Pioneer Junior College

H2 Mathematics

2016 JC 2 Preliminary Examination Paper 1 Solution

1

Let the passcode be *xyz*.

$x+y+z=14$........(1)

$100z+10y+x=100x+10y+z+495$

$99z-99x=495$......(2)

$y-x=3$.................(3)

Using the GC, $x=2, y=5, z=7$

$\therefore$ the passcode is 257

2

Let the radius and height of the cone be *r* and *h* respectively.

Let the radius of the circular card be *x* and angle *ACB* be θ.

By Pythagoras Theorem,

$x^2=r^2+h^2 \Rightarrow r^2=x^2-h^2$

$V=\frac{1}{3}\pi r^2 h \quad =\frac{1}{3}\pi\left(x^2-h^2\right)h \quad =\frac{1}{3}\pi\left(x^2 h-h^3\right)$

$\frac{\mathrm{d}V}{\mathrm{d}h}=\frac{1}{3}\pi\left(x^2-3h^2\right)=0$

$h^2=\frac{x^2}{3} \Rightarrow r^2=\frac{2}{3}x^2$

Consider the circumference of the circle without sector:

$2\pi r=\frac{2\pi-\theta}{2\pi}\left(2\pi x\right)$

$2\pi\sqrt{\frac{2}{3}}x=(2\pi-\theta)(x)$

$\theta=2\left(1-\sqrt{\frac{2}{3}}\right)\pi$

Alternatively, consider the curve surface area of the cone,

$\pi x^2\left(\frac{2\pi-\theta}{2\pi}\right)=\pi r x$

$\pi x^2-\pi x^2\left(\frac{\theta}{2\pi}\right)=\pi\sqrt{\frac{2}{3}x^2}\,x$

$$1-\left(\frac{\theta}{2\pi}\right)=\sqrt{\frac{2}{3}}$$

$$\theta=2\left(1-\sqrt{\frac{2}{3}}\right)\pi$$

$$\frac{\mathrm{d}^2V}{\mathrm{d}h^2}=\frac{1}{3}\pi\left(-6h\right)=-2\pi h<0 \text{ (Max)}$$

3

(i) $$\frac{4}{4r^2+12r+5}=\frac{4}{\left(2r+1\right)\left(2r+5\right)}=\frac{A}{2r+1}+\frac{B}{2r+5}$$

$$4=A\left(2r+5\right)+B\left(2r+1\right)$$

$$\text{when } r=-\frac{5}{2}:\quad 4=B\left[2\left(-\frac{5}{2}\right)+1\right]\quad \Rightarrow B=-1$$

$$\text{when } r=-\frac{1}{2}:\quad 4=A\left[2\left(-\frac{1}{2}\right)+5\right]\quad \Rightarrow A=1$$

$$\therefore \frac{4}{4r^2+12r+5}=\frac{1}{2r+1}-\frac{1}{2r+5}$$

(ii) $$\sum_{r=1}^{n-1}\frac{2}{4r^2+12r+5}=\frac{1}{2}\sum_{r=1}^{n-1}\frac{4}{4r^2+12r+5}$$

$$\begin{aligned}
=\frac{1}{2}\Bigg\{&\frac{1}{3}-\frac{1}{7}\\
+\;&\frac{1}{5}-\frac{1}{9}\\
+\;&\frac{1}{7}-\frac{1}{11}\\
+\;&\frac{1}{9}-\frac{1}{13}\\
+\;&\cdots\cdots\cdots\\
+\;&\frac{1}{2n-5}-\frac{1}{2n-1}\\
+\;&\frac{1}{2n-3}-\frac{1}{2n+1}\\
+\;&\frac{1}{2n-1}-\frac{1}{2n+3}\Bigg\}
\end{aligned}$$

$$=\frac{1}{2}\left(\frac{1}{3}+\frac{1}{5}-\frac{1}{2n+1}-\frac{1}{2n+3}\right)=\frac{1}{2}\left(\frac{8}{15}-\frac{4n+4}{4n^2+8n+3}\right)=\frac{4}{15}-\frac{2(n+1)}{4n^2+8n+3}$$

