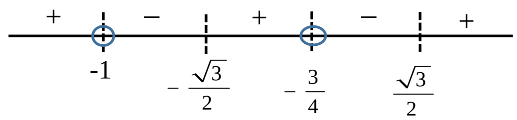
Innova Junior College
H2 Mathematics
JC2 Preliminary Examinations Paper 2
Solutions

1	**Solution**
	$\dfrac{3}{4x+3} \le \dfrac{x}{x+1}$ $\dfrac{3}{4x+3} - \dfrac{x}{x+1} \le 0$ $\dfrac{3x+3-4x^2-3x}{(4x+3)(x+1)} \le 0$ $\dfrac{-4x^2+3}{(4x+3)(x+1)} \le 0 \quad ---(*)$ $\dfrac{4x^2-3}{(4x+3)(x+1)} \ge 0$ $\dfrac{(2x-\sqrt{3})(2x+\sqrt{3})}{(4x+3)(x+1)} \ge 0$  Hence $x < -1$ or $-\dfrac{\sqrt{3}}{2} \le x < -\dfrac{3}{4}$ or $x \ge \dfrac{\sqrt{3}}{2}$
	For $\dfrac{3}{4e^x+3} > \dfrac{e^x}{e^x+1}$, making use of the result in above part, $-1 < e^x < -\dfrac{\sqrt{3}}{2}$ or $-\dfrac{3}{4} < e^x < \dfrac{\sqrt{3}}{2}$ (no solns since e^x is always positive) Hence, $e^x < \dfrac{\sqrt{3}}{2} \Rightarrow x < \ln\left(\dfrac{\sqrt{3}}{2}\right)$

2 **Solution**

(i)

Hour	Start of hour	End of hour
1	1	1 + 3 = 4
2	4	4 + 12 = 16
3	16	16 + 48 = 64
4	…	…

$$4(4)^{n-1} > 200\,000$$
$$(4)^{n-1} > 50000$$
$$n-1 > 7.80482$$
$$n > 8.80842$$

Number of complete hours = 9

<u>Alternative Solution</u>

$$4(4)^{n-1} > 200\,000$$

n	Total
8	65536 < 200 000
9	262144 > 200 000
10	11048576 > 200 000

Number of complete hours = 9

(ii)

Day	Start of day	End of day
1	$200000-x$	$1.02(200000-x)$
2	$1.02(200000-x)$ $-x$	$1.02[1.02(200000-x)$ $-x]$ $=1.02^2(200000)$ $-1.02x-1.02^2x$
3	…	…

At the end of day n, the number comments
$=1.02^n(200000)-\left(1.02x+1.02^2x+...+1.02^nx\right)$ --- (*)

	$=1.02^{n}(200000)-x\left(\dfrac{1.02\left(1.02^{n}-1\right)}{0.02}\right)$ $=1.02^{n}(200000)-51x\left(1.02^{n}-1\right)$
(iii)	$1.02^{30}(200\,000)-51x\left(1.02^{30}-1\right)<0$ $x>\dfrac{1.02^{30}(200\,000)}{51\left(1.02^{30}-1\right)}$ $x\geq 8755$ (to nearest integer)
	Day 1: no. of comments removed $=15000$ Day 2: no. of comments removed $=15000(0.9)$ Day 3: no. of comments removed $=15000(0.9)^{2}$ As $n\to\infty$, no. of comments removed $=\dfrac{15000}{1-0.9}=150\,000$ Software Y is unable to remove all the comments because eventually it is only able to remove 150 000 comments.

3	**Solution**
(i)	Method 1: $\int \frac{1}{x(5-x)}\,\mathrm{d}x = \int \frac{1}{10}\ \mathrm{d}t \quad ---(*)$ Doing partial fractions $\frac{1}{x(5-x)} = \frac{A}{x} + \frac{B}{5-x}$ $= \frac{A(5-x)+B(x)}{x(5-x)}$ $A = \frac{1}{5}$ $B = \frac{1}{5}$ $\int \frac{1}{5x} + \frac{1}{5(5-x)}\,\mathrm{d}P = \int \frac{1}{10}\ \mathrm{d}t$ $\frac{1}{5}\left[\ln\lvert x\rvert - \ln\lvert 5-x\rvert\right] = \frac{1}{10}t + c$ $\ln\left\lvert \frac{x}{5-x} \right\rvert = \frac{1}{2}t + c$ $\frac{x}{5-x} = A\mathrm{e}^{\frac{1}{2}t}$, where $A = \pm\mathrm{e}^{c}$ Given $x = 1$ when $t = 0$, $\frac{1}{5-1} = A\mathrm{e}^{0} \Rightarrow A = \frac{1}{4}$ $x = \frac{5}{4}\mathrm{e}^{\frac{1}{2}t} - \frac{1}{4}x\mathrm{e}^{\frac{1}{2}t}$ $x(4+\mathrm{e}^{\frac{1}{2}t}) = 5\mathrm{e}^{\frac{1}{2}t}$ $x = \frac{5\mathrm{e}^{\frac{1}{2}t}}{4+\mathrm{e}^{\frac{1}{2}t}}$

