Pioneer Junior College

H2 Mathematics

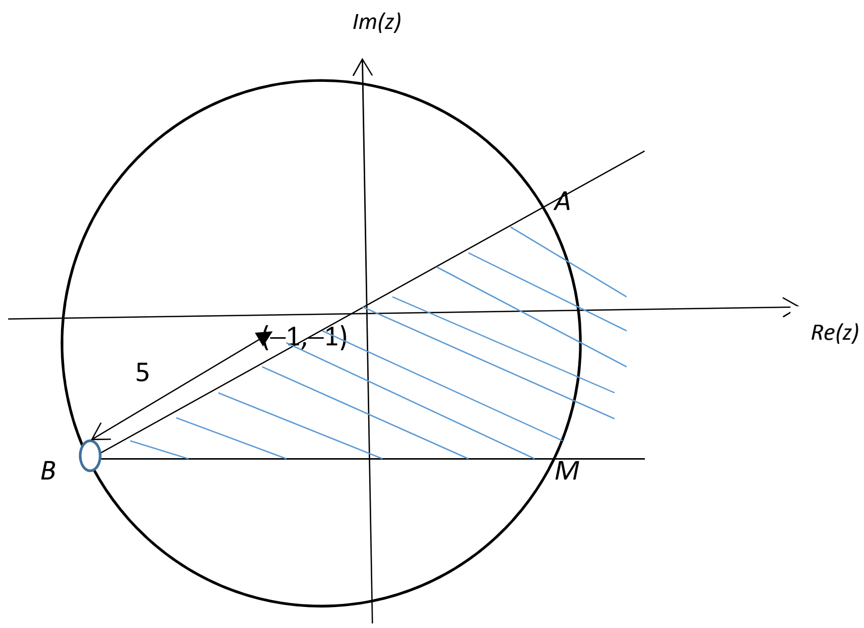
JC 2 Preliminary Examination Paper 2 Solution

1

(i) $|2z-a-b|=|a-b|$

Centre of circle C $= \dfrac{(2+3\mathrm{i})+(-4-5\mathrm{i})}{2} = -1-\mathrm{i}$

Radius of circle C $= \dfrac{\sqrt{(2+4)^2+(3+5)^2}}{2} = 5$



(ii)*AM* is the common region satisfies both (i) and (ii).

$\angle$AMB = 90º since AB is diameter and angle in semicircle is a right angle

Max arg(z) = arg(a) = $\tan^{-1}(3/2)$ = 0.983 radians

Min arg(z) = arg(2 – 5i) = $-\tan^{-1}(5/2)$ = –1.19 radians

So range required is $-1.19 \le \arg(z) \le 0.983$

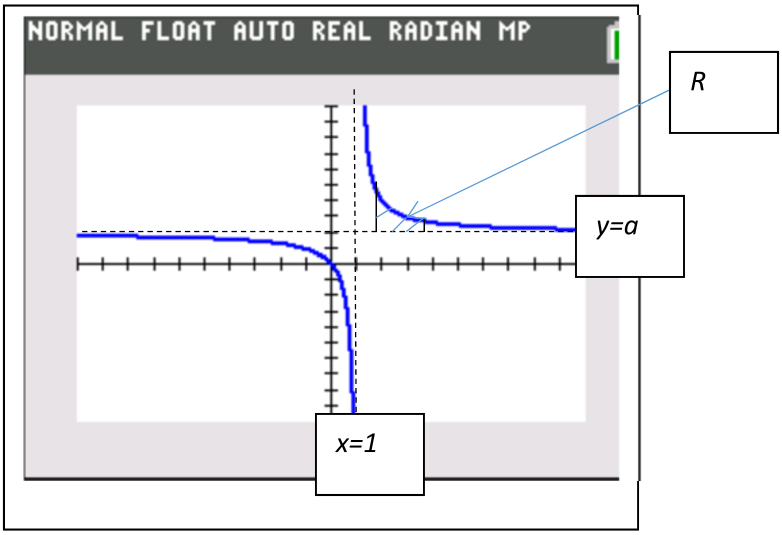
2

(i) $y=\dfrac{ax}{x-1}=a+\dfrac{a}{x-1}$

$y=\dfrac{1}{x}$ is translated 1 unit in the direction of x-axis, followed by a scaling of a units parallel to the y axis and is translated a units in the direction of y-axis

