

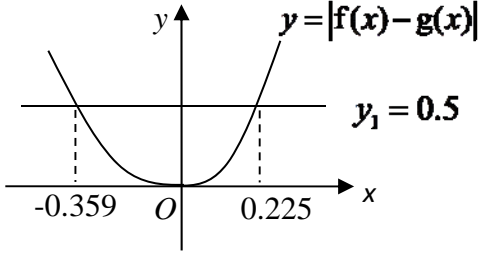


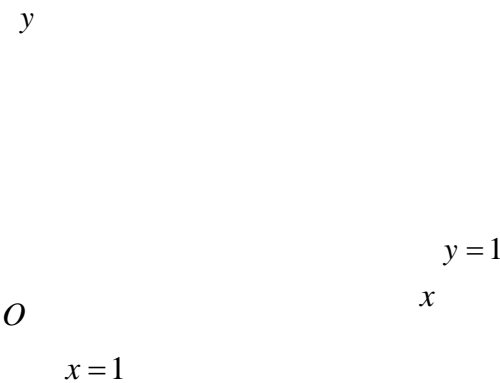
Jurong Junior College
2016 JC2 H2 Mathematics Prelim Paper 1 Solutions

Qn	Solution
1	$a + b + c = 24$ $5a + 2b - 3c = 79$ $a + b = 4(c + 1)$ $a = 17, b = 3, c = 4$
2(i)	<p>Let P_n be the statement that $u_n = \frac{n}{4n^2 - 1}$ for all $n \in \mathbb{N}^+$</p> <p>When $n = 1$,</p> <p>LHS = $u_1 = \frac{1}{3}$ (given)</p> <p>RHS = $\frac{1}{4 \times 1^2 - 1} = \frac{1}{3}$</p> <p>Hence P_1 is true.</p> <p>Assume P_k is true for some $k \in \mathbb{N}^+$, i.e. $u_k = \frac{k}{4k^2 - 1}$.</p> <p>We want to prove that P_{k+1} is true, i.e. $u_{k+1} = \frac{k+1}{4(k+1)^2 - 1}$</p> <p>LHS = u_{k+1}</p> $= u_k - \frac{1}{(2k-1)(2k+3)}$ $= \frac{k}{(2k-1)(2k+1)} - \frac{1}{(2k-1)(2k+3)}$ $= \frac{k(2k+3) - (2k+1)}{(2k-1)(2k+1)(2k+3)}$ $= \frac{2k^2 + k - 1}{(2k-1)(2k+1)(2k+3)}$ $= \frac{(2k-1)(k+1)}{(2k-1)(2k+1)(2k+3)}$ $= \frac{k+1}{(2k+1)(2k+3)}$ $= \frac{k+1}{4k^2 + 8k + 3}$ $= \frac{k+1}{4(k+1)^2 - 1} = \text{RHS}$ <p>Hence P_k is true $\Rightarrow P_{k+1}$ is true.</p> <p>Since P_1 is true & P_k is true $\Rightarrow P_{k+1}$ is true, by mathematical induction, P_n is true for all $n \in \mathbb{N}^+$.</p>