(i)	Method 2: $$\int \frac{1}{x(5-x)}\,\mathrm{d}x = \int \frac{1}{10}\ \mathrm{d}t \quad ---(*)$$ $$\int \frac{1}{\frac{25}{4}-(x-\frac{5}{2})^2}\,\mathrm{d}x = \int \frac{1}{10}\ \mathrm{d}t$$ $$\frac{1}{2\left(\frac{5}{2}\right)}\ln\left\lvert\frac{\frac{5}{2}+(x-\frac{5}{2})}{\frac{5}{2}-(x-\frac{5}{2})}\right\rvert = \frac{1}{10}t+c$$ $$\ln\left\lvert\frac{x}{5-x}\right\rvert = \frac{1}{2}t+c$$ $$\frac{x}{5-x} = A\mathrm{e}^{\frac{1}{2}t}\ , \text{ where } A=\mathrm{e}^{\pm c}$$ Given $x=1$ when $t=0$, $\frac{1}{5-1}=A\mathrm{e}^{0} \ \Rightarrow\ A=\frac{1}{4}$ $$x=\frac{5}{4}\mathrm{e}^{\frac{1}{2}t}-\frac{1}{4}x\mathrm{e}^{\frac{1}{2}t}$$ $$x(4+\mathrm{e}^{\frac{1}{2}t})=5\mathrm{e}^{\frac{1}{2}t}$$ $$x=\frac{5\mathrm{e}^{\frac{1}{2}t}}{4+\mathrm{e}^{\frac{1}{2}t}} \quad \text{or} \quad x=\frac{5}{4\mathrm{e}^{-\frac{1}{2}t}+1}$$
(ii)	When $x = 2$, $\frac{5\mathrm{e}^{\frac{1}{2}t}}{4+\mathrm{e}^{\frac{1}{2}t}}=2$ $$8+2\mathrm{e}^{\frac{1}{2}t}=5\mathrm{e}^{\frac{1}{2}t}$$ $$3\mathrm{e}^{\frac{1}{2}t}=8$$ $$\mathrm{e}^{\frac{1}{2}t}=\frac{8}{3}$$ $$t=2\ln\left(\frac{8}{3}\right)$$ It takes $t=2\ln\left(\frac{8}{3}\right)$ years.
(iii)	As $t\to\infty$, $x\to 5$. $\therefore$ The population of wild boars will increase and stabilise at 500 eventually.

4	**Solution**
(i)	$l: \ \mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ -7 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}$, where λ is a real parameter. $p: \ \mathbf{r} \bullet \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = 2$ $\sin\theta = \dfrac{\left\| \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\|}{\left\| \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} \right\| \left\| \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\|} = \dfrac{6}{\sqrt{21}\sqrt{2}}$ $\therefore \theta = 67.8^\circ$ (1 dec pl)
(ii)	For the point of intersection between *l* and *p*, $\begin{pmatrix} 1-2\lambda \\ \lambda \\ -7+4\lambda \end{pmatrix} . \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = 2$ $1-2\lambda+7-4\lambda = 2$ $\lambda = 1$ The position vector of point of intersection is $\begin{pmatrix} 1 \\ 0 \\ -7 \end{pmatrix} + \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ -3 \end{pmatrix}$. Coordinates of point of intersection are $(-1, 1, -3)$.
(iii)	The line perpendicular to *p* passing through $(1, 0, -7)$ is $\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ -7 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \ \mu \in \mathbb{R}$ $\begin{pmatrix} 1+\mu \\ 0 \\ -7-\mu \end{pmatrix} . \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = 2$ $1+\mu+7+\mu = 2$