(ii) The equations of asymptotes are $x = 1$ and $y = a$

The intercepts are (0,0)



(iii) The volume required $=\pi\int_2^4\left(a+\dfrac{a}{x-1}\right)^2\mathrm{d}x-\pi(a)^2(2)$

$$=a^2\pi\int_2^4\left(1+\frac{2}{x-1}+\frac{1}{(x-1)^2}\right)\mathrm{d}x-2\pi a^2$$

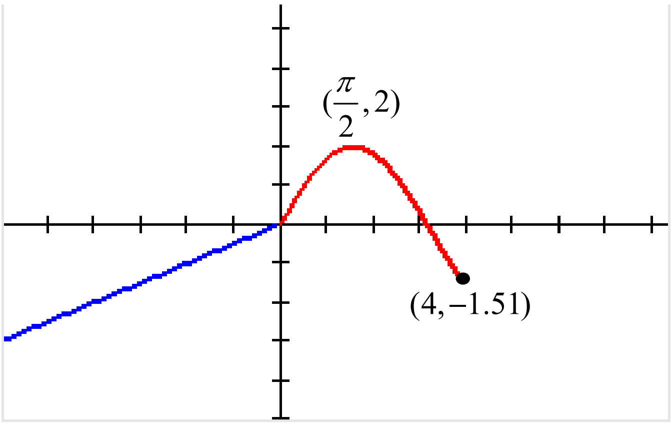
$$=a^2\pi\left[x+2\ln|x-1|-\frac{1}{x-1}\right]_2^4-2\pi a^2$$

$$=\left(\frac{2}{3}+2\ln 3\right)\pi a^2$$

(iv) Using part (i), the area S is the same as the area R found in (iii). To rotate S about the line $y = -a$ is the same as to rotate R about the x-axis. So the volume obtained is $\left(\dfrac{2}{3}+2\ln 3\right)\pi a^2$

3

(i)



(ii) From the graph, take $k = \frac{\pi}{2}$

(iii)

$$f(x) = \begin{cases} \frac{x}{2} & \text{if } x \le 0 \\ 2\sin x & \text{if } 0 < x \le \frac{\pi}{2} \end{cases}$$

$$\text{Let } y_1 = \frac{x}{2}$$

$$x = 2y_1$$

$$\text{Let } y_2 = 2\sin x$$

$$x = \sin^{-1}\left(\frac{y_2}{2}\right)$$

$$f^{-1}(x) = \begin{cases} 2x & \text{if } x \le 0 \\ \sin^{-1}\left(\frac{x}{2}\right) & \text{if } 0 < x \le 2 \end{cases}$$

(iv) $f^{-1}(x) = f(x)$ is the same as solving $f(x) = x$

$\frac{x}{2} = x \Rightarrow x = 0 \quad \text{if } x \le 0$

$2\sin x = x$, $x = 1.90 > \frac{\pi}{2}$, so only solution is $x = 0$

(v) Using $R_g = (-\infty, 0)$, $fg(x) = \frac{-x^3}{2}$

4

(i)

$$\frac{d}{dx}\left(\tan^{-1}\left(\frac{\sqrt{3}}{2}x\right)\right)=\frac{1}{1+\left(\frac{\sqrt{3}}{2}x\right)^2}\left(\frac{\sqrt{3}}{2}\right)$$

$$=\left(\frac{\sqrt{3}}{2}\right)\frac{1}{1+\frac{3}{4}x^2}$$

$$=\left(\frac{\sqrt{3}}{2}\right)\frac{4}{3x^2+4}$$

$$=\frac{2\sqrt{3}}{3x^2+4}$$

(ii)

$$\frac{1}{3x^2+4}=(3x^2+4)^{-1}=4^{-1}\left(1+\frac{3x^2}{4}\right)^{-1}$$

$$=\frac{1}{4}\left[1+(-1)\left(\frac{3x^2}{4}\right)+\frac{-1(-2)}{2}\left(\frac{3x^2}{4}\right)^2+\frac{-1(-2)(-3)}{6}\left(-\frac{3x^2}{4}\right)^3+\ldots\right]$$

$$=\frac{1}{4}\left(1-\frac{3}{4}x^2+\frac{9}{16}x^4-\frac{27}{64}x^6+\ldots\right)$$

$$\approx\frac{1}{4}-\frac{3}{16}x^2+\frac{9}{64}x^4-\frac{27}{256}x^6$$

Range of validity: $\left|\frac{3x^2}{4}\right|<1\Rightarrow x^2<\frac{4}{3}\Rightarrow-\frac{2}{\sqrt{3}}<x<\frac{2}{\sqrt{3}}$