(iii) $$S_{n-1}\geq 0.99S_{\infty}$$

$$\frac{2n+2}{4n^2+8n+3} \le \left(\frac{1}{100}\right)\left(\frac{4}{15}\right)$$

$$\frac{1500(2n+2)-4(4n^2+8n+3)}{(1500)(4n^2+8n+3)} \le 0$$

$$\frac{-4n^2+742n+747}{(1500)(4n^2+8n+3)} \le 0$$

$$\frac{(-n+186.501)(n+1.001)}{1500(2n+1)(2n+3)} \le 0$$

$-n+186.501 \le 0$ since $(2n+1)>0, (2n+3)>0, (n+1.001)>0$

$n \ge 186.501$

Alternatively

$$\frac{4}{15}-\frac{2n+2}{4n^2+8n+3} \ge 0.99\left(\frac{4}{15}\right)$$

$$\frac{2n+2}{4n^2+8n+3} \le \left(\frac{1}{100}\right)\left(\frac{4}{15}\right)$$

$4n^2+8n+3 \ge 750n+750$ (Since n is positive integer)

$4n^2-742n-747 \ge 0$

$n \ge 186.5$

minimum n = 187 (Alternative solution)

4 (i)

$x = 2a\cos^3\theta$

$$\frac{dx}{dt} = 2a(2\cos^2\theta)(-\sin\theta)$$

$y = a\sin^3\theta$

$$\frac{dy}{dt} = a(3\sin^2\theta)(\cos\theta)$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta}$$

$$= \frac{3a\cos\theta\sin^2\theta}{-6a\cos^2\theta\sin\theta}$$

$$= -\frac{1}{2}\tan\theta$$

$$\frac{dy}{dx} = -\frac{1}{2}$$

$$\theta = \frac{\pi}{4}$$

Point P: $\left(2a\cos^3\left(\frac{\pi}{4}\right), a\sin^3\left(\frac{\pi}{4}\right)\right) = \left(\frac{a\sqrt{2}}{2}, \frac{a\sqrt{2}}{4}\right)$

(ii) The equation of tangent at Q is

$$y - a\sin^3 t = -\frac{1}{2}\tan t(x - 2a\cos^3 t)$$

$$y = -\left(\frac{1}{2}\tan t\right)x + a\sin t\cos^2 t + a\sin^3 t$$

$$y = -\left(\frac{1}{2}\tan t\right)x + a\sin t$$

$R(2a\cos t, 0), S(0, a\sin t)$

Midpoint of $RS = \left(a\cos t, \frac{1}{2}a\sin t\right)$

$$x = a\cos t \Rightarrow \cos t = \frac{x}{a}$$

$$y = \frac{1}{2}a\sin t \Rightarrow \sin t = \frac{2y}{a}$$

$$\cos^2 t + \sin^2 t = 1$$

$$\left(\frac{x}{a}\right)^2 + \left(\frac{2y}{a}\right)^2 = 1$$

$$x^2 + 4y^2 = a^2$$

Since $0 < t < \frac{\pi}{2}, 0 < x < a$ or $0 < y < \frac{a}{2}$

5 (i) $S_1 = \frac{1}{2!} = \frac{1}{2}$

$$S_2 = \frac{1}{2!} + \frac{2}{3!} = \frac{5}{6}$$

$$S_3 = \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} = \frac{23}{24}$$

$$S_4 = \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \frac{4}{5!} = \frac{119}{120}$$

(ii)

$$S_1 = \frac{1}{2} = 1 - \frac{1}{2}$$

$$S_2 = \frac{5}{6} = 1 - \frac{1}{6}$$

$$S_3 = \frac{23}{24} = 1 - \frac{1}{24}$$

$$S_4 = \frac{119}{120} = 1 - \frac{1}{120}$$

$$S_n = 1 - \frac{1}{(n+1)!}$$

(iii)

