SERANGOON JUNIOR COLLEGE

2016 JC2 PRELIMINARY EXAMINATION

MATHEMATICS

Higher 2 **9740/1**

15 Sept 2016

3 hours

Additional materials: Writing paper

List of Formulae (MF15)

TIME : 3 hours

READ THESE INSTRUCTIONS FIRST

Write your name and class on the cover page and on all the work you hand in.
Write in blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphic calculator.
Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets **[ ]** at the end of each question or part question.
At the end of the examination, fasten all your work securely together.

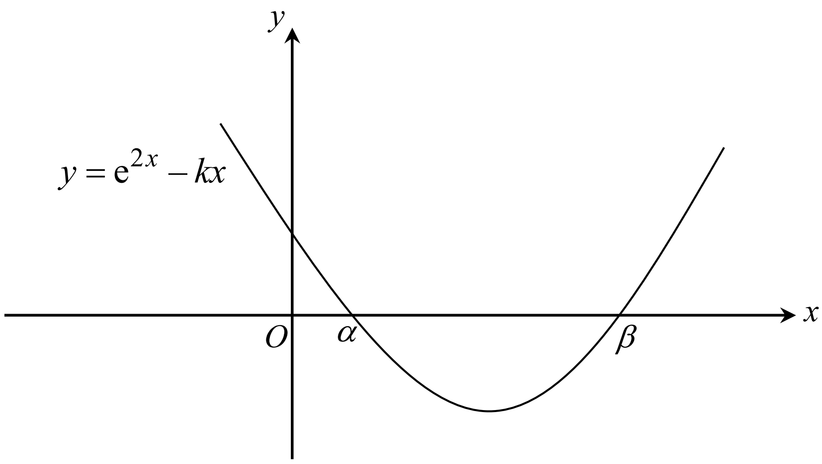
Total marks for this paper is 100 marks

This question paper consists of 6 printed pages (inclusive of this page) and 2 blank page.

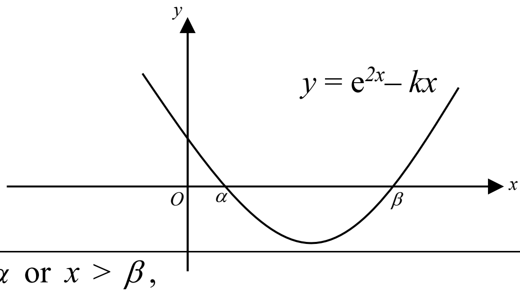
[TURN OVER]

Answer all questions [100 marks].

1	State a sequence of 3 transformations which transform the graph of $g(x)=e^{(6x+2)^2}+1$ to the graph of $f(x)=e^{(2x-2)^2}$.	[3]
	Solution	
	I) Translation in the negative y-direction by 1 unit $y=e^{(6x+2)^2}+1 \to y=e^{(6x+2)^2}$	
	II) Stretch parallel to the x-axis by a scale factor 3. $y=e^{(6x+2)^2} \to y=e^{\left(6\left(\frac{x}{3}\right)+2\right)^2}=e^{(2x+2)^2}$	
	III) Translation in the positive x-direction by 2 units y= $e^{(2x+2)^2} \to y=e^{(2(x-2)+2)^2}=e^{(2x-2)^2}$	
	Alternatively	
	I) Translation in the negative y-direction by 1 unit $y=e^{(6x+2)^2}+1 \to y=e^{(6x+2)^2}$	
	II) Translation in the positive x-direction by 2/3 units $y=e^{(6x+2)^2} \to y=e^{\left(6\left(x-\frac{2}{3}\right)+2\right)^2}=e^{(6x-2)^2}$	
	III) Stretch parallel to the x-axis by a scale factor 3. $y=e^{(6x-2)^2} \to y=e^{\left(6\left(\frac{x}{3}\right)-2\right)^2}=e^{(2x-2)^2}$	
	Note: The translation can be step 1, 2 or 3.	
2	Using the standard series expansions, obtain the Maclaurin series of $\ln\left[(1+x)(1-2x)^3\right]$ in ascending powers of x, up to and including the term in x^3.	[2]
	(i) Find the set of values of x for which the above expansion is valid.	[1]
	(ii) Hence, find the range of values of x for which the expansion $\left[e^{\ln\left[\ln\frac{1}{(1+2x)^3}\right]}-\ln(1-x)\right](2+7x)^5$ is valid.	[2]
	Solution	
	$\ln[(1+x)(1-2x)^3]=\ln(1+x)+3\ln(1-2x)$ $=x-\frac{x^2}{2}+\frac{x^3}{3}+3\left(-2x-2x^2-\frac{8x^3}{3}\right)+\cdots$	

	$= -5x - \frac{13x^2}{2} - \frac{23x^3}{3} + \cdots$	
	(i) So $-1 < -2x \le 1$ and $-1 < x \le 1$	
	$-\frac{1}{2} \le x < \frac{1}{2}$ and $-1 < x \le 1$	
	$\left\{ x \in \mathbb{R} : -\frac{1}{2} \le x < \frac{1}{2} \right\}$	
	(ii) $\left[e^{\ln\left[\ln \frac{1}{(1+2x)^3}\right]} - \ln(1-x) \right](2+7x)^3 = \left[-3\ln(1+2x) - \ln(1-x)\right](2+7x)^3$	
	$= -\left[3\ln(1+2x) + \ln(1-x)\right](2+7x)^3$	
	From **(i)**, replace x by $-x$,	
	$-\frac{1}{2} < x \le \frac{1}{2}$ and $x \in \mathbb{R}$	
	$\therefore$ Range of values of x is $-\frac{1}{2} < x \le \frac{1}{2}$.	
3	**(a)** 	
	The diagram shows the graph of $y = e^{2x} - kx$, where k is a positive real number. The two roots of the equation $e^{2x} - kx = 0$ are denoted by α and β, where $\alpha < \beta$.	
	It is given that there is a sequence of real numbers $x_1, x_2, x_3 ...$ that satisfies the recurrence relation, $x_{n+1} = \frac{1}{k} e^{2x_n}$, for $n \ge 1$.	
	By considering $x_{n+1} - x_n$, prove that $x_{n+1} > x_n$ if $x_n < \alpha$ or $x_n > \beta$.	[2]
	(b) Prove by the method of mathematical induction that $\sum_{r=1}^{n} \cos r\theta = \frac{\sin(n+\frac{1}{2})\theta - \sin\frac{1}{2}\theta}{2\sin\frac{1}{2}\theta}$, for all positive integers n.	[5]

