YISHUN JUNIOR COLLEGE
2016 JC2 PRELIMINARY EXAM PAPER 2
H2 MATHEMATICS
SOLUTIONS

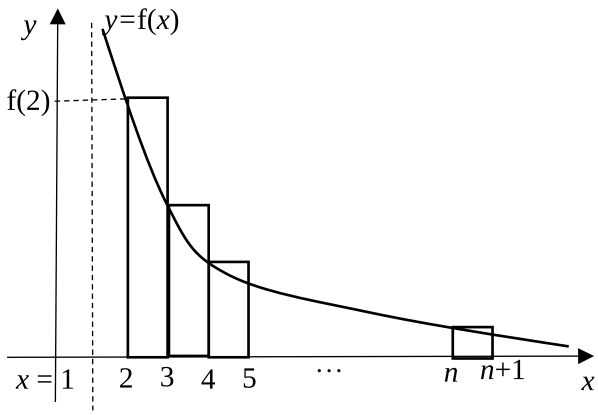
Qn	Solution
1	Let P_n be the statement "$u_n = (n-1)!\sum_{r=0}^{n-1}\frac{1}{r!}$" When $n=1$, L.H.S. = u_1 = 1 (Given) R.H.S = $(1-1)!\left(\frac{1}{0!}\right)$ = 1 Hence, P_1 is true. Assume that P_k is true for some $k \in \mathbf{Z}^+$, ie, $u_k = (k-1)!\sum_{r=0}^{k-1}\frac{1}{r!}$. To show P_{k+1} is true, ie, $u_{k+1} = (k)!\sum_{r=0}^{k}\frac{1}{r!}$. $u_{k+1} = ku_k + 1$ $= k(k-1)!\sum_{r=0}^{k-1}\frac{1}{r!} + 1$ $= k!\sum_{r=0}^{k-1}\frac{1}{r!} + \frac{k!}{k!}$ $= k!\left(\sum_{r=0}^{k-1}\frac{1}{r!} + \frac{1}{k!}\right)$ $= k!\sum_{r=0}^{k}\frac{1}{r!}$ Therefore, P_k is true $\Rightarrow$ P_{k+1} is true By Mathematical Induction, P_n is true for all $n \geq 1$. $\therefore \frac{u_n}{(n-1)!} = \sum_{r=0}^{n-1}\frac{1}{r!}$ $\lim_{n\to\infty}\frac{u_n}{(n-1)!} = \sum_{r=0}^{\infty}\frac{1}{r!} = e^x\|_{x=1} = e$

2

$$\mathrm{f}(x)=\frac{5x-2}{x(x-1)(x+2)}=\frac{1}{x-1}+\frac{1}{x}-\frac{2}{x+2}$$

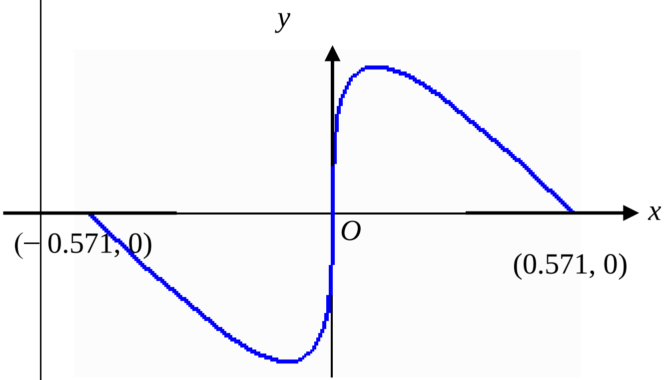
$$\sum_{r=2}^{n}\mathrm{f}(r)=\sum_{r=2}^{n}\left(\frac{1}{r-1}+\frac{1}{r}-\frac{2}{r+2}\right)$$

