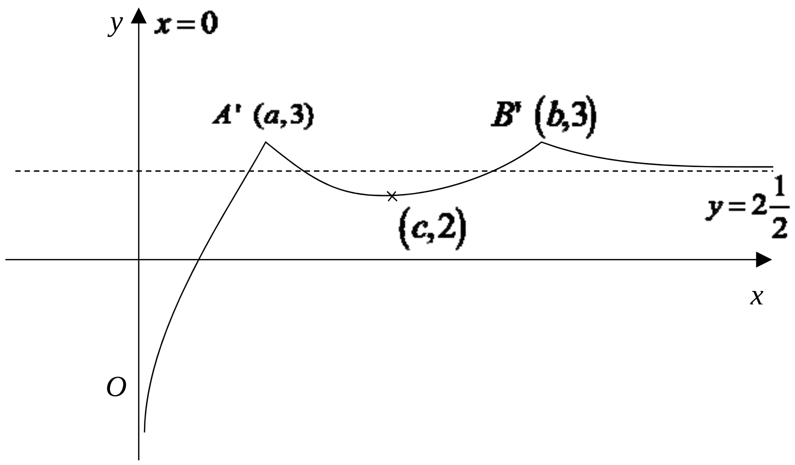
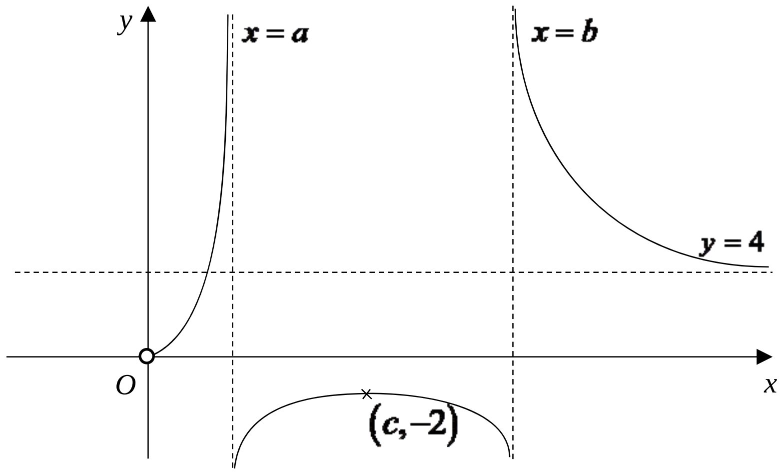
Innova Junior College
H2 Mathematics
JC2 Preliminary Examinations Paper 1
Solutions

1	**Solution**
	Let \$*x*, \$*y* and \$*z* be the cost of a ticket for a senior citizen, adult and child respectively. $2x+19y+9z=1982$ $10y+3z=908$ $x+7y+4z=778$ Using GC, $x=36$ $y=74$ $z=56$ Thus, the cost of a ticket for a senior citizen is \$36, for an adult is \$74 and for a child is \$56. 4(36) + 5(74) + 1(56) = 570 Therefore, the total cost for Group *D* = \$570

<table>
<tr><th>2</th><th>Solution</th></tr>
<tr><td>(a)</td><td>
</td></tr>
<tr><td>(b)</td><td>
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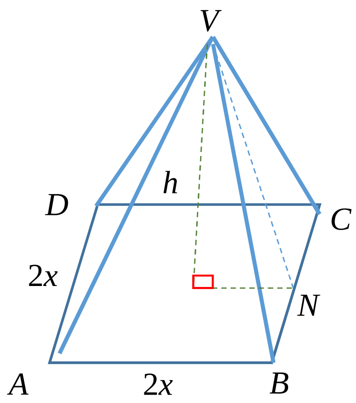
3	**Solution**
	Using cosine rule, $AC^2 = 1^2 + 4^2 - 2(1)(4)\cos\theta$ $= 1 + 16 - 8\cos\theta$ $= 17 - 8\cos\theta$ $\approx 17 - 8\left(1 - \frac{\theta^2}{2}\right)$ $= 9 + 4\theta^2$ $AC \approx \left(9 + 4\theta^2\right)^{\frac{1}{2}} \quad (\because\ AC > 0)$ $AC \approx \left(9 + 4\theta^2\right)^{\frac{1}{2}}$ $= 9^{\frac{1}{2}}\left(1 + \frac{4}{9}\theta^2\right)^{\frac{1}{2}}$ $= 3\left(1 + \frac{1}{2}\left(\frac{4}{9}\theta^2\right) + \ldots\right)$ $\approx 3\left(1 + \frac{2}{9}\theta^2\right)$ $= 3 + \frac{2}{3}\theta^2$ Therefore, $a = 3$ and $b = \frac{2}{3}$