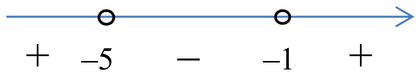
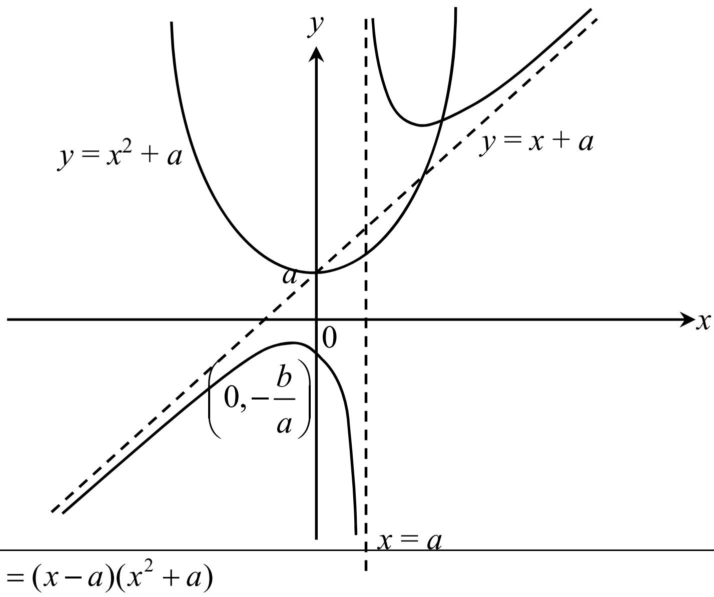
[TURN OVER]

	Solution	
	(a) $x_{n+1} - x_n$ $= \dfrac{1}{k}e^{2x_n} - x_n$ $= \dfrac{e^{2x_n} - kx_n}{k}$  	
	From given sketch, if $x < \alpha$ or $x > \beta$, $e^{2x} - kx > 0$ So if $x_n < \alpha$ or $x_n > \beta$, $x_{n+1} - x_n = \dfrac{e^{2x_n} - kx_n}{k} > 0$ Therefore, $x_{n+1} > x_n$ (Proven)	
	(b) Let P_n be the statement $\cos\theta + \cos 2\theta + \cos 3\theta + ... + \cos n\theta = \dfrac{\sin(n+\frac{1}{2})\theta - \sin\frac{1}{2}\theta}{2\sin\frac{1}{2}\theta}$ for $n \in \square^{+}$.	
	To show P_1 is true,	
	$\text{LHS} = \sum_{r=1}^{1}\cos r\theta = \cos\theta$ $\text{RHS} = \dfrac{\sin\frac{3}{2}\theta - \sin\frac{1}{2}\theta}{2\cos\frac{1}{2}\theta}$ $= \dfrac{2\cos\theta\sin(\frac{1}{2}\theta)}{2\sin\frac{1}{2}\theta}$ $= \cos\theta = \text{LHS}$ $\therefore P_1$ is true.	
	Suppose P_k is true for some $k \in \mathbf{Z}^+$, i.e. $\sum_{r=1}^{k}\cos r\theta = \dfrac{\sin(k+\frac{1}{2})\theta - \sin\frac{1}{2}\theta}{2\sin\frac{1}{2}\theta}$	
	To show P_{k+1} is true, i.e. $\sum_{r=1}^{k+1}\cos r\theta = \dfrac{\sin(k+\frac{3}{2})\theta - \sin\frac{1}{2}\theta}{2\sin\frac{1}{2}\theta}$ $\text{LHS} = \sum_{r=1}^{k+1}\cos r\theta$ $= \sum_{r=1}^{k}\cos r\theta + \cos(k+1)\theta$	
	$= \dfrac{\sin(k+\frac{1}{2})\theta - \sin\frac{1}{2}\theta}{2\sin\frac{1}{2}\theta} + \cos(k+1)\theta$	
	$= \dfrac{\sin(k+\frac{1}{2})\theta - \sin\frac{1}{2}\theta + 2\cos(k+1)\theta\sin\frac{1}{2}\theta}{2\sin\frac{1}{2}\theta}$	

	$= \dfrac{\sin(k+\frac{1}{2})\theta - \sin\frac{1}{2}\theta + \sin(k+\frac{3}{2})\theta - \sin(k+\frac{1}{2})\theta}{2\sin\frac{1}{2}\theta}$ $= \dfrac{\sin(k+\frac{3}{2})\theta - \sin\frac{1}{2}\theta}{2\sin\frac{1}{2}\theta} = \text{RHS}$	
	Hence, P_{k+1} is true if P_k is true. (Missing is "P")	
	Since P_1 is true and P_{k+1} is true when P_k is true, by Mathematical Induction, P_n is true for all positive integers n.	
4	Andy and his fiancée signed up for a new 4-room flat in Boon Keng. They take up a housing loan of $450,000 provided by BEST bank for the purchase. The couple pay a fixed monthly instalment of $$A$ on the first day of each month. Interest is charged on the last day of each year at a fixed rate of 1.6% of the remaining loan amount at the beginning of that year. If the first instalment is paid in January 2016,	
	(i) Show that the amount the couple owe the bank at the end of 2017 is $\$[464515.2 - 24.192A]$.	[1]
	(ii) Given that A is 1500, find the date and amount of the final repayment to the nearest cent.	[5]
	Solution	
	(i) Amt owe at the end of 2017 = (1.016)(1.016)(450000) – 1.016(12A) – 12A $= \$[464515.2 - 24.192A]$	
	(ii)	

