YISHUN JUNIOR COLLEGE
2016 JC2 PRELIMINARY EXAM PAPER 1
H2 MATHEMATICS
SOLUTIONS

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Let S, B and W be the number of strawberry, blueberry and walnut muffins purchased respectively.

$S+B+W=30$

$1.6S+1.75B+2.2W=53.40$

$S=2W \Rightarrow S-2W=0$

From GC, $S=12,\ B=12,\ W=6$

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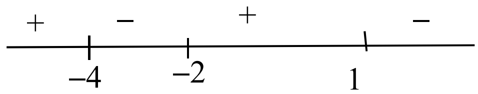
$\dfrac{2}{x+2} \geq \dfrac{x+1}{3}$

$\dfrac{2}{x+2} - \dfrac{x+1}{3} \geq 0$

$\dfrac{6-(x+1)(x+2)}{3(x+2)} \geq 0$

$\dfrac{-x^2-3x+4}{3(x+2)} \geq 0$

$\dfrac{-(x+4)(x-1)}{3(x+2)} \geq 0$



Hence, $x \leq -4$ or $-2 < x \leq 1$

For $\dfrac{2}{x+3} \geq \dfrac{x+2}{3}$

From above, $x+1 \leq -4$ or $-2 < x+1 \leq 1$

$x \leq -5$ or $-3 < x \leq 0$

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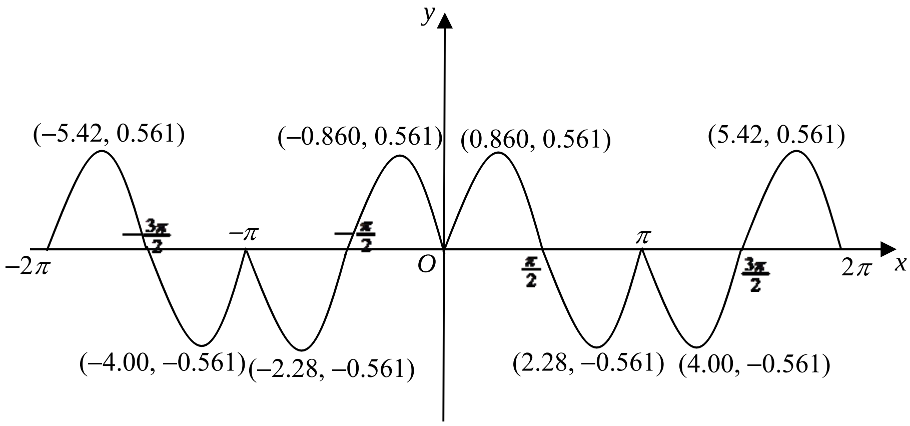
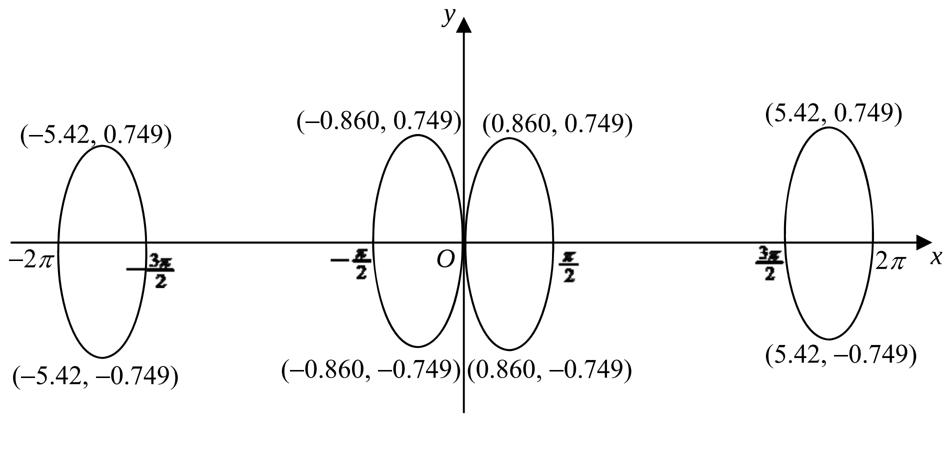
3(i)	$\frac{\mathrm{d}}{\mathrm{d}x}\left(x\mathrm{e}^{-x}\right)=-x\mathrm{e}^{-x}+\mathrm{e}^{-x}$ $=\mathrm{e}^{-x}\left(1-x\right)$
(ii)	$\int\frac{x-x^2}{\mathrm{e}^x}\,\mathrm{d}x=\int x\frac{\left(1-x\right)}{\mathrm{e}^x}\,\mathrm{d}x$ $=x^2\mathrm{e}^{-x}-\int x\mathrm{e}^{-x}\mathrm{d}x$ $=x^2\mathrm{e}^{-x}-\left[-x\mathrm{e}^{-x}+\int\mathrm{e}^{-x}\mathrm{d}x\right]$ $=x^2\mathrm{e}^{-x}+x\mathrm{e}^{-x}+\mathrm{e}^{-x}+C$
4(i)	
4(ii)	