(iii)

$$y=\tan^{-1}\left(\frac{\sqrt{3}}{2}x\right)=2\sqrt{3}\int\frac{1}{3x^2+4}dx$$

$$=2\sqrt{3}\int\left(\frac{1}{4}-\frac{3}{16}x^2+\frac{9}{64}x^4-\frac{27}{256}x^6\right)dx$$

$$=2\sqrt{3}\left(\frac{1}{4}x-\frac{1}{16}x^3+\frac{9}{320}x^5-\frac{27}{1792}x^7\right)+C$$

when $x=0$, $y=0$, $C=0$

$$\therefore\tan^{-1}\left(\frac{\sqrt{3}}{2}x\right)=\frac{\sqrt{3}}{2}x-\frac{\sqrt{3}}{8}x^3+\frac{9\sqrt{3}}{160}x^5-\frac{27\sqrt{3}}{896}x^7$$

5 (i) Number the list of patients from 1 to N. $k = N/0.05N = 20$ (Randomly select a number from 1 to 20, and let every 20th patient after first patient chosen try the new drug. For example, if a number 5 is chosen, then survey every 5th, 25th, 45th patient and so on, until the sample size of 5% patients is obtained.

(ii) Disadvantage: The sample is not representative of the population of diabetic patients as age and gender may affect the drug. More appropriate method is stratified sampling.

6

(i)

$$P(B') = 4P(B)$$

$$P(B') = 4[1 - P(B')]$$
$$P(B') = 4 - 4P(B')$$
$$5P(B') = 4$$
$$P(B') = \frac{4}{5} \text{ (shown)}$$

Alternatively,

$$1 - P(B) = 4P(B)$$
$$5P(B) = 1$$
$$P(B) = \frac{1}{5}$$
$$P(B') = \frac{4}{5} \text{ (shown)}$$

(ii)

$$P(A \mid B') = 3P(A \mid B)$$
$$\frac{P(A \cap B')}{P(B')} = \frac{3P(A \cap B)}{P(B)}$$
$$\frac{P(A \cap B')}{P(B')} = \frac{3P(A \cap B)}{1 - P(B')}$$
$$\frac{P(A \cap B')}{4/5} = \frac{3P(A \cap B)}{1/5}$$
$$P(A \cap B') = 12P(A \cap B) ----(*)$$
$$P(A) - P(A \cap B) = 12P(A \cap B)$$
$$P(A) = 13P(A \cap B) ----(**)$$
$$P(B' \mid A) = \frac{P(A \cap B')}{P(A)}$$
$$= \frac{12P(A \cap B)}{13P(A \cap B)}$$
$$= \frac{12}{13} \text{ or } 0.923 \text{ (3 s.f.)}$$

7 (i)Number of teams $= {}^{2}C_{1}\,{}^{15}C_{5} = 6006$

(i) 4 cases: 4F3M, 5F2M , 6F1M and 7F

Number of teams $= {}^{10}C_{4}\,{}^{8}C_{3} + {}^{10}C_{5}\,{}^{8}C_{2} + {}^{10}C_{6}\,{}^{8}C_{1} + {}^{10}C_{7} = 20616$

(ii) Total $-$ teams from A and B $-$ teams from B and C $-$ teams from A and C

$$= {}^{18}C_{7} - {}^{12}C_{7} \times 3$$
$$= 29448$$

8 (i) X – mass in kilograms of an Atlantic salmon

$$X \sim \mathrm{N}\left(\mu, \sigma^2\right)$$
$$\mathrm{P}(X < 22) = 0.159$$
$$\mathrm{P}\left(Z < \frac{22-\mu}{\sigma}\right) = 0.159$$
$$\frac{22-\mu}{\sigma} = -0.99858$$
$$\mu - 0.99858\sigma = 22 \quad \text{---(1)}$$
$$\mathrm{P}(X > 31) = 0.106$$
$$\mathrm{P}\left(Z > \frac{31-\mu}{\sigma}\right) = 0.106$$
$$\frac{31-\mu}{\sigma} = 1.2481$$
$$\mu + 1.2481\sigma = 31 \quad \text{---(2)}$$