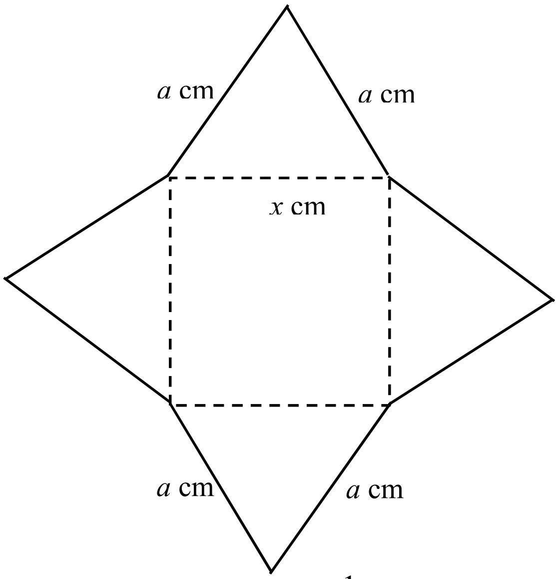
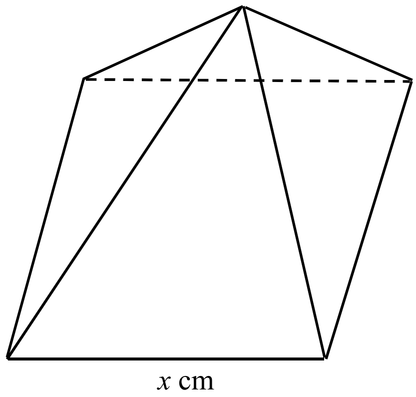
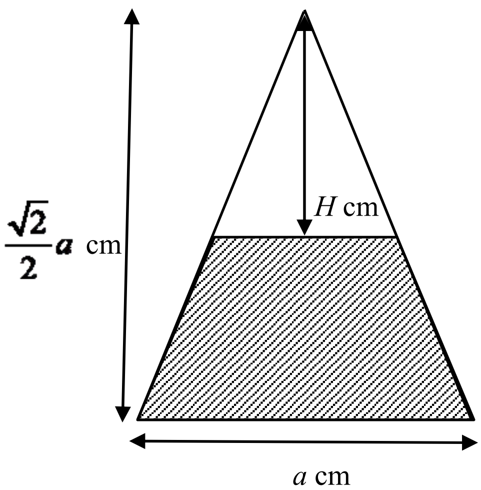
year	Amt owed at the beginning	Amt owed at the end of the year after paying 18000
1st	450000	1.016(450000) -18000
2nd	1.016(450000) -12A	(1.016)(1.016)(450000) – 1.016(18000) – 18000
3rd	$(1.016^2)(450000)$ – 1.016(12A) – 12A	$(1.016)(1.016^2)(450000)$ – (1.016)(1.016)(18000) – (1.016)(18000) – 18000
	…	…
nth		$(1.016^n)(450000)$ –

	$(1.016^{n-1})(18000)$ – $(1.016^{n-2})(18000)$ – …… – 18000	
	Amount of money owe at the end of nth year $= 450000(1.016)^n - 18000(1 + 1.016 + 1.016^2 + ... + 1.016^{n-1})$	
	Consider $450000(1.016)^n - 18000\left(\dfrac{1\left(1.016^n - 1\right)}{1.016 - 1}\right) \leq 0$	
	$450000(1.016)^n - 1125000\left(1.016^n - 1\right) \leq 0$	
	Using G.C, $n \geq 32.2$	
	When $n = 32$,	
	Amount owe at the end of 32 years $= \$450000(1.016)^{32} - \$1125000\left(1.016^{32} - 1\right) = \$3233.601$	
	Since they will be paying \$1500 each month, they will finished the payment on 1st March 2048. The last payment is \$233.60.	
5	**(a)** It is given that $y = \dfrac{x^2 - x - 1}{x + 1}$, $x \in \mathbb{R}$, $x \neq -1$. Without using a graphic calculator, find the set of values that y cannot take.	[3]
	(b) The curve C has equation $y = \dfrac{x^2 + b}{x - a}$, where $a > 0$, $b > a$ and $x \neq a$	
	(i) Draw a sketch of the curve C, label clearly the equation(s) of its asymptote(s) and the coordinates of any intersection with the axes.	[3]
	(ii) By drawing an additional graph on the diagram drawn in **(i)**, state the number of real root(s) of the equation $x^2 + b = (x - a)\left(x^2 + a\right)$.	[2]
	Solution	
	(a) Consider any horizontal line $y = k$, $k \in \mathbb{R}$. Consider the intersection of the graphs $y = \dfrac{x^2 - x - 1}{x + 1}$ and $y = k$, i.e. $\dfrac{x^2 - x - 1}{x + 1} = k$ $\Rightarrow x^2 - x - 1 = k(x + 1)$ $\Rightarrow x^2 + x(-1 - k) + (-1 - k) = 0$	
	For the equation to have no real solutions, Discriminant < 0 $\Rightarrow (-1 - k)^2 - 4(1)(-1 - k) < 0$	

	$\Rightarrow k^2+6k+5<0$ $(k+1)(k+5)<0$  $\therefore$ The set of values that y cannot take is $\{y\in\mathbb{R} : -5<y<-1\}$	
	(bi) When $x=0$, $y=-\dfrac{b}{a}$ $y=\dfrac{x^2+b}{x-a}=(x+a)+\dfrac{a^2+b}{x-a}$ $x=a$ and $y=x+a$, are equations of the asymptotes	
	(ii)  	
	$x^2+b=(x-a)(x^2+a)$ $\Rightarrow \dfrac{x^2+b}{x-a}=x^2+a$	
	By adding an additional graph in (i), i.e. $y=x^2+a$, no. of real root is 1.	
6	**(a)** The equations of two planes p_1 and p_2 are $x+4y+2z=7$, $3x+\lambda y+4z=\mu$, respectively, where λ and μ are constants.	
	(i) Given that the two planes intersect in a line l, with a vector equation given by $\mathbf{r}=\begin{pmatrix}1\\1\\1\end{pmatrix}+s\begin{pmatrix}-2\\1\\-1\end{pmatrix},\ s\in\mathbf{R},$ show that the value of λ is 10 and find the value of μ .	[3]
	(ii) If plane p_3 is the reflection of p_1 in p_2 , find the acute angle between p_1 and p_3 .	[2]

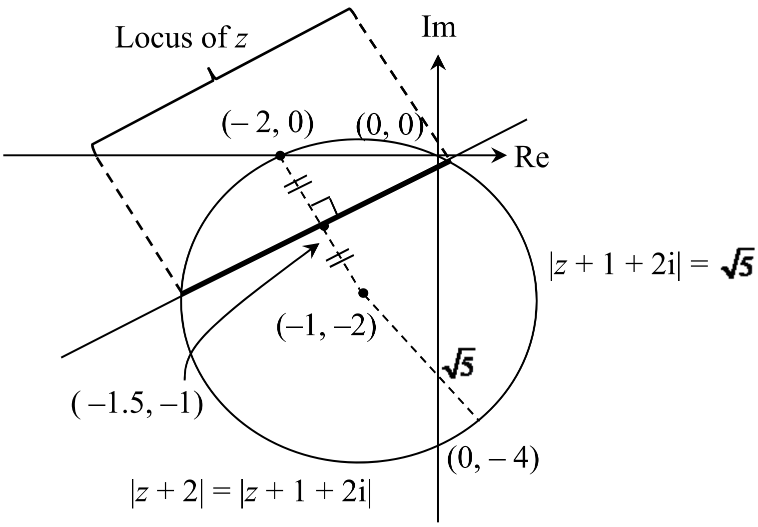
[TURN OVER]