(ii)	<p>Sum of 1st n terms of</p> $\frac{1}{5 \times 9} + \frac{1}{7 \times 11} + \frac{1}{9 \times 13} + \dots + \frac{1}{(2n+3)(2n+7)}$ $= \sum_{r=3}^{n+2} \frac{1}{(2r-1)(2r+3)}$ $= \sum_{r=3}^{n+2} [u_r - u_{r+1}]$ $= \begin{array}{l} u_3 - u_4 \\ + u_4 - u_5 \\ + u_5 - u_6 \\ + \dots \\ + \dots \\ + u_{n+2} - u_{n+3} \end{array}$ $= u_3 - u_{n+3}$ $= \frac{3}{35} - \frac{n+3}{4(n+3)^2 - 1}$
(iii)	<p>As $n \rightarrow \infty$, $\frac{n+3}{4(n+3)^2 - 1} \rightarrow 0$</p> <p>Hence, the series is convergent and $\sum_{r=3}^{\infty} \frac{1}{(2r-1)(2r+3)} = \frac{3}{35}$</p>
3 (i)	<p>Required Area = $\int_{-\frac{2}{3}}^0 \left(3 - \frac{12}{(3x+2)^2 + 4} \right) dx$</p> $= \int_{-\frac{2}{3}}^0 \left(3 - \frac{12}{(3x+2)^2 + 2^2} \right) dx$ $= \left[3x - 2 \tan^{-1} \left(\frac{3x+2}{2} \right) \right]_{-\frac{2}{3}}^0$ $= 0 - 2 \left(\frac{\pi}{4} \right) - [-2 - 0]$ $= 2 - \frac{\pi}{2}$

(ii)	$y = \frac{12}{(3x+2)^2+4} \Rightarrow 3x = -2 \pm \sqrt{\frac{12}{y}-4} = -2 \pm \sqrt{\frac{12-4y}{y}}$ <p>Since $x \geq -\frac{2}{3}$, $x = -\frac{2}{3} + \frac{1}{3}\sqrt{\frac{12-4y}{y}}$</p> <p>Required volume = $\pi \int_{\frac{3}{2}}^3 x^2 dy$</p> $= \pi \int_{\frac{3}{2}}^3 \left(-\frac{2}{3} + \frac{1}{3}\sqrt{\frac{12-4y}{y}} \right)^2 dy$ $= 0.5125$
4(i)	$y = \tan\left(2\tan^{-1}x + \frac{\pi}{4}\right)$ $\frac{dy}{dx} = \sec^2\left(2\tan^{-1}x + \frac{\pi}{4}\right) \frac{2}{1+x^2}$ $(1+x^2)\frac{dy}{dx} = 2\left(1 + \tan^2\left(2\tan^{-1}x + \frac{\pi}{4}\right)\right)$ $(1+x^2)\frac{dy}{dx} = 2(1+y^2)$ <p><u>Alternative Method</u></p> $\tan^{-1} y = 2 \tan^{-1} x + \frac{\pi}{4}$ <p>Differentiate with respect to x,</p> $\frac{1}{1+y^2} \frac{dy}{dx} = \frac{2}{1+x^2}$ $\Rightarrow (1+x^2)\frac{dy}{dx} = 2(1+y^2)$
(ii)	<p>Differentiate with respect to x,</p> $\Rightarrow (1+x^2)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = 4y\frac{dy}{dx}$ $\Rightarrow (1+x^2)\frac{d^2y}{dx^2} + (2x-4y)\frac{dy}{dx} = 0$ <p>When $x = 0$, $y = \tan \frac{\pi}{4} = 1$</p> $(1+0)\frac{dy}{dx} = 2(1+1) \Rightarrow \frac{dy}{dx} = 4$ $(1+0)\frac{d^2y}{dx^2} + (0-4)(4) = 0 \Rightarrow \frac{d^2y}{dx^2} = 16$ $y = \tan\left[2 \tan^{-1} x + \frac{\pi}{4}\right]$ $= 1 + 4x + \frac{16}{2!}x^2 + \dots$ $= 1 + 4x + 8x^2 + \dots$

