HCI H3 Physics Prelim 2015 Suggested Solutions

Section A

1 **(i)** At Brewster's angle, $\tan\theta_1 = n_2 / n_1$ [1]

Angle between the reflected and refracted ray is 90^o, $\theta_1 + \theta_2 = 90^o$

$\tan(90^o - \theta_2) = n_2 / n_1$

$\cot\theta_2 = n_2 / n_1$

$\tan\theta_2 = n_1 / n_2$ [1]

(ii) At Brewster's angle, $\tan\theta_3 = n_3 / n_2$ [1]

And $\Phi + (90^o + \theta_2) + (90^o - \theta_3) = 180^o$ [1]

$\Phi + \theta_2 - \theta_3 = 0$

$\Phi + \arctan(n_1/n_2) - \arctan(n_3/n_2) = 0$

$\Phi = -\arctan(n_1/n_2) + \arctan(n_3/n_2)$ [1]

2 **(a)** *Proper length is defined as the distance between two spatial coordinates in a reference frame measured* ***at the same time*** *by observers who are* ***at rest relative to it****.* [2]

(b) *Consider two observers O and O' measuring the length of the object where O' is at rest relative to the object,*

For an observer O who is moving at speed v relative to the object,

Length of an object is obtained by taking the difference between two ends of the object. The spatial coordinates of the two ends are (x_1, t_1) *and* (x_2, t_2)

where $\boldsymbol{t_1 = t_2}$**.** [1]

Using Lorentz transformation eqn, the spatial coordinates of the two ends as measured by the observer O' who is at rest relative to the object is given by

$x_1{}' = \gamma(x_1 - vt_1)$ *and* $x_2{}' = \gamma(x_2 - vt_2)$

Taking the difference,

$$\boldsymbol{L}' = \boldsymbol{x_2}{}' - \boldsymbol{x_1}{}' = \gamma\big(x_2 - x_1 - v(t_2 - t_1)\big) = \gamma L$$ [1]

Since O' is at rest relative to the object, his measurement is the proper length [1] *and hence* $L' = L_0 = \gamma L$*.*

(c) *Since* $L_o = \gamma_A L_A$ *and* $L_o = \gamma_B L_B$ -------------------- *(1)* [1]

May write $k = \frac{L_B}{L_A} = \frac{\gamma_A}{\gamma_B} = \sqrt{\frac{1-\left(\frac{u}{c}\right)^2}{1-4\left(\frac{u}{c}\right)^2}}$

Hence $k^2\left(1 - 4\left(\frac{u}{c}\right)^2\right) = 1 - \left(\frac{u}{c}\right)^2$

$\left(\frac{u}{c}\right)^2 = \frac{1-k^2}{1-4k^2}$ -- *(1)* [1]

Substitute into $L_o = \gamma_A L_A$ ----------------------------- *(1)* [1]

Obtain $\frac{L_A}{L_0} = \sqrt{1 - 4\frac{1-k^2}{1-4k^2}} = \sqrt{\frac{3}{4k^2-1}}$ ------------------------- *(1)* [1]

(d) ***Satellite A will see a more prolate cross-section as compared to satellite B.*** [1]

This is because the ***polar radius remains the same*** *for both observers but* [1]

the ***equatorial radius is more contracted for satellite A*** *as it is moving faster.* [1]

3 **(a)** (i)