Let P_n be the statement $S_n = 1 - \dfrac{1}{(n+1)!}$ for $n = 1, 2, 3, ...$

when $n = 1$

$\text{LHS} = S_1 = \dfrac{1}{2}$

$\text{RHS} = 1 - \dfrac{1}{(1+1)!} = \dfrac{1}{2}$

$\therefore P_1$ is true

Assume P_k is true for some $k = 1, 2, 3, ...$

$S_k = 1 - \dfrac{1}{(k+1)!}$

We want to prove that P_{k+1} is also true

$S_{k+1} = 1 - \dfrac{1}{(k+2)!}$

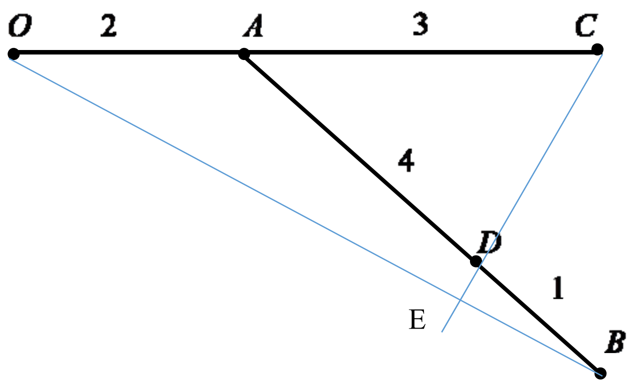
$$
\begin{aligned}
\text{LHS} &= S_{k+1} \\
&= S_k + \frac{k+1}{(k+2)!} \\
&= 1 - \frac{1}{(k+1)!} + \frac{k+1}{(k+2)!} \\
&= 1 - \left[\frac{(k+2)-(k+1)}{(k+2)!}\right] \\
&= 1 - \frac{1}{(k+2)!} \\
&= \text{RHS}
\end{aligned}
$$

$\therefore P_{k+1}$ is true

Since P_1 is true and P_k is true $\Rightarrow P_{k+1}$ is true ,
by mathematical induction P_n is true for all $n = 1, 2, 3, ...$

6

(i) $\dfrac{OA}{OC}=\dfrac{2}{5}$



$\dfrac{OC}{OA}=\dfrac{5}{2}$

$\overrightarrow{OC}=\dfrac{5}{2}\overrightarrow{OA}=\dfrac{5}{2}\mathbf{a}$

By ratio theorem,

$\overrightarrow{OD}=\dfrac{\mathbf{a}+4\mathbf{b}}{5}$

$\overrightarrow{OD}=\dfrac{1}{5}\mathbf{a}+\dfrac{4}{5}\mathbf{b}$

(ii) $\overrightarrow{CD}=\dfrac{1}{5}\mathbf{a}+\dfrac{4}{5}\mathbf{b}-\dfrac{5}{2}\mathbf{a}=-\dfrac{23}{10}\mathbf{a}+\dfrac{4}{5}\mathbf{b}$

$l_{CD}:\mathbf{r}=\dfrac{5}{2}\mathbf{a}+\lambda\left(-\dfrac{23}{10}\mathbf{a}+\dfrac{4}{5}\mathbf{b}\right)\quad \lambda\in\mathbf{R}$

(iii) Since E is a point on CD produced,

$\overrightarrow{OE}=\dfrac{5}{2}\mathbf{a}+\lambda\left(-\dfrac{23}{10}\mathbf{a}+\dfrac{4}{5}\mathbf{b}\right)\quad \lambda\in\mathbf{R}$

Since E is a point on OB,

$\overrightarrow{OE}=\alpha\mathbf{b}\quad \alpha\in\mathbf{R}$

$\dfrac{5}{2}\mathbf{a}+\lambda\left(-\dfrac{23}{10}\mathbf{a}+\dfrac{4}{5}\mathbf{b}\right)=\alpha\mathbf{b}$

$\left(\dfrac{5}{2}-\dfrac{23}{10}\lambda\right)\mathbf{a}+\dfrac{4}{5}\lambda\mathbf{b}=\alpha\mathbf{b}$

$\dfrac{5}{2}-\dfrac{23}{10}\lambda=0\Rightarrow\lambda=\dfrac{25}{23}$

$\dfrac{4}{5}\lambda=\alpha\Rightarrow\alpha=\dfrac{20}{23}$

$\therefore\overrightarrow{OE}=\dfrac{20}{23}\mathbf{b}$

$OE:EB=20:3$

7

$$\frac{\mathrm{d}T}{\mathrm{d}t} = -k\left(T - T_{\mathrm{o}}\right), \quad k > 0$$

$$\int \frac{1}{T - T_{\mathrm{o}}}\mathrm{d}T = \int -k\mathrm{d}t$$

Note that no modulus required since T > T_0

$\ln\left(T - T_{\mathrm{o}}\right) = -kt + C$, where C is an arbitary constant

$T - T_{\mathrm{o}} = \mathrm{e}^{-kt+C}$

$T - T_{\mathrm{o}} = \mathrm{e}^{-kt}\mathrm{e}^{C}$

$T - T_{\mathrm{o}} = A\mathrm{e}^{-kt}$, where $A = \mathrm{e}^{C}$

$\therefore T = T_{\mathrm{o}} + A\mathrm{e}^{-kt}$ (shown)