<table>
<tr><th>4</th><th>Solution</th></tr>
<tr><td>(i)</td><td>


Volume of the pyramid = $\frac{8}{3}$
$\Rightarrow \frac{1}{3}(2x)^2 h = \frac{8}{3}$
$\Rightarrow h = \frac{2}{x^2}$
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<tr><td>(ii)</td><td>
In triangle VBC, height = VN = $\sqrt{h^2+x^2}$
Area of the triangle $VBC = \frac{1}{2}(2x)\sqrt{h^2+x^2}$
$= x\sqrt{\frac{4}{x^4}+x^2} \;=\; x\sqrt{x^2\left(\frac{4}{x^6}+1\right)}$
$= x^2\sqrt{1+4x^{-6}}$

Hence total surface area of the pyramid,
S = base area + 4 × area of triangle VBC
$= (2x)^2 + 4\left(x^2\sqrt{1+4x^{-6}}\right)$
$S = 4x^2\left[1+\sqrt{\left(1+\frac{4}{x^6}\right)}\right]$ (shown)
</td></tr>
<tr><td>(iii)</td><td>
$S = 4x^2\left[1+\sqrt{(1+4x^{-6})}\right]$
$\frac{dS}{dx} = 4x^2\left(\frac{1}{2}\left(1+4x^{-6}\right)^{-\frac{1}{2}}\left(-24x^{-7}\right)\right)+(8x)\left[1+\sqrt{(1+4x^{-6})}\right]$
$= -48x^{-5}\left(1+4x^{-6}\right)^{-\frac{1}{2}}+8x\left[1+\sqrt{(1+4x^{-6})}\right]$
$= -8x\left[6x^{-6}\left(1+4x^{-6}\right)^{-\frac{1}{2}}-1-\sqrt{(1+4x^{-6})}\right]$
At the stationary value of S, $\frac{dS}{dx}=0.$
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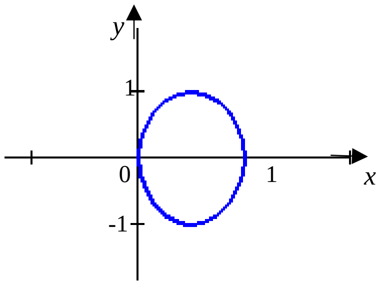
	$\therefore \quad -8x\left[6x^{-6}\left(1+4x^{-6}\right)^{-\frac{1}{2}}-1-\sqrt{(1+4x^{-6})}\right]=0$ By G.C., $x=0.89090=0.89$ (to 2dp)