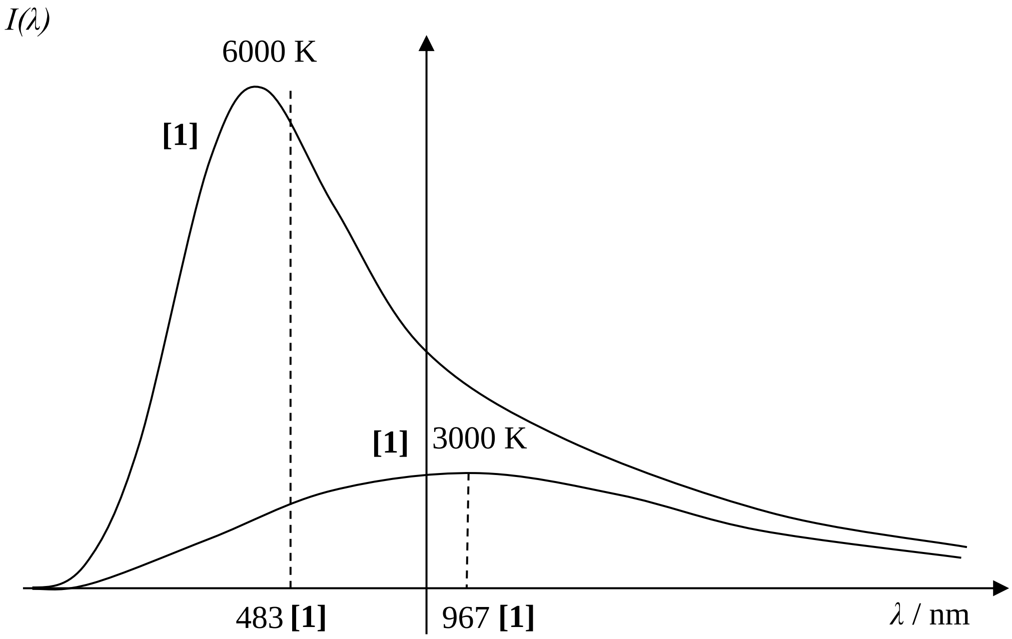
Classical theory postulates that all possible modes of standing waves in a cavity can emit electromagnetic radiation when the particles along the cavity walls vibrate. Based on equipartition theory, each mode has the same probability to radiate. Based on Rayleigh-Jeans Law which is derived by counting the number of standing wave modes in a cavity, the number of modes per frequency range increases with frequency. Hence the intensity of blackbody radiation should be infinite in the limit of high frequencies. [1]

Experimental results show that the intensity initially increases as frequency increases, reaches a peak intensity, then begin to decrease and drops to zero at very high frequencies. This observation contradicts the classical prediction [1].

(ii)

Max Planck postulated that the allowed energies of a blackbody oscillator is quantized in steps proportional to frequency. The energy for high frequency quanta is therefore large [1]
The probability for an oscillator to absorb a quantum of energies decreases with increasing frequency. Hence the intensity decreases at high frequencies, thereby avoiding the UV catastrophe predicted by classical theory [1]

(b) (i) . [2]



(ii)

Correct shape of the graph at 6000 K [2]
Peak value at 483 nm [1]

(c)

Differentiate $I(\lambda)$ with respect to λ, and equate the equation to zero to find λ_{max} , where $I(\lambda_{max})$ is a maximum. [2]

(d) (i)

The wave function ψ (x, t) of a particle is also called the probability amplitude function associated with a particle. [1]
The square of the wave function gives the probability of finding a particle at position x and time t. [1]

(ii)

$\Psi_k(x,t) = Ae^{i(kx-\omega t)} = A\{\cos(kx-\omega t) + i\sin(kx-\omega t)\}$ [1]

For a free particle, the particle's location is completely unknown at all times. Thus, by Uncertainty Principle, this implies that the momentum and energy are known precisely as $p = \hbar k$ and $E = \hbar\omega$, where k is the wavenumber and ω is the angular frequency of the free particle matter wave. [2]

(iii)

For constructive interference, the condition is path difference = $n\lambda$, where n is an integer.

Constructive interference occurs when
path difference between 1st plane and the Nth plane inside the crystal = $n\lambda$
And physical distance difference between 1st plane and the Nth plane
= N (2X0.505 λ)

i.e. $N(1.01\lambda) = n\lambda$, where N is the Nth atomic plane. **[1]**

Since N and n has to be an integer, the lowest integer for both N and n are 100 and 101 respectively. **[1]**

Thus, for destructive interference, the path difference should the mid of zero λ and 101λ. Since C.I. corresponds to 1^{st} and 100th planes, D.I. will correspond to 1^{st} and 50th planes. **[1]**

The 50^{th} atomic plane produces a wave that cancels the wave from the surface reflection.

4 **(a)** **(i)** No. The wave function must approach zero as $x \rightarrow +\infty$ and as $x \rightarrow -\infty$. [1]

(ii) No. The wave function must approach zero as $x \rightarrow +\infty$ and as $x \rightarrow -\infty$. [1]

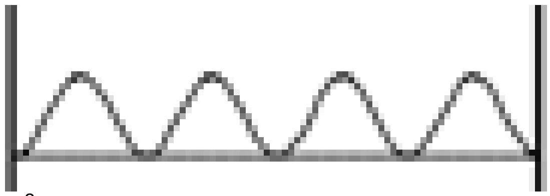
(iii) Yes. [1]