$T_{\mathrm{o}} = 30^{\circ}C$

At $t = 0$: $100 = 30 + A\mathrm{e}^{-k(0)}$

$A = 70$

At $t = 15$: $70 = 30 + 70\mathrm{e}^{-15k}$

$40 = 70\mathrm{e}^{-15k}$

$\mathrm{e}^{-15k} = \frac{4}{7}$

$k = -\frac{1}{15}\ln\frac{4}{7} \approx 0.0373077$

To find time taken for pot of dessert to cool to at most $35^{\circ}C$:

$30 + 70\mathrm{e}^{-kt} \leq 35$

$70\mathrm{e}^{-kt} \leq 5$

$\mathrm{e}^{-kt} \leq \frac{5}{70}$

$-kt \leq \ln\frac{5}{70}$

$$t \geq \frac{\ln(5/70)}{-\frac{1}{15}\ln(4/7)}$$

$t \geq 70.74$

$t = 71$ minutes

It takes at least 71 minutes for the pot of dessert to cool to $35^{\circ}C$ and 30 minutes to cook. Hence Nurul must start preparing the pot of dessert at 7.19pm the latest.

8

(i) Let L be the distance covered by the lion.
$a = 2.5$ and $d = -0.05$

$$L = \frac{n}{2}\left[2a + \left(n-1\right)d\right]$$
$$= \frac{n}{2}\left[2\left(2.5\right) + (n-1)\left(-0.05\right)\right]$$
$$= -\frac{1}{40}n^2 + \frac{101}{40}n$$

(ii) Let P be the distance covered by the prey.
$a = 1.5$ and $r = 0.95$

$$P = \frac{1.5\left(1-0.95^n\right)}{1-0.95}$$
$$= 30(1-0.95^n)$$

When $n \to \infty$, $P \to 30$
So the distance covered by the prey can never exceed 30m

(iii) In order for the lion to catch its prey,

$L \geq P+25$

$$-\frac{1}{40}n^2 + \frac{101}{40}n \geq 30(1-0.95^n) + 25$$
$$-\frac{1}{40}n^2 + \frac{101}{40}n + 30(0.95^n) \geq 55$$
$$n = 24, -\frac{1}{40}n^2 + \frac{101}{40}n + 30(0.95^n) = 54.96 < 55$$
$$n = 25, -\frac{1}{40}n^2 + \frac{101}{40}n + 30(0.95^n) = 55.822 > 55$$
$$n = 26, -\frac{1}{40}n^2 + \frac{101}{40}n + 30(0.95^n) = 56.556 > 55$$

least $n = 25$
Hence, the lion will need at least 25 leaps to catch its prey.

(iv) Let the initial distance be k

In order for the prey to escape the hunt,

$P+k \geq L$

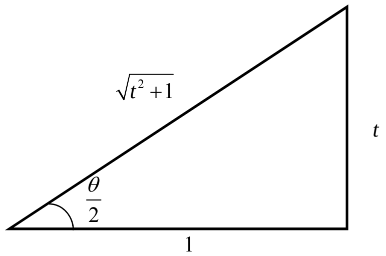
$$30(1-0.95^{30}) + k \geq -\frac{1}{40}(30^2) + \frac{101}{40}(30)$$
$$23.561 + k \geq 53.25$$
$$k \geq 29.689$$

$\therefore$ the shortest distance is 30 m.

9

(a) **(i)** $t=\tan\frac{\theta}{2}$

$$\tan\theta=\frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}=\frac{2t}{1-t^2}$$



by triangle rule:

$$\sin\theta=\frac{2t}{1+t^2}\quad\text{(shown)}$$

Alternatively RHS = $\frac{2t}{1+t^2}=\frac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}=\frac{2\tan\frac{\theta}{2}}{\sec^2\frac{\theta}{2}}=\frac{2\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\times\cos^2\frac{\theta}{2}=\sin\theta$ =LHS

Alternatively

Use double angle formula: $\sin\theta=2\sin\frac{\theta}{2}\cos\frac{\theta}{2}=2\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\cos^2\frac{\theta}{2}=\frac{2\tan\frac{\theta}{2}}{\sec^2\frac{\theta}{2}}=\frac{2t}{1+t^2}$