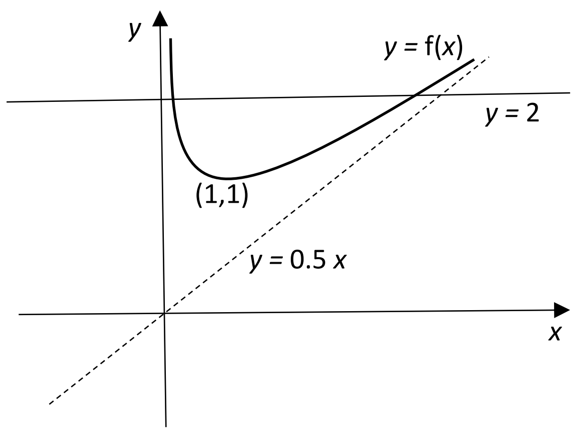
$$\begin{aligned}
&=\frac{1}{1}+\frac{1}{2}-\frac{2}{4}\\
&+\frac{1}{2}+\frac{1}{3}-\frac{2}{5}\\
&+\frac{1}{3}+\frac{1}{4}-\frac{2}{6}\\
&+\frac{1}{4}+\frac{1}{5}-\frac{2}{7}\\
&+\frac{1}{5}+\frac{1}{6}-\frac{2}{8}\\
&+\cdots\\
&+\frac{1}{n-4}+\frac{1}{n-3}-\frac{2}{n-1}\\
&+\frac{1}{n-3}+\frac{1}{n-2}-\frac{2}{n}\\
&+\frac{1}{n-2}+\frac{1}{n-1}-\frac{2}{n+1}\\
&+\frac{1}{n-1}+\frac{1}{n}-\frac{2}{n+2}\\
&=\frac{8}{3}-\left(\frac{1}{n}+\frac{2}{n+1}+\frac{2}{n+2}\right)
\end{aligned}$$



From the above diagram,
Sum of areas of (n–1) rectangles
> Area under graph of y = f(x) from x = 2 to n+1

$$(1)\mathrm{f}(2)+(1)\mathrm{f}(3)+(1)\mathrm{f}(4)+\ldots+(1)\mathrm{f}(n)>\int_2^{n+1}\mathrm{f}(x)\,\mathrm{d}x \quad\therefore\quad \sum_{r=2}^{n}\mathrm{f}(r)>\int_2^{n+1}\mathrm{f}(x)\,\mathrm{d}x$$

3(i)	$x = t - a\sin t \quad y = t\cos t$ $\frac{dx}{dt} = 1 - a\cos t \quad \frac{dy}{dt} = -t\sin t + \cos t$ $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$ $= \frac{-t\sin t + \cos t}{1 - a\cos t}$ normal parallel to x-axis $\Rightarrow 1 - a\cos t = 0$ $\Rightarrow a = \frac{1}{\cos t}$ --- (1) When $x = 0$, $t - a\sin t = 0$ ----- (2) Sub (1) into (2): $t - \tan t = 0$ From GC, $t = 0$ (since $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$) Sub into (1): $a = \frac{1}{\cos 0} = 1$ (shown)
(ii)	
(iii)	Area of region $= \left\| \int_{1-\frac{\pi}{2}}^{0} y \, dx \right\| + \int_{0}^{\frac{\pi}{2}-1} y \, dx$ $= \left\| \int_{-\frac{\pi}{2}}^{0} t\cos t\left(1-\cos t\right) dt \right\| + \int_{0}^{\frac{\pi}{2}} t\cos t\left(1-\cos t\right) dt$ $= 0.408$

4(i)	gf exists when $R_f \subseteq D_g$. Since $R_f = [1,\infty) \subseteq (0,\infty) = D_g$, therefore gf exists. $\text{gf}(x) = \text{g}\left(\dfrac{x^2+1}{2x}\right)$ $= \dfrac{2x}{x^2+1}$ $\text{gf} : x \mapsto \dfrac{2x}{x^2+1}, \quad x > 0$ $(0,\infty) \xrightarrow{f} [1,\infty) \xrightarrow{g} (0,1]$ $\therefore R_{gf} = (0,1]$
4(ii)	 The line $y = 2$ cuts the graph of f more than once. Hence, f is not a one–one function. Therefore, f does not have an inverse.
4(iii)	From the graph, least value of $k = 1$ $x^2 - 2yx + 1 = 0$ $x = \dfrac{2y \pm \sqrt{4y^2-4}}{2} = y \pm \sqrt{y^2-1}$ Since $x \geq 1$, $x = y + \sqrt{y^2-1}$ $f^{-1}(x) = x + \sqrt{x^2-1}, x \geq 1$

