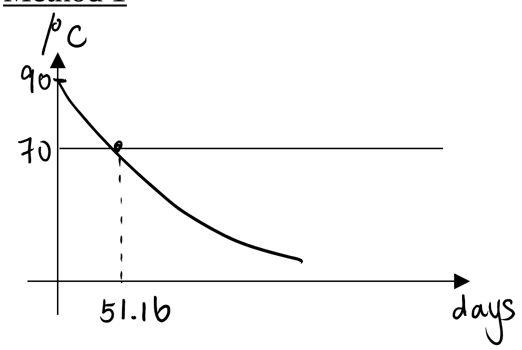
2016 Prelim Paper 2 Solutions

Qn	Solution
1 (i)	1^{st} day: $0.97(92)$ 2^{nd} day: $0.97\left(0.97(92)+2\right)$ 3^{rd} day: $0.97\left(0.97\left(0.97(92)+2\right)+2\right)$ $=0.97^3(92)+0.97^2(2)+0.97(2)$ $=87.788$ $=87.8\ ^\circ\text{C}$ (3 s.f.)
(ii)	n^{th} day: $=0.97^n(92)+0.97^{n-1}(2)+...+0.97(2)$ $=0.97^n(92)+2\left(\dfrac{0.97\left(1-0.97^{n-1}\right)}{1-0.97}\right)$ <u>Method 1</u>  <u>Method 2</u> Using GC, When $n=51$, temperature $=70.025°\text{C}$ When $n=52$, temperature $=69.865°\text{C}$ $\therefore$ 52 days
(iii)	No, temperature will not drop infinitely since $r=0.97<1$. As $n\to\infty$, temperature approaches $0+\dfrac{2\times 0.97}{1-0.97}=64.7\ ^\circ\text{C}$ in the long run.
2 (a)	$2\left(-\dfrac{\text{i}}{2}\right)^3+(\text{i}-8)\left(-\dfrac{\text{i}}{2}\right)^2+a\left(-\dfrac{\text{i}}{2}\right)+13\text{i}=0$ $\dfrac{1}{4}\text{i}+2-\dfrac{1}{4}\text{i}-\dfrac{a\text{i}}{2}+13\text{i}=0$ $a=26-4\text{i}$ $2z^3+(\text{i}-8)z^2+(26-4\text{i})z+13\text{i}=(2z+\text{i})(z^2+bz+13)$ Comparing the coefficient of z^2, $\text{i}-8=2b+\text{i}$ $\therefore b=-4$ $(2z+\text{i})(z^2-4z+13)=0$ $\therefore z=-\dfrac{\text{i}}{2}$ or $z=2\pm 3\text{i}$.

	Replace z with iw, $2(\mathrm{i}w)^3+(\mathrm{i}-8)(\mathrm{i}w)^2+a(\mathrm{i}w)+13\mathrm{i}=0$ $-2\mathrm{i}w^3+(8-\mathrm{i})w^2+a\mathrm{i}w+13\mathrm{i}=0$ Dividing throughout by $-\mathrm{i}$, $2w^3+(1+8\mathrm{i})w^2-aw-13=0.$ $\mathrm{i}w=-\frac{\mathrm{i}}{2}$ or $\mathrm{i}w=2\pm3\mathrm{i}.$ $\therefore w=-\frac{1}{2}$ or $w=\pm3-2\mathrm{i}$
(b)	$z^6=-729=729\mathrm{e}^{\mathrm{i}\pi}=729\mathrm{e}^{(\pi+2k\pi)\mathrm{i}}$ $z=3\mathrm{e}^{\left(\frac{\pi}{6}+\frac{2k\pi}{6}\right)\mathrm{i}}, k=0,\pm1,\pm2,-3$ OR $z=3\mathrm{e}^{-\frac{5\pi}{6}\mathrm{i}},3\mathrm{e}^{\frac{5\pi}{6}\mathrm{i}},3\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}},3\mathrm{e}^{\frac{\pi}{2}\mathrm{i}},3\mathrm{e}^{-\frac{\pi}{6}\mathrm{i}},3\mathrm{e}^{\frac{\pi}{6}\mathrm{i}}$ $z_1=3\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}$ $\arg\left(\frac{z_1{}^n}{z_1{}^*}\right)=n\arg z_1+\arg z_1=(n+1)\frac{\pi}{6}$ Positive real number $(n+1)\frac{\pi}{6}=2k\pi$ $\therefore (n+1)\frac{\pi}{6}=2\pi$ Minimum $n=11.$
3 (i)	$x=3\sin\theta+1$ $\frac{\mathrm{d}x}{\mathrm{d}\theta}=3\cos\theta$ $\int\frac{x}{\sqrt{9-(x-1)^2}}\,\mathrm{d}x,$ $=\int\frac{3\sin\theta+1}{\sqrt{9-(3\sin\theta)^2}}(3\cos\theta)\,\mathrm{d}\theta$ $=\int(3\sin\theta+1)\,\mathrm{d}\theta$ $=-3\cos\theta+\theta+C$ $=-\sqrt{9-(x-1)^2}+\sin^{-1}\frac{x-1}{3}+C$

