw=\int -1\ dx$ $\frac{2}{3}\ln\left\lvert\frac{3}{2}w+2\right\rvert=-x+A$ where A is an arbitrary constant $\ln\left\lvert\frac{3}{2}w+2\right\rvert=-\frac{3}{2}x+\frac{3}{2}A$ $\left\lvert\frac{3}{2}w+2\right\rvert=e^{-\frac{3}{2}x+\frac{3}{2}A}=e^{\frac{3}{2}A}.e^{-\frac{3}{2}x}$ $\frac{3}{2}w+2=\pm e^{\frac{3}{2}A}.e^{-\frac{3}{2}x}=Be^{-\frac{3}{2}x}$ where $B=\pm e^{\frac{3}{2}A}$ $w=\frac{2}{3}Be^{-\frac{3}{2}x}-\frac{4}{3}$ $w=Ce^{-\frac{3}{2}x}-\frac{4}{3}$ where $C=\frac{2}{3}B$
8(ii)	$\frac{dy}{dx}=Ce^{-\frac{3}{2}x}-\frac{4}{3}$ $\int \frac{dy}{dx}\ dx=\int\left(Ce^{-\frac{3}{2}x}-\frac{4}{3}\right)dx$ $y=-\frac{2}{3}Ce^{-\frac{3}{2}x}-\frac{4}{3}x+E$ where E is an arbitrary constant $y=De^{-\frac{3}{2}x}-\frac{4}{3}x+E$ where $D=-\frac{2}{3}C$
8(iii)	When $D = 0$ (or $C = 0$) and $E = 0$, $y=-\frac{4}{3}x$

When $D = 1$ (or $C = -\frac{3}{2}$) [or any value of D or C] and $E = 0$ [or any value of E corresponding to the choice of E above],

$$y = \mathrm{e}^{-\frac{3}{2}x} - \frac{4}{3}x$$



9 (a) (i)

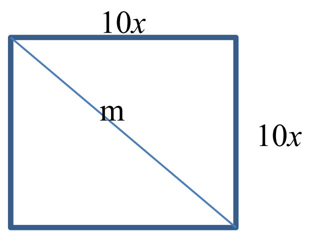
$$\begin{aligned} z^3 &= 1 - \mathrm{i}\sqrt{3} \\ &= 2\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{3} + 2k\pi\right)}, \ k \in \mathbb{Z} \\ z &= 2^{\frac{1}{3}}\mathrm{e}^{\mathrm{i}\frac{1}{3}\left(-\frac{\pi}{3} + 2k\pi\right)}, \ k = 0, \pm 1 \\ &= 2^{\frac{1}{3}}\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{9} + \frac{2k\pi}{3}\right)}, \ k = 0, \pm 1 \end{aligned}$$

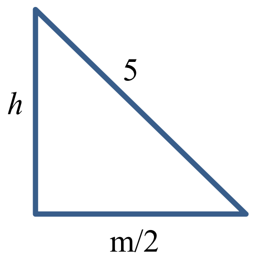
$$z = 2^{\frac{1}{3}}\mathrm{e}^{\mathrm{i}\frac{5\pi}{9}}, 2^{\frac{1}{3}}\mathrm{e}^{-\mathrm{i}\frac{\pi}{9}}, 2^{\frac{1}{3}}\mathrm{e}^{-\mathrm{i}\frac{7\pi}{9}}$$

(ii)

$$\begin{aligned} &\left(z^n - 2\mathrm{e}^{\mathrm{i}\theta}\right)\left(z^n - 2\mathrm{e}^{-\mathrm{i}\theta}\right) \\ &= z^{2n} - 2z^n\left(\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta}\right) + 4 \\ &= z^{2n} - 2z^n\left(\cos\theta + \mathrm{i}\sin\theta + \cos(-\theta) + \mathrm{i}\sin(-\theta)\right) + 4 \\ &= z^{2n} - 2z^n\left(\cos\theta + \mathrm{i}\sin\theta + \cos\theta - \mathrm{i}\sin\theta\right) + 4 \\ &= z^{2n} - 4z^n\cos\theta + 4 \quad \text{(shown)} \end{aligned}$$