	$2\mu=-6$ $\mu=-3$ $\overrightarrow{ON}=\begin{pmatrix}1-3\\0\\-7+3\end{pmatrix}=\begin{pmatrix}-2\\0\\-4\end{pmatrix}$
(iv)	<u>Method 1:</u> Let the coordinates of *A* be $(1,0,-7)$. Let *A'* be the reflected point of *A* in *p*. Using ratio theorem, $\overrightarrow{ON}=\frac{\overrightarrow{OA}+\overrightarrow{OA'}}{2}$ $\Rightarrow \overrightarrow{OA'}=2\overrightarrow{ON}-\overrightarrow{OA}=2\begin{pmatrix}-2\\0\\-4\end{pmatrix}-\begin{pmatrix}1\\0\\-7\end{pmatrix}=\begin{pmatrix}-5\\0\\-1\end{pmatrix}$ The reflected line contains the point *A'* and point of intersection between *l* and *p*. The direction vector of the reflected line is $\begin{pmatrix}-1\\1\\-3\end{pmatrix}-\begin{pmatrix}-5\\0\\-1\end{pmatrix}=\begin{pmatrix}4\\1\\-2\end{pmatrix}$ $\therefore \mathbf{r}=\begin{pmatrix}-1\\1\\-3\end{pmatrix}+\alpha\begin{pmatrix}4\\1\\-2\end{pmatrix}, \alpha\in\mathbb{R}$
(iv)	<u>Method 2:</u> Let the coordinates of *A* be $(1,0,-7)$.Let *A'* be the reflected point of *A* in *p*. Let the coordinates of *B* be $(-1,1,-3)$. $\overrightarrow{BN}=\frac{\overrightarrow{BA}+\overrightarrow{BA'}}{2}$ $\overrightarrow{BA'}=2\overrightarrow{BN}-\overrightarrow{BA}$ $=2\begin{pmatrix}-2\\0\\-4\end{pmatrix}-2\begin{pmatrix}-1\\1\\-3\end{pmatrix}-\begin{pmatrix}1\\0\\-7\end{pmatrix}+\begin{pmatrix}-1\\1\\-3\end{pmatrix}=\begin{pmatrix}-4\\-1\\2\end{pmatrix}$ $\therefore \mathbf{r}=\begin{pmatrix}-1\\1\\-3\end{pmatrix}+\alpha\begin{pmatrix}-4\\-1\\2\end{pmatrix}, \alpha\in\mathbb{R}$

5	**Solution**
(i)	Systematic sampling
(ii)	(slower, more difficult to collect) Systematic sampling is a more tedious process to select the employees, whereas quota sampling is quick and easy. Another possible reason: might miss out a certain group of people due to different reporting times.
(iii)	The interviewer could consider transport mode of the employees as the stratum. A possible quota for each stratum is as follows: By private transport: 10; By public transport: 10; By walking: 10 The interviewer can then stand at the entrance of the building and select the sample until the above quota is met.

By private transport	By public transport	By walking
10	10	10

6	**Solution**
	$\mathrm{E}(X)=1.93 \quad \mathrm{Var}(X)=1.4$ Since n is large, by Central Limit Theorem, $\overline{X} \sim \mathrm{N}\left(1.93, \frac{1.4}{n}\right)$ approximately. Given that $\mathrm{P}\left(\overline{X} > 2\right) < 0.24$ --- (*)
	<u>Method 1: Using GC to set up table</u> when $n=142$, $\mathrm{P}\left(\overline{X} > 2\right) = 0.24041 \ (>0.24)$ when $n=143$, $\mathrm{P}\left(\overline{X} > 2\right) = 0.23964 \ (<0.24)$ when $n=144$, $\mathrm{P}\left(\overline{X} > 2\right) = 0.23887 \ (<0.24)$ $\therefore$ least n is 143.
	<u>Method 2: Using algebraic method via standardization</u> $\mathrm{P}\left(\overline{X} \le 2\right) > 0.76$ $\mathrm{P}\left(Z \le \frac{2-1.93}{\sqrt{1.4/n}}\right) > 0.76$

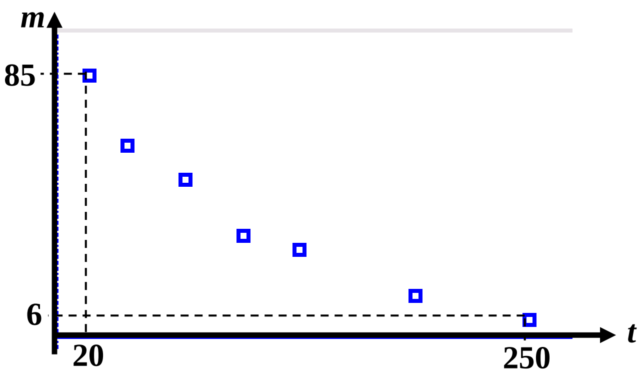
	From GC, $$\frac{2-1.93}{\sqrt{1.4/n}} > 0.70630 \quad ---(**)$$ $$\sqrt{n} > \frac{0.70630}{0.07}\sqrt{1.4}$$ $$\sqrt{n} > 11.939$$ $$n > 142.53$$ $\therefore$ least n is 143.

