2016 HCI Prelim Paper 1 Solutions

Qn	Solution
1	* Let P_n be statement $U_n = \sin(nx)$ for all $n \in \mathbb{Z}^+$. When $n=1$, LHS $= U_1 = \sin x$, RHS $= \sin x$ $\therefore P_1$ is true. * Assume P_k is true for some $k \in \mathbb{Z}^+$, i.e. $U_k = \sin(kx)$. Want to prove that P_{k+1} is true, i.e. $U_{k+1} = \sin(k+1)x$. LHS $= U_{k+1}$ $= U_k + 2\cos\frac{(2k+1)x}{2}\sin\frac{x}{2}$ $= \sin(kx) + 2\cos\left(\frac{2k+1}{2}\right)x\sin\left(\frac{1}{2}\right)x$ $= \sin(kx) + \sin(k+1)x - \sin(kx)$ $= \sin(k+1)x =$ RHS *Since P_1 is true, P_k is true implies P_{k+1} is true, by MI P_n is true for all $n \in \mathbb{Z}^+$.
2	$\frac{2}{4(x+1)^2+1} > 1$ $\frac{-(2x+1)(2x+3)}{4(x+1)^2+1} > 0$ Since $4(x+1)^2+1>0$ for all x, $(2x+1)(2x+3)<0$ $\therefore -\frac{3}{2} < x < -\frac{1}{2}$ $\int_{-1}^{\frac{\sqrt{3}}{2}-1}\left\lvert 1-\frac{2}{4(x+1)^2+1}\right\rvert \mathrm{d}x$ $= \int_{-1}^{-\frac{1}{2}}\left(-1+\frac{2}{4(x+1)^2+1}\right)\mathrm{d}x + \int_{-\frac{1}{2}}^{\frac{\sqrt{3}}{2}-1}\left(1-\frac{2}{4(x+1)^2+1}\right)\mathrm{d}x$ $= \left[-x+\tan^{-1}(2x+2)\right]_{-1}^{-\frac{1}{2}} + \left[x-\tan^{-1}(2x+2)\right]_{-\frac{1}{2}}^{\frac{\sqrt{3}}{2}-1}$ $= \left[\frac{1}{2}+\tan^{-1}1-1\right] + \left[\frac{\sqrt{3}}{2}-1-\tan^{-1}\sqrt{3}+\frac{1}{2}+\tan^{-1}1\right]$ $= \frac{\pi}{6}+\frac{\sqrt{3}}{2}-1$

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(a)

$$\overrightarrow{OP} = \underset{\sim}{a} + 3\overrightarrow{AB} = \underset{\sim}{a} + 3\left(\underset{\sim}{b} - \underset{\sim}{a}\right) = 3\underset{\sim}{b} - 2\underset{\sim}{a}$$

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = 2\underset{\sim}{a} - \left(3\underset{\sim}{b} - 2\underset{\sim}{a}\right) = 4\underset{\sim}{a} - 3\underset{\sim}{b}$$

$$l_{PQ} : \underset{\sim}{r} = 2\underset{\sim}{a} + \lambda\left(4\underset{\sim}{a} - 3\underset{\sim}{b}\right),\ \lambda \in \mathbb{R}$$

$$l_{OB} : \underset{\sim}{r} = \mu\underset{\sim}{b},\ \mu \in \mathbb{R}$$

At point of intersection, $2\underset{\sim}{a} + \lambda\left(4\underset{\sim}{a} - 3\underset{\sim}{b}\right) = \mu\underset{\sim}{b}$

Comparing coefficients of $\underset{\sim}{a}$ and $\underset{\sim}{b}$, $\lambda = -\frac{1}{2},\ \mu = \frac{3}{2}$

$\therefore$ position vector of the point of intersection $= \frac{3}{2}\underset{\sim}{b}$

(b)

$$\underset{\sim}{a} \times \underset{\sim}{b} = \begin{pmatrix} 2 \\ -4 \\ 2 \end{pmatrix} = 2\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \Rightarrow \underset{\sim}{n} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

Let F be the foot of perpendicular.

Method 1

$$l_{FC} : \underset{\sim}{r} = \begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix} + s\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix},\ s \in \mathbb{R} \quad , \ \Pi_{OAB} : \underset{\sim}{r} \cdot \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = 0$$

$$\left[\begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix} + s\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}\right] \cdot \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = 0$$

$$-6 + 4 + s\left(1 + 4 + 1\right) = 0$$

$$s = \frac{1}{3}$$

$$\therefore \overrightarrow{OF} = \begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 0+1 \\ 9-2 \\ 12+1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 \\ 7 \\ 13 \end{pmatrix}$$