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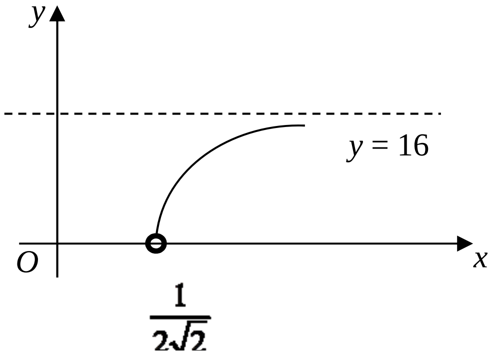
<tr><td>(ii)
(a)</td><td>


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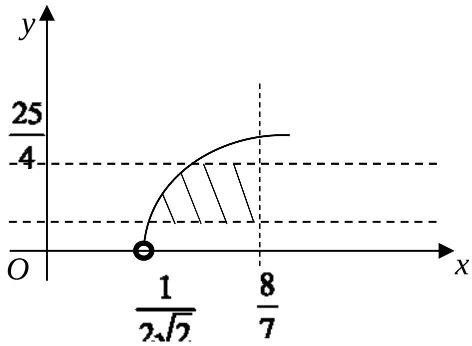

$$\text{Area} = -\int_1^{\frac{25}{4}} x\,\mathrm{d}y + \left(\frac{25}{4} - 1\right) \times \frac{8}{7}$$

$$= -\int_1^{\frac{5}{2}} \frac{2t}{\sqrt{9-(t-1)^2}}\,\mathrm{d}t + 6$$

$$= -2\left[-\sqrt{9-(t-1)^2} + \sin^{-1}\frac{t-1}{3}\right]_1^{\frac{5}{2}} + 6$$

$$= -2\left[(-\sqrt{\frac{27}{4}} + \sin^{-1}\frac{1}{2}) - (-\sqrt{9} + \sin^{-1}0)\right] + 6$$

$$= 3\sqrt{3} - \frac{\pi}{3}$$
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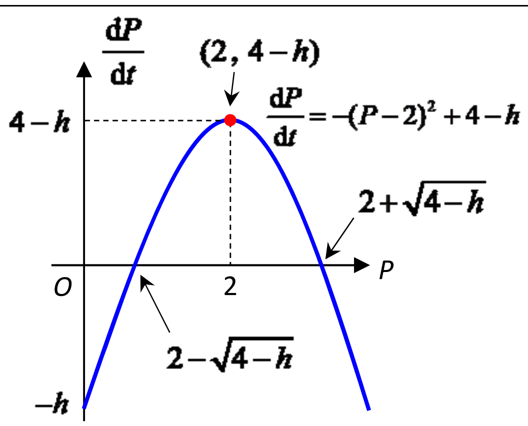
<tr><td>4(i)</td><td>

$$\frac{\mathrm{d}P}{\mathrm{d}t} = P(4-P) - h$$
$$= -P^2 + 4P - h$$
$$= -(P^2 - 4P) - h$$
$$= -(P-2)^2 + 4 - h$$