7	**Solution**
(i)	*Method 1:* Required probability $=1-\dfrac{{}^{13}C_4}{{}^{24}C_4}-\dfrac{{}^{11}C_4}{{}^{24}C_4}=0.902\ \text{(3 sig fig)}$ *Method 2:* Required probability $=1-\dfrac{13\times12\times11\times10}{24\times23\times22\times21}-\dfrac{11\times10\times9\times8}{24\times23\times22\times21}=0.902\ \text{(3 sig fig)}$ *Method 3:* Required probability $=\dfrac{{}^{11}C_1\times{}^{13}C_3+{}^{11}C_2\times{}^{13}C_2+{}^{11}C_3\times{}^{13}C_1}{{}^{24}C_4}$ $=0.902\ \text{(3 sig fig)}$
(ii)	*Method 1:* Required probability $=\dfrac{{}^{11}C_2\times2!\times{}^{22}C_2\times2!}{{}^{24}C_4\times4!}=0.199\ \text{(3 sig fig)}$ *Method 2:* Required probability $=\dfrac{11\times10\times22\times21}{24\times23\times22\times21}=0.199\ \text{(3 sig fig)}$

8	**Solution**
(i)	Unbiased estimate of the population mean $\bar{x} = \dfrac{573.39}{13} = 44.10692308 = 44.1$ (3 s.f.) Unbiased estimate of the population variance $s^2 = \dfrac{42.22}{12} = 3.518333333 = 3.52$ (3 s.f.)
(ii)	The battery life of a PI-99 calculator is assumed to be normally distributed.
(iii)	Let X be the r.v. denoting the battery life of a randomly chosen PI-99 calculator. Let μ be the population mean battery life of the PI-99 calculators. $H_0: \quad \mu = k$ $H_1: \quad \mu \neq k$ where H_0 is the null hypothesis and H_1 is the alternative hypothesis.
(iv)	To test at 5% level of significance. Under H_0 , the test statistic is $T = \dfrac{\bar{X} - k}{\frac{S}{\sqrt{13}}} \sim t_{(12)}$. Since the null hypothesis is not rejected, *t*-value falls outside critical region. $\therefore \quad -2.178812 < t - \text{value} < 2.178812$ $-2.178812 < \dfrac{\bar{x} - k}{s/\sqrt{13}} < 2.178812$ --- (*) $\bar{x} - 2.178812\left(\dfrac{s}{\sqrt{13}}\right) < k < \bar{x} + 2.178812\left(\dfrac{s}{\sqrt{13}}\right)$ where $\bar{x} = 44.10692$ and $s = \sqrt{3.51833}$ $\therefore \quad 42.97 < k < 45.24$ The required set is $\{k \in \mathbb{R} : 42.97 < k < 45.24\}$

9	**Solution**
(i)	The average number of faults detected by each system (for the track and the train) is constant from one day to another.
(ii)	Let X be the r.v. denoting the total number of faults detected by the two systems in a periods of 10 days. $X \sim \text{Po}\big((0.25+0.15)\times 10\big)$, i.e. $X \sim \text{Po}(4)$ $\therefore\ \text{P}\big(X \le 4\big) = 0.6288369 = 0.629$ (3 sig fig)
(iii)	Let Y be the r.v. denoting the total number of faults detected by the two systems in a period of n days. $Y \sim \text{Po}(0.4n)$ Given $\text{P}\big(Y=0\big) < 0.05$, *Method 1: Algebraic method* $\mathrm{e}^{-0.4n} < 0.05$ **(o.e.** $\left(\mathrm{e}^{-0.25n}\right)\left(\mathrm{e}^{-0.15n}\right) < 0.05$**)** $n > 7.489$ $\therefore$ the smallest number of days required is 8. *Method 2: GC table* When $n=7$, $\text{P}\big(Y=0\big) = 0.06081$ (>0.05) When $n=8$, $\text{P}\big(Y=0\big) = 0.04076$ (<0.05) When $n=9$, $\text{P}\big(Y=0\big) = 0.02732$ (<0.05) $\therefore$ the smallest number of days required is 8.
(iv)	Let W and V be the r.v. denoting the number of faults detected on the track and on the track in a period of 10 days respectively. $W \sim \text{Po}(2.5)$ and $V \sim \text{Po}(1.5)$ Required probability $= \text{P}\big(W \ge 3 \mid V+W \le 4\big)$ $= \dfrac{\text{P}\big(W \ge 3 \cap V+W \le 4\big)}{\text{P}\big(V+W \le 4\big)}$ $= \dfrac{\text{P}(W=3)\text{P}(V=0)+\text{P}(W=3)\text{P}(V=1)+\text{P}(W=4)\text{P}(V=0)}{\text{P}\big(V+W \le 4\big)}$ $= \dfrac{\text{P}(W=3)\text{P}(V \le 1)+\text{P}(W=4)\text{P}(V=0)}{\text{P}\big(V+W \le 4\big)}$ $= 0.237$ (3 sig fig)