5	**Solution**
(i)	$(\mathbf{b}-\mathbf{a})\cdot(\mathbf{b}-\mathbf{a}) = \lvert\mathbf{b}\rvert^2 + \lvert\mathbf{a}\rvert^2 - 2\mathbf{a}.\mathbf{b}$ $= \lvert\mathbf{b}\rvert^2 + 9\lvert\mathbf{b}\rvert^2 - 2\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\cos 60^\circ$ $\lvert\mathbf{b}-\mathbf{a}\rvert^2 = 10\lvert\mathbf{b}\rvert^2 - 2\left(3\lvert\mathbf{b}\rvert\right)\lvert\mathbf{b}\rvert\frac{1}{2}$ $\lvert\mathbf{b}-\mathbf{a}\rvert = \sqrt{7}\,\lvert\mathbf{b}\rvert$ Therefore, $k=\sqrt{7}$.
(ii)	$\mathbf{c} = \frac{1}{3}\mathbf{a}$ $\overrightarrow{CA} = \frac{2}{3}\mathbf{a}$ Shortest distance of C to l = $\frac{\left\lvert \frac{2}{3}\mathbf{a}\times(\mathbf{b}-\mathbf{a})\right\rvert}{\lvert\mathbf{b}-\mathbf{a}\rvert}$ $= \frac{\left\lvert \frac{2}{3}\mathbf{a}\times\mathbf{b} - \frac{2}{3}\mathbf{a}\times\mathbf{a}\right\rvert}{\lvert\mathbf{b}-\mathbf{a}\rvert}$ $= \frac{2\lvert\mathbf{a}\times\mathbf{b}\rvert}{3\lvert\mathbf{b}-\mathbf{a}\rvert} \qquad \because \mathbf{a}\times\mathbf{a}=\mathbf{0}$ $= \frac{2\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\sin 60^\circ}{3\lvert\mathbf{b}-\mathbf{a}\rvert}$ $= \frac{6\lvert\mathbf{b}\rvert^2\,\frac{\sqrt{3}}{2}}{3\sqrt{7}\,\lvert\mathbf{b}\rvert}$ $= \frac{\sqrt{3}\,\lvert\mathbf{b}\rvert}{\sqrt{7}}$ $= \sqrt{\frac{3}{7}}\,\lvert\mathbf{b}\rvert$

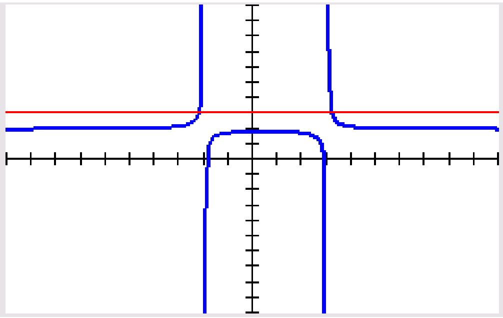
6	Solution
(i)	$x=\cos^2\theta, \quad y=\sin 2\theta, \quad \text{for } -\frac{\pi}{2}<\theta\le\frac{\pi}{2}.$ 
(ii)	$x=\cos^2\theta$ $\mathrm{d}x=-2\cos\theta\sin\theta\,\mathrm{d}\theta$ When $y=0, \quad \sin 2\theta=0$ $2\theta=0,\pi$ $\theta=0,\frac{\pi}{2}$ When $\theta=0, \quad x=1$; When $\theta=\frac{\pi}{2}, \quad x=0$ Volume of the solid formed $=\pi\int_0^1 y^2\ \mathrm{d}x$ $=\pi\int_{\frac{\pi}{2}}^0(\sin 2\theta)^2(-2\cos\theta\sin\theta\,\mathrm{d}\theta)$ $=\pi\int_{\frac{\pi}{2}}^0(\sin 2\theta)^2(-\sin 2\theta\,\mathrm{d}\theta)$ $=\pi\int_0^{\frac{\pi}{2}}\sin^3 2\theta\ \mathrm{d}\theta$ (shown) $\therefore\ a=0, \quad b=\frac{\pi}{2}.$ Let $u=\cos 2\theta$. $\therefore \mathrm{d}u=-2\sin 2\theta\ \mathrm{d}\theta$ When $\theta=0, \quad u=1$; When $\theta=\frac{\pi}{2}, \quad u=-1$ Volume of the solid formed $=\pi\int_0^{\frac{\pi}{2}}\sin^3 2\theta\,\mathrm{d}\theta$