$$\therefore F\left(\frac{1}{3}, \frac{7}{3}, \frac{13}{3}\right)$$

	<u>Method 2</u> $\overrightarrow{FC}=\left(\overrightarrow{OC}\cdot\hat{\underset{\sim}{n}}\right)\hat{\underset{\sim}{n}}$ $=\left(\begin{pmatrix}0\\3\\4\end{pmatrix}\cdot\frac{\begin{pmatrix}1\\-2\\1\end{pmatrix}}{\left\|\begin{pmatrix}1\\-2\\1\end{pmatrix}\right\|}\right)\frac{\begin{pmatrix}1\\-2\\1\end{pmatrix}}{\left\|\begin{pmatrix}1\\-2\\1\end{pmatrix}\right\|}$ $=\frac{-6+4}{\sqrt{6}}\frac{\begin{pmatrix}1\\-2\\1\end{pmatrix}}{\sqrt{6}}$ $=-\frac{1}{3}\begin{pmatrix}1\\-2\\1\end{pmatrix}$ $\overrightarrow{OF}=\overrightarrow{OC}+\overrightarrow{CF}$ $=\begin{pmatrix}0\\3\\4\end{pmatrix}+\frac{1}{3}\begin{pmatrix}1\\-2\\1\end{pmatrix}=\frac{1}{3}\begin{pmatrix}0+1\\9-2\\12+1\end{pmatrix}=\frac{1}{3}\begin{pmatrix}1\\7\\13\end{pmatrix}$ $\therefore F\left(\frac{1}{3},\frac{7}{3},\frac{13}{3}\right)$
4	<u>Method 1</u> $\frac{2n+1}{\sqrt{n^2+2n}+\sqrt{n^2-1}}=\frac{2n+1}{\sqrt{n^2+2n}+\sqrt{n^2-1}}\times\frac{\sqrt{n^2+2n}+\sqrt{n^2-1}}{\sqrt{n^2+2n}+\sqrt{n^2-1}}$ $=\frac{(2n+1)\left(\sqrt{n^2+2n}-\sqrt{n^2-1}\right)}{\left(n^2+2n\right)-\left(n^2-1\right)}$ $=\sqrt{n^2+2n}-\sqrt{n^2-1}$

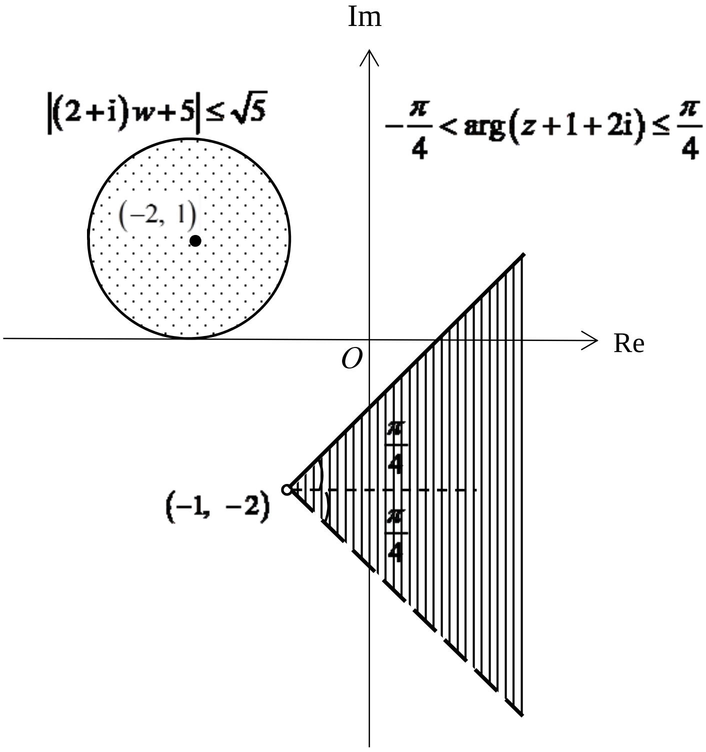
	<u>Method 2</u> $\left(\sqrt{n^2+2n}-\sqrt{n^2-1}\right)\left(\sqrt{n^2+2n}+\sqrt{n^2-1}\right)$ $=\left(n^2+2n-\left(n^2-1\right)\right)$ $=2n+1$ $\therefore \dfrac{2n+1}{\sqrt{n^2+2n}+\sqrt{n^2-1}}=\sqrt{n^2+2n}-\sqrt{n^2-1}$
	$\displaystyle\sum_{n=1}^{N}\frac{2n+1}{\sqrt{n^2+2n}+\sqrt{n^2-1}}$ $\displaystyle=\sum_{n=1}^{N}\left(\sqrt{n^2+2n}-\sqrt{n^2-1}\right)$ $=\begin{bmatrix} \sqrt{3}-\sqrt{0} \\ +\sqrt{8}-\sqrt{3} \\ \cdots \\ +\sqrt{N^2+2N}-\sqrt{N^2-1} \end{bmatrix}$ $=\sqrt{N^2+2N}$
(a)	Replace n by $n+1$, $\displaystyle\sum_{n=2}^{N}\frac{2n-1}{\sqrt{n^2-1}+\sqrt{n(n-2)}}$ $\displaystyle=\sum_{n=1}^{N-1}\frac{2n+1}{\sqrt{n^2+2n}+\sqrt{n^2-1}}$ $=\sqrt{(N-1)^2+2(N-1)}$ $=\sqrt{N^2-1}$
(b)	Notice that $\sqrt{n^2+2n}>n$ and $\left(\sqrt{n^2-1}\right)^2-(n-1)^2=2n-2\geq 0$. $\Rightarrow \sqrt{n^2-1}\geq n-1$ $\Rightarrow \sqrt{n^2+2n}+\sqrt{n^2-1}>2n-1$ $\Rightarrow \dfrac{1}{\sqrt{n^2+2n}+\sqrt{n^2-1}}<\dfrac{1}{2n-1}$ $\displaystyle\therefore \sum_{n=1}^{N}\frac{2n+1}{2n-1}>\sum_{n=1}^{N}\frac{2n+1}{\sqrt{n^2+2n}+\sqrt{n^2-1}}=\sqrt{N^2+2N}$