(ii)

$$\int_0^{\frac{\pi}{2}}\frac{\tan\frac{\theta}{2}+1}{\sin\theta+1}\,\mathrm{d}\theta$$
$$=\int_0^1\frac{\mathrm{t}+1}{\frac{2t}{1+t^2}+1}\left(\frac{2}{1+t^2}\mathrm{dt}\right)$$
$$=\int_0^1\frac{\mathrm{t}+1}{\frac{2t+1+t^2}{1+t^2}}\left(\frac{2}{1+t^2}\mathrm{dt}\right)$$
$$=\int_0^1\frac{2(\mathrm{t}+1)}{2t+1+t^2}\mathrm{dt}$$
$$=\int_0^1\frac{2t+2}{t^2+2t+1}\mathrm{dt}=\int_0^1\frac{2}{t+1}\mathrm{dt}$$
$$=2\left[\ln(t+1)\right]_0^1$$
$$=2\ln 2$$

$t=\tan\frac{\theta}{2}$

when $\theta=\frac{\pi}{2}$: $t=\tan\frac{\pi/2}{2}$=1

when $\theta=0$: $t=\tan\frac{0}{2}$=0

$$\tan^{-1}t=\frac{\theta}{2}$$
$$\frac{1}{1+t^2}\frac{\mathrm{d}t}{\mathrm{d}\theta}=\frac{1}{2}$$
$$\frac{\mathrm{d}t}{\mathrm{d}\theta}=\frac{1+t^2}{2}$$
$$\frac{\mathrm{d}\theta}{\mathrm{d}t}=\frac{2}{1+t^2}$$

(b)

$$\int e^{2v}\cos 3v\,dv$$

$$= \frac{1}{3}e^{2v}\sin 3v - \int \frac{2}{3}e^{2v}\sin 3v\,dv$$

$$u = e^{2v} \qquad \frac{dy}{dv} = \cos(3v)$$

$$\frac{du}{dv} = 2e^{2v} \qquad y = \frac{1}{3}\sin(3v)$$

$$= \frac{1}{3}e^{2v}\sin 3v - \frac{2}{3}\left[-\frac{1}{3}e^{2v}\cos(3v) + \int \frac{2}{3}e^{2v}\cos(3v)dv\right]$$

$$u = e^{2v} \qquad \frac{dy}{dv} = \sin(3v)$$

$$\frac{du}{dv} = 2e^{2v} \qquad y = -\frac{1}{3}\cos(3v)$$

$$= \frac{1}{3}e^{2v}\sin 3v + \frac{2}{9}e^{2v}\cos(3v) - \int \frac{4}{9}e^{2v}\cos(3v)dv$$

$$\frac{13}{9}\int e^{2v}\cos 3v\,dv = \frac{1}{3}e^{2v}\sin 3v + \frac{2}{9}e^{2v}\cos(3v)$$

$$\int e^{2v}\cos 3v\,dv = \frac{3}{13}e^{2v}\sin 3v + \frac{2}{13}e^{2v}\cos(3v) + c$$

Alternatively

$$\int e^{2v}\cos 3v\,dv$$

$$= \frac{1}{2}e^{2v}\cos 3v + \int \frac{3}{2}e^{2v}\sin 3v\,dv$$

$$u = \cos(3v) \qquad \frac{dy}{dv} = e^{2v}$$

$$\frac{du}{dv} = -3\sin(3v) \qquad y = \frac{1}{2}e^{2v}$$

$$= \frac{1}{2}e^{2v}\cos 3v + \frac{3}{2}\left[\frac{1}{2}e^{2v}\sin(3v) - \int \frac{3}{2}e^{2v}\cos(3v)dv\right]$$

$$u = \sin(3v) \qquad \frac{dy}{dv} = e^{2v}$$

$$\frac{du}{dv} = 3\cos(3v) \qquad y = \frac{1}{2}e^{2v}$$

$$= \frac{1}{2}e^{2v}\cos 3v + \frac{3}{4}e^{2v}\sin(3v) - \frac{9}{4}\int e^{2v}\cos(3v)dv$$

$$\frac{13}{4}\int e^{2v}\cos 3v\,dv = \frac{1}{2}e^{2v}\cos 3v + \frac{3}{4}e^{2v}\sin(3v)$$

$$\int e^{2v}\cos 3v\,dv = \frac{3}{13}e^{2v}\sin 3v + \frac{2}{13}e^{2v}\cos(3v) + c$$