(iv) No. The wave function must be continuous. [1]

(b) **(i)** The wave function inside the box **must go to zero at the boundaries to be continuous** with the wave function outside the box. [1]
This boundary condition leads to **a solution of the Schrödinger equation** for only certain allowable energies. [1]

(ii) **1.** $n = 2$ [1]

2. $n = 1$ [1]
The probability density is the **square of the wave function.**
At $x = L/2$, the $n = 2$ state has a probability density of zero. [1]

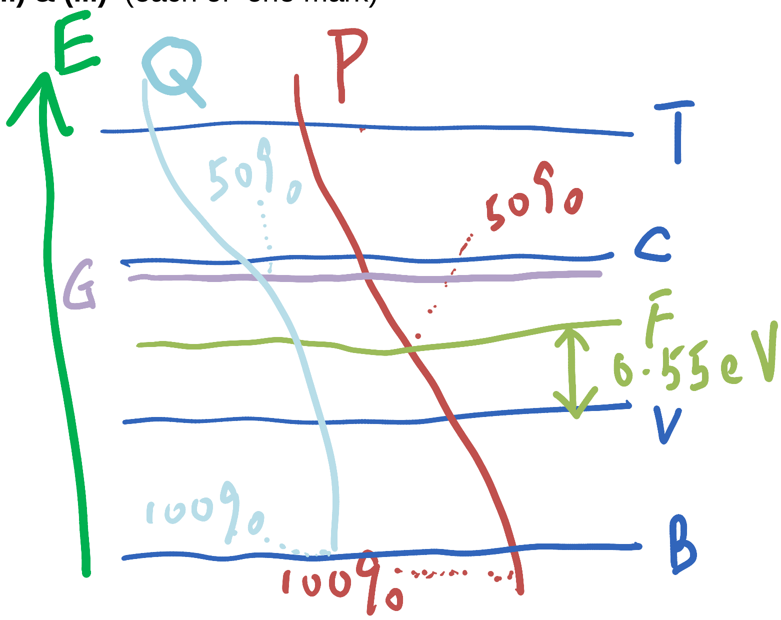
3.  [1]

$\sin^2$ function with four peaks.

(iii) No. A quantum particle in a box cannot be at rest. If the particle were at rest at the centre of the box then **both its position and momentum would be known with no uncertainty**. [1]
This would violate **Heisenberg's uncertainty principle.** [1]

5

(a) **(i), (ii) & (iii)** (each of one mark)



(b) **(i) & (ii)** (each of one mark)

(iii)
1. K.E. = 3/2 kT = 0.04 eV
2. 0.04 eV [1]

At 23 °C , majority of the extra electrons from the donor atoms becomes mobile electrons in the conduction band, implying that the occupancy of electrons at the donor energy level is low and the Fermi energy level should be near and below it. [1]

5 (c) When the supply voltage is 10.0 V, zener breakdown should happen.
i.e. p.d. across protective resistor = 4 V [1]
current in the electrical load = 6.0 / 100 = 60 mA [1]
min resistance of protective resistor = 4 / 60X10^{-3} = 67 Ω [1]
When supply voltage is 12.0 V,
Current in the protective resistor = 6.0 / 67 = 90 mA [1]
Current in the zener diode = 90 – 60 = 30 mA [1]
minimum power rating of zener diode
= 6 X 30 x10^{-3} = 0.18 W [1]

Section B

6 (a) (i) *Rest-mass energy refers to the energy possessed by a particle when **it is at rest**. It is given by **mc²**.* [2]

(ii) *Total energy refers to the **sum** of a particle's **rest-mass energy** and its **relativistic kinetic energy**.* [2]

(b) (i) *Let M denote the mass of the new particle.*
By conservation of mass-energy,

$$2\gamma m_p c^2 = Mc^2$$ [1]

$$M = \frac{2}{\sqrt{1-0.981^2}}938 = 9670MeV/c^2$$ [1] [1]

(ii) From conservation of momentum, the new particle will be moving with a speed v after the collision. [1]
We may write,
$\gamma_p m_p u = \gamma M v$.. ……………. (1) [1]

