SAINT ANDREW'S JUNIOR COLLEGE

Preliminary Examination

MATHEMATICS
Higher 2 **9740/01**

Paper 1

Monday **29 August 2016** **3 hours**

Additional materials : Answer paper
List of Formulae (MF15)
Cover Sheet

READ THESE INSTRUCTIONS FIRST

Write your name, civics group and index number on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.

Answer **all** the questions. Total marks : **100**

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphic calculator.
Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.
At the end of the examination, fasten all your work securely together.

This document consists of 7 printed pages including this page.

[Turn over

1 The sum $\sum_{r=1}^{n} \frac{1}{(3r-2)(3r+1)}$ is denoted by S_n.

(i) By using the method of differences, find an expression for S_n in terms of n. **[3]**

(ii) Hence find the value of S_n as n tends to infinity. **[1]**

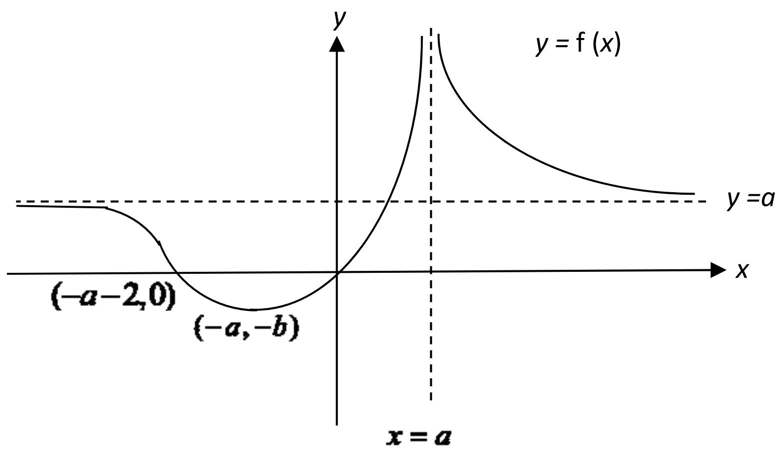
(iii) Find the smallest value of n for which S_n is within 2×10^{-4} of the sum to infinity. **[3]**

(iv) Using your answer in part **(i)**, find $\sum_{r=0}^{n} \frac{1}{(3r+1)(3r+4)}$ and deduce that

$\sum_{r=0}^{n} \frac{1}{(3r+4)^2} < \frac{1}{3}$. **[4]**

2 **(a)** By completing the square, or otherwise, state precisely a sequence of geometrical transformations which would transform the graph of $y = \ln(4x^2 - 16x + 15)$ onto the graph of $y = \frac{1}{2}\ln(4x^2 - 1)$. **[3]**

(b) The diagram shows the graph of $y = \mathrm{f}(x)$. The graph passes through the origin and the point $(-a-2, 0)$. It has a minimum point at $(-a, -b)$, $a > 1$, $b > 1$. The graph also has a vertical asymptote $x = a$ and a horizontal asymptote $y = a$.



Sketch, on separate diagrams, the graph of:

(i) $y = \frac{1}{\mathrm{f}(x)}$; **[2]**

(ii) $y = \mathrm{f}'(x)$, **[2]**

showing clearly all the asymptotes, turning points and axes intercepts.

3 **(a)** John deposits $\$x$ into a bank at the beginning of each year. The bank pays interest at a fixed rate of 5% of the amount at the end of each year. John then withdraws the interest as soon as it is added. Find, in terms of x and N, the total amount of interest he will collect at the beginning of $(N+1)$th year. **[3]**

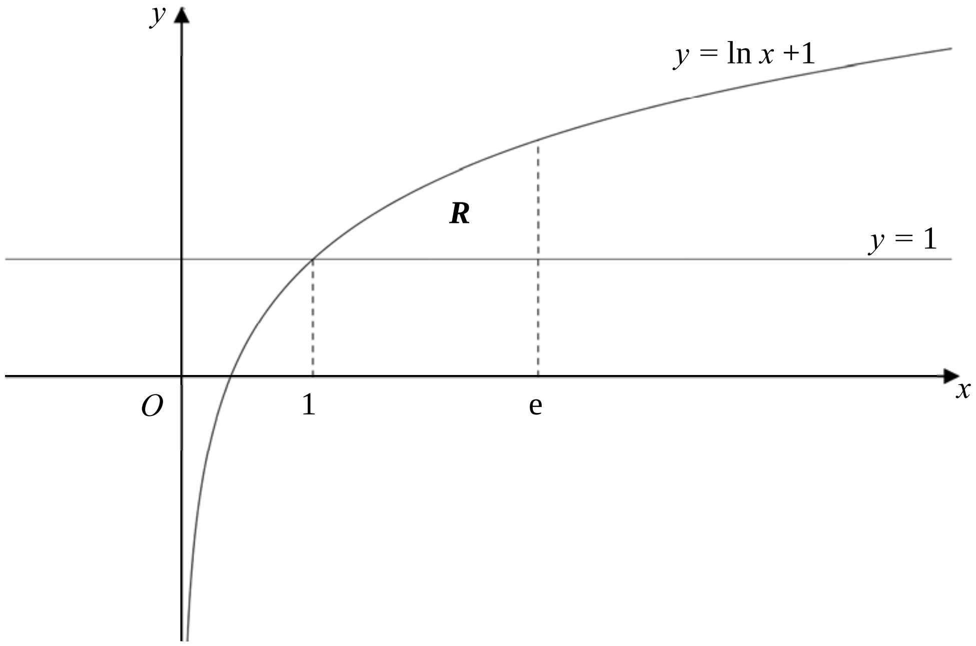
(b) An agricultural farm has 2000kg of vegetables. At the end of each week, the farm sells 10% of the vegetables and grows another 80kg on the farm.