	$=\frac{1}{2}\pi\int_0^{\frac{\pi}{2}}\sin^2 2\theta(2\sin 2\theta)\,d\theta$ $=\frac{1}{2}\pi\int_0^{\frac{\pi}{2}}\left(1-\cos^2 2\theta\right)\left(2\sin 2\theta\,d\theta\right)$ $=\frac{1}{2}\pi\int_1^{-1}\left(1-u^2\right)(-du)$ $=-\frac{1}{2}\pi\left[u-\frac{u^3}{3}\right]_1^{-1}$ $=-\frac{1}{2}\pi\left[\left(-1+\frac{1}{3}\right)-\left(1-\frac{1}{3}\right)\right]$ $=-\frac{1}{2}\pi\left(-\frac{4}{3}\right)$ $=\frac{2}{3}\pi$ units^3

7	**Solution**
(i)	$3y^3-8y^2+10y=4-5x$ Differentiate wrt x, $\left(9y^2-16y+10\right)\frac{\mathrm{d}y}{\mathrm{d}x}=-5$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-5}{9y^2-16y+10}$ When $x=\frac{4}{5}$, $3y^3-8y^2+10y=0$. $\therefore y=0$ and $\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{1}{2}$. Eqn of tangent : $y-0=-\frac{1}{2}\left(x-\frac{4}{5}\right)$, ie $y=-\frac{1}{2}x+\frac{2}{5}$
(ii)	$\left(9y^2-16y+10\right)\frac{\mathrm{d}y}{\mathrm{d}x}=-5$ Differentiate wrt x, $\left(9y^2-16y+10\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\left(18y-16\right)\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2=0$. When $x=0$, $y=\frac{2}{3},\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{3}{2},\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{27}{10}$. $\therefore \quad y=\frac{2}{3}-\frac{3}{2}x+\frac{27}{20}x^2+\ldots$
(iii)	$y=\frac{2}{3}-\frac{3}{2}x$

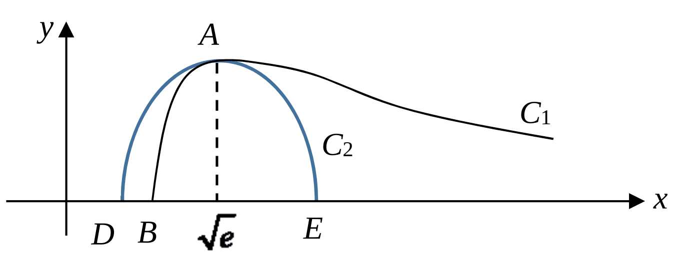
8	**Solution**
(a)	$\arg\left(w^5\right)=5\arg\left(w\right)=0,\pm\pi,\pm2\pi\ldots$ $\arg\left(w\right)=0,\frac{\pi}{5},-\frac{\pi}{5},\frac{2\pi}{5},\frac{-2\pi}{5},\ldots$ Since $k<0$, $\arg\left(w\right)=-\frac{\pi}{5}$ or $-\frac{2\pi}{5}$. $\frac{k}{\sqrt{3}}=\tan\left(-\frac{\pi}{5}\right)$ or $\frac{k}{\sqrt{3}}=\tan\left(-\frac{2\pi}{5}\right)$ $k=\sqrt{3}\tan\left(-\frac{\pi}{5}\right)$ or $k=\sqrt{3}\tan\left(-\frac{2\pi}{5}\right)$ $n=-\frac{1}{5}$ or $-\frac{2}{5}$
(bi)	**Method 1** $1-z^2=1-\left(\cos\theta+\mathrm{i}\sin\theta\right)^2$ $=1-\left(\cos^2\theta+2\mathrm{i}\cos\theta\sin\theta+\left(\mathrm{i}\sin\theta\right)^2\right)$ $=1-\left(1-\sin^2\theta+2\mathrm{i}\sin\theta\cos\theta-\sin^2\theta\right)$ $=1-1+2\sin^2\theta-2\mathrm{i}\sin\theta\cos\theta$ $=2\sin^2\theta-2\mathrm{i}\sin\theta\cos\theta$ $=2\sin\theta\left(\sin\theta-\mathrm{i}\cos\theta\right)$ **Method 2** $1-z^2=1-\left(\cos\theta+\mathrm{i}\sin\theta\right)^2$ $=1-\left(\cos2\theta+\mathrm{i}\sin2\theta\right)$ $=1-\cos2\theta-\mathrm{i}\sin2\theta$ $=1-\left(1-2\sin^2\theta\right)-2\mathrm{i}\sin\theta\cos\theta$ $=2\sin^2\theta-2\mathrm{i}\sin\theta\cos\theta$ $=2\sin\theta\left(\sin\theta-\mathrm{i}\cos\theta\right)$
(bii)	**Method 1** $\left\|1-z^2\right\|=\left\|2\sin\theta\left(\sin\theta-\mathrm{i}\cos\theta\right)\right\|$ $=2\sin\theta\sqrt{\sin^2\theta+\cos^2\theta}$ $=2\sin\theta$ Given that $0\le\theta\le\frac{\pi}{2}$