	Hence, $z^6-2z^3+4=0$ $z^6-4\left(\frac{1}{2}\right)z^3+4=0$ $z^6-4\left(\cos\frac{\pi}{3}\right)z^3+4=0$ $\left(z^3-2\mathrm{e}^{\mathrm{i}\frac{\pi}{3}}\right)\left(z^3-2\mathrm{e}^{-\mathrm{i}\frac{\pi}{3}}\right)=0$ The roots of are $z^3 = 2\mathrm{e}^{-\mathrm{i}\frac{\pi}{3}}$ $z=2^{\frac{1}{3}}\mathrm{e}^{\mathrm{i}\frac{5\pi}{9}},2^{\frac{1}{3}}\mathrm{e}^{-\mathrm{i}\frac{\pi}{9}},2^{\frac{1}{3}}\mathrm{e}^{-\mathrm{i}\frac{7\pi}{9}}$ (from (i)) Since the coefficients of the equation are all real, complex roots occur in conjugate pairs. Therefore, for $z^3 = 2\mathrm{e}^{\mathrm{i}\frac{\pi}{3}}$, the roots are $z=2^{\frac{1}{3}}\mathrm{e}^{-\mathrm{i}\frac{5\pi}{9}},2^{\frac{1}{3}}\mathrm{e}^{\mathrm{i}\frac{\pi}{9}},2^{\frac{1}{3}}\mathrm{e}^{\mathrm{i}\frac{7\pi}{9}}$
9(b)	$1-z^2=1-(\cos 2\theta+\mathrm{i}\sin 2\theta)$ $=1-\cos 2\theta-\mathrm{i}(2\sin\theta\cos\theta)$ $=2\sin^2\theta-\mathrm{i}(2\sin\theta\cos\theta)$ $=(-2i\sin\theta)(\cos\theta+\mathrm{i}\sin\theta)$ $=(-2\mathrm{i}\sin\theta)z$ (shown) Alternatively :

	$1-z^2=1-(e^{i2\theta})$ $=e^{i\theta}(e^{-i\theta}-e^{i\theta})$ $=e^{i\theta}(\cos\theta-i\sin\theta-\cos\theta-i\sin\theta)$ $=z(-2i\sin\theta)$ (Shown) $\left\|1-z^2\right\|=\left\|-2i\sin\theta\right\|\left\|z\right\|=2\sin\theta$ $\arg(1-z^2)=\arg(-2i\sin\theta)+\arg(z)$ $=\arg(2\sin\theta)+\arg(-i)+\arg(z)$ $=\theta-\dfrac{\pi}{2}$
10(i)	Let V be the total volume of the solid and h be the height of the pyramid. Height of trapezium, l $=\sqrt{25x^2-9x^2}$ $=\sqrt{16x^2}$ $=4x$ $V_{\text{base}}=\frac{1}{2}(4x+10x)(4x)(10x)$ $=280x^3$  $m=\sqrt{2(10x)^2}$ $=10\sqrt{2}x$ $\dfrac{m}{2}=5\sqrt{2}x$



$$h = \sqrt{25 - \left(5\sqrt{2}x\right)^2}$$
$$= \sqrt{25 - 50x^2}$$

$$V_{pyramid} = \frac{1}{3}(10x)^2\left(\sqrt{25-50x^2}\right)$$
$$= \frac{100x^2}{3}\left(\sqrt{25-50x^2}\right)$$

$$V = 280x^3 + \frac{100x^2}{3}\sqrt{25-50x^2}$$

$$\frac{\mathrm{d}V}{\mathrm{d}x} = 840x^2 + \frac{100}{3}\left\{x^2\left(\frac{1}{2\sqrt{25-50x^2}}\right)(-100x) + 2x\sqrt{25-50x^2}\right\}$$
$$= 840x^2 + \frac{100}{3}\left\{2x\sqrt{25-50x^2} - \frac{50x^3}{\sqrt{25-50x^2}}\right\}$$
$$= 840x^2 + \frac{100}{3}(2x)\left\{\sqrt{25-50x^2} - \frac{25x^2}{\sqrt{25-50x^2}}\right\}$$
$$= \frac{20}{3}x\left\{126x + 10\left[\sqrt{25-50x^2} - \frac{25x^2}{\sqrt{25-50x^2}}\right]\right\}$$