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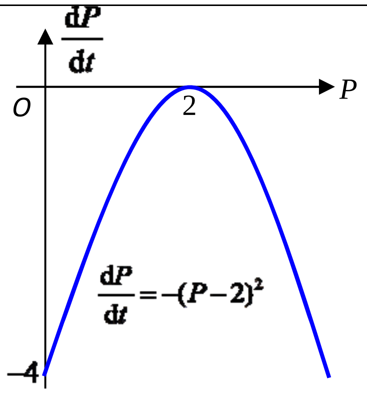
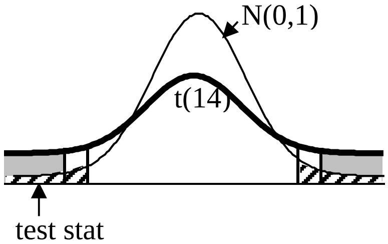
(ii)	From graph in (i), when $h=4$, $\frac{dP}{dt}=-(P-2)^2=0$ at $P=2$. $\therefore$ largest $h=4$ where the population P remains constant at $P=2$. Hence $MSY=4$ million 
(iii)	$\frac{dP}{dt}=P(4-P)-3=-\left[(P^2-4P)+3\right]=-\left[(P-2)^2-1\right]$ <u>Method 1</u> $\int\frac{1}{(P-2)^2-1}\,dP=-\int 1\,dt$ $\frac{1}{2}\ln\left\|\frac{(P-2)-1}{(P-2)+1}\right\|=-t+C$ $\frac{P-3}{P-1}=\pm e^{-2t+2C}=Ae^{-2t}$ where $A=\pm e^{2C}$ In 2015, let $t=0$, $P=3.2$; hence $A=\frac{1}{11}$ $\therefore\ P-3=\frac{1}{11}e^{-2t}(P-1)$ $11P-33=Pe^{-2t}-e^{-2t}$ Hence $P=\frac{33-e^{-2t}}{11-e^{-2t}}=\frac{33e^{2t}-1}{11e^{2t}-1}$ <u>Method 2</u> $\frac{dP}{dt}=1-(P-2)^2$ $\int\frac{1}{1-(P-2)^2}\,dP=\int 1\,dt$ $\frac{1}{2}\ln\left\|\frac{1+(P-2)}{1-(P-2)}\right\|=t+C$ $\frac{P-1}{3-P}=\pm e^{2t+2C}=Ae^{2t}$ where $A=\pm e^{2C}$ In 2015, let $t=0$, $P=3.2$; hence $A=-11$ $P-1=-11e^{2t}(3-P)$ $P-1=-33e^{2t}+11Pe^{2t}$ $P=\frac{1-33e^{2t}}{1-11e^{2t}}$ or $P=\frac{33e^{2t}-1}{11e^{2t}-1}$