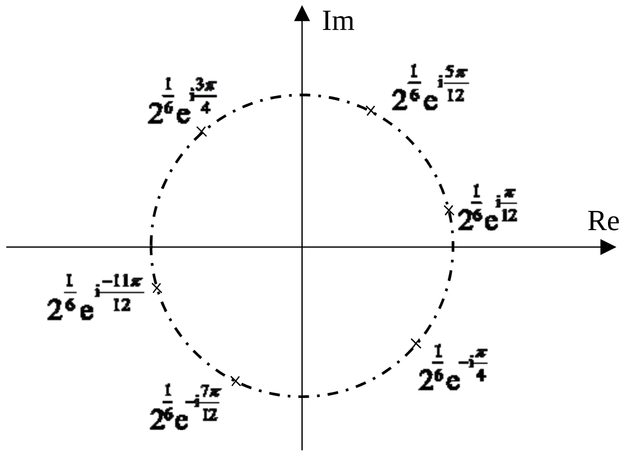
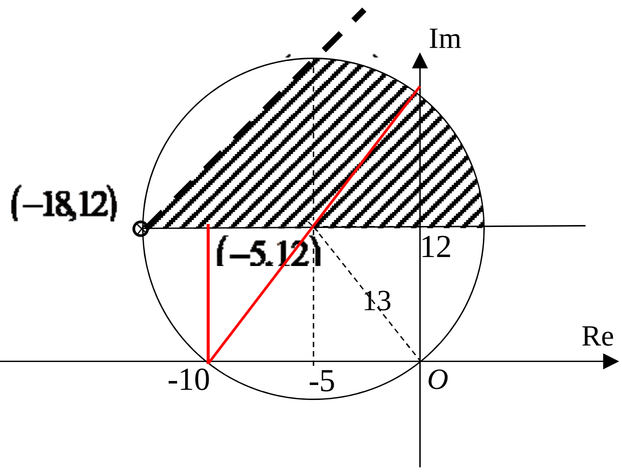
	$\arg\left(1-z^2\right)=\arg\left[2\sin\theta\left(\sin\theta-\mathrm{i}\cos\theta\right)\right]$ $=\arg\left(2\sin\theta\right)+\arg\left(\sin\theta-\mathrm{i}\cos\theta\right)$ $=0-\tan^{-1}\left(\dfrac{\cos\theta}{\sin\theta}\right)$ $=-\tan^{-1}\left(\tan\left(\dfrac{\pi}{2}-\theta\right)\right)$ $=-\left(\dfrac{\pi}{2}-\theta\right)$ $=\theta-\dfrac{\pi}{2}$ **Method 2** $1-z^2=2\sin\theta\left(\sin\theta-\mathrm{i}\cos\theta\right)$ $=2\sin\theta\left(-\mathrm{i}\right)\left(\cos\theta+\mathrm{i}\sin\theta\right)$ $=\left(-2\mathrm{i}\sin\theta\right)\mathrm{e}^{\mathrm{i}\theta}$ $\left\|1-z^2\right\|=\left\|\left(-2\mathrm{i}\sin\theta\right)\mathrm{e}^{\mathrm{i}\theta}\right\|$ $=2\sin\theta$ $\arg\left(1-z^2\right)=\arg\left(\left(-2\mathrm{i}\sin\theta\right)\mathrm{e}^{\mathrm{i}\theta}\right)$ $=\arg\left(-2\mathrm{i}\sin\theta\right)+\arg\left(\mathrm{e}^{\mathrm{i}\theta}\right)$ $=-\dfrac{\pi}{2}+\theta$ **Method 3** $1-z^2=2\sin\theta\left(\sin\theta-\mathrm{i}\cos\theta\right)$ $=2\sin\theta\left(\cos\left(\dfrac{\pi}{2}-\theta\right)-\mathrm{i}\sin\left(\dfrac{\pi}{2}-\theta\right)\right)$ $=2\sin\theta\left(\cos\left(\theta-\dfrac{\pi}{2}\right)+\mathrm{i}\sin\left(\theta-\dfrac{\pi}{2}\right)\right)$ $\left\|1-z^2\right\|=2\sin\theta$ $\arg\left(1-z^2\right)=\theta-\dfrac{\pi}{2}$