5(i)	$3x+4y-7z=2$ $x-2y=4$ $5x-4y+2z=3$ From GC, $x=-\frac{15}{31}, y=-\frac{139}{62}, z=-\frac{55}{31}$ $\therefore$ the point of intersection is $\left(-\frac{15}{31},-\frac{139}{62},-\frac{55}{31}\right)$
(ii)	Let F be the foot of perpendicular from C to plane. Equation of CF: $\mathbf{r}=\begin{pmatrix}-1\\2\\1\end{pmatrix}+\lambda\begin{pmatrix}1\\-2\\0\end{pmatrix}, \lambda\in\mathbb{R}$ $\overrightarrow{OF}=\begin{pmatrix}-1\\2\\1\end{pmatrix}+\lambda\begin{pmatrix}1\\-2\\0\end{pmatrix}$, for some $\lambda\in\mathbb{R}$ F also lies on plane p_2 $\overrightarrow{OF}\cdot\begin{pmatrix}1\\-2\\0\end{pmatrix}=4$ $\begin{pmatrix}-1+\lambda\\2-2\lambda\\1\end{pmatrix}\cdot\begin{pmatrix}1\\-2\\0\end{pmatrix}=4$ $-1+\lambda-4+4\lambda=4$ $5\lambda=9$ $\lambda=\frac{9}{5}$ $\therefore\overrightarrow{OF}=\begin{pmatrix}-1\\2\\1\end{pmatrix}+\frac{9}{5}\begin{pmatrix}1\\-2\\0\end{pmatrix}=\begin{pmatrix}4/5\\-8/5\\1\end{pmatrix}$ the coordinates is $\left(\frac{4}{5},-\frac{8}{5},1\right)$
(iii)	$3x+4y-7z=2$ $x-2y=4$ From GC, p_1 and p_2 intersect at the line $\mathbf{r}=\begin{pmatrix}2\\-1\\0\end{pmatrix}+\mu\begin{pmatrix}14\\7\\10\end{pmatrix}, \mu\in\mathbb{R}$ No common points $\Rightarrow$ p_3 must be parallel to the line.