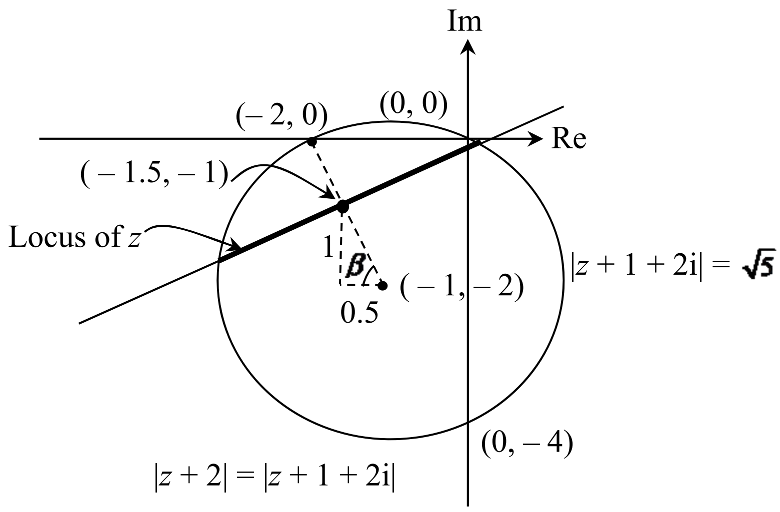
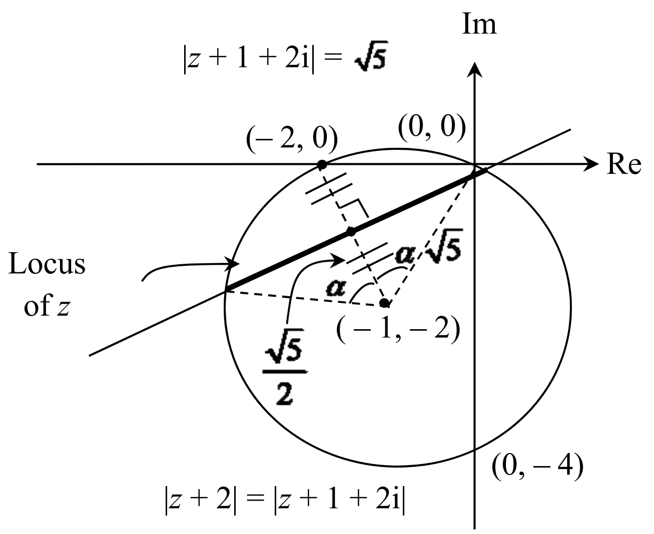
	(b) Relative to the origin O, the points A, B, C and D have position vectors **a**, **b**, **c** and **d** respectively. It is given that λ and μ are non-zero numbers such that $\lambda\mathbf{a}+\mu\mathbf{b}-\mathbf{c}=\mathbf{0}$ and $\lambda+\mu=1$,	
	(i) Show that A, B and C are collinear.	[3]
	(ii) If O is not on the line AC and $\lvert\mathbf{c}\times\mathbf{a}\rvert(\mathbf{b}-\mathbf{a})=(\mathbf{c}\cdot\mathbf{d})\,\mathbf{d}$, determine the relationship between $\overrightarrow{AC}$ and $\overrightarrow{OD}$, explaining your answer clearly.	[2]
	Solution	
	(ai) $\begin{pmatrix}-2\\1\\-1\end{pmatrix}\cdot\begin{pmatrix}3\\\lambda\\4\end{pmatrix}=0$	
	$\lambda=10$	
	$\mu=\begin{pmatrix}1\\1\\1\end{pmatrix}\bullet\begin{pmatrix}3\\10\\4\end{pmatrix}=17$	
	(ii) Let θ be the angle between p_1 and p_2. $\cos\theta=\dfrac{\begin{pmatrix}1\\4\\2\end{pmatrix}\bullet\begin{pmatrix}3\\10\\4\end{pmatrix}}{\sqrt{1^2+4^2+2^2}\sqrt{3^2+10^2+4^2}}$	
	$\cos\theta=\dfrac{51}{\sqrt{21}\sqrt{125}}$	
	$\theta=5.4869^\circ$	
	Acute angle between p_1 and p_3 = $2(5.4869^\circ)=11.0^\circ$	
	(bi) $\overrightarrow{AB}=\mathbf{b}-\mathbf{a}$ $\overrightarrow{AC}=\mathbf{c}-\mathbf{a}$ $=\lambda\mathbf{a}+\mu\mathbf{b}-\mathbf{a}$	
	$=(\lambda-1)\mathbf{a}+\mu\mathbf{b}$	
	$=-\mu\mathbf{a}+\mu\mathbf{b}$	
	$=\mu(\mathbf{b}-\mathbf{a})$	
	Since $\overrightarrow{AC}=\mu\,\overrightarrow{AB}$ for some $\mu\in\mathbb{R}\setminus\{0\}$, A, B, C are collinear.	
	(ii) $\lvert\mathbf{c}\times\mathbf{a}\rvert(\mathbf{b}-\mathbf{a})=(\mathbf{c}\cdot\mathbf{d})\,\mathbf{d}\Rightarrow\overrightarrow{AB}=k\,\overrightarrow{OD}$ for some $k\in\mathbb{R}$ as $\lvert\mathbf{c}\times\mathbf{a}\rvert\neq0$ since O is not on AC	
	since $\overrightarrow{AB}=\mu\,\overrightarrow{AC}$ for some $\mu\in\mathbb{R}$ so $\overrightarrow{AC}$ is parallel to $\overrightarrow{OD}$.	