Solving (1) and (2):

$\mu = 26.00022 \approx 26.0$ (shown)

$\sigma = 4.00591 \approx 4.01$ (shown)

(ii) Let W be the number of Atlantic salmon with more than 31 kg, out of 40

$W \sim \mathrm{B}(40, 0.106)$

$n = 40$ large, $np = 40(0.106) = 4.24 < 5$

so $W \sim \mathrm{Po}(4.24)$ approx.

Required prob = $\mathrm{P}(W \leq 5) = 0.74659 \approx 0.747$

(iii) Y – mass in kilograms of an Bluefin tuna.

$$Y \sim \mathrm{N}\left(380, 10^2\right)$$

Let T be the mass of 2 Bluefin tuna and 3 Atlantic salmon

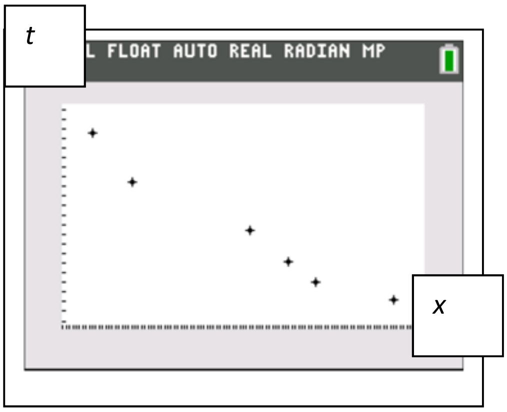
$$T = X_1 + X_2 + X_3 + Y_1 + Y_2 \sim \mathrm{N}(838, 248.2403)$$
$$\frac{X_1 + X_2 + X_3 + Y_1 + Y}{5} = \overline{T} \sim \mathrm{N}\left(\frac{838}{5}, \frac{248.2403}{25}\right) \text{ exactly}$$
$$\mathrm{P}\left(\overline{T} \leq 170\right) = 0.777 \text{ (3 s.f.)}$$

Alternatively

$$\mathrm{P}(T \leq 170 \times 5) = 0.777$$

9

(i)



(ii) The time for swimming cannot decrease forever as there is a limit on how fast a swimmer can swim and from the scatter diagram, as x increases, t decreases with decreasing amount, so linear model is not appropriate.

(iii)

$\ln t = a + bx$: $r = -0.9851$

$\frac{1}{t} = a + bx$: $r = 0.9877$

(iv) Since $|r|$ for $\frac{1}{t} = a + bx$ is higher than that of $\ln t = a + bx$, $\frac{1}{t} = a + bx$ is the preferred model.

(v) Let the timing be t

$$\frac{1}{t} = -0.09836 + \left(5.96846 \times 10^{-5}\right) x$$

Only value that satisfies the equation is $\left(\bar{x}, \overline{\left(\frac{1}{t}\right)}\right)$.

$$\overline{\frac{1}{t}} = -0.09836 + \left(5.96846 \times 10^{-5}\right) \bar{x} = -0.09836 + 0.0000596846(1956) = 0.018383$$

So $\frac{1}{7}\left(\frac{1}{65.6} + \frac{1}{60.4} + \frac{1}{55.4} + \frac{1}{52.2} + \frac{1}{49.99} + \frac{1}{48.18} + \frac{1}{t}\right) = 0.018383$ =

$t = 52.87 \approx 52.9$

So the timing at 1964 is 52.9 second

10

(i) Let X be the carbon emission of "Green Leaf".
From GC, unbiased estimate of population mean $= \bar{x} = 80.35$,
Unbiased estimate of population variance $= s^2 = (1.3089)^2 = 1.7132$

(ii) Since n is small and σ^2 is unknown, we use the t-test.
Assumption: The carbon emission of the "Green Leaf" is normally distributed.

$H_0 : \mu = 80$ vs $H_1 : \mu > 80$

Test Statistic, $t = 1.1959$

From GC, p-value = 0.12323 > 0.1

Since the p-value is more than the level of significance, we do not reject H_0. There is insufficient evidence, at the 10% level, to indicate that the manufacturer's claim is not true.