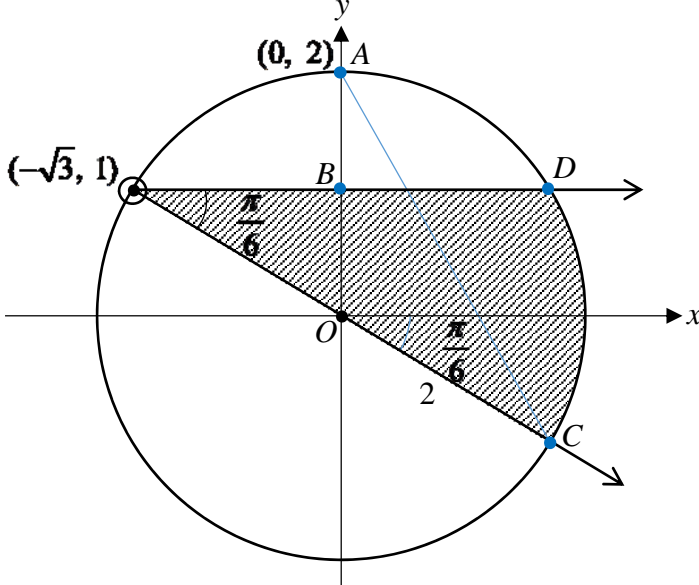
(iii)	<p>Sketch $y = f(x) - g(x)$ and $y_1 = 0.5$</p> 
	<p>For $f(x) - g(x) < 0.5$, $-0.359 < x < 0.225$</p>
5(i)	<p>$x = 2 \sin 2t$, $y = \cos 2t$, for $0 \leq t < \pi$.</p> $\frac{dx}{dt} = 4 \cos 2t \quad \frac{dy}{dt} = -2 \sin 2t$ $\frac{dy}{dx} = -\frac{\sin 2t}{2 \cos 2t}$ <p>Equation of normal at P, $t = \theta$:</p> $y - \cos 2\theta = \frac{2 \cos 2\theta}{\sin 2\theta} (x - 2 \sin 2\theta)$ $(\sin 2\theta) y - \cos 2\theta \sin 2\theta = (2 \cos 2\theta) x - 4 \cos 2\theta \sin 2\theta$ $(2 \cos 2\theta) x - (\sin 2\theta) y = 3 \cos 2\theta \sin 2\theta \text{ (shown)}$ <p>i.e. $m = 3$</p>
(ii)	<p>At the x-axis, $y = 0$</p> $(2 \cos 2\theta) x = 3 \sin 2\theta \cos 2\theta$ $x = \frac{3}{2} \sin 2\theta \quad \text{i.e. } A\left(\frac{3}{2} \sin 2\theta, 0\right)$ <p>At the y-axis, $x = 0$</p> $-(\sin 2\theta) y = 3 \sin 2\theta \cos 2\theta$ $y = -3 \cos 2\theta \quad \text{i.e. } B(0, -3 \cos 2\theta)$ <p>mid-point of AB: $\left(\frac{3}{4} \sin 2\theta, -\frac{3}{2} \cos 2\theta\right)$</p> $x = \frac{3}{4} \sin 2\theta \Rightarrow \sin 2\theta = \frac{4}{3} x$ $y = -\frac{3}{2} \cos 2\theta \Rightarrow \cos 2\theta = -\frac{2}{3} y$ <p>Cartesian equation of the locus of the mid-point of AB:</p> $\sin^2 2\theta + \cos^2 2\theta = 1$ $\frac{16x^2}{9} + \frac{4y^2}{9} = 1 \quad \text{i.e. } 16x^2 + 4y^2 = 9$