5(i) $z^6 = -64 = 64\,\mathrm{e}^{\mathrm{i}\pi}$

$$= 64\,\mathrm{e}^{\mathrm{i}(2k+1)\pi},\ k \in \mathbf{Z}$$

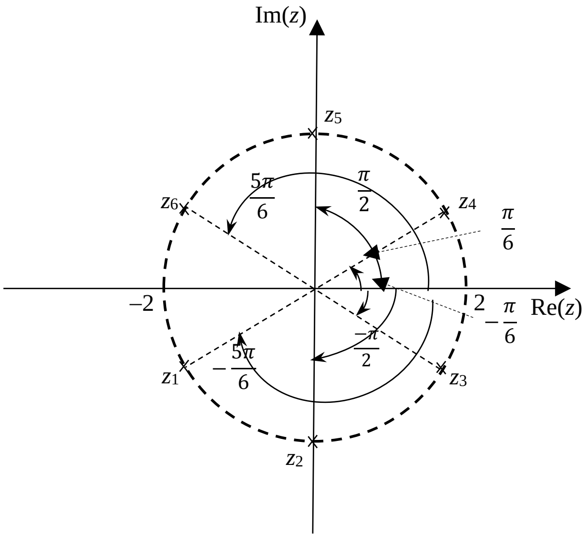
$$z = 2\,\mathrm{e}^{\mathrm{i}\left(\frac{2k+1}{6}\right)\pi},\ k = 0, \pm 1, \pm 2, -3$$

$$z = 2\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{6}\right)}, 2\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{2}\right)}, 2\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{6}\right)}, 2\mathrm{e}^{\mathrm{i}\left(\frac{5\pi}{6}\right)}, 2\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{2}\right)}, 2\mathrm{e}^{\mathrm{i}\left(-\frac{5\pi}{6}\right)}$$

(ii)

$$z_1 = 2\mathrm{e}^{\mathrm{i}\left(-\frac{5\pi}{6}\right)}, z_2 = 2\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{2}\right)}, z_3 = 2\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{6}\right)}$$

$$z_4 = 2\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{6}\right)}, z_5 = 2\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{2}\right)}, z_6 = 2\mathrm{e}^{\mathrm{i}\left(\frac{5\pi}{6}\right)}$$

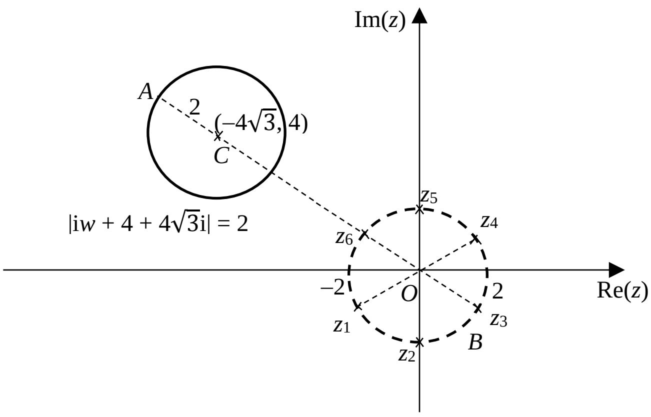


The 6 points representing the 6 roots are the vertices of a regular hexagon.

(iii) $|\mathrm{i}w + 4 + 4\sqrt{3}\,\mathrm{i}| = 2$

$|\,\mathrm{i}\,|\,|w - \mathrm{i}(4 + 4\sqrt{3}\,\mathrm{i})| = 2$

$|w - (-4\sqrt{3} + 4\mathrm{i})| = 2$

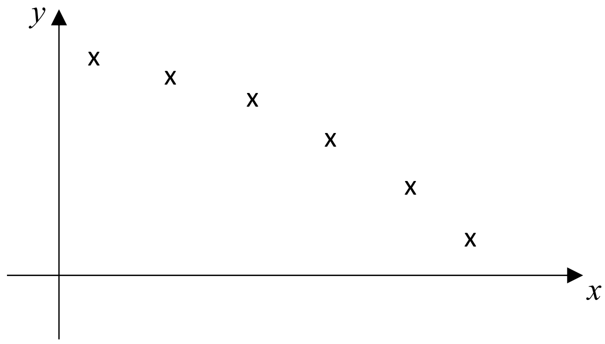
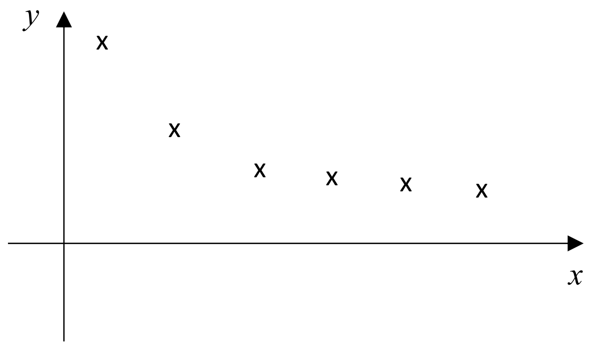
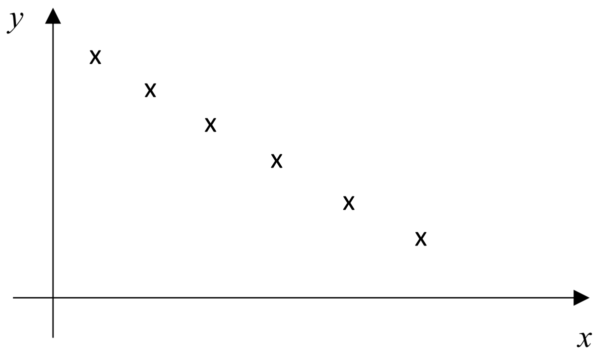