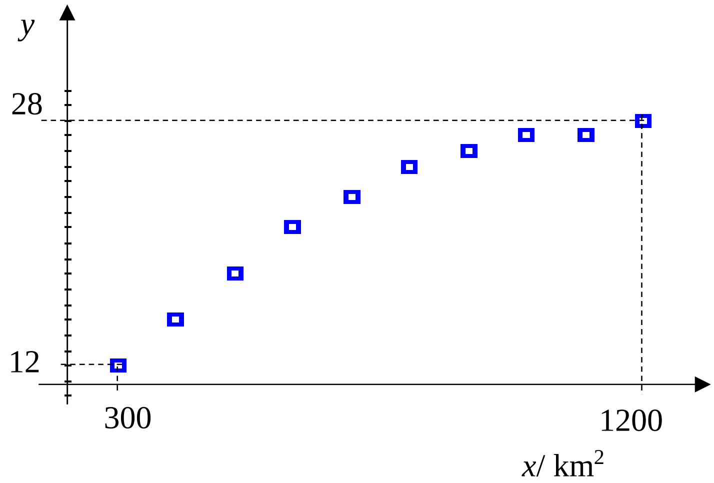
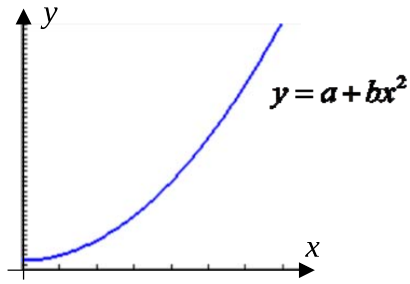
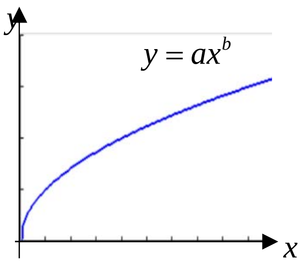
	Method 3 $-\int \frac{1}{P^2-4P+3}\,\mathrm{d}P = \int 1\,\mathrm{d}t$ $-\int \frac{1}{(P-3)(P-1)}\,\mathrm{d}P = \int 1\,\mathrm{d}t$ $-\int \frac{1}{2(P-3)}\,\mathrm{d}P + \int \frac{1}{2(P-1)}\,\mathrm{d}P = \int 1\,\mathrm{d}t$ (using partial fractions) $\frac{1}{2}\ln\left\|\frac{P-1}{P-3}\right\| = t + C$ $\frac{P-1}{P-3} = \pm e^{2t+2C} = Ae^{2t}$ In 2015, let $t=0$, $P=3.2$; hence $A=11$ $P-1 = 11e^{2t}(P-3)$ $P-1 = 11Pe^{2t} - 33e^{2t}$ $P = \frac{1-33e^{2t}}{1-11e^{2t}}$ or $P = \frac{33e^{2t}-1}{11e^{2t}-1}$ In 2016, $t=1$ Hence $P = \frac{33e^2-1}{11e^2-1} = 3.02$ $\therefore$ the population of wild salmon is 3.02 million in 2016.
(iv)	There are no external factors such as marine pollution or climate change that drastically affect the population of wild salmon in that region.
5 (i)	Simple Random Sampling: Using a random number generator to generate 400 numbers and use select the voting slips corresponding to these 400 numbers Systematic Sampling: Consider N registered voter in the electoral division such that the sampling interval $\frac{N}{400}$ is an integer. Using a random number generator, select a number from 1 to k and take every kth number thereafter until a sample of 400 is obtained. Choose the voting slips corresponding to the numbers. Stratified Sampling: Use each polling station as the stratum. The number of votes in each stratum is calculated by $\frac{\text{no. of voters polling at the station}}{\text{total number of voters in the electoral division}} \times 400$. The votes is then obtained from each stratum using simple random sampling. Quota Sampling: Consider N registered voter in the electoral division such that the sampling interval $\frac{N}{400}$ is an integer. Using a random number generator, select a number from 1 to k and take every kth number thereafter until a sample of 400 is obtained. Choose the voting slips corresponding to the numbers.

	Use each polling station as the stratum, decide on the number of votes to be obtained from each stratum. For example if there are 4 polling stations, obtain 100 votes from each polling station. The voting slips are then selected based on the officer's discretion.
(ii)	Simple random sampling: Advantage: The sample obtained is free from bias The sampling procedures are easy to follow Disadvantage The sample obtained might not be a good representation of the electoral division Systematic Sampling: Advantage: It is easy execute because only the first number needs to be chosen The electoral division will be evenly sampled as the voting slips is chosen at regular intervals Disadvantage: If there is a periodic trend like every *k*th voters are of the same gender, systematic sampling may produce a biased sample Stratified Sampling: Advantage: Stratified sampling will provide a sample of voter that is representative of electoral division The results in each polling station can be analysed separately. Easy to conduct as the sampling frame (registered voters) is known. Disadvantage: It is time consuming to carry out stratified sampling Quota Sampling: Advantage: The sample of voters can be obtained quickly Disadvantage: The sample of voters obtained may not be a good representative of the electoral division
6	$\mathrm{P}(X=r-1)=\mathrm{P}(X=r)$ $\binom{n}{r-1}p^{r-1}(1-p)^{n-r+1}=\binom{n}{r}p^{r}(1-p)^{n-r}$ $\frac{n!}{(r-1)!(n-r+1)!}p^{r-1}(1-p)^{n-r+1}=\frac{n!}{(r)!(n-r)!}p^{r}(1-p)^{n-r}$ $\frac{r!(n-r)!}{(r-1)!(n-r+1)!}=\frac{p^{r}(1-p)^{n-r}}{p^{r-1}(1-p)^{n-r+1}}$ $\frac{r}{n-r+1}=\frac{p}{1-p}$ $\frac{\mathrm{P}(X=r)}{\mathrm{P}(X=r-1)}=\left(\frac{n-r+1}{r}\right)\frac{p}{1-p}$ (shown)