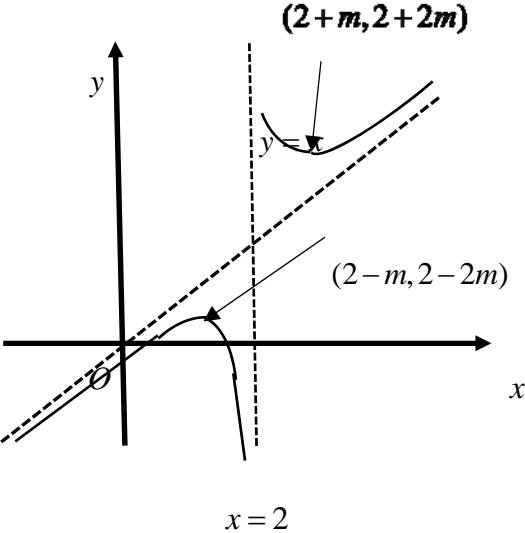
Qn	Solution
6(i)	
(ii)	<p>Let $y = g(x) = \sqrt{\frac{x^2+2}{x^2-1}}$, $x > 1$</p> <p>Then $y^2 = \frac{x^2+2}{x^2-1} = 1 + \frac{3}{x^2-1}$</p> $y^2 - 1 = \frac{3}{x^2-1}$ $x^2 - 1 = \frac{3}{y^2-1}$ $x^2 = 1 + \frac{3}{y^2-1} = \frac{y^2+2}{y^2-1}$ $x = g^{-1}(y) = \sqrt{\frac{y^2+2}{y^2-1}} \text{ since } x > 1 > 0$ $\Rightarrow g^{-1} : x \mapsto \sqrt{\frac{x^2+2}{x^2-1}}, x > 1$ <p>g is self-inverse as $g(x) = g^{-1}(x)$ and $D_{g^{-1}} = D_g$</p>
(iii)	<p>$g^2(x) = gg(x) = gg^{-1}(x) = x.$</p> <p>$g^3(x) = gg^2(x) = g(x).$ (shown)</p> <p>It follows that $g^{50}(x) = x$ and $g^{51}(x) = g(x)$, $x > 1$</p> <p>For $4 - g^{50}(x) = [g^{51}(x)]^2$</p> <p>Then $4 - x = \frac{x^2+2}{x^2-1}$, $x > 1$</p> $(4-x)(x^2-1) = x^2+2, \quad x > 1$ $x^3 - 3x^2 - x + 6 = 0, \quad x > 1$ $(x-2)(x^2 - x - 3) = 0$ $x = 2 \text{ or } x = \frac{1 \pm \sqrt{1+12}}{2}$ <p>since $x > 1$, $\Rightarrow x = 2$ or $x = \frac{1+\sqrt{13}}{2}$ (ans)</p>

Qn	Solution
7 (a)	$u = \cos(\ln x) \quad \frac{dv}{dx} = \frac{1}{x^2}$ $\frac{du}{dx} = -\frac{\sin(\ln x)}{x} \quad v = -\frac{1}{x}$ $\int \frac{\cos(\ln x)}{x^2} dx = -\frac{\cos(\ln x)}{x} - \int \frac{\sin(\ln x)}{x^2} dx$ $= -\frac{\cos(\ln x)}{x} + \frac{\sin(\ln x)}{x} - \int \frac{\cos(\ln x)}{x^2} dx$ $\Rightarrow 2 \int \frac{\cos(\ln x)}{x^2} dx = \frac{\sin(\ln x)}{x} - \frac{\cos(\ln x)}{x}$ $\therefore \int \frac{\cos(\ln x)}{x^2} dx = \frac{1}{2x} [\sin(\ln x) - \cos(\ln x)] + c$
(b)	$u = \sqrt{x+3} \Rightarrow u^2 = x+3$ <p>Differentiating w.r.t. x, $2u \frac{du}{dx} = 1$</p> <p>When $x = 1$, $u = 2$; When $x = 6$, $u = 3$</p> $\int_1^6 \frac{x-2}{x\sqrt{x+3}} dx = \int_2^3 \frac{u^2-5}{(u^2-3)u} (2u du)$ $= 2 \int_2^3 \left(1 - \frac{2}{u^2-3} \right) du$ $= \left[2u - \frac{4}{2\sqrt{3}} \ln \left(\frac{u-\sqrt{3}}{u+\sqrt{3}} \right) \right]_2^3$ $= 2(3) - \frac{2}{\sqrt{3}} \ln \left(\frac{3-\sqrt{3}}{3+\sqrt{3}} \right) - 2(2) + \frac{2}{\sqrt{3}} \ln \left(\frac{2-\sqrt{3}}{2+\sqrt{3}} \right)$ $= 2 + \frac{2}{\sqrt{3}} \ln \left(\frac{3-\sqrt{3}}{3+\sqrt{3}} \right) \quad \text{i.e. } a = b = 2, c = d = 3$