(i) Find the amount of vegetables the farm has at the end of nth week, expressing your answer in the form $A(B^n)+C$, where A, B and C are constants to be determined. **[3]**

(ii) At which week will the amount of vegetables in the farm be first less than 835kg? **[2]**

4 **(i)** Find $\int u^2 e^u \, du$. **[3]**

(ii) The curve C has equation $y = \ln x + 1$ as shown below.



The region R is bounded by the curve C and the lines $x = 1$, $x = \mathrm{e}$ and $y = 1$.

Write down the equation of the curve by translating C one unit in the negative y-direction.

Hence, using the substitution $u = \ln x$, evaluate the exact volume generated when R is rotated completely about the line $y = 1$ by 2π radians. **[4]**

5 The functions f and g are defined by

$$\mathrm{f} : x \mapsto \mathrm{e}^{2x} - 2\mathrm{e}^{x} + 3, \quad x \in \mathbf{R}$$

$$\mathrm{g} : x \mapsto \ln(2 - x), \quad x \in \mathbf{R},\ x < 2$$

(i) By sketching a graph, explain why the inverse function f^{-1} does not exist. **[2]**

(ii) Given that the domain of f is restricted to $(-\infty, a]$, state the maximum value of a for which f^{-1} exist. **[1]**

(iii) Using the value of a found in **(ii)** and by completing the square, find the inverse function f^{-1}. **[3]**

(iv) Find the exact range of gf^{-1}. **[2]**

6 Given that $y = \sqrt{4 + \sin 2x}$, show that $y\dfrac{\mathrm{d}y}{\mathrm{d}x} = \cos 2x$. **[1]**

(i) By further differentiation of the above result, find the Maclaurin series for y in ascending powers of x up to and including the term in x^2 . **[3]**

(ii) Verify the correctness of the series found in **(i)** by using an appropriate standard series expansion. **[2]**

(iii) Deduce from part **(i)** the approximate value of $\int_0^{0.1} \sqrt{4 - \sin 2x}\, \mathrm{d}x$, giving your answer to 5 significant figures. **[2]**

7 Relative to the origin O, the position vectors of two points A and B are $\mathbf{a}$ and $\mathbf{b}$ respectively, where $\mathbf{a}$ and $\mathbf{b}$ are non-zero and non-parallel vectors. The vector $\mathbf{a}$ is a unit vector which is perpendicular to $\alpha\mathbf{a} + \beta\mathbf{b}$, where $\alpha > 1$ and $\beta > 1$ and the angle between $\mathbf{a}$ and $\mathbf{b}$ is $\dfrac{5\pi}{6}$.

(i) Show that $|\mathbf{b}| = \dfrac{2\sqrt{3}}{3}\left(\dfrac{\alpha}{\beta}\right)$. **[3]**

(ii) Give the geometrical interpretation of $\left|\mathbf{a}\square\mathbf{b}\right|$ and find its value in terms of α and β . **[3]**

(iii) The point M divides AB in the ratio $\lambda : 1-\lambda$ where $0 < \lambda < 1$. The point N is such that $OMNB$ is a parallelogram. Find $\overrightarrow{ON}$ in terms of $\mathbf{a}$ and $\mathbf{b}$ and the area of the triangle OAN in terms of λ, α and β . **[5]**

8 The variables w, x and y are connected by the following differential equations:

$$\frac{\mathrm{d}w}{\mathrm{d}x} = -\frac{3}{2}w - 2 \quad \text{(A)}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = w \quad \text{(B)}$$

(i) Solve equation (A) to find w in terms of x. **[3]**

(ii) Hence find y in terms of x. **[2]**

(iii) The result in part **(ii)** represents a family of curves. Some members of the family are straight lines. Write down the equation of one of these lines. On a single diagram, sketch your line together with a non-linear member of the family of curves that has your line as an asymptote, indicating clearly any axes intercepts. **[3]**