5

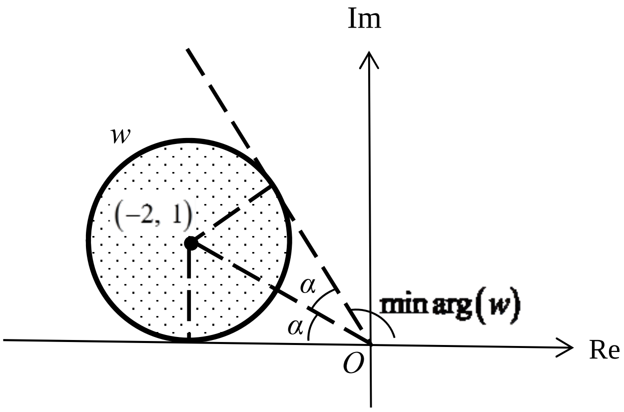
$$\left|(2+\mathrm{i})w+5\right| \le \sqrt{5}$$

$$\left|2+\mathrm{i}\right|\left|w+\frac{5}{2+\mathrm{i}}\right| \le \sqrt{5}$$

$\left|w+2-\mathrm{i}\right| \le 1 \Rightarrow$ circle centre $(-2,\ 1)$, radius 1



(i)



$$\alpha = \sin^{-1}\frac{1}{\sqrt{(-2)^2+1^2}} = \sin^{-1}\frac{1}{\sqrt{5}}$$

$$\min\arg(w) = \pi - 2\alpha = 2.2143 = 2.21(3\text{sf})$$

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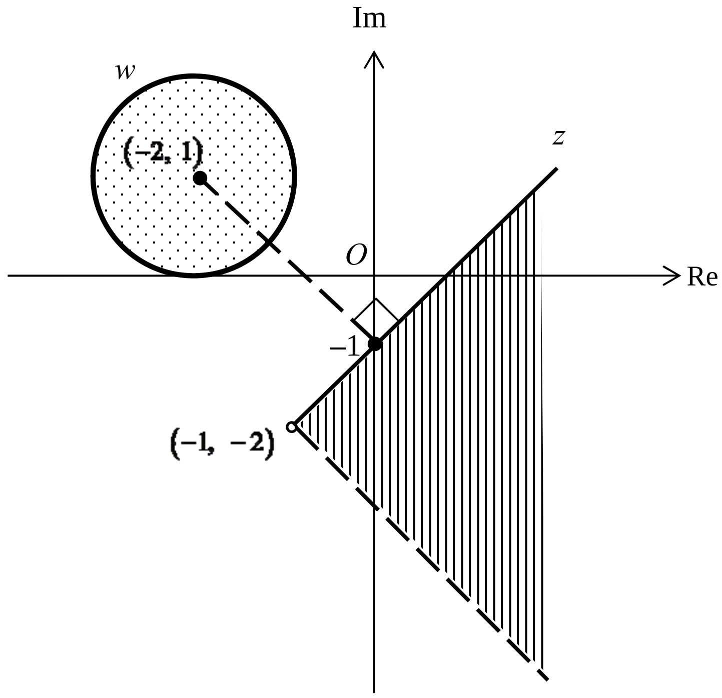