For stationary values of V,

$$\frac{\mathrm{d}V}{\mathrm{d}x} = 0$$

Since $x > 0$,

$$126x + 10\left[\sqrt{25-50x^2} - \frac{25x^2}{\sqrt{25-50x^2}}\right] = 0$$

	$126x = -10\left[\sqrt{25-50x^2} - \dfrac{25x^2}{\sqrt{25-50x^2}}\right]$ $126x\sqrt{25-50x^2} = -10\left[\left(25-50x^2\right)-25x^2\right]$ $126x\sqrt{25-50x^2} = -10\left[25-75x^2\right]$ $= 750x^2 - 250$ Squaring both sides, $\left(126x\sqrt{25-50x^2}\right)^2 = \left(750x^2-250\right)^2$ $15876x^2\left(25-50x^2\right) = 562500x^4 - 375000x^2 + 62500$ $396900x^2 - 793800x^4 = 562500x^4 - 375000x^2 + 62500$ $3969x^2 - 7938x^4 = 5625x^4 - 3750x^2 + 625$ $13563x^4 - 7719x^2 + 625 = 0$
(ii)	Using G.C., since $x > 0$, the two values of x are $0.6865562 \approx 0.68656$ or $0.3126699 \approx 0.31267$ (to 5 dp).
(iii)	$\dfrac{\mathrm{d}V}{\mathrm{d}x} = \dfrac{20}{3}x\left\{126x + 10\left[\sqrt{25-50x^2} - \dfrac{25x^2}{\sqrt{25-50x^2}}\right]\right\}$ For $x = 0.6865562$, $\dfrac{\mathrm{d}V}{\mathrm{d}x} = \dfrac{20}{3}(0.6865562)\{126(0.6865562)$ $+10\left[\sqrt{25-50(0.6865562)^2} - \dfrac{25(0.6865562)^2}{\sqrt{25-50(0.6865562)^2}}\right]\}$ $= 0.03789$ ≈ 0 For $x = 0.3126699$, $\dfrac{\mathrm{d}V}{\mathrm{d}x} = \dfrac{20}{3}(0.3126699)\{126(0.3126699)$ $+10\left[\sqrt{25-50(0.3126699)^2} - \dfrac{25(0.3126699)^2}{\sqrt{25-50(0.3126699)^2}}\right]\}$ $= 164.24$ $\neq 0$

	Hence $x_1 = 0.68656$
11 (i)	$l_1 : \mathbf{r} = \begin{pmatrix} 5 \\ -1 \\ 4 \end{pmatrix} + \gamma \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ $p_1 : \mathbf{r} \cdot \begin{pmatrix} a \\ 0 \\ 1 \end{pmatrix} = 5a + 4$ Acute $\angle$ between l_1 and $p_1 = \frac{\pi}{6}$ $\Rightarrow$ Acute $\angle$ between $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} a \\ 0 \\ 1 \end{pmatrix} = \frac{\pi}{3}$ $\cos\left(\frac{\pi}{3}\right) = \frac{\left\| \begin{pmatrix} a \\ 0 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\|}{\sqrt{a^2+1}\sqrt{2}}$ $\frac{1}{2} = \frac{a}{\sqrt{2\left(a^2+1\right)}}$ $2\left(a^2+1\right) = 4a^2$ $2a^2 = 2$ $a = \pm 1$ Since $a > 0$, $a = 1$. (proven)
(ii)	$p_2 : \quad \mathbf{r} = \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ Normal vector of $p_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ $p_2 : \mathbf{r} \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 0$ Using GC, solving $x + z = 9$ and $x - y + z = 0$,

	$x = 9 - z$ $y = 9$ $z = z$ Therefore equation of l_2, $\underset{\sim}{r} = \begin{pmatrix} 9 \\ 9 \\ 0 \end{pmatrix} + s\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, s \in \mathbb{R}$
(iii)	$x + z = 9 \cdots (1)$ $x + z = y \cdots (2)$ $5x + 4y + \alpha z = \beta \cdots (3)$ (1) and (2) are the cartesian equations of planes p_1 and p_2 respectively. Let $p_3 : \mathbf{r} \cdot \begin{pmatrix} 5 \\ 4 \\ \alpha \end{pmatrix} = \beta$ Since the system of equations is known to have more than one solution, p_1, p_2 and p_3 intersect at l_2. Therefore, l_2 lies in p_3. $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 4 \\ \alpha \end{pmatrix} = 0$ $-5 + \alpha = 0$ $\alpha = 5$ $\begin{pmatrix} 9 \\ 9 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 4 \\ 5 \end{pmatrix} = \beta$ $\beta = 81$
	Since the system of equations is known to have exactly one solution, p_1, p_2 and p_3 intersect at a point. Therefore, l_2 intersects p_3 at a point. $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 4 \\ \alpha \end{pmatrix} \neq 0$ $-5 + \alpha \neq 0$ $\alpha \neq 5$

	$\beta \in \mathbb{R}$
	Since $\alpha = 5$ and $\beta \neq 81$, the planes do not intersect at a common point. Since the 3 planes are not parallel to each other, the 3 planes form a triangular prism.