$$u = \sin(\ln x) \quad \frac{dv}{dx} = \frac{1}{x^2}$$

$$\frac{du}{dx} = \frac{\cos(\ln x)}{x} \quad v = -\frac{1}{x}$$

Qn	Solution
8 (i)	$ z ^2 \leq 4 \Rightarrow z \leq 2$ $-\frac{\pi}{6} \leq \arg(z + \sqrt{3} - i) \leq 0 \Rightarrow -\frac{\pi}{6} \leq \arg[z - (-\sqrt{3} + i)] \leq 0$ 
(ii)	<p>Minimum value of $z - 2i$ $= AB$ $= 1$ Maximum value of $z - 2i$ $= AC$ $= \sqrt{2^2 + 2^2 - 2(2)(2)\cos\frac{2\pi}{3}}$ $= 2\sqrt{3}$</p>
(iii)	$z^{100} = 2^{100} = 2^{100} e^{i0}$ $\Rightarrow z = 2e^{i\left(\frac{0+2k\pi}{100}\right)}, k = 0, \pm 1, \pm 2, \dots, \pm 49, 50$ $\Rightarrow z = 2e^{i\frac{k\pi}{50}}$ Roots are found in region R (along the minor arc CD) if $-\frac{\pi}{6} \leq \frac{k\pi}{50} \leq \frac{\pi}{6}$ $\Rightarrow -8\frac{1}{3} \leq k \leq 8\frac{1}{3}$ $\Rightarrow k = -8, -7, -6, \dots, 8$ \therefore Number of roots found in region $R = 17$.

Qn	Solution
9(i)	$f(x) = x + \frac{m^2}{x-2}$ <p>Let $\frac{df}{dx} = 1 - \frac{m^2}{(x-2)^2} = 0$</p> $(x-2)^2 - m^2 = 0$ $x = 2 \pm m$ <p>When $x = 2 + m$, $f(x) = 2 + m + \frac{m^2}{2+m-2} = 2 + 2m$</p> <p>When $x = 2 - m$, $f(x) = 2 - m + \frac{m^2}{2-m-2} = 2 - 2m$</p> <p>The stationary points are $(2 + m, 2 + 2m)$ and $(2 - m, 2 - 2m)$</p>
	 <p style="text-align: right;">$y = f(x) = x + \frac{m^2}{x-2}$</p>
(ii)	$f(x) = \frac{x(x-2) + m^2}{x-2}$ <p>when $x = 0$, $f(x) = -\frac{m^2}{2}$</p> $x^2 - (2+k)x + (m^2 + 2k) = 0$ $x^2 - 2x + m^2 = k(x-2)$ $x + \frac{m^2}{x-2} = k$ <p>By inserting a horizontal line $y = k$ on the graph of C, by observation, to have two distinct positive roots,</p> <p>then $-\frac{m^2}{2} < k < 2 - 2m$ or $k > 2 + 2m$ (ans)</p>

(iii)	$y = f(x) = x + \frac{m^2}{x-2}$ <p>After A: $y = f(x+2) = x+2 + \frac{m^2}{x}$</p> <p>After B: $y = f(2x+2) = 2x+2 + \frac{m^2}{2x}$</p> <p>After C: $y = f(2x+2) - 2 = 2x + \frac{m^2}{2x}$</p> <p>Given that $2x + \frac{m^2}{2x} = 2x + \frac{1}{8x}$</p> $\Rightarrow \frac{m^2}{2} = \frac{1}{8}$ $\Rightarrow m = \frac{1}{2} \text{ since } 0 < m < 1 \text{ (ans)}$
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Qn	Solution
10 (i)	<p>Let F be the foot of the perpendicular.</p> $\overrightarrow{AF} \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = 0$ $\Rightarrow \left[\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} - \begin{pmatrix} 4 \\ -3 \\ 0 \end{pmatrix} \right] \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = 0$ $\Rightarrow -2 + 3\lambda - 1 = 0$ $\Rightarrow \lambda = 1$ $\therefore \overrightarrow{OF} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix}.$
(ii)	<p>Let B be $(2, -3, 1)$.</p> <p>A normal to π_1 is $\overrightarrow{BA} \times \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}.$</p> $\begin{pmatrix} 4 \\ -3 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = 1$ <p>A cartesian equation of π_1 is $x + y + 2z = 1$.</p>

