Candidate Name: ________________________ Class: ____________

JC2 PRELIMINARY EXAM
Higher 2

PIONEER JUNIOR COLLEGE
While I Live, I Learn

MATHEMATICS
Paper 1

9740/01
14 Sep 2016
3 hours

Additional Materials: Cover page
Answer papers
List of Formulae (MF15)

READ THESE INSTRUCTIONS FIRST

Write your full name and class on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

1 The police wish to crack a 3-digit passcode. The sum of the digits is 14. When the digits in the number are reversed, the new number becomes 495 more than the original number. The digit in the tens position is 3 more than the digit in the hundreds position. What is the passcode? [4]

2

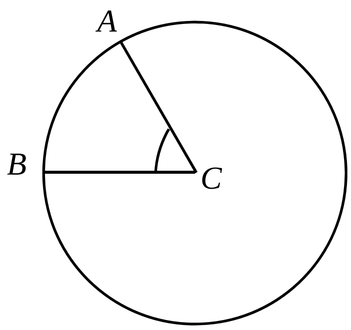

Fig. 1

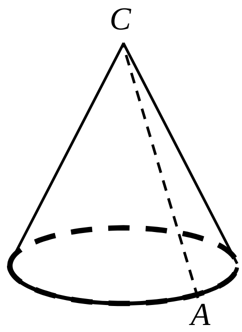

Fig. 2

Fig. 1 shows a circular card with centre C. A sector CAB is removed from the card, and the remaining card is folded such that AC and BC meet without overlapping to form a cone, as shown in Fig. 2 (A will meet B). Use differentiation to find the angle ACB exactly such that the volume of the cone is as large as possible. [6]

[It is given that a cone with radius r and height h has volume $\frac{1}{3}\pi r^2 h$ and curve surface area $\pi r l$ where l is the slant height.]

3 **(i)** Show that $\dfrac{4}{4r^2+12r+5}$ can be expressed as $\dfrac{A}{2r+1}+\dfrac{B}{2r+5}$, where A and B are constants to be determined. [2]

(ii) Hence, find an expression for $\displaystyle\sum_{r=1}^{n-1}\frac{2}{4r^2+12r+5}$ in terms of n. [3]

(iii) Hence, find the smallest value of n for which $\displaystyle\sum_{r=1}^{n-1}\frac{2}{4r^2+12r+5}$ is at least 99% of its sum to infinity. [3]

4 A curve C has parametric equations

$$x = 2a\cos^3\theta, \quad y = a\sin^3\theta,$$

where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ and a is a positive constant.

(i) A point P lies on C. Find, in terms of a, the exact coordinates of P, whose tangent is parallel to the line $2y = -x$. [4]

(ii) The tangent to C at the point $Q\left(2a\cos^3 t, \ a\sin^3 t\right)$, where $0 < t < \frac{\pi}{2}$, meets the x- and y-axes at R and S respectively. Find a cartesian equation of the locus of the mid-point of RS as t varies. [4]

5 The sum, S_n, of the first n terms of a sequence is given by

$$S_n = \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \ldots + \frac{n}{(n+1)!}.$$

(i) Find the values of S_1, S_2, S_3 and S_4. [2]

(ii) By expressing S_n in the form $\left[1 - \mathrm{f}(n)\right]$ for $n = 1, 2, 3, 4$, find a conjecture for S_n in terms of n. [2]

(iii) Hence prove by mathematical induction the result of S_n for all positive integers n. [4]

6 Referred to the origin O, points A and B have position vectors **a** and **b** respectively. Point C lies on OA produced such that $OA : OC = 2 : 5$. Point D is on AB, between A and B such that $AD : DB = 4 : 1$.

(i) Find the position vectors $\overrightarrow{OC}$ and $\overrightarrow{OD}$, giving your answers in terms of **a** and **b**. [2]

(ii) Find a vector equation of line CD. [2]

(iii) Point E lies on CD produced, and it is also on OB, between O and B. Find $\overrightarrow{OE}$ and the ratio $OE: EB$. [5]

7 Newton's law of cooling states that the rate of cooling in t minutes is proportional to the difference between the body temperature $T\,^{o}C$ and its immediate surrounding temperature $T_o\,^{o}C$. Show that $T = T_o + Ae^{-kt}$, where A and k are positive constants.