$\min|z-w| = 2\sqrt{2}-1$
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<td>(iii)</td>
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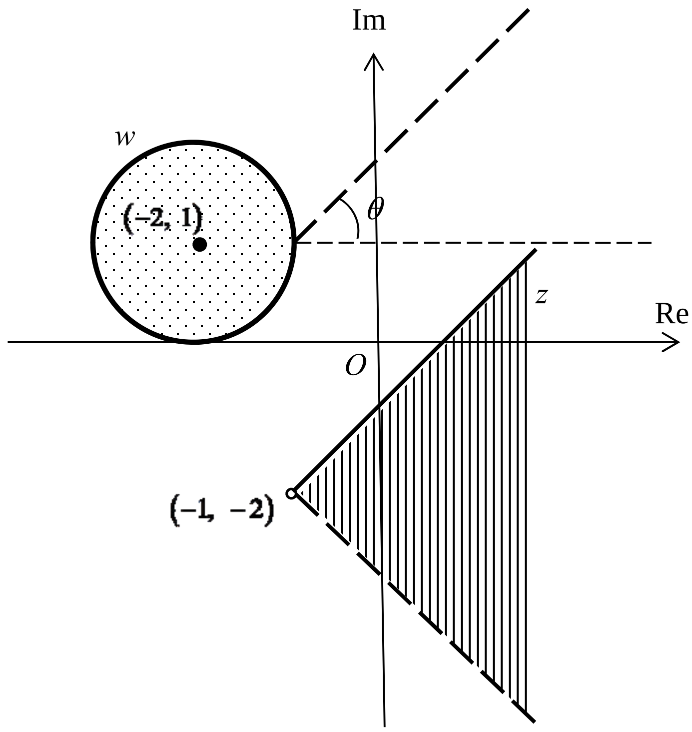

$\theta = \dfrac{\pi}{4}$
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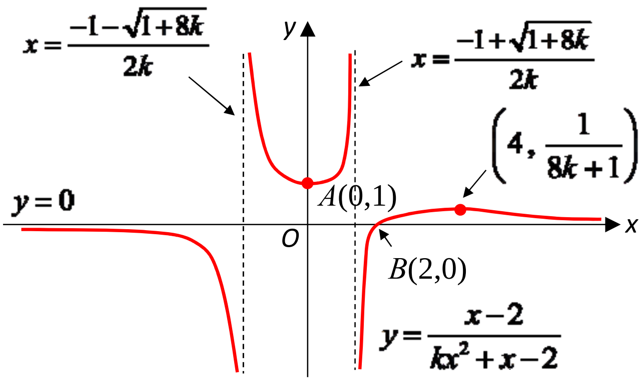
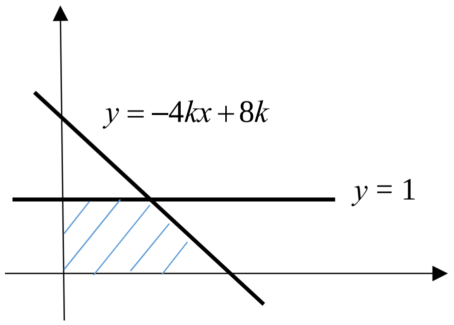
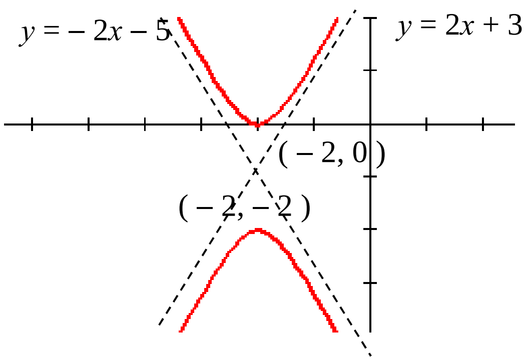
6 (i)	$\overrightarrow{OD}=\begin{pmatrix}0\\0\\2\end{pmatrix},\ \overrightarrow{OE}=\begin{pmatrix}3\\1.5\\2\end{pmatrix}$ $\overrightarrow{DE}=\begin{pmatrix}3\\1.5\\2\end{pmatrix}-\begin{pmatrix}0\\0\\2\end{pmatrix}=\begin{pmatrix}3\\1.5\\0\end{pmatrix}=1.5\begin{pmatrix}2\\1\\0\end{pmatrix}$ $l_{DE}:\underset{\sim}{r}=\begin{pmatrix}0\\0\\2\end{pmatrix}+\lambda\begin{pmatrix}2\\1\\0\end{pmatrix},\ \lambda\in\mathbb{R}$
(ii)	$\overrightarrow{AD}=\begin{pmatrix}0\\0\\2\end{pmatrix}-\begin{pmatrix}3\\0\\0\end{pmatrix}=\begin{pmatrix}-3\\0\\2\end{pmatrix}$ $\overrightarrow{DE}\times\overrightarrow{AD}=1.5\begin{pmatrix}2\\1\\0\end{pmatrix}\times\begin{pmatrix}-3\\0\\2\end{pmatrix}=1.5\begin{pmatrix}2\\-4\\3\end{pmatrix}\Rightarrow\underset{\sim}{n}=\begin{pmatrix}2\\-4\\3\end{pmatrix}$ $\Pi_{ADE}:\underset{\sim}{r}\cdot\begin{pmatrix}2\\-4\\3\end{pmatrix}=\begin{pmatrix}0\\0\\2\end{pmatrix}\cdot\begin{pmatrix}2\\-4\\3\end{pmatrix}=6$ $\therefore 2x-4y+3z=6$
(iii)	$\underset{\sim}{n}_{OABC}=\begin{pmatrix}0\\0\\1\end{pmatrix},\ \underset{\sim}{n}_{ADE}=\begin{pmatrix}2\\-4\\3\end{pmatrix}$ angle between planes $=\cos^{-1}\dfrac{\left\|\begin{pmatrix}0\\0\\1\end{pmatrix}\cdot\begin{pmatrix}2\\-4\\3\end{pmatrix}\right\|}{\left\|\begin{pmatrix}0\\0\\1\end{pmatrix}\right\|\left\|\begin{pmatrix}2\\-4\\3\end{pmatrix}\right\|}$ $=\cos^{-1}\dfrac{3}{\sqrt{4+16+9}}$ $=\cos^{-1}\dfrac{3}{\sqrt{29}}$ $=56.1°(1\text{dp})$