(iv)	As arg $(-4\sqrt{3}+4\text{i}) = \frac{5\pi}{6}$, O, C and B are collinear. Hence, maximum $\lvert w - z_n\rvert = \lvert w - z_3\rvert = AB$ $= OC + CA + OB$ $= \sqrt{\left(4\sqrt{3}\right)^2 + 4^2} + 2 + 2$ $= 12$
6(i) **(ii)**	Number the employees from 1 to 40 000. Randomly select 800 numbers using a random number generator. The employees corresponding to these 800 numbers are selected for the survey. The manager can survey the first 400 male employees and first 400 female employees who step into the canteen on a particular day.
7(i) **(ii)** **(iii)** **(iv)**	No of ways = $9\times8\times7\times6$ $= 3024$ No of ways = $3024 - C_4^9$ $= 2898$ No of ways = $C_1^4 \times C_3^5 \times 4!$ $= 960$ No of ways= $\sum_{r=1}^{9} r^3$ $= 2025$
8(a) **(b)** **(i)**	$\mathrm{P}(X \cap Y') = \mathrm{P}(X) - \mathrm{P}(X \cap Y)$ $= \mathrm{P}(X) - \mathrm{P}(X)\times\mathrm{P}(Y)$ since X and Y are ind. $= P(X)[1 - P(Y)]$ $= \mathrm{P}(X)\times\mathrm{P}(Y')$ Since $\mathrm{P}(X \cap Y') = \mathrm{P}(X)\times\mathrm{P}(Y')$, events X and Y' are independent. P(A’|B’)=0.3 $\frac{\mathrm{P}(A' \cap B')}{\mathrm{P}(B')} = 0.3$ $\mathrm{P}(A' \cap B') = 0.3\times(1-0.6)$ $=0.12$ $\mathrm{P}(A \cup B) = 1 - 0.12 = 0.88$ $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ $\mathrm{P}(A \cap B)$ =0.5+0.6-0.88 =0.22

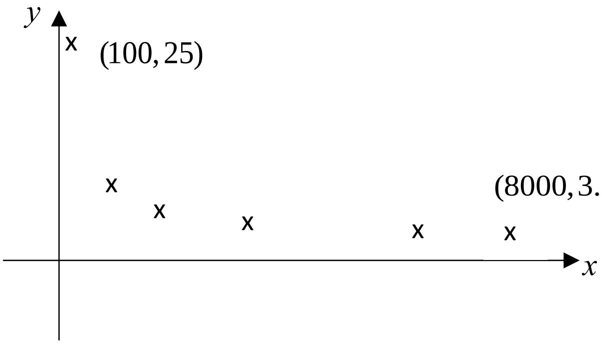
(ii)	$\mathrm{P}(B'\cap A)=\mathrm{P}(A)-\mathrm{P}(A\cap B)$ $=0.5-0.22=0.28$ $\mathrm{P}(B' \mid A)=\dfrac{\mathrm{P}(B'\cap A)}{\mathrm{P}(A)}$ $=\dfrac{0.28}{0.5}=0.56$
(iii)	Since $\mathrm{P}(A\cap B)=0.22\neq 0$, events A and B are not mutually exclusive
(iv)	$\mathrm{P}(A\cap B)=0.22$ $\mathrm{P}(A)\times\mathrm{P}(B)=0.5\times 0.6=0.3$ Since $\mathrm{P}(A\cap B)\neq\mathrm{P}(A)\times\mathrm{P}(B)$, events A and B are not independent.
9(i)	(1) The probability of Rickie finding a seat on the train is constant every weekday. (2) The event of Rickie finding a seat on the train on one weekday is independent of the other weekdays.
(ii)	A~B(5, 0.65) $\mathrm{P}(A=2\text{ or }3)=\mathrm{P}(A\leq 3)-\mathrm{P}(A\leq 1)$ = 0.51756 Let X be the number of weeks, out of 52, that Rickie is contented. Then $X\sim\mathrm{B}(52,\ 0.51756)$ Since n = 52 is large, np = 52(0.51756)=26.91312 > 5, nq = 52(1−0.51756)=25.08688 > 5, $X\sim\mathrm{N}(26.91312,\ 52\times 0.51756\times(1-0.51756))$ approx. i.e. $X\sim\mathrm{N}(26.91312, 12.98397)$ approx. $\mathrm{P}(X\leq 30)\to\mathrm{P}(X<30.5)$ using continuity correction ≈ 0.840 (3 s.f)