7	A piece of metal with negligible thickness has been cut into a shape that is made up of four isosceles triangles each with base x cm and fixed sides a cm. Their bases frame to a form a square with sides of length x cm. A right pyramid is formed by folding along the dotted lines as shown in the diagram below.   [Volume of a pyramid = $\frac{1}{3} \times$ base area $\times$ height]	
	(i) Show that the volume of the pyramid is $\frac{x^2}{3}\sqrt{a^2 - \frac{x^2}{2}}$ cm^3.	[2]
	(ii) Find the value of x, in terms of a, that will give maximum volume for the pyramid.	[4]
	(iii)  To make the pyramid into a paperweight with negligible thickness, a viscous fluid is pumped into the interior at a rate of 1 cm^3/s. Given that H cm is the perpendicular distance from the apex of the pyramid to the viscous fluid surface,	

	$x=a$ and the height of the pyramid is $\frac{\sqrt{2}}{2}a$ cm, find the rate at which H is changing when $H=\frac{a}{2}$, giving your answer in terms of a. [The diagram above shows the cross sectional area of the pyramid.]	[3]
	Solution	
	(i) Let the height of the isosceles triangle be k cm. $k^2+\frac{x^2}{4}=a^2$ $k^2=a^2-\frac{x^2}{4}$	
	Therefore, height of the pyramid $=\sqrt{a^2-\frac{x^2}{4}-\frac{x^2}{4}}=\sqrt{a^2-\frac{x^2}{2}}$	
	Volume of pyramid, $V=\frac{x^2}{3}\sqrt{a^2-\frac{x^2}{2}}$	
	(ii) $\frac{\mathrm{d}V}{\mathrm{d}x}=\frac{4x}{3}\sqrt{a^2-\frac{x^2}{2}}+\frac{x^2}{3}\left(\frac{1}{2}\right)\frac{-x}{\sqrt{a^2-\frac{x^2}{2}}}$	
	$=\frac{4x\left(a^2-\frac{x^2}{2}\right)-x^3}{6\sqrt{a^2-\frac{x^2}{2}}}$ $=\frac{4a^2x-3x^2}{6\sqrt{a^2-\frac{x^2}{2}}}$ $=\frac{2x\left(a-\frac{\sqrt{3}}{2}x\right)\left(a+\frac{\sqrt{3}}{2}x\right)}{3\sqrt{a^2-\frac{x^2}{2}}}$	
	When $\frac{\mathrm{d}V}{\mathrm{d}x}=0$, $\frac{2x\left(a-\frac{\sqrt{3}}{2}x\right)\left(a+\frac{\sqrt{3}}{2}x\right)}{3\sqrt{a^2-\frac{x^2}{2}}}=0$	
	$\frac{2x}{3}\left(a^2-\frac{x^2}{2}\right)=\frac{x^3}{6}$	

	$x=\frac{2\sqrt{3}}{3}a$ or $-\frac{2\sqrt{3}}{3}a$ (rejected $\because x>0$)	
	When $x=\frac{2\sqrt{3}}{3}a^-$, $\frac{1}{\sqrt{a^2-\frac{x^2}{2}}}>0$, $\frac{2}{3}x>0$, $a-\frac{\sqrt{3}}{2}x>0$ and $a+\frac{\sqrt{3}}{2}x>0$, thus $\frac{\mathrm{d}V}{\mathrm{d}x}>0$. When $x=\frac{2\sqrt{3}}{3}a^+$, $\frac{1}{\sqrt{a^2-\frac{x^2}{2}}}>0$, $\frac{2}{3}x>0$, $a-\frac{\sqrt{3}}{2}x<0$ and $a+\frac{\sqrt{3}}{2}x>0$, thus $\frac{\mathrm{d}V}{\mathrm{d}x}<0$. Therefore $x=\frac{2\sqrt{3}}{3}a$ gives maximum volume.	
	(iii) Volume of pyramidal empty space, W cm^3, in the pyramid as it is being filled up $=\frac{1}{3}b^2H$, where b is the length of the square base $\frac{b}{H}=\sqrt{2}$ Therefore $W=\frac{2}{3}H^3 \Rightarrow \frac{\mathrm{d}W}{\mathrm{d}H}=2H^2$	
	When $H=\frac{a}{2}$	
	$\frac{\mathrm{d}W}{\mathrm{d}H}\times\frac{\mathrm{d}H}{\mathrm{d}t}=\frac{\mathrm{d}W}{\mathrm{d}t}$ $2\left(\frac{a}{2}\right)^2\times\frac{\mathrm{d}H}{\mathrm{d}t}=-1$	
	$\frac{\mathrm{d}H}{\mathrm{d}t}=-\frac{2}{a^2}$	
	H is decreasing at a rate of $\frac{2}{a^2}$ cm/s.	
8	**(i)** Show that $\left(0,-\frac{1}{4}\right)$ lies on the locus $\lvert z+2\rvert=\lvert z+1+2\mathrm{i}\rvert$.	[1]
	(ii) Sketch on a single Argand diagram the loci $\lvert z+1+2\mathrm{i}\rvert=\sqrt{5}$ and $\lvert z+2\rvert=\lvert z+1+2\mathrm{i}\rvert$.	[4]

	(iii) Hence indicate clearly on the Argand diagram the locus of z that satisfies the relations $\lvert z+1+2\mathrm{i}\rvert \le \sqrt{5}$ and $\lvert z+2\rvert = \lvert z+1+2\mathrm{i}\rvert$.	[1]
	(iv) Find the greatest and least possible values of $\arg(z+1+2\mathrm{i})$, giving your answers in radians correct to 3 decimal places.	[4]
	Solution	
	(i) When $z=-\frac{1}{4}\mathrm{i}$ $\text{L.H.S}=\left\lvert -\frac{1}{4}\mathrm{i}+2\right\rvert=\sqrt{4\frac{1}{16}}$ $\text{R.H.S}=\left\lvert -\frac{1}{4}\mathrm{i}+1+2\mathrm{i}\right\rvert=\left\lvert 1+\frac{7}{4}\mathrm{i}\right\rvert=\sqrt{4\frac{1}{16}}$ Hence $\left(0,-\frac{1}{4}\right)$ lies on $\lvert z+2\rvert = \lvert z+1+2\mathrm{i}\rvert$.	
	(ii) & (iii)  	
	(iv)	

		
	$\beta = \tan^{-1}(\frac{1}{0.5}) = \tan^{-1}(2)$	
		
	$\alpha = \cos^{-1}\left[\frac{\left(\frac{\sqrt{5}}{2}\right)}{\sqrt{5}}\right] = \cos^{-1}(\frac{1}{2}) = \frac{\pi}{3}$	
	Greatest value of arg (z +1+2i) = $\pi - (\beta - \alpha)$ = 3.08164 = 3.082	
	Least value of arg (z +1+2i) = $\pi - \beta - \alpha$ = 0.987246 = 0.987	
	(iv) **Alternative Method**	