By conservation of mass-energy,
$(\gamma_p + 1)m_p c^2 = \gamma M c^2$……..(2) [1]
Taking eqn (1)/(2)
Obtain $\left(\frac{\gamma_p}{\gamma_p+1}\right)u = v$ [1]
Substitute into (2) to remove v

$$(\gamma_p + 1)^2\left(1 - \left(\frac{\gamma_p}{\gamma_p+1}\right)^2\left(\frac{u}{c}\right)^2\right) = \left(\frac{M}{m_p}\right)^2$$

$$\left((\gamma_p + 1)^2 - (\gamma_p)^2\left(\frac{u}{c}\right)^2\right) = \left(\frac{M}{m_p}\right)^2$$

$$\left(1 - \left(\frac{u}{c}\right)^2\right)\gamma_p^2 + 2\gamma_p + 1 - \left(\frac{M}{m_p}\right)^2 = 0$$

$$1 + 2\gamma_p + 1 - \left(\frac{M}{m_p}\right)^2 = 0$$

$$\gamma_p = \frac{1}{2}\left(\frac{9670}{938}\right)^2 - 1 = 52.1 = \frac{1}{\sqrt{1-\left(\frac{u}{c}\right)^2}}$$ [1] [1]

Hence $= 0.999816c$.

(iii) *Kinetic energy of a particle is given by* $(\gamma - 1)mc^2$ [1]
For Method 1,total KE = $2(\gamma-1)mc^2 = 2\left(\frac{1}{\sqrt{1-0.981^2}} - 1\right)938MeV = 7794MeV$ [1]
For Method 2, total KE = $(\gamma-1)mc^2 = \left(\frac{1}{\sqrt{1-0.999816^2}} - 1\right)938MeV = 47961MeV$ [1][1]
Hence the ratio of KE of method 2 to 1 is given by 6.2 .

(iv) *Speed of proton as observed by observer O' is given by*

$$v' = \frac{v-u}{1-\frac{uv}{c^2}} = \frac{0.981c-(-0.981c)}{1-\frac{(-0.981c)(0.981c)}{c^2}} = 0.999816c$$ [2]

This is exactly the speed that was used in the calculations in b(ii). Hence the ratio of the kinetic energies of same two particles in the experiment as seen by the two different observers O and O' varies by a factor of 6.2. Hence kinetic energy is not an invariant quantity under Lorentz transformation. [1]

7 (a)

(i) $k = \frac{2\pi}{\lambda} = 5\times10^{10}$. Therefore de Broglie wavelength, λ = 0.126 nm [1]

(ii) momentum, $p = \frac{\hbar}{\lambda} = 5.26\times10^{-24}\, kg.m.s^{-1}$. [1]

(iii) energy $E = \frac{p^2}{2m} = 95\,eV$ [1]

(b) (i) Based on classical theory, if the x-ray behaves like a wave, the scattered wave frequency at a given angle relative to the incoming radiation should show a distribution of Doppler shifted values. [1]
However, if x-rays behave like a particle, the distribution is given by the Compton shifted equation [1] with the photon transferring energy and momentum to the electron [1] [3]
Based on classical theory, if the x-ray behaves like a wave, the frequency of a scattered wave at a given angle relative to the incoming radiation should show a distribution of Doppler shifted values. However, if x-rays behave like a particle, the distribution is given by the Compton shifted equation which agrees with experimental measurements.

(ii) Note: This is a high energy photon. We have to use the relativistic expressions for momentum and energy because the electron will emerge with an energy large enough to require relativistic treatment. [4]

Let
Subscript g represents photon and e represents electron.
Subscript i represents initial and f represents final state.

By conservation of momentum ,

$$p_{\gamma i} = p_{\gamma f} + p_{ef} \quad \cdots\cdots \quad (1)$$

By conservation of energy ,

$$E_{\gamma i} + m_e c^2 = E_{\gamma f} + E_{ef} \quad \cdots\cdots \quad (2)$$

Rewrite equation (1) to express momenta in terms of energies

$$\left(p_{ef}\right)^2 = \left(p_{\gamma i} - p_{\gamma f}\right)^2$$

$$\left(\frac{E_{ef}}{c}\right)^2 - m_e^2 c^2 = \left(\frac{E_{\gamma i}}{c} + \frac{E_{\gamma f}}{c}\right)^2 \quad \cdots\cdots \quad (3)$$

Substitute E_{ef} in equation (2) into (3)

$$\left(\frac{E_{\gamma i}}{c}-\frac{E_{\gamma f}}{c}+m_e c\right)^2 - m_e^2 c^2 = \left(\frac{E_{\gamma i}}{c}+\frac{E_{\gamma f}}{c}\right)^2 \quad \cdots\cdots \quad (4)$$