(iii)	$\mathbf{n}_1 \times \mathbf{n}_2 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ 7 \end{pmatrix} = \begin{pmatrix} 7 \\ -5 \\ -1 \end{pmatrix}$ <p>A direction vector of L is $\begin{pmatrix} -7 \\ 5 \\ 1 \end{pmatrix}$.</p> <p>$\pi_1: x + y + 2z = 1$ $\pi_2: x + 7z = c$</p> <p>Let $z = 0$. Then $x = c$ and $y = 1 - c$. A point on L is $(c, 1 - c, 0)$.</p> <p>\therefore A vector equation of L is $\mathbf{r} = \begin{pmatrix} c \\ 1 - c \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -7 \\ 5 \\ 1 \end{pmatrix}$, where $\mu \in \mathbb{R}$.</p>
(iv) (a)	<p>For the 3 planes to meet in the line L,</p> $\begin{pmatrix} -7 \\ 5 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ d \end{pmatrix} = 0 \text{ and } \begin{pmatrix} c \\ 1 - c \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ d \end{pmatrix} = 5.$ $\Rightarrow -14 - 5 + d = 0 \text{ and } 2c - 1 + c = 5$ $\Rightarrow d = 19 \text{ and } c = 2$
(b)	<p>For the 3 planes to have only one point in common,</p> $\begin{pmatrix} -7 \\ 5 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ d \end{pmatrix} \neq 0.$ $\Rightarrow d \neq 19$

Qn	Solution		
11 (i)	Instalment	Outstanding amount at the beginning of month	Total amount with interest owed at the end of month
		50000	50000×1.002
	1	$50000 \times 1.002 - 1000$	$50000 \times 1.002^2 - 1000 \times 1.002$
	2	$50000 \times 1.002^2 - 1000 \times 1.002 - 1000$	$50000 \times 1.002^3 - 1000 \times 1.002^2 - 1000 \times 1.002$

<p>Amount owed at the end of n instalments</p> $= 50000 \times 1.002^{n+1} - 1000 \times 1.002^n - 1000 \times 1.002^{n-1} - \dots - 1000 \times 1.002$ $= 50000 \times 1.002^{n+1} - \frac{1000 \times 1.002 \times [1.002^n - 1]}{1.002 - 1}$ $= 50000 \times 1.002^{n+1} - 1000 \times 501 \times (1.002^n - 1) \quad \text{-----} (*)$ $= 50000 \times 1.002^{n+1} - 501000 \times (1.002^n - 1) \text{ (shown)}$			
(ii)	$50000 \times 1.002^{n+1} - 501000 \times (1.002^n - 1) \leq 0$ By using G.C., least integer $n = 53$ i.e. no of instalments required = 53		
(iii)	Amount paid at the 53th instalment = Amount owed at the end of 52 instalments $= 50000 \times 1.002^{53} - 501000 \times (1.002^{52} - 1)$ $= 733.12$ (to 2 d.p.)		
(iv)	Let the amount need to be paid for each instalment be k . Then from (*) in (i) $50000 \times 1.002^{20} - k \times 501 [1.002^{19} - 1] \leq 0$ $k \geq \frac{50000 \times 1.002^{20}}{501 [1.002^{19} - 1]} = 2684.53$ Least value of $k = 2685$ (to the nearest dollars)		