<table>
<tr><td></td><td>

Equation of circle: $(x + 1)^2 + (y + 2)^2 = 5$

$$y = -2 \pm\sqrt{5-(x+1)^2} \text{ ------------------ (1)}$$

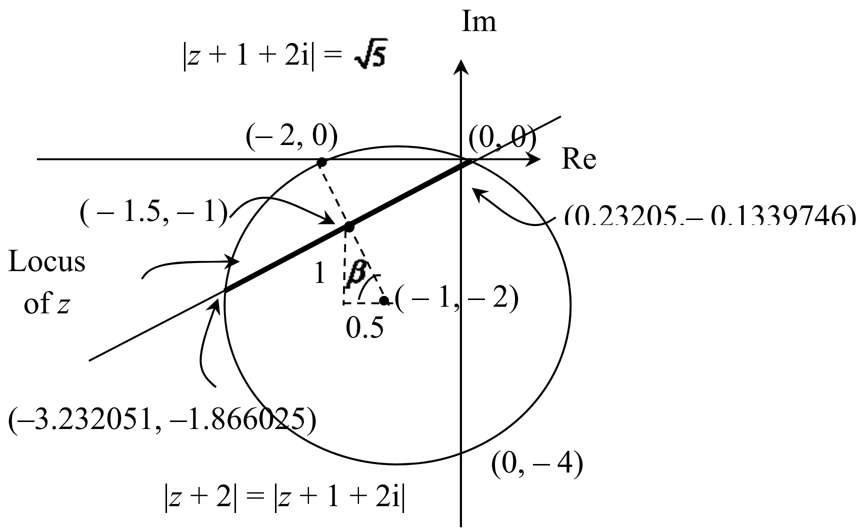
Gradient of the line passing through the $(-2, 0)$ and $(-1, -2)$ $= \dfrac{0-(-2)}{(-2)-(-1)} = -2$

Gradient of the perpendicular bisector $= -\dfrac{1}{-2} = \dfrac{1}{2}$

Equation of the perpendicular bisector: $y + 1 = \dfrac{1}{2}(x + 1.5)$

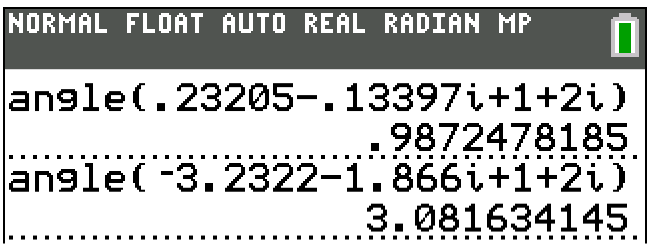
$$y = \frac{1}{2}x - \frac{1}{4} \qquad \text{------------------ (2)}$$

Using GC, the points of intersection are $(-3.232051, -1.866025)$ and $(0.23205, -0.1339746)$.



Least value of $\arg(z + 1 + 2i)$ when $z = 0.23205 - 0.1339746i$ is 0.987.

Greatest value of $\arg(z + 1 + 2i)$ when $z = -3.232051 - 1.866025i$ is 3.082



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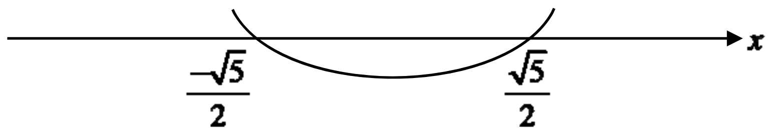
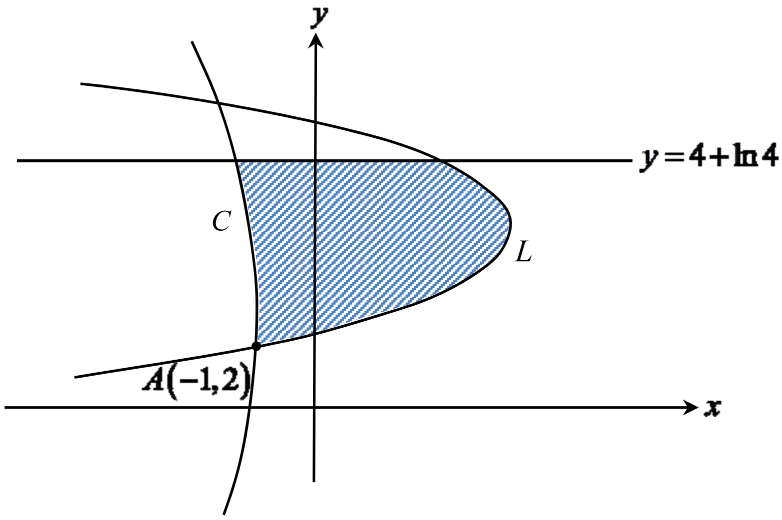
<tr><td></td><td></td><td></td></tr>
<tr><td>9</td><td>The path travelled by an object measured with respect to the origin in the horizontal and vertical directions, at time t seconds, is denoted by the variables x and y respectively.</td><td></td></tr>
</table>

	It is given that when $t=0,\ x=1,\ y=0$ and $\frac{\mathrm{d}x}{\mathrm{d}t}=1$. The variables are related by the differential equations $\frac{\mathrm{d}y}{\mathrm{dt}}-y+\sqrt{e^{2t}-4y^2}=0$ and $\frac{\mathrm{d}^2x}{\mathrm{d}t^2}=\cos^2 2t$.	
	(i) Using the substitution $y=w\mathrm{e}^t$, show $\frac{\mathrm{d}w}{\mathrm{d}t}=-\sqrt{1-4w^2}$ and hence find y in terms of t.	[6]
	(ii) Find x in terms of t.	[4]
	Solution	
	(i) $\frac{\mathrm{d}y}{\mathrm{dt}}-y+\sqrt{e^{2t}-4y^2}=0$	
	$y=w\mathrm{e}^t$ $\frac{\mathrm{d}y}{\mathrm{d}t}=\frac{\mathrm{d}w}{\mathrm{d}t}\mathrm{e}^t+w\mathrm{e}^t$	
	$\frac{\mathrm{d}y}{\mathrm{d}t}=\mathrm{e}^t\left(\frac{\mathrm{d}w}{\mathrm{d}t}+w\right)$	
	$\mathrm{e}^t\left(\frac{\mathrm{d}w}{\mathrm{d}t}+w\right)-w\mathrm{e}^t+\sqrt{\mathrm{e}^{2t}-4\left(w\mathrm{e}^t\right)^2}=0$	
	$\mathrm{e}^t\left(\frac{\mathrm{d}w}{\mathrm{d}t}+w\right)-w\mathrm{e}^t+\mathrm{e}^t\sqrt{1-4w^2}=0$ $\frac{\mathrm{d}w}{\mathrm{d}t}+w-w+\sqrt{1-4w^2}=0$ $\frac{\mathrm{d}w}{\mathrm{d}t}=-\sqrt{1-4w^2}\ (\text{shown})$	
	$\frac{1}{\sqrt{1-4w^2}}\frac{\mathrm{d}w}{\mathrm{d}t}=-1$	
	$\frac{1}{2}\sin^{-1}(2w)=-t+C$	
	$w=\frac{1}{2}\sin(-2t+2C)$	
	$y=\frac{1}{2}\mathrm{e}^t\sin(-2t+C)$	
	When $t=0, y=0$ $\frac{1}{2}\sin(-2C)=0$ $C=0$ $\therefore y=\frac{1}{2}\mathrm{e}^t\sin(-2t)$ or $y=-\frac{1}{2}\mathrm{e}^t\sin(2t)$	
	(ii) $\frac{\mathrm{d}^2x}{\mathrm{d}t^2}=\cos^2 2t$	