Simplifying equation (4) gives

$$2m_e c\left(\frac{E_{\gamma i}}{c}-\frac{E_{\gamma f}}{c}\right) = 4\frac{E_{\gamma i}}{c}\frac{E_{\gamma f}}{c} \quad \cdots\cdots \quad (5)$$

Multiplying both sides by $1/\left(2m_e E_{\gamma i} E_{\gamma f}\right)$

$$\frac{1}{E_{\gamma f}/c}-\frac{1}{E_{\gamma i}/c}=\frac{2}{m_e c}$$

Note that

$$\frac{1}{E_{\gamma i}/c}=p_{\gamma i}=\frac{h}{\lambda_i}$$

Hence

$$\frac{\lambda_f}{h}-\frac{\lambda_i}{h}=\frac{2}{m_e c}$$

$$\Delta\lambda=\frac{2h}{m_e c}$$

(c)

(i) A population inversion occurs for the two energy levels involved in the lasing process when the number of atoms in the higher-energy level state is greater than the number of atoms in the lower-energy level state. [2]

(ii) A metastable level is an excited energy level with a relatively long lifetime as compared to other excited energy levels. [2]

(d)

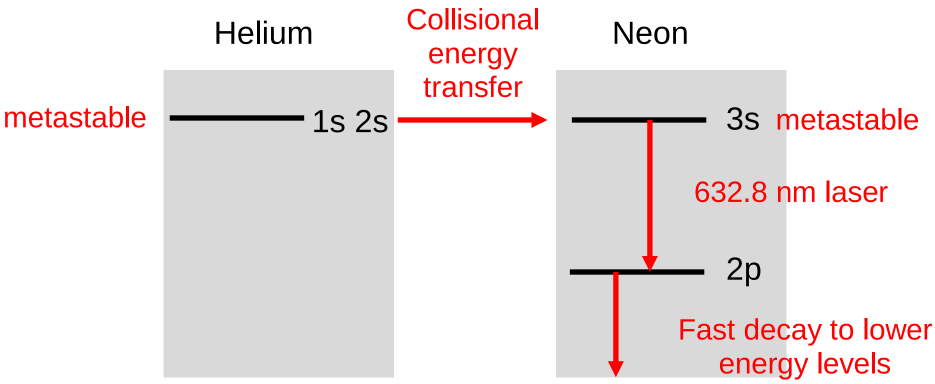
Population distribution of atoms with energy E_1 is given by [2]

$$N_1 = N_o e^{-\frac{(E_1-E_o)}{kT}}$$

$$\frac{E_1-E_o}{kT}=\frac{hc}{\lambda}\frac{1}{kT}=\frac{\left(6.63\times10^{-34}\right)\left(3.00\times10^{8}\right)}{580\times10^{-9}}\frac{1}{\left(1.38\times10^{-23}\right)(300)}=82.833$$

$$N_1=\left(4.0\times10^{20}\right)e^{-82.833}=4.2\times10^{-16}\approx 0$$

(e) [4]



Indicate metastable states (Helium 1s2s) [1]

An electrical discharge excites some helium atoms to a metastable 2s state that cannot radiatively return to the ground state (selection rules forbid it). The excited helium atoms, however, can lose energy by energy-exchange collisions with neon atoms [1]

The neon atoms that were initially in a ground state, gain energy from the collision and get excited into higher energy 3s levels. [1]

Transfer of energy from Helium 2s to Neon 3s by collisions 'pump' a large number of neon atoms into the excited 3s state so that a population inversion exists between the 3s levels and the almost always empty 3p lower levels (fast decay, extremely short lifetime). [1]

The diagram above shows one such possible transition.

8 (a) The energy for the n = 1 state is

$$E_1 = \frac{h^2}{8mL^2} = \frac{(6.63\times10^{-34}\ \text{Js})^2}{8(9.11\times10^{-31}\ \text{kg})(5.0\times10^{-10}\ \text{m})^2} = 2.41\times10^{-19}\ \text{J} = 1.51\,\text{eV}.$$ [1]

The energy for the n = 2 state is

$$E_2 = \frac{2^2 h^2}{8mL^2} = 4E_1 = 6.03\,\text{eV}.$$ [1]

Hence, the energy required to jump from the n = 1 to the n = 2 level is

$$\Delta E = E_2 - E_1 = 6.03 - 1.51 = 4.52\ \text{eV}\ (2\ \text{or}\ 3\,\text{s.f.}).$$ [1]