	$\begin{pmatrix} 5 \\ -4 \\ a \end{pmatrix} \cdot \begin{pmatrix} 14 \\ 7 \\ 10 \end{pmatrix} = 0$ $42 + 10a = 0$ $a = -4.2$
6(i)	$z = \frac{3+\mathrm{i}}{2-\mathrm{i}} = \frac{(3+\mathrm{i})(2+\mathrm{i})}{2^2+1} = \frac{1}{5}(5+5\mathrm{i}) = 1+\mathrm{i}$ Therefore, $\|z\| = \sqrt{2}$ [Or $\|z\| = \left\|\frac{3+\mathrm{i}}{2-\mathrm{i}}\right\| = \frac{\sqrt{10}}{\sqrt{5}} = \sqrt{2}$] arg $z = \frac{\pi}{4}$
(ii)	$\mathrm{e}^{x+\mathrm{i}2y} = z$ $\mathrm{e}^{x}\mathrm{e}^{\mathrm{i}2y} = \sqrt{2}\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}$ $\Rightarrow \mathrm{e}^{x} = \sqrt{2}$ or $\mathrm{e}^{\mathrm{i}2y} = \mathrm{e}^{\mathrm{i}\frac{\pi}{4}}$ or $\mathrm{e}^{\mathrm{i}\left(-\frac{7\pi}{4}\right)}$ $\Rightarrow x = \ln\sqrt{2} = \frac{1}{2}\ln 2$ or $y = \frac{\pi}{8}$ or $-\frac{7\pi}{8}$
(iii)	For $\left(\frac{z^2}{z^*}\right)^n$ to be purely imaginary, $\arg\left(\frac{z^2}{z^*}\right)^n = (2k+1)\frac{\pi}{2}, k \in \mathbf{Z}$ $n\left[2\arg z - \arg z^*\right] = (2k+1)\frac{\pi}{2}$ $n\left[\frac{\pi}{2} + \frac{\pi}{4}\right] = (2k+1)\frac{\pi}{2}$ $n = \frac{2}{3}(2k+1)$ Hence, the smallest positive integer $n = 2$

7	Let A and V be the external surface area and volume of the container respectively. $A=300y+150\pi x+2xy+2\pi\left(\frac{x}{2}\right)^2$ $V=150xy+150\pi\left(\frac{x}{2}\right)^2$ $7200=150xy+\frac{75}{2}\pi x^2$ $y=\frac{48}{x}-\frac{\pi x}{4}$ Substitute $y=\frac{48}{x}-\frac{\pi x}{4}$ into surface area equation: $A=300\left(\frac{48}{x}-\frac{\pi x}{4}\right)+150\pi x+2x\left(\frac{48}{x}-\frac{\pi x}{4}\right)+\frac{\pi x^2}{2}$ $=\frac{14400}{x}+96+75\pi x$ $\frac{\mathrm{d}A}{\mathrm{d}x}=-14400x^{-2}+75\pi$ To find least surface area, $\frac{\mathrm{d}A}{\mathrm{d}x}=0$ $-\frac{14400}{x^2}+75\pi=0$ $x=\sqrt{\frac{192}{\pi}}$ or $x=-\sqrt{\frac{192}{\pi}}$ (rejected $\because x>0$) By 2nd Derivative Test $\frac{\mathrm{d}^2A}{\mathrm{d}x^2}=\frac{28800}{x^3}$ When $x=\sqrt{\frac{192}{\pi}}$, $\frac{\mathrm{d}^2A}{\mathrm{d}x^2}>0$, A is minimum.
8(a)	Note that $\theta\geq 20$ or $\theta-20\geq 0$ $\frac{\mathrm{d}\theta}{\mathrm{d}t}=-k(\theta-20)$, $k>0$ Given when $t=0$, $\theta=80$, $\frac{\mathrm{d}\theta}{\mathrm{d}t}=-4$ $-4=-k(80-20)\Rightarrow k=\frac{1}{15}$ $\frac{\mathrm{d}\theta}{\mathrm{d}t}=-\frac{1}{15}(\theta-20)$