	$=\dfrac{1+\cos 4t}{2}$	
	$\dfrac{\mathrm{d}x}{\mathrm{d}t}=\dfrac{1}{2}\left(t+\dfrac{\sin 4t}{4}\right)+D_1$	
	$x=\dfrac{1}{2}\left(\dfrac{t^2}{2}-\dfrac{\cos 4t}{16}\right)+D_1t+D_2$	
	When $t=0, x=1, \dfrac{\mathrm{d}x}{\mathrm{d}t}=1$	
	$D_1=1, D_2=\dfrac{33}{32}$	
	$x=\dfrac{t^2}{4}-\dfrac{\cos 4t}{32}+t+\dfrac{33}{32}$	
10	Given that $\mathrm{f}(r)=\dfrac{3^r}{r+1}$, show that $\mathrm{f}(r+2)-\mathrm{f}(r)=\dfrac{(8r+6)3^r}{(r+1)(r+3)}$.	[1]
	(i) Find $\displaystyle\sum_{r=1}^{n}\frac{(4r+3)3^r}{(r+1)(r+3)}$ in terms of n.	[2]
	(ii) Hence find $\displaystyle\sum_{r=1}^{n}\frac{(4r+11)3^r}{(r+3)(r+5)}$ in terms of n.	[4]
	(iii) Using the result in **(ii)**, show that $\displaystyle\sum_{r=0}^{n}\left[\frac{r\cdot 3^r}{(r+5)^2}\right]-\frac{3^{n+1}}{4}<-\frac{51}{160}$.	[3]
	Solution	
	$\mathrm{f}(r+2)-\mathrm{f}(r)=\dfrac{3^{r+2}}{r+3}-\dfrac{3^r}{r+1}$	
	$=\dfrac{r\cdot 3^{r+2}+3^{r+2}-3^r\cdot r-3^{r+1}}{(r+1)(r+3)}$	
	$=\dfrac{8r\cdot 3^r+3^{r+1}(3-1)}{(r+1)(r+3)}$	
	$=\dfrac{(8r+6)3^r}{(r+1)(r+3)}$	
	(i) $\displaystyle\sum_{r=1}^{n}\frac{(4r+3)3^r}{(r+1)(r+3)}=\frac{1}{2}\sum_{r=1}^{n}\left[\mathrm{f}(r+2)-\mathrm{f}(r)\right]$	

	$=\dfrac{1}{2}\left[\begin{array}{l} \mathrm{f}(3)-\mathrm{f}(1) \\ +\mathrm{f}(4)-\mathrm{f}(2) \\ +\mathrm{f}(5)-\mathrm{f}(3) \\ +\ldots\ldots\ldots\ldots \\ +\mathrm{f}(n)-\mathrm{f}(n-2) \\ +\mathrm{f}(n+1)-\mathrm{f}(n-1) \\ +\mathrm{f}(n+2)-\mathrm{f}(n) \end{array}\right] \quad =\dfrac{1}{2}\left[-\mathrm{f}(1)-\mathrm{f}(2)+\mathrm{f}(n+1)+\mathrm{f}(n+2)\right]$	]
	$=-\dfrac{9}{4}+\dfrac{3^{n+1}}{2}\left[\dfrac{1}{n+2}+\dfrac{3}{n+3}\right]=-\dfrac{9}{4}+\dfrac{3^{n+1}}{2}\left[\dfrac{4n+9}{(n+2)(n+3)}\right]$	]
	(ii) $\displaystyle\sum_{r=1}^{n}\frac{(4r+11)3^{r}}{(r+3)(r+5)}=\frac{1}{9}\sum_{r=1}^{n}\frac{(4r+11)3^{r+2}}{(r+3)(r+5)}$	
	$\displaystyle=\frac{1}{9}\sum_{r=3}^{n+2}\frac{(4r+3)3^{r}}{(r+1)(r+3)}$	
	$\displaystyle=\frac{1}{9}\left[\sum_{r=1}^{n+2}\frac{(4r+3)3^{r}}{(r+1)(r+3)}-\frac{21}{8}-\frac{99}{15}\right]$	
	$\displaystyle=\frac{1}{9}\left[-\frac{9}{4}+\frac{3^{n+3}(4n+17)}{2(n+4)(n+5)}-\frac{21}{8}-\frac{99}{15}\right]$	
	$\displaystyle=-\frac{51}{40}+\frac{3^{n+1}(4n+17)}{2(n+4)(n+5)}$	
	(iii) $\displaystyle\sum_{r=0}^{n}\frac{r\cdot 3^{r}}{(r+5)^{2}}=\frac{1}{4}\sum_{r=1}^{n}\frac{4r\cdot 3^{r}}{(r+5)^{2}}$	
	$\displaystyle<\frac{1}{4}\sum_{r=1}^{n}\frac{(4r+11)\cdot 3^{r}}{(r+5)^{2}}$	
	$\displaystyle<\frac{1}{4}\sum_{r=1}^{n}\frac{(4r+11)\cdot 3^{r}}{(r+3)(r+5)}$	
	So $\displaystyle\sum_{r=0}^{n}\frac{r\cdot 3^{r}}{(r+5)^{2}}<\frac{1}{4}\left[-\frac{51}{40}+\frac{3^{n+1}(4n+17)}{2(n+4)(n+5)}\right]$	
	$\displaystyle\Rightarrow\sum_{r=0}^{n}\frac{r\cdot 3^{r}}{(r+5)^{2}}-\frac{3^{n+1}}{4}<-\frac{51}{160}+\frac{3^{n+1}(4n+17)}{8(n+4)(n+5)}-\frac{3^{n+1}}{4}$	
	$\displaystyle=-\frac{51}{160}-\frac{3^{n+1}}{4}\left[1-\frac{4n+17}{2(n+4)(n+5)}\right]$	
	$\displaystyle<-\frac{51}{160}$ since $\displaystyle\frac{4n+17}{2(n+4)(n+5)}\leq\frac{17}{18}$ for all $n\geq 0$	