[3]

Nurul is the chef of a dessert shop and she leaves her work place at 9pm daily. Before she leaves, she is required to cook a big pot of dessert and leave it to cool, before placing it in the refrigerator for the next business day. She takes 30 minutes to cook the pot of dessert to $100\,^{o}C$, and then leaves it to cool. After 15 minutes, the pot of dessert cools to $70\,^{o}C$.

The room temperature of the kitchen is $30\,^{o}C$, and the refrigerator can only accommodate items with temperature of at most $35\,^{o}C$. By what time, correct to the nearest minute, must Nurul start to cook the pot of dessert so that she will be able to leave her work place on time? [5]

8 A lion eyes its prey which is k m away and starts its chase with a leap of 2.5 m. Each subsequent leap of the lion is shorter than its preceding leap by 0.05 m. Its prey notices the lion's chase and runs away with a first leap of 1.5 m, with each subsequent leap 5% less than the previous leap. You may assume that the lion and the prey start running at the same moment and they complete the same number of leaps after the first leap.

(i) Find the total distance covered by the lion after n leaps. [2]

(ii) Find the total distance covered by the prey after n leaps. Deduce that the distance covered by the prey can never be greater than 30 m. [3]

(iii) Given $k = 25$, find the least number of leaps the lion needs to take to catch its prey. [3]

(iv) Assuming that the lion can cover a maximum of 30 leaps, find the least integer k, so that the prey will survive the hunt. [3]

9 **(a)** **(i)** If $t=\tan\dfrac{\theta}{2}$, show that $\sin\theta=\dfrac{2t}{1+t^2}$. [2]

(ii) Use the substitution $t=\tan\dfrac{\theta}{2}$ to find the exact value of

$$\int_0^{\frac{\pi}{2}}\left(\frac{\tan\frac{\theta}{2}+1}{\sin\theta+1}\right)\mathrm{d}\theta.$$ [5]

(b) Find $\int \mathrm{e}^{2v}\cos 3v\,\mathrm{d}v$. [4]

10 The point A has coordinates $(18,2,0)$. The plane p_1 has the equation $x+3y+z=a$, where a is a constant. It is given that p_1 contains the line l_1 with equation $\dfrac{x-1}{2}=y=\dfrac{z-1}{-5}$.

.

(i) Show that $a=2$. [2]

(ii) Find the coordinates of the foot of perpendicular from the point A to p_1.

[3]

(iii) B is given to be a general point on l_1. Find an expression for the distance between the point A and B. Hence find the position vector of B that is nearest to A. [4]

The planes p_2 and p_3 have the equations $x+z=1$ and $2x+by+z=4$ respectively, where b is a constant.

(iv) Given that p_2 and p_3 intersect at l_2, show that l_2 is parallel to the vector $\begin{pmatrix}-b\\1\\b\end{pmatrix}$. By finding a point that lies on both planes, find a vector equation of l_2.

[3]

11 **(a)** The complex number w is such that $w = r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $0 < \theta \leq \dfrac{\pi}{2}$. The complex conjugate of w is denoted by w^*. Given that $\dfrac{w^2}{w^*} = -3$, find the exact values of r and θ. Hence find the three smallest positive integer n for which w^n is a real number. [5]

(b) The complex number z is such that $z^5 - 1 - \mathrm{i} = 0$.

(i) Find the modulus and argument of each of the possible values of z. [5]

(ii) Two of these values are z_1 and z_2, where $\dfrac{\pi}{2} < \arg z_1 < \pi$ and $-\pi < \arg z_2 < -\dfrac{\pi}{2}$. Find the exact value of $\arg(z_1 - z_2)$ in terms of π and illustrate the locus $\arg(z - z_1) = \arg(z_1 - z_2)$ on an Argand diagram. [5]