(iii) Since n is small and σ^2 is unknown, we use the t-test.

For H_0 to be rejected, Test Statistic > 2.8214

Unbiased estimate of population variance $s^2 = \frac{10}{9}m^2$

Test Statistic, $t = \frac{80.6 - 80}{m/3} > 2.8214$

$m < 0.638$

11 (i) X – number of T4-cells in 0.01 mm^3 of blood

$X \sim \text{Po}(5)$

Y – number of T8-cells in 0.01 mm^3 of blood

$Y \sim \text{Po}(1.5)$

$$\begin{aligned}\text{P}(\text{healthy}) &= \text{P}(X \geq 4)\text{P}(Y \geq 1) \\ &= \left[1 - \text{P}(X \leq 3)\right]\left[1 - \text{P}(Y = 0)\right] \\ &= 0.57098 \approx 0.571 \text{ (3 s.f.)}\end{aligned}$$

(ii) Req prob $= \left[\text{P}(\text{healthy})\right]\left[\text{P}(\text{unhealthy})\right]^2 \times \frac{3!}{2!}$

$= (0.57098)(1 - 0.57098)^2 \times \frac{3!}{2!} = 0.315$ (3 s.f.)

Alternatively,

A – number of patients who are healthy out of 3 patients

$A \sim \text{B}(3, 0.57098)$

$\text{P}(A = 1) = 0.31528 \approx 0.315$ (3 s.f.)

(iii) $\text{P}(\text{susceptible})$

$= \text{P}(X < 3) = \text{P}(X \leq 2)$

$= 0.12465$

W – number of patients who are susceptible to infection out of 100 patients

$W \sim \text{B}(100, 0.12465)$

Since n is large and

$np = (100)(0.12465) = 12.465 > 5$ and $n(1-p) = (100)(1 - 0.12465) = 87.535 > 5$

$W \sim \text{N}(12.465, 10.911)$ approx

$\mathrm{P}(20 \le W \le 50)$
$= \mathrm{P}(19.5 < W < 50.5)$ (c.c)
$= 0.016595 \approx 0.0166$ (3 s.f.)

12 (i)The occurrences of faulty gearbox must be independent of one another

The probability of a faulty gearbox is constant

(ii) C – number of cars that has gearbox issues out of 20 cars

$C \sim \mathrm{B}(20, 0.02)$

$\mathrm{P}(2 < C < 6) = \mathrm{P}(C \le 5) - \mathrm{P}(C \le 2)$
$= 0.0070667 = 0.00707$ (3 s.f.)

Alternatively,

$\mathrm{P}(2 < C < 6) = \mathrm{P}(C = 3) + \mathrm{P}(C = 4) + \mathrm{P}(C = 5)$
$= 0.0070667 = 0.00707$ (3 s.f.)

(iii) $C \sim B(n, 0.02)$

$\mathrm{P}(C < 2) \le 0.95$

$\mathrm{P}(C \le 1) \le 0.95$

$\mathrm{P}(C = 0) + \mathrm{P}(C = 1) \le 0.95$

${}^{n}C_{0}(0.02)^{0}(0.98)^{n-0} + {}^{n}C_{1}(0.02)^{1}(0.98)^{n-1} \le 0.95$

$(0.98)^{n} + n\left(\frac{1}{49}\right)(0.98)^{n} \le 0.95$

$(0.98)^{n}\left(1 + \frac{n}{49}\right) \le 0.95$

$(0.98)^{n}\left(1 + \frac{n}{49}\right) - 0.95 \le 0$

$n \ge 18.0977$

Hence the least number of cars Mr Ouyang has to sample is 19

Alternatively, By GC using table

n	$(0.98)^{n}\left(1+\frac{n}{49}\right)$
17	0.95541
18	0.95049
19	0.94538

Hence the least number of cars Mr Ouyang has to sample is 19.

(iv) C – number of cars that has gearbox issues out of 20 cars

$C \sim \mathrm{B}(20, 0.02)$

$\mathrm{E}(C) = 0.4$ $\mathrm{Var}(C) = 0.392$

Since $n = 100$ large, by CLT, $\overline{C} \sim N\left(0.4, \frac{0.392}{100}\right)$ approx.

$\mathrm{P}(\overline{C} \ge 0.3) = 0.945$