	$\int \frac{1}{\theta - 20}\,\mathrm{d}\theta = -\frac{1}{15}\int 1\,\mathrm{d}t$ $\ln(\theta - 20) = -\frac{1}{15}t + C$ $(\because \theta - 20 > 0)$ $\theta - 20 = \mathrm{e}^{-\frac{1}{15}t + C}$ $\theta - 20 = A\mathrm{e}^{-\frac{1}{15}t}$, where $A = \mathrm{e}^{c}$ $\Rightarrow \theta = 20 + A\mathrm{e}^{-\frac{1}{15}t}$ Alternatively: $\ln\lvert\theta - 20\rvert = -\frac{1}{15}t + C$ $\lvert\theta - 20\rvert = \mathrm{e}^{-\frac{1}{15}t + C}$ $\theta - 20 = A\mathrm{e}^{-\frac{1}{15}t}$, where $A = \pm\mathrm{e}^{c}$ $\Rightarrow \theta = 20 + A\mathrm{e}^{-\frac{1}{15}t}$ When $t = 0$, $\theta = 80$, $\Rightarrow 80 = 20 + A$ i.e. $A = 60$ $\therefore \theta = 20 + 60\,\mathrm{e}^{-\frac{1}{15}t}$ $40 = 20 + 60\,\mathrm{e}^{-\frac{1}{15}t}$ $\mathrm{e}^{-\frac{1}{15}t} = \frac{1}{3}$ $-\frac{1}{15}t = \ln\frac{1}{3} = -\ln 3$ $t = 15\ln 3$
(b)	$u = x + y$ $\Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 1 + \frac{\mathrm{d}y}{\mathrm{d}x}$ $\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x} - 1$ $\frac{\mathrm{d}y}{\mathrm{d}x} = (x + y)^2$ $\Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} - 1 = u^2$ $\Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 1 + u^2$ $\int \frac{1}{1 + u^2}\,du = \int 1\,dx$ $\tan^{-1}(u) = x + C$ $\tan^{-1}(x + y) = x + C$ When $x = 0$, $y = 0$, $C = 0$ $\therefore x + y = \tan x$ Hence, $y = \tan x - x$

9(a)	$y=\ln(\cos x)$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-\sin x}{\cos x}=-\tan x$ $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-\sec^2 x=-\left(1+\tan^2 x\right)$ $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-\left[1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right]$ $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2=-1$
(i)	$\frac{\mathrm{d}^3y}{\mathrm{d}x^3}+2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=0$ $\frac{\mathrm{d}^4y}{\mathrm{d}x^4}+2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\frac{\mathrm{d}^3y}{\mathrm{d}x^3}+2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=0$ When $x=0$, $y=0$, $\frac{\mathrm{d}y}{\mathrm{d}x}=0$, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-1$, $\frac{\mathrm{d}^3y}{\mathrm{d}x^3}=0$, $\frac{\mathrm{d}^4y}{\mathrm{d}x^4}=-2$ $y=\ln(\cos x)=\frac{x^2}{2!}(-1)+\frac{x^4}{4!}(-2)+...$ $=-\frac{1}{2}x^2-\frac{1}{12}x^4+...$
(ii)	$\tan x=-\frac{\mathrm{d}y}{\mathrm{d}x}$ $=-\frac{\mathrm{d}}{\mathrm{d}x}\left[-\frac{1}{2}x^2-\frac{1}{12}x^4+...\right]$ $=-\left(-x-\frac{1}{3}x^3+...\right)$ $=x+\frac{1}{3}x^3+...$
(b)	$\mathrm{f}\left(x\right)=\ln\left(k+x\right)^n$ $=n\ln\left(k+x\right)$ $=n\ln\left(k\left(1+\frac{x}{k}\right)\right)$ $=n\left[\ln k+\ln\left(1+\frac{x}{k}\right)\right]$ $=n\ln k+n\ln\left(1+\frac{x}{k}\right)$ $=n\ln k+n\left(\frac{x}{k}-\frac{1}{2}\left(\frac{x}{k}\right)^2+\cdots\right)$ $=n\ln k+\frac{nx}{k}-\frac{nx^2}{2k^2}+\cdots$