11	The functions f and g are defined as follows: $\mathrm{f}: x \mapsto -\left\|x^2+2x\right\|, \quad a < x \le 0$ $\mathrm{g}: x \mapsto -\sqrt{x+1}, \quad x > -1$	
	(i) State the least value of a for the inverse function of f to exist. Hence, find f^{-1} in similar form.	[4]
	For the following parts, use the value of a found in part **(i)**.	
	(ii) Write down ff^{-1} in similar form.	[1]
	(iii) Find the rule for gf in the form $bx + c$, where $b, c \in$. State its range.	[3]
	(iv) Find the exact range of x for which $\mathrm{f}\left(x-\frac{3}{2}\right) > \mathrm{g\,f}\left(x-\frac{3}{2}\right)$.	[3]
	Solution	
	(i) Least $a=-1$	
	$\mathrm{f}(x) = -\left(-\left(x^2+2x\right)\right) = x^2+2x$	
	Let $y = \mathrm{f}(x) = x^2+2x$	
	$y = (x+1)^2 - 1$	
	$y+1 = (x+1)^2$	
	$x = -1 \pm \sqrt{y+1}$	
	$x = -1 + \sqrt{y+1} \left(\because -1 < x \le 0\right)$	
	$\mathrm{f}^{-1}: x \mapsto -1+\sqrt{x+1}, \quad -1 < x \le 0$	
	(ii) $\mathrm{ff}^{-1}: x \mapsto x, \quad -1 < x \le 0$	
	(iii) $\mathrm{gf}(x) = -\sqrt{x^2+2x+1}$	
	$= -\sqrt{(x+1)^2}$	
	$= -\left\|x+1\right\|$	
	$= -x-1 \quad \left(\because D_{\mathrm{gf}} = (-1, 0]\right)$	
	$R_{\mathrm{gf}} = [-1, 0)$	
	(iv) $\mathrm{f}\left(x-\frac{3}{2}\right) > \mathrm{g\,f}\left(x-\frac{3}{2}\right)$	
	$\left(x-\frac{3}{2}\right)^2 + 2\left(x-\frac{3}{2}\right) > -\left(x-\frac{3}{2}\right) - 1$	
	$x^2 - 3x + \frac{9}{4} + 3x - 3 + 1 - \frac{3}{2} > 0$	
	$x^2 - \frac{5}{4} > 0$	
	$\left(x-\frac{\sqrt{5}}{2}\right)\left(x+\frac{\sqrt{5}}{2}\right) > 0$	

		
	$x<-\frac{\sqrt{5}}{2}$ or $x>\frac{\sqrt{5}}{2}$	
	But after translation by 1.5 units in the positive x – direction, $D_{\mathrm{f}}=D_{\mathrm{gf}}=\left(\frac{1}{2},\frac{3}{2}\right]$	
	$\therefore \frac{\sqrt{5}}{2}<x\le\frac{3}{2}$	
12	**(a)(i)** Find $\frac{\mathrm{d}}{\mathrm{d}x}\left[(\ln x)^2\right]$.	[1]
	(ii) The curve C is defined by the parametric equations $x=\ln t-t,\quad y=2t+\ln\left(t^2\right)$ where $t>0$.	
	Another curve L is defined by the equation $(4-y)^2=3-x$. The graphs of C and L intersect at the point $A(-1,2)$ as shown in the diagram below.	
		
	Find the exact area of the shaded region bounded by C, L and the line $y=4+\ln 4$.	[6]
	(b) The region R is the finite region enclosed by the curve $(y-1)^2=1-x$ and the y-axis. The region S is the region in the 2nd quadrant enclosed by the curve $y=2\tan\left(x+\frac{\pi}{4}\right)$ and the axes.	
	Find the total volume generated when region R and S is rotated through 2π radians about the x-axis, leaving your answers in exact form.	[4]
	Solution	
	(ai) $\frac{\mathrm{d}}{\mathrm{d}x}\left[(\ln x)^2\right]=(2\ln x)\left(\frac{1}{x}\right)=\frac{2\ln x}{x}$	

	(ii) Area = $\int_2^{4+\ln 4}\left[3-(4-y)^2\right]\,\mathrm{d}y-\int_2^{4+\ln 4}x\ \mathrm{d}y$	
	$=\left[3y-\dfrac{(4-y)^3}{(-3)}\right]_2^{4+\ln 4}-\int_1^2(\ln t-t)\left(2+\dfrac{2}{t}\right)\mathrm{d}t$	
	$=3(2+\ln 4)-\dfrac{1}{3}\left[(-\ln 4)^3-8\right]-\int_1^2\left(2\ln t+\dfrac{2\ln t}{t}-2t-2\right)\mathrm{d}t$	
	$=\dfrac{10}{3}+6\ln 2-\dfrac{(\ln 4)^3}{3}-2\int_1^2\ln t\ \mathrm{d}t-\left[(\ln t)^2\right]_1^2+\left[t^2\right]_1^2+\left[2t\right]_1^2$	
	$=\dfrac{10}{3}+6\ln 2-\dfrac{(\ln 4)^3}{3}-2\left[t\ln t\right]_1^2+\int_1^2 2\ \mathrm{d}t-(\ln 2)^2+3+2$	
	$=\dfrac{25}{3}+6\ln 2-\dfrac{(\ln 4)^3}{3}-4\ln 2+\left[2t\right]_1^2-(\ln 2)^2$	
	$=\dfrac{31}{3}+2\ln 2-\dfrac{(\ln 4)^3}{3}-(\ln 2)^2$	
	(b) Vol = $\pi\int_{-\frac{\pi}{4}}^0 4\tan^2\left(x+\dfrac{\pi}{4}\right)\mathrm{d}x+\pi\int_0^1\left[\left(1+\sqrt{1-x}\right)^2-\left(1-\sqrt{1-x}\right)^2\right]\mathrm{d}x$	
	$=4\pi\int_{-\frac{\pi}{4}}^0\left(\sec^2\left(x+\dfrac{\pi}{4}\right)-1\right)\mathrm{d}x+\pi\int_0^1 4\sqrt{1-x}\ \mathrm{d}x$	
	$=4\pi\left[\tan\left(x+\dfrac{\pi}{4}\right)-x\right]_{-\frac{\pi}{4}}^0+4\pi\left[\dfrac{(1-x)^{\frac{3}{2}}}{-\frac{3}{2}}\right]_0^1$	
	$=4\pi\left(1-\dfrac{\pi}{4}\right)+\dfrac{8}{3}\pi=\dfrac{20}{3}\pi-\pi^2$	

END OF PAPER