10(i) (a)	Let X and Y be the r.v. "number of drug trafficking and cigarette trafficking cases in 8 weeks respectively". Then $X \sim \text{Po}(1.6)$, $Y \sim \text{Po}(5.6)$ $\text{P}(X > 6) = 1 - \text{P}(X \le 6)$ ≈ 0.00134 (3 s.f)
(i) (b)	$X + Y \sim \text{Po}(7.2)$ $\text{P}(X + Y < 5) = \text{P}(X + Y \le 4)$ ≈ 0.156 (3 s.f)
(ii)	Let W be the number of drug trafficking cases in a period of n weeks. Then $W \sim \text{Po}(0.2n)$ $\text{P}(W < 2) < 0.01$ $\text{P}(W \le 1) < 0.01$ $\text{e}^{-0.2n} + 0.2n(\text{e}^{-0.2n}) < 0.01$ Using GC, If n = 33, $\text{P}(W \le 1) = 0.0103 > 0.01$ If n = 34, $\text{P}(W \le 1) = 0.00869 < 0.01$ $\therefore$ smallest possible integer value of n is 34. Alternative method: Plot graph Using GC, $n > 33.19176$ $\therefore$ smallest possible integer value of n is 34.
(iii)	The number of drug and cigarette trafficking cases per week may decrease after police's raids so the trafficking cases may not occur with a constant average rate. A person can be both a drug and cigarette trafficker so the drug trafficking cases and cigarette trafficking cases may not be independent.
11(i)	Let A and B be the mass of a randomly chosen towel from Alpha and Bravo respectively. $\text{P}(A > 380) = 0.0668$ $\text{P}(A \le 380) = 0.9332$ $\text{P}(Z \le \frac{380 - \mu}{20}) = 0.9332$ Using GC, $\frac{380 - \mu}{20} = 1.5000556$

	$380-\mu=30.001112$ $\mu=349.998888$ $=350$ (3 s.f)
(ii)	$T=1.6(A_1+A_2+A_3+A_4)+1.5(B_1+B_2)$ $\sim \mathrm{N}(1.6\times4\times350+1.5\times2\times275,\ 1.6^2\times4\times20^2+1.5^2\times2\times15^2)$ i.e. $T\sim \mathrm{N}(3065,\ 5108.5)$ $\mathrm{P}(T>3000)=0.81843...$ ≈ 0.818 (3 s.f)
12(i)	Unbiased estimate of population mean, $\bar{x}=\dfrac{\sum(x-4)}{80}+4$ $=\dfrac{25}{80}+4=4.3125$ Unbiased estimate of population variance, $s^2=\dfrac{1}{80-1}\left(140-\dfrac{25^2}{80}\right)$ =1.673259… $\approx$ 1.67 (3 s.f)
(ii)	$H_0:\mu=4$ $H_1:\mu>4$ Under H_0, the test statistic is $Z=\dfrac{\overline{X}-\mu}{S/\sqrt{n}}\sim N(0,1)$ approx. (by CLT), where $\mu=4,\ \bar{x}=4.3125,\ s^2=1.6733,\ n=80$ Level of significance=1% Using GC, p-value=0.015357 (5 s.f) Since p-value=0.015357>0.01, we do not reject H_0 and conclude that at the 1% level, there is no sufficient evidence to doubt Burger Queen's claim.
(iii)	Critical value=1.28155 In order not to reject H_0, $\dfrac{\bar{x}-4}{1.5/\sqrt{80}}<1.28155$ $\bar{x}<4.2149...$ Required set = $\{\bar{x}\in\mathbb{R}^+ : \bar{x}<4.21\}$

13(i)	(A) $y = a + bx^2$, where a is positive and b is negative  (B) $y = c + \frac{d}{x}$, where both c and d are positive  (C) $y = e + fx$, where e is positive and f is negative 

(ii)



(iii)

Case (B) is the most appropriate because as x increases, y decreases at a decreasing rate.

From GC, $r = 0.994981\ldots$

≈ 0.995 (3s.f)

(iv)

From GC, $y = 3.62896 + \dfrac{2154.915}{x}$

$y = 3.63 + \dfrac{2150}{x}$ (3 sf)

When $y = 10$, $10 = 3.62896 + \dfrac{2154.915}{x}$

$x \approx 338$ (3s.f)

The estimate is likely to be reliable as $y = 10$ is within the range of given y values and r value is close to 1.