10 (i)	$\int \frac{1}{1+(3-y)^2}\,\mathrm{d}y = (-1)\tan^{-1}\left(\frac{3-y}{1}\right)+C$ $= -\tan^{-1}(3-y)+C$
(ii)	$y = 3-\frac{x}{\sqrt{4-x^2}}$ $\frac{x}{\sqrt{4-x^2}} = 3-y$ $\frac{x^2}{4-x^2} = (3-y)^2$ $x^2 = (3-y)^2(4-x^2)$ $x^2 = 4(3-y)^2 - x^2(3-y)^2$ $x^2 + x^2(3-y)^2 = 4(3-y)^2$ $x^2 = \frac{4(3-y)^2}{1+(3-y)^2}$ $= 4-\frac{4}{1+(3-y)^2}$ Volume of revolution about the y-axis $=\pi\int_1^3 x^2\,\mathrm{d}y$ $=\pi\int_1^3 4-\frac{4}{1+(3-y)^2}\,\mathrm{d}y$ $=\pi\int_1^3 4\,\mathrm{d}y - 4\pi\int_1^3 \frac{1}{1+(3-y)^2}\,\mathrm{d}y$ $=\pi[4y]_1^3 - 4\pi\left[-\tan^{-1}(3-y)\right]_1^3$ $=8\pi-4\pi\tan^{-1}(2)$
	Using the substitution $x = 2\sin\theta$ $\frac{\mathrm{d}x}{\mathrm{d}\theta} = 2\cos\theta$

	When $x=1, \sin\theta=\frac{1}{2}\Rightarrow\theta=\frac{\pi}{6}$ When $x=0, \sin\theta=0\Rightarrow\theta=0$ Area under the curve $=\int_0^1 3-\frac{x}{\sqrt{4-x^2}}\,\mathrm{d}x$ $=\int_0^{\frac{\pi}{6}}\left(3-\frac{2\sin\theta}{\sqrt{4-4\sin^2\theta}}\right)(2\cos\theta)\,\mathrm{d}\theta$ $=\int_0^{\frac{\pi}{6}} 6\cos\theta-\frac{4\sin\theta\cos\theta}{2\cos\theta}\,\mathrm{d}\theta$ $=\left[6\sin\theta+2\cos\theta\right]_0^{\frac{\pi}{6}}$ $=1+\sqrt{3}$
11	Distance travelled per lap is in AP: $a_1 = 50$, $d = 2 \times 1 = 2$. Given total distance travelled > 1500 $\frac{n}{2}[\,2(50)+(n-1)2] > 1500$ $n^2 + 49n - 1500 > 0$ $(n + 70.33)(n - 21.33) > 0$ $n < -70.33$ or $n > 21.33$ Since $n \in \mathbf{Z}^+$, least $n = 22$ Time taken per lap is in GP:

	$a_1 = 20$, $r = 1.15$ Required least time taken $= \dfrac{20\left((1.15)^{22} - 1\right)}{1.15 - 1}$ ≈ 2752.6 s ≈ 46 min $\dfrac{20\left((1.15)^{n} - 1\right)}{1.15 - 1} \geq 900$ $(1.15)^n \geq 7.75$ $n \geq \dfrac{\ln 7.75}{\ln 1.15}$ $n \geq 14.65$ Number of complete laps = 14 $S_{14} = \dfrac{20\left((1.15)^{14} - 1\right)}{1.15 - 1}$ $= 810.094$ s He needs to run for another $900 - 810.094 = 89.906$ s Distance $T_{15} = 50 + (15 - 1)2 = 78$ $\dfrac{89.906}{20(1.15)^{14}} \times 78 = 49.555$ He is running <u>towards</u> *S* and at a distance $78 - 49.555 \approx$ <u>28.4 m</u> away from *S*.