9	**Solution**
(i)	 From graph, the horizontal line $y = 3$ cuts the graph at two points. Hence f is not a **one-one** function, hence f^{-1} does not exist.
(ii)	$\mathrm{f}(x) = \frac{1}{x^2 - x - 6} + 2$ $\mathrm{f}'(x) = -\left(x^2 - x - 6\right)^{-2}(2x - 1)$ $= \frac{1 - 2x}{\left(x^2 - x - 6\right)^2}$ For the function to be decreasing, $\mathrm{f}'(x) \le 0$. $1 - 2x \le 0$ $1 \le 2x$ $x \ge 0.5$ $\{x \in \mathbb{R}:\ x \ge 0.5,\ x \ne 3\}$
(iii)	$\mathrm{gf}(x) = 2 - \frac{1}{x^2 - x - 6} - 2$ $= -\frac{1}{x^2 - x - 6},\quad x \le \frac{1}{2}$ $(\mathrm{gf})^{-1}\left(\frac{1}{4}\right) = x$ $\mathrm{gf}(x) = \frac{1}{4}$ $-\frac{1}{x^2 - x - 6} = \frac{1}{4}$ $x^2 - x - 6 = -4$ $x^2 - x - 2 = 0$ $(x - 2)(x + 1) = 0$ $x = 2$ (rejected) or $x = -1$ $x = -1$

10	**Solution**
(i)	Let P_n be the statement $u_n = \dfrac{n}{(n+1)!}$ for $n \in \mathbb{Z}^+$. P_1 is true since $u_1 = \dfrac{1}{2!} = \dfrac{1}{2}$. Assume that P_k is true for some $k \in \mathbb{Z}^+$, i.e. $u_k = \dfrac{k}{(k+1)!}$ Consider P_{k+1}: i.e. $u_{k+1} = \dfrac{k+1}{(k+2)!}$ $u_{k+1} = \dfrac{k}{(k+1)!} - \dfrac{k^2+k-1}{(k+2)!}$ $= \dfrac{k(k+2)-\left(k^2+k-1\right)}{(k+2)!}$ $= \dfrac{k^2+2k-k^2-k+1}{(k+2)!}$ $= \dfrac{k+1}{(k+2)!}$ Thus, P_k is true $\Rightarrow$ P_{k+1} is true. Since P_1 is true, and P_k is true $\Rightarrow$ P_{k+1} is true, by mathematical induction, P_n is true for all $n \in \mathbb{Z}^+$.
(ii)	$\sum_{n=1}^{N} \dfrac{n^2+n-1}{(n+2)!} = \sum_{n=1}^{N} \left(u_n - u_{n+1}\right)$ $[\ u_1 - \cancel{u_2}$ $+ \cancel{u_2} - \cancel{u_3}$ $+ \cancel{u_3} - \cancel{u_4}$ $+ \cancel{\ldots}$ $+ \cancel{u_{N-1}} - \cancel{u_N}$ $+ \cancel{u_N} - u_{N+1}\ ]$ $= u_1 - u_{N+1} = \dfrac{1}{2} - \dfrac{N+1}{(N+2)!}$