	Angle $= 180° - 2\cos^{-1}\dfrac{3}{\sqrt{29}}$ $= 67.7°$ (1 d.p.)
7 (i)	$\dfrac{dy}{dx} = \dfrac{(kx^2+x-2)-(x-2)(2kx+1)}{(kx^2+x-2)^2} = \dfrac{-kx^2+4kx}{(kx^2+x-2)^2}$ When $x=0$, $\dfrac{dy}{dx} = \dfrac{0}{(-2)^2} = 0$ and $y = \dfrac{-2}{-2} = 1$ Hence required equation of tangent is $y=1$.
(ii)	For axial intercepts, when $y=0$, $x=2$. when $x=0$, $y=1$. For vertical asymptotes, $kx^2+x-2=0$ $\therefore\ x = \dfrac{-1\pm\sqrt{1+8k}}{2k}$ For turning points, $\dfrac{dy}{dx}=0$ $-kx^2+4kx=0$ $-kx(x-4)=0$ $\therefore\ x=0$ or $x=4$ 
(iii)	At $x=2$, $\dfrac{dy}{dx} = \dfrac{-4k+8k}{(4k)^2} = \dfrac{4k}{16k^2} = \dfrac{1}{4k}$ $\therefore$ gradient of normal $=-4k$ Hence required equation of normal is $y-0=-4k(x-2)$ $y=-4kx+8k$

(iv)	 When $y=1$, $1=-4kx+8k \Rightarrow x=\dfrac{8k-1}{4k}$ $\therefore$ required area $=\dfrac{1}{2}\left(\dfrac{8k-1}{4k}+2\right)(1)$ $=\dfrac{16k-1}{8k}$ $=2-\dfrac{1}{8k}$ $>2-\dfrac{1}{8}$ (since $k>1$) $>\dfrac{15}{8}$
8 (i)	$\text{Area}=2\int_0^{\pi} x^2 \sin x \,\mathrm{d}x$ $=2\left[\left[-x^2\cos x\right]_0^{\pi}+\int_0^{\pi} 2x\cos x\,\mathrm{d}x\right]$ $=2\left[\pi^2+2(\left[x\sin x\right]_0^{\pi}-\int_0^{\pi}\sin x\,\mathrm{d}x)\right]$ $=2\left[\pi^2+2\left[\cos x\right]_0^{\pi}\right]$ $=2(\pi^2-4)$ units^2
(ii)	

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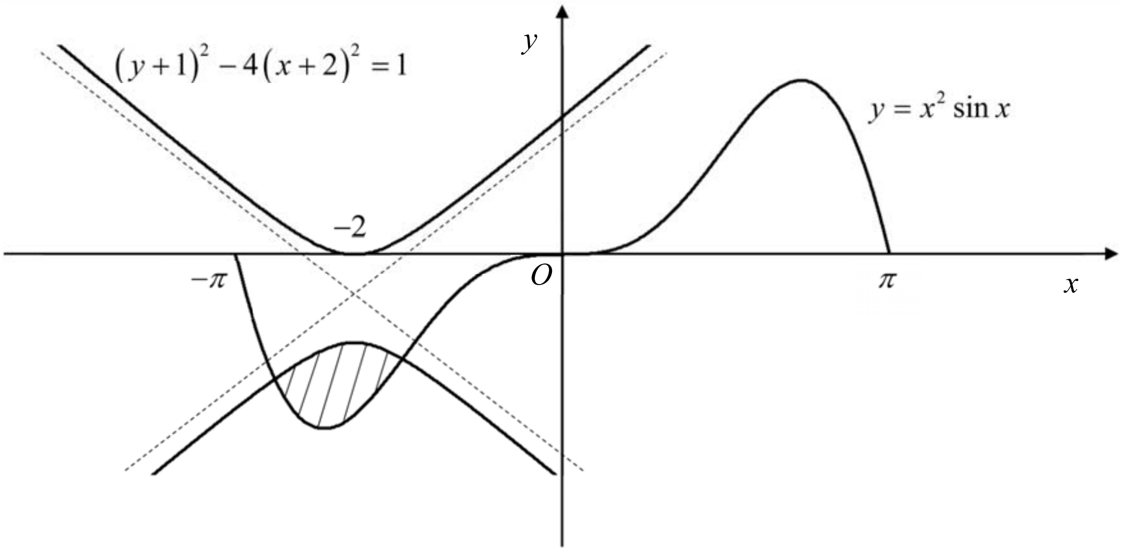
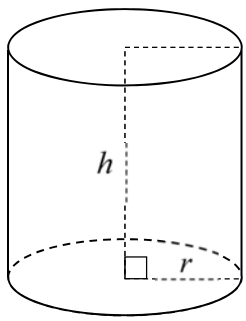
<tr><td>(iii)</td><td>
</td></tr>
<tr><td></td><td>Coordinates of the points of intersections of the 2 curves are $(-1.5374,-2.3623)$ and $(-2.7626,-2.8238)$.