12 (i)	$l:\mathbf{r}=\begin{pmatrix}0\\-1\\2\end{pmatrix}+\lambda\begin{pmatrix}0\\1\\4\end{pmatrix},\lambda\in\mathbb{R}$. $\mathbf{n}_2=\begin{pmatrix}-2\\3\\1\end{pmatrix}\times\begin{pmatrix}0\\1\\4\end{pmatrix}$ $=\begin{pmatrix}11\\8\\-2\end{pmatrix}$ $q:\mathbf{r}\cdot\begin{pmatrix}11\\8\\-2\end{pmatrix}=\begin{pmatrix}0\\-1\\2\end{pmatrix}\cdot\begin{pmatrix}11\\8\\-2\end{pmatrix}$ $=-12$ $\therefore$ the Cartesian equation is $11x+8y-2z=-12$
(ii)	$-2x+3y+z=-4$ $11x+8y-2z=-12$ From GC, $x=-\frac{4}{49}+\frac{2}{7}\mu, y=-\frac{68}{49}-\frac{1}{7}\mu, z=\mu$ $\therefore\mathbf{r}=\begin{pmatrix}-\frac{4}{49}\\-\frac{68}{49}\\0\end{pmatrix}+\mu\begin{pmatrix}2\\-1\\7\end{pmatrix},\mu\in\mathbb{R}$
(iii)	$\begin{pmatrix}0\\-1+\lambda\\2+4\lambda\end{pmatrix}=\begin{pmatrix}-4/49+2\mu\\-68/49-\mu\\7\mu\end{pmatrix}$ From GC, $\lambda=-\frac{3}{7},\mu=\frac{2}{49}$ Substitute $\lambda=-\frac{3}{7}$ into *l*: $\mathbf{r}=\begin{pmatrix}0\\-1\\2\end{pmatrix}-\frac{3}{7}\begin{pmatrix}0\\1\\4\end{pmatrix}=\begin{pmatrix}0\\-\frac{10}{7}\\\frac{2}{7}\end{pmatrix}$ $\therefore$ the coordinates of *C* is $\left(0,-\frac{10}{7},\frac{2}{7}\right)$.

(iv)	$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}$ $=\begin{pmatrix}x\\0\\0\end{pmatrix}-\begin{pmatrix}0\\-1\\2\end{pmatrix}$ $=\begin{pmatrix}x\\1\\-2\end{pmatrix}$ $\overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA}$ $=\begin{pmatrix}0\\-\frac{10}{7}\\\frac{2}{7}\end{pmatrix}-\begin{pmatrix}0\\-1\\2\end{pmatrix}$ $=\begin{pmatrix}0\\-\frac{3}{7}\\-\frac{12}{7}\end{pmatrix}$ Area of ABC, $R=\frac{1}{2}\left\lvert\overrightarrow{AB}\times\overrightarrow{AC}\right\rvert$ $=\frac{1}{2}\left\lvert\begin{pmatrix}x\\1\\-2\end{pmatrix}\times\begin{pmatrix}0\\-3/7\\-12/7\end{pmatrix}\right\rvert$ $=\frac{1}{2}\left\lvert\begin{pmatrix}-18/7\\12x/7\\-3x/7\end{pmatrix}\right\rvert=\frac{3}{14}\left\lvert\begin{pmatrix}-6\\4x\\-x\end{pmatrix}\right\rvert$ $=\frac{3}{14}\sqrt{(-6)^2+(4x)^2+(-x)^2}$ $=\frac{3}{14}\sqrt{36+16x^2+x^2}$ $=\frac{3}{14}\sqrt{36+17x^2}$

	$\frac{dR}{dx} = \frac{3}{14} \times \frac{1}{2}\left(36+17x^2\right)^{-\frac{1}{2}}\left(34x\right)$ $= \frac{102x}{28\sqrt{36+17x^2}}$ $= \frac{51x}{14\sqrt{36+17x^2}}$ when $x = \sqrt{5}, \frac{dx}{dt} = \frac{dx}{dR} \times \frac{dR}{dt}$ $= \frac{14(11)}{51\sqrt{5}} \times 17$ $= \frac{154}{3\sqrt{5}}$ units per second