10

(i) $\ell_1 : \mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ -5 \end{pmatrix}, \quad \lambda \in \mathbf{R}$

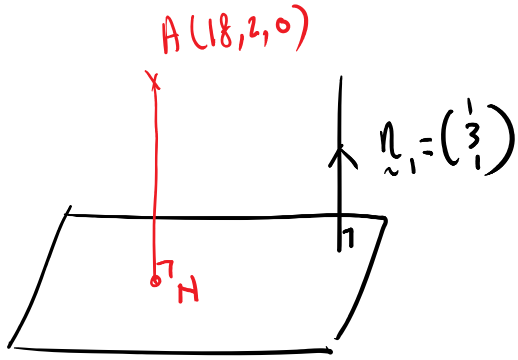
Since (1, 0, 1) is on ℓ_1 and p_1

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} = a$$

$1 + 0 + 1 = a$

$a = 2$ (shown)

(ii) Let N be the foot of perpendicular from A to p_1



$$\ell_{AN} : \mathbf{r} = \begin{pmatrix} 18 \\ 2 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}, \quad \alpha \in \mathbf{R}$$

let $\overrightarrow{ON} = \begin{pmatrix} 18+\alpha \\ 2+3\alpha \\ \alpha \end{pmatrix}$ for some value of α

Since N is a point on p_1

$$\begin{pmatrix} 18+\alpha \\ 2+3\alpha \\ \alpha \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} = 2$$

$18 + \alpha + 6 + 9\alpha + \alpha = 2$

$24 + 11\alpha = 2$

$11\alpha = -22$

$\alpha = -2$

$$\overrightarrow{ON} = \begin{pmatrix} 18-2 \\ 2-6 \\ -2 \end{pmatrix} = \begin{pmatrix} 16 \\ -4 \\ -2 \end{pmatrix} \therefore N(16, -4, -2)$$

(iii) Since B is on ℓ_1

$$\overrightarrow{OB} = \begin{pmatrix} 1+2\lambda \\ \lambda \\ 1-5\lambda \end{pmatrix}$$

$$\overrightarrow{AB} = \begin{pmatrix} 1+2\lambda \\ \lambda \\ 1-5\lambda \end{pmatrix} - \begin{pmatrix} 18 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} -17+2\lambda \\ -2+\lambda \\ 1-5\lambda \end{pmatrix}$$

$$\left|\overrightarrow{AB}\right| = \sqrt{(-17+2\lambda)^2 + (-2+\lambda)^2 + (1-5\lambda)^2}$$
$$\left|\overrightarrow{AB}\right| = \sqrt{(289-68\lambda+4\lambda^2)+(4-4\lambda+\lambda^2)+(1-10\lambda+25\lambda^2)}$$
$$\left|\overrightarrow{AB}\right| = \sqrt{294-72\lambda+30\lambda^2}$$
$$\left|\overrightarrow{AB}\right|^2 = 294-72\lambda+30\lambda^2$$

For shortest distance from A to ℓ_1

$\left|\overrightarrow{AB}\right|^2$ must be minimum

$$\therefore \left|\overrightarrow{AB}\right|^2 = 30\lambda^2 - 72\lambda + 294$$
$$2\left|\overrightarrow{AB}\right|\frac{\mathrm{d}\left|\overrightarrow{AB}\right|^2}{\mathrm{d}\lambda} = 60\lambda - 72$$
$$\frac{\mathrm{d}\left|\overrightarrow{AB}\right|^2}{\mathrm{d}\lambda} = 0$$
$$60\lambda - 72 = 0$$
$$\lambda = \frac{6}{5}$$
$$\overrightarrow{OB} = \begin{pmatrix} 1+2\lambda \\ \lambda \\ 1+5\lambda \end{pmatrix} = \begin{pmatrix} 17/5 \\ 6/5 \\ 7 \end{pmatrix} \text{ or } \frac{1}{5}\begin{pmatrix} 17 \\ 6 \\ 35 \end{pmatrix}$$

(iv) direction vector of $\ell_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ b \\ 1 \end{pmatrix} = \begin{pmatrix} -b \\ -(1-2) \\ b \end{pmatrix} = \begin{pmatrix} -b \\ 1 \\ b \end{pmatrix}$