Volume of solid generated
$=\pi\int_{-2.7626}^{-1.5374}(x^2\sin x)^2\,\mathrm{d}x-\pi\int_{-2.7626}^{-1.5374}(-1-\sqrt{1+4(x+2)^2})^2\,\mathrm{d}x=26.8$ units^3</td></tr>
<tr><td>9
(i)</td><td>Let A cm^2 be the surface area of the cylindrical container.
Let r cm and h cm be the radius and height of the cylindrical container respectively.



Volume $=\pi r^2 h=k$
$\therefore\ h=\dfrac{k}{\pi r^2}$
$A=2\pi rh+\pi r^2$
$\quad=2\pi r\left(\dfrac{k}{\pi r^2}\right)+\pi r^2$
$\quad=\dfrac{2k}{r}+\pi r^2$

Can also express r in terms of h and find A in terms of h, then let $\dfrac{\mathrm{d}A}{\mathrm{d}h}=0$ to obtain h and subsequently r.

Hence $\dfrac{\mathrm{d}A}{\mathrm{d}r}=-\dfrac{2k}{r^2}+2\pi r$
When $\dfrac{\mathrm{d}A}{\mathrm{d}r}=0$,</td></tr>
</table>

	$-\frac{2k}{r^2}+2\pi r=0$ $r^3=\frac{k}{\pi}$ $r=\sqrt[3]{\frac{k}{\pi}}$ $\therefore\ h=\frac{k}{\pi r^2}=\frac{k}{\pi\left[\left(\frac{k}{\pi}\right)^{\frac{1}{3}}\right]^2}=\sqrt[3]{\frac{k}{\pi}}$ Hence $h:r=\sqrt[3]{\frac{k}{\pi}}:\sqrt[3]{\frac{k}{\pi}}=1:1$ (shown) $\frac{\mathrm{d}^2A}{\mathrm{d}r^2}=\frac{4k}{r^3}+2\pi>0$ since $p>0$ and $k>0$ Hence A is a minimum when $r=\sqrt[3]{\frac{k}{\pi}}$
(ii)	From (i), $h:r=1:1$ Hence $A=2\pi rh+\pi r^2=2\pi r(r)+\pi r^2=3\pi r^2$ For new design, $h:r=5:2$ Hence new $A=2\pi rh+\pi r^2=2\pi r\left(\frac{5}{2}r\right)+\pi r^2=6\pi r^2$ $\therefore$ required ratio is $6\pi r^2:3\pi r^2=2:1$
(b)	<u>Method 1</u> Let V cm^3 be the volume of the cylindrical container. $V=\pi r^2h=\pi r^2(\frac{5}{2}r)=\frac{5}{2}\pi r^3$ $A=2\pi rh+\pi r^2=2\pi r\left(\frac{5}{2}r\right)+\pi r^2=6\pi r^2$ $\frac{\mathrm{d}V}{\mathrm{d}r}=\frac{15}{2}\pi r^2$ Can also find $\frac{\mathrm{d}V}{\mathrm{d}h}$ and $\frac{\mathrm{d}A}{\mathrm{d}h}$ and use $\frac{\mathrm{d}A}{\mathrm{d}t}=\frac{\mathrm{d}A}{\mathrm{d}h}\times\frac{\mathrm{d}h}{\mathrm{d}V}\times\frac{\mathrm{d}V}{\mathrm{d}t}$ $\frac{\mathrm{d}A}{\mathrm{d}r}=12\pi r$ $\frac{\mathrm{d}A}{\mathrm{d}t}=\frac{\mathrm{d}A}{\mathrm{d}r}\times\frac{\mathrm{d}r}{\mathrm{d}V}\times\frac{\mathrm{d}V}{\mathrm{d}t}$ $=12\pi r\times\frac{2}{15\pi r^2}\times 80$ $=\frac{128}{r}$ When $h=50$, $r=\frac{2}{5}(50)=20$