	$r(1-p)=p(n-r+1)$ $r-pr=np-pr+p$ $r=(n+1)p$ X will have two modes when $(n+1)p$ is a positive integer. $r=(n+1)p$, $r=(n+1)p-1$
7 (i)	No of ways that the single women are all separated $= {}^{6}C_{4}\times 4!\ \times(6-1)!$ =43200 Probability = $\dfrac{43200}{9!}=\dfrac{5}{42}=0.119$
(ii)	Probability that the single women are next to one another $=\ \mathrm{P}(S)=\dfrac{(7-1)!\ \times\ 4!}{9!}=\dfrac{1}{21}$ Probability that the single men are next to each other $=\ \mathrm{P}(B)=\dfrac{(9-1)!\ \times\ 2!}{9!}=\dfrac{2}{9}$ Probability that the single women are next to one another and the single men are next to each other $=\ \mathrm{P}(S\cap B)=\dfrac{(6-1)!\ \times\ 2!\ \times\ 4!}{9!}=\dfrac{1}{63}$ Therefore probability = $\mathrm{P}(S)+\mathrm{P}(B)-2\mathrm{P}(S\cap B)=\dfrac{1}{21}+\dfrac{2}{9}-\dfrac{2}{63}=\dfrac{5}{21}=0.238$
	No of ways = $9!\times 2!=725760$
8 (i)	$A, B\sim \mathrm{N}(71,8^2)$ $A-B\sim \mathrm{N}(0,128)$ $\mathrm{P}(0\le A-B\le 2)=0.0702$ (3.s.f)
(ii)	$X\ \sim \mathrm{N}(62,\sigma^2)$, $Y\sim \mathrm{N}(71,8^2)$ Let $M=\dfrac{X+Y}{2}\sim \mathrm{N}(66.5,\ \dfrac{\sigma^2+8^2}{4})$ $\mathrm{P}(M\ge 75)=0.15$ $\mathrm{P}(M\le 75)=0.85$ $\mathrm{P}(Z\le \dfrac{75-66.5}{\sqrt{\dfrac{\sigma^2+64}{4}}})=0.85$ $\dfrac{8.5}{\sqrt{\dfrac{\sigma^2+64}{4}}}=1.03643338$ $\sigma=14.319$ $\sigma=14.3$
(iii)	$X-Y\sim \mathrm{N}(-9,269.0389)$ $\mathrm{P}(X>Y)=\mathrm{P}(X-Y>0)$ $=0.292$ (3 s.f.)
(iv)	Not valid because X and Y are not be independent for the same student.