To find a common point between p_2 and p_3 by letting $y=0$:

$x + z = 1$ --- (1)

$2x + z = 4$ --- (2)

Solve (1) and (2):

$x = 3, \quad z = -2$

Hence ℓ_2: $\mathbf{r} = \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} + \mu\begin{pmatrix} -b \\ 1 \\ b \end{pmatrix}, \quad \mu \in \mathbf{R}$ (shown)

11

(i) $w = re^{i\theta}$

$w^* = re^{-i\theta}$

$$\frac{w^2}{w^*} = \frac{\left(re^{i\theta}\right)^2}{re^{-i\theta}}$$

$$= \frac{r^2e^{i2\theta}}{re^{-i\theta}}$$

$$= re^{i3\theta} = -3 = 3e^{i\pi}$$

$$3\theta = \pi \Rightarrow \theta = \frac{\pi}{3} (0 < \theta \le \frac{1}{2}\pi)$$

$r = 3$

$w = 3e^{i\frac{\pi}{3}}$, $w^n = 3^n e^{i\frac{n\pi}{3}}$

w^n is real $,\Rightarrow \dfrac{n\pi}{3} = 0, \pi, 2\pi, \ldots$, so n = 3,6, 9,…

(b) **(i)** $z^5 = 1 + i$

$$= \sqrt{2}e^{\left(2k\pi + \frac{\pi}{4}\right)i}$$

$z = 2^{\frac{1}{10}} e^{\left(\frac{2k\pi}{5} + \frac{\pi}{20}\right)i}$, $k = 0\ , \pm 1\ , \pm 2$

$z = 2^{\frac{1}{10}} e^{\frac{\pi}{20}i},\ 2^{\frac{1}{10}} e^{\frac{9\pi}{20}i},\ 2^{\frac{1}{10}} e^{-\frac{7\pi}{20}i},\ 2^{\frac{1}{10}} e^{\frac{17\pi}{20}i},\ 2^{\frac{1}{10}} e^{-\frac{3\pi}{4}i}$

So $|z| = 2^{\frac{1}{10}}$ for all z

$\arg(z) = \dfrac{\pi}{20}, \dfrac{9\pi}{20}, -\dfrac{7\pi}{20}, \dfrac{17\pi}{20}, -\dfrac{3\pi}{4}$

(ii) $z_1 = 2^{\frac{1}{10}} e^{\frac{17\pi}{20}i}$

$z_2 = 2^{\frac{1}{10}} e^{-\frac{15\pi}{20}i}$

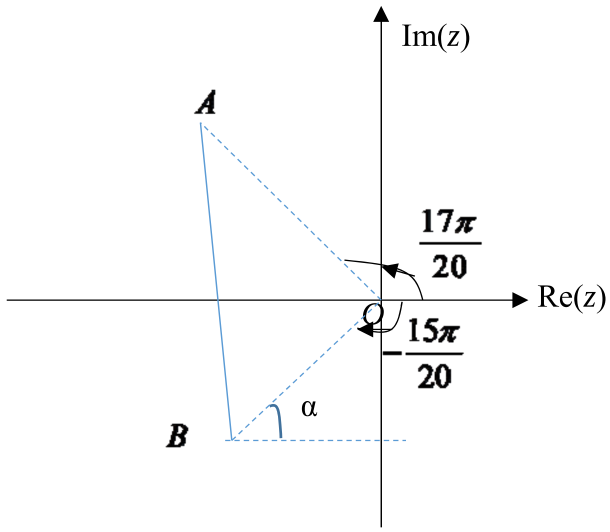
Let Point A and B represent z_1 and z_2 respectively.

$|z_1| = |z_2| \Rightarrow OAB$ is an isosceles triangle.

$$\angle AOB = \frac{2\pi}{5}$$

$$\angle OAB = \angle OBA = \frac{1}{2}[\pi - \angle AOB]$$

$$= \frac{1}{2}\left[\pi - \frac{2\pi}{5}\right] = \frac{3\pi}{10}$$



$$
\begin{aligned}
\arg\left(z_1 - z_2\right) &= \alpha + \angle OBA \\
&= \left(\pi - \frac{15\pi}{20}\right) + \frac{3\pi}{10} \\
&= \frac{11\pi}{20}
\end{aligned}
$$