	Hence $\frac{\mathrm{d}A}{\mathrm{d}t}=\frac{128}{20}=6.4\ \text{ cm}^2/\text{s}$ <u>Method 2</u> $A=6\pi r^2 \quad \therefore\ r=\sqrt{\frac{A}{6\pi}}$ (reject $r=-\sqrt{\frac{A}{6\pi}}$ since $r\ge 0$) Hence $V=\pi r^2 h=\pi r^2\left(\frac{5}{2}r\right)=\frac{5}{2}\pi r^3$ $=\frac{5}{2}\pi\left(\sqrt{\frac{A}{6\pi}}\right)^3$ $=\frac{5A^{\frac{3}{2}}}{2(6)^{\frac{3}{2}}\pi^{\frac{1}{2}}}$ $\frac{\mathrm{d}V}{\mathrm{d}A}=\frac{15A^{\frac{1}{2}}}{4(6)^{\frac{3}{2}}\pi^{\frac{1}{2}}}$ When $h=50$, $r=\frac{2}{5}(50)=20$ $\therefore\ A=6\pi(20)^2=2400\pi$ Hence $\frac{\mathrm{d}A}{\mathrm{d}t}=\frac{\mathrm{d}A}{\mathrm{d}V}\times\frac{\mathrm{d}V}{\mathrm{d}t}$ $=\frac{4(6)^{\frac{3}{2}}\pi^{\frac{1}{2}}}{15A^{\frac{1}{2}}}\times 80$ $=\frac{4(6)^{\frac{3}{2}}\pi^{\frac{1}{2}}}{15(2400\pi)^{\frac{1}{2}}}\times 80$ $=6.4\quad \text{cm}^2/\text{s}$
10 (a) (i)	$\frac{\sin\theta}{x}=\frac{\sin\left(\pi-\frac{\pi}{6}-\theta\right)}{y}$ $\frac{x}{y}=\frac{\sin\theta}{\sin\left(\frac{5\pi}{6}-\theta\right)}$ $\frac{x}{y}=\frac{\sin\theta}{\sin\frac{5\pi}{6}\cos\theta-\sin\theta\cos\frac{5\pi}{6}}$ $\frac{x}{y}=\frac{\sin\theta}{\frac{1}{2}\cos\theta+\frac{\sqrt{3}}{2}\sin\theta}=\frac{2\sin\theta}{\cos\theta+\sqrt{3}\sin\theta}$ (shown)

(a) (ii)	$\frac{x}{y}=\frac{2\sin\theta}{\cos\theta+\sqrt{3}\sin\theta}$ $\frac{x}{y}=\frac{2\left(\theta-\frac{\theta^3}{3!}+...\right)}{1+\sqrt{3}\theta-\frac{\theta^2}{2}+...}$ $\frac{x}{y}\approx 2\left(\theta-\frac{\theta^3}{3!}\right)\left(1+\left(\sqrt{3}\theta-\frac{\theta^2}{2}\right)\right)^{-1}$ $\frac{x}{y}\approx 2\left(\theta-\frac{\theta^3}{3!}\right)\left(\begin{array}{l}1+(-1)\left(\sqrt{3}\theta-\frac{\theta^2}{2}\right)\\ +\frac{(-1)(-2)}{2!}\left(\sqrt{3}\theta-\frac{\theta^2}{2}\right)^2\end{array}\right)$ $\frac{x}{y}\approx 2\left(\theta-\frac{\theta^3}{3!}\right)\left(1-\sqrt{3}\theta+\frac{\theta^2}{2}+3\theta^2\right)$ $\frac{x}{y}\approx 2\theta-2\sqrt{3}\theta^2+\frac{20}{3}\theta^3$
(b) (i)	Using sine rule, $\frac{\sin\theta}{x}=\frac{\sin\frac{\pi}{6}}{\frac{1}{6}}=3 \quad \therefore \theta=\sin^{-1}3x$
(b) (ii)	<u>Method 1</u> $\sin\theta=3x$ $\cos\theta\frac{\mathrm{d}\theta}{\mathrm{d}x}=3$ ---- (1) $\cos\theta\frac{\mathrm{d}^2\theta}{\mathrm{d}x^2}-\sin\theta\left(\frac{\mathrm{d}\theta}{\mathrm{d}x}\right)^2=0$ --- (2) $\cos\theta\frac{\mathrm{d}^3\theta}{\mathrm{d}x^3}-\sin\theta\frac{\mathrm{d}\theta}{\mathrm{d}x}\frac{\mathrm{d}^2\theta}{\mathrm{d}x^2}-2\sin\theta\frac{\mathrm{d}\theta}{\mathrm{d}x}\frac{\mathrm{d}^2\theta}{\mathrm{d}x^2}-\cos\theta\left(\frac{\mathrm{d}\theta}{\mathrm{d}x}\right)^3$ $=0---(3)$ When $x=0$, $\theta=0,\ \frac{\mathrm{d}\theta}{\mathrm{d}x}=3,\ \frac{\mathrm{d}^2\theta}{\mathrm{d}x^2}=0,\ \frac{\mathrm{d}^3\theta}{\mathrm{d}x^3}=27$ $\theta=3x+\frac{27}{3!}x^3+...=3x+\frac{9}{2}x^3+...$