9(a)	Let X be the total number of speeding incidents caught at Junctions A and B in an hour. $X \sim \text{Po}(\frac{23}{12})$ $\text{P}(X \geq 2) = 1 - \text{P}(X \leq 1) = 0.571$
(b)	$A+B+C \sim \text{Po}\left(\frac{23}{12}+\lambda\right)$ Required Probability $= \text{P}\left(C \geq 1 \mid A+B+C=2\right)$ $= \frac{\text{P}(A+B=1)\text{P}(C=1)+\text{P}(A+B=0)\text{P}(C=2)}{\text{P}(A+B+C=2)}$ $= \frac{\left(\frac{23}{12}\text{e}^{-\frac{23}{12}}\right)\left(\lambda \text{e}^{-\lambda}\right)+\left(\text{e}^{-\frac{23}{12}}\right)\left(\text{e}^{-\lambda}\frac{\lambda^2}{2}\right)}{\left(\text{e}^{-\frac{23}{12}-\lambda}\right)\frac{\left(\frac{23}{12}+\lambda\right)^2}{2}}$ $= \frac{\frac{\text{e}^{-\frac{23}{12}-\lambda}}{2}\left(\frac{23}{6}\lambda+\lambda^2\right)}{\frac{\text{e}^{-\frac{23}{12}-\lambda}}{2}\left(\frac{23}{12}+\lambda\right)^2}$ $= \frac{\frac{\lambda}{6}(23+6\lambda)}{\frac{1}{144}(23+12\lambda)^2}$ $= \frac{24\lambda(23+6\lambda)}{(23+12\lambda)^2}$
(c)	Let A be the number of speeding incidents caught at Junctions A, and B be the number of speeding incidents caught at Junction B in a day $A \sim \text{Po}(16)$, $B \sim \text{Po}(30)$ Since both 16 and 30 are greater than 10, $A \sim \text{N}(16,16)$ and $B \sim \text{N}(30,30)$ approximately $\Rightarrow A-B \sim \text{N}(-14, 46)$ $\text{P}(A-B>0) \xrightarrow{c.c} \text{P}(A-B>0.5)$ = 0.0163 The occurrence of speeding incidents caught at Junction A and Junction B are independent of each other.
10 (i)	Since n is small and population variance unknown, the nutritionist should use t-test. It is assumed that the calories count of the energy bar follows a normal distribution.

(ii)	$s^2 = \frac{15}{14}(20.74) = 22.221$ Let H_0 be the null hypothesis, H_1 be the alternative hypothesis. Let μ be the population mean number of calories in an energy bar and $\bar{X}$ be the sample mean. $\mathrm{H}_0 : \mu = 350$ $\mathrm{H}_1 : \mu \neq 350$ Under H_0, Test statistic, $T = \dfrac{\bar{X} - 350}{\sqrt{\dfrac{22.221}{15}}} \sim t_{14}$ *p* value = 0.0373 < 0.05 Since the *p* – value < 0.05, reject H_0. There is sufficient evidence at 5% level of significance to conclude that the mean number of calories in an energy bar is not 350.
(iii)	Since H_0 is rejected for *t*-test, $p_t < 0.05$ and since $p_z < p_t < 0.05$. Therefore H_0 will be rejected under *z* test. So the conclusion will not be different. Alternative method  H_0 is rejected under t test $\Rightarrow$ test statistic is inside critical region for t-test $\Rightarrow$ test statistic is inside critical region for Z-test $\Rightarrow$ H_0 is rejected under Z test The conclusion would be the same.
(iv)	Since H_0 is not rejected, $\left\lvert \dfrac{\bar{x} - 350}{\sqrt{1.3827}} \right\rvert < 1.95996$ $\Rightarrow -1.95996 < \dfrac{\bar{x} - 350}{\sqrt{1.3827}} < 1.95996$ $-2.3047 < \bar{x} - 350 < 2.3047$ $347.7 < \bar{x} < 352.3$
11(i)	$\bar{x} = 750$ $\bar{y} = 0.01758(750) + 9.018$ $\quad = 22.203$ $22.203(10) = k + 199$ $\therefore k = 23.03 \approx 23$

(ii)	
(iii)	$r = 0.958$. Though r is close to 1, the shape of the scatter plot is curvilinear therefore a linear model is not appropriate
(iv)	  Model B is a more appropriate model as graph concave downwards like the scatter diagram $\ln y = -0.96684 + 0.61722 \ln x$ $\ln y = -0.967 + 0.617 \ln x$ $\ln(24) = -0.96684 + 0.61722 \ln x$ $0.61722 \ln x = \ln(24) + 0.96684$ $x = 825$ Since $r = 0.982$ is close to 1 and $y = 24$ is within the data range, the prediction is appropriate