10	**Solution**
(i)	Ben's performance (i.e. whether he loses or wins) in a game is independent of any other games that he plays with Alex.
(ii)	Let X be the r.v. denoting the number of games that Ben loses out of 10 games. $X \sim \mathrm{B}(10, 0.7)$ $\mathrm{P}(X > 5) = 1 - \mathrm{P}(X \le 5)$ $= 0.84973$ ≈ 0.850 (3 sig fig)
(iii)	Let Y be the r.v. denoting the number of games that Ben loses out of n games. $Y \sim \mathrm{B}(n, 0.3)$ $\mathrm{P}(Y > 8) \le 0.01$ $1 - \mathrm{P}(Y \le 8) \le 0.01$ Using GC, When $n = 13$, $\mathrm{P}(Y > 8) = 0.00403$ (< 0.01) When $n = 14$, $\mathrm{P}(Y > 8) = 0.00829$ (< 0.01) When $n = 15$, $\mathrm{P}(Y > 8) = 0.01524$ (> 0.01) $\therefore$ the greatest value of n is 14.
(iv)	Let W be the r.v. denoting the number of games that Ben loses out of 50 games. $W \sim \mathrm{B}(50, 0.3)$ As n = 50 is large, $np = 15$ (> 5) and $nq = 35$ (> 5), $\therefore W \sim \mathrm{N}(15, 10.5)$ approximately $\mathrm{P}(10 \le W \le 20) = \mathrm{P}(9.5 \le W \le 20.5)$ $---(*)$ $= 0.910$ (3 sig fig)

11	**Solution**
(i)	 From the scatter diagram, a curvinlinear correlation is observed between *m* and *t* (i.e. as *t* increases, *m* decreases at a decreasing rate), and hence a linear model with equation of the form $m = a + bt$ cannot be used to model the relationship between *m* and *t*.
(ii) (a)	Product moment correlation coefficient between *m* and $t^2 = -0.8454$.
(b)	Product moment correlation coefficient between *m* and $\ln t = -0.9961$.
(iii)	Since the absolute value of the correlation coefficient between *m* and $\ln t$ (i.e. case (b)) is <u>**closer**</u> to 1, this indicates that the linear correlation between the variables *m* and $\ln t$ is <u>**stronger**</u> as compared to that between the variables for case (a). $\therefore$ case (b) is the better model for the relationship between *m* and *t*. $m = 179.026 - 31.2175 \ln t$ $\Rightarrow m = 179 - 31.2 \ln t$ (3 sig fig)
(iv)	When $t = 150$, $m = 179.026 - 31.2175 \ln 150 = 22.61$ (2 dec pl) The estimate obtained is reliable, because the given value of $t = 150$ lies within the given sample data range for *t* and the product moment correlation coefficient between *m* and $\ln t$ is very close to -1, hence indicating a strong negative linear correlation between the variables *m* and $\ln t$.

12	**Solution**
(i)	$X+Y \sim \mathrm{N}(50\,,32.65)$ $\mathrm{P}(X+Y>45)=0.809224=0.809$ (3 sig fig)
(ii)	$\mathrm{E}(X+Y-S)=12$ $\mathrm{Var}(X+Y-S)=39.41$ $\therefore\ X+Y-S \sim \mathrm{N}(12\,,39.41)$ P(method A is faster than method B by more than 5 mins) $=\mathrm{P}(S+15-(X+Y)>5)$ $=\mathrm{P}(X+Y-S<10)$ $=0.375020=0.375$ (3 sig fig)
(iii)	Let $A=X+Y$ and $B=S+15$. $A \sim \mathrm{N}(50\,,32.65)$ and $B \sim \mathrm{N}(53\,,2.6^2)$ Let $W=\dfrac{A_1+A_2+A_3+A_4+B_1+...+B_6}{10}$ $\therefore\quad \mathrm{E}(W)=\dfrac{50\times 4+53\times 6}{10}=51.8$ & $\quad \mathrm{Var}(W)=\dfrac{32.65\times 4+2.6^2\times 6}{10^2}=1.7116$ $\therefore W \sim \mathrm{N}(51.8\,,1.7116)$ Required probability $=\mathrm{P}(W>50)$ $=0.915566=0.916$ (3 sig fig)