(iii)	As $N \to \infty$, $\frac{N+1}{(N+2)!} \to 0$ $\sum_{n=1}^{\infty} \frac{n^2+n-1}{(n+2)!} \to \frac{1}{2}$ which is a constant, hence it is a convergent series.

11	**Solution**
(i)	$y=\dfrac{\ln x}{x^2}$, $x\ge 1$ $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{x^2\cdot\frac{1}{x}-\ln x\cdot 2x}{x^4}$ $=\dfrac{x-2x\ln x}{x^4}$ When $\dfrac{\mathrm{d}y}{\mathrm{d}x}=0$ and since $x\neq 0$, $1-2\ln x=0$ $2\ln x=1$ $\ln x=\dfrac{1}{2}$ $x=\mathrm{e}^{\frac{1}{2}}=\sqrt{\mathrm{e}}$ When $x=\sqrt{\mathrm{e}}$, $y=\dfrac{\ln\sqrt{\mathrm{e}}}{\left(\sqrt{\mathrm{e}}\right)^2}$ $=\dfrac{1}{2\mathrm{e}}$ Hence the coordinates of A is $\left(\sqrt{\mathrm{e}},\dfrac{1}{2\mathrm{e}}\right)$.
(ii)	$B(1,0)$, $D\left(\sqrt{\mathrm{e}}-1,0\right)$ and $E\left(\sqrt{\mathrm{e}}+1,0\right)$ 
(iii)	Area $=\displaystyle\int_{\sqrt{\mathrm{e}}-1}^{\sqrt{\mathrm{e}}}\frac{\sqrt{1-\left(x-\sqrt{\mathrm{e}}\right)^2}}{2\mathrm{e}}\,\mathrm{d}x-\int_{1}^{\sqrt{\mathrm{e}}}\frac{\ln x}{x^2}\,\mathrm{d}x$ $=0.14446942-0.09020401$ $=0.05426541$ $=0.0543$ (correct to 3 s.f.)

12	**Solution**
(ai)	$z^6 - 2\mathrm{i} = 0$ $z^6 = 2\mathrm{i}$ $z^6 = 2\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{2}+2k\pi\right)},\ k = 0, \pm1, \pm2, -3$ $z = 2^{\frac{1}{6}}\mathrm{e}^{\mathrm{i}\frac{-11\pi}{12}}, 2^{\frac{1}{6}}\mathrm{e}^{\mathrm{i}\frac{-7\pi}{12}}, 2^{\frac{1}{6}}\mathrm{e}^{\mathrm{i}\frac{-\pi}{4}}, 2^{\frac{1}{6}}\mathrm{e}^{\mathrm{i}\frac{\pi}{12}}, 2^{\frac{1}{6}}\mathrm{e}^{\mathrm{i}\frac{5\pi}{12}}, 2^{\frac{1}{6}}\mathrm{e}^{\mathrm{i}\frac{3\pi}{4}}$
(aii)	
(aiii)	Since *ABCDEF* is a regular hexagon, the triangles *OAB*, *OBC* … are equilateral triangles. Perimeter of the polygon $= 6 \times 2^{\frac{1}{6}}$ $= 6.735$ (to 3 d.p.)
(bi)	
(bii)	Minimum $\lvert w+10 \rvert = 12$ Maximum $\lvert w+10 \rvert = 26$ (diameter of circle)