Method 2

$\theta = \sin^{-1}(3x)$

$\frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{3}{\sqrt{1-9x^2}} = 3\left(1-9x^2\right)^{-\frac{1}{2}}$

$\frac{\mathrm{d}^2\theta}{\mathrm{d}x^2} = 3\left(-\frac{1}{2}\right)\left(1-9x^2\right)^{-\frac{3}{2}}(-18x) = 27x\left(1-9x^2\right)^{-\frac{3}{2}}$

$\frac{\mathrm{d}^3\theta}{\mathrm{d}x^3} = -\frac{81}{2}x(-18x)\left(1-9x^2\right)^{-\frac{5}{2}} + 27\left(1-9x^2\right)^{-\frac{3}{2}}$

$= 729x^2\left(1-9x^2\right)^{-\frac{5}{2}} + 27\left(1-9x^2\right)^{-\frac{3}{2}}$

When $x = 0$,

$\theta = 0$, $\frac{\mathrm{d}\theta}{\mathrm{d}x} = 3$, $\frac{\mathrm{d}^2\theta}{\mathrm{d}x^2} = 0$, $\frac{\mathrm{d}^3\theta}{\mathrm{d}x^3} = 27$

$\theta = 3x + \frac{27}{3!}x^3 + \ldots = 3x + \frac{9}{2}x^3 + \ldots$

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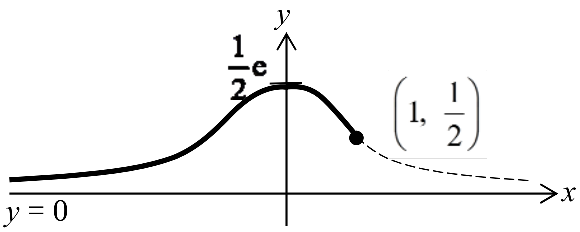
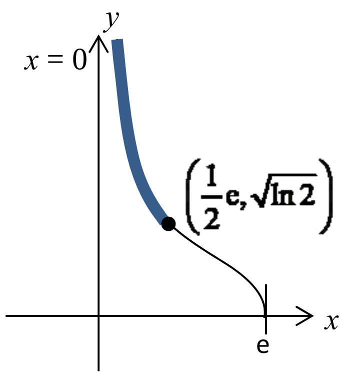
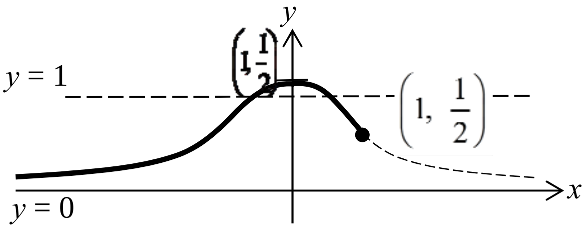
(i)



Since $R_\mathrm{f} = \left(0, \frac{1}{2}\mathrm{e}\right] \subseteq D_\mathrm{g} = (0, \mathrm{e}]$, $R_\mathrm{f} \subseteq D_\mathrm{g}$ and gf exists.



$R_\mathrm{gf} = \left[\sqrt{\ln 2}, \infty\right)$

(ii)	 Since a horizontal line y = 1 cuts the graph of y = f(x) twice, f is not a one-to-one function and f^{-1} does not exist.
(iii)	$b=0$ Let $y=\mathrm{f}(x)$ $y=\frac{1}{2}\mathrm{e}^{1-x^2}$ $\ln(2y)=1-x^2$ $x=\pm\sqrt{1-\ln(2y)}$ Since $x\le 0,\ x=-\sqrt{1-\ln(2y)}$ $\mathrm{f}^{-1}:x\mapsto -\sqrt{1-\ln(2x)},\ x\in\mathbb{R},\ 0<x\le\frac{1}{2}\mathrm{e}$
(iv)	$y=\sqrt{1-\ln x}\xrightarrow{\text{Step 1}}y=\sqrt{1-\ln\left(\frac{x}{2}\right)}$ $y=\sqrt{1-\ln\left(\frac{x}{2}\right)}\xrightarrow{\text{Step 2}}y=-\sqrt{1-\ln\left(\frac{x}{2}\right)}$ $0<x\le\mathrm{e}\rightarrow 0<\frac{x}{2}\le\mathrm{e}$ $\therefore 0<x\le 2\mathrm{e}$ $\mathrm{h}:x\mapsto -\sqrt{1-\ln\left(\frac{x}{2}\right)},\ x\in\mathbb{R},\ 0<x\le 2\mathrm{e}$