9 **(a)** **(i)** Solve $z^3 = 1 - \mathrm{i}\sqrt{3}$, giving your answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$, and $-\pi < \theta \le \pi$. **[3]**

(ii) Show that

$$\left(z^n - 2\mathrm{e}^{\mathrm{i}\theta}\right)\left(z^n - 2\mathrm{e}^{-\mathrm{i}\theta}\right) = z^{2n} - 4z^n\cos\theta + 4,$$

Hence find the roots of the equation

$z^6 - 2z^3 + 4 = 0$ in the form of $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$, and $-\pi < \theta \le \pi$. **[3]**

(b) Given that $z = \cos\theta + \mathrm{i}\sin\theta$, show that $1 - z^2 = (-2\mathrm{i}\sin\theta)z$. Given also that $0 < \theta < \pi$, find the modulus and argument of $1 - z^2$ in terms of θ. **[5]**

10

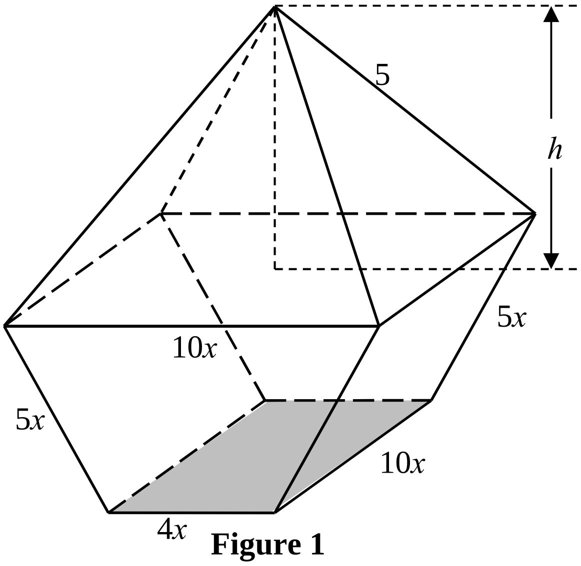


Figure 1

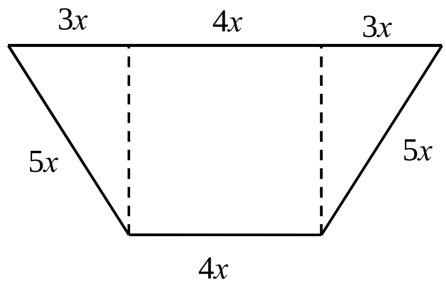


Figure 2

A designer decided to build a model as shown in **Figure 1** above, consisting of a base and a top. The base is made up of a prism with a cross-section of a trapezium where the length of the parallel sides are $4x$ cm and $10x$ cm (**Figure 2**). The top is a right pyramid with a square base of sides $10x$ cm, height h cm and a fixed slant height of 5 cm.

(i) Find an expression for the volume of the model, V, in terms of x. Given that $x = x_1$ is the value of x which gives the maximum value of V, show that x_1 satisfies the equation $13563x^4 - 7719x^2 + 625 = 0$. **[6]**

(ii) Find the two solutions to the equation in part **(i)** for which $x > 0$, giving your answers correct to 5 decimal places. **[2]**

(iii) Using both the solutions found in part **(ii)**, show that one of the values does not give a stationary value of V. Hence, write down the value of x_1. **[3]**

[Area of trapezium = $\frac{1}{2}(\text{sum of parallel sides}) \times \text{height}$;

Volume of pyramid = $\frac{1}{3}$base area $\times$ height]

11 The line l_1 and the planes p_1 and p_2 have equations as follows:

$$l_1 \ : \ x-5=-y-1, z=4;$$

$$p_1 \ : \ xa+z=5a+4,$$

$$p_2: \quad \mathbf{r}=\lambda\begin{pmatrix}1\\1\\0\end{pmatrix}+\mu\begin{pmatrix}0\\1\\1\end{pmatrix}$$

where a is a positive constant and λ and μ are real numbers.

(i) Given that the acute angle between l_1 and p_1 is $\frac{\pi}{6}$, show that $a=1$. **[2]**

(ii) The planes p_1 and p_2 meet in the line l_2. Find a vector equation of l_2. **[2]**

(iii) Hence, find the values of α and β such that the system of equations

$$x+z=9$$
$$x+z=y$$
$$5x+4y+\alpha z=\beta$$

has

(a) more than one solution;

(b) exactly one solution.

If $\alpha=5$, $\beta=10$, give a geometrical interpretation of the relationship between the 3 equations. Explain your answer. **[6]**

End of Paper