(b) (i) Assuming that the potential $U(x) = 0$ inside the box, the time-independent Schrödinger equation reduces to

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} = E\psi(x) = 0.$$ [1]

Differentiating $\psi(x) = Ax + B$ twice with respect to x gives $d^2\psi(x)/dx^2 = 0$, so the left side of the equation is zero and so $\psi(x) = Ax + B$ is a solution of this time-independent Schrödinger equation for $E = 0$. [1]

(ii) 1. Applying the boundary condition at $x = 0$ gives $\psi(0) = B = 0$, so $\psi(x) = Ax$. [1]

Applying the boundary condition at $x = L$ gives $\psi(L) = AL = 0$, so $A = 0$. [1]

2. $\psi(x) = 0$ both inside the box ($0 \le x \le L$) *and outside*: there is *zero* probability of finding the particle anywhere with this wave function and so $\boldsymbol{\psi(x) = Ax + B}$ **is *not* a physically valid wave function.** [1]

(c) (i) For the potential $U(x) = U_0$, the time-independent Schrödinger equation becomes

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} = (E - U_0)\psi(x).$$ [1]

Reordering gives

$$\frac{d^2\psi(x)}{dx^2} = \frac{2m(U_0 - E)}{\hbar^2}\psi(x).$$

We must show that $\psi(x) = Ce^{\kappa x} + De^{-\kappa x}$ satisfies this equation. Evaluating the left-hand side, we get

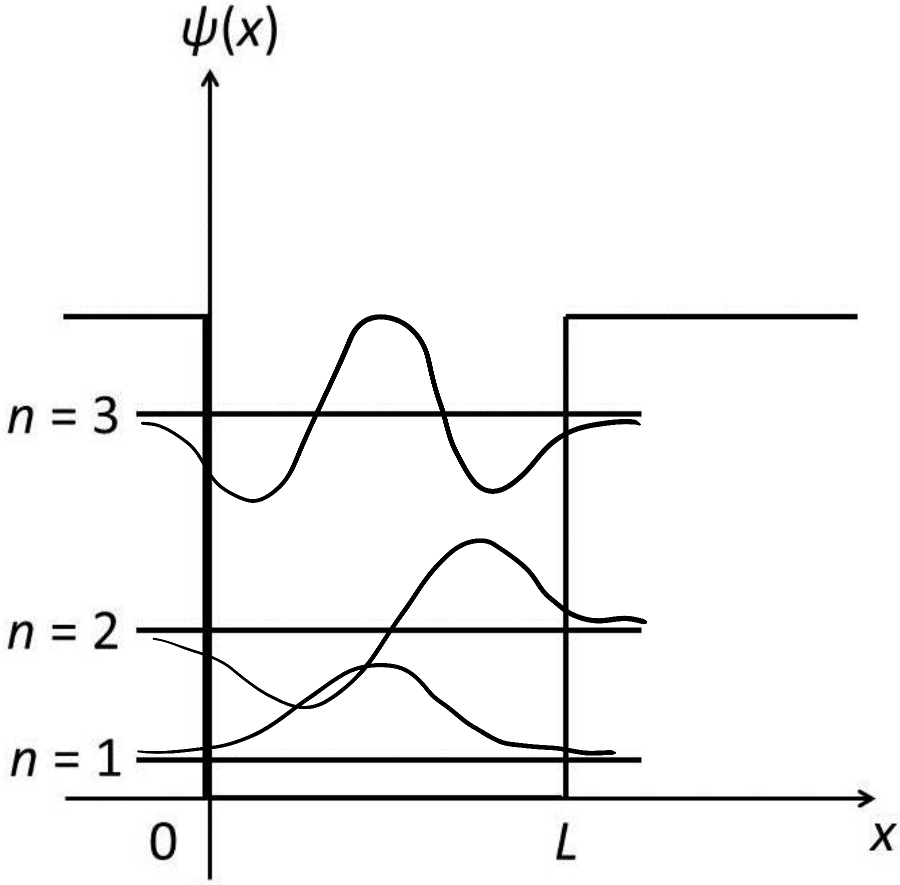
$$\frac{d^2\psi(x)}{dx^2} = \frac{d^2}{dx^2}\left(Ce^{\kappa x}\right) + \frac{d^2}{dx^2}\left(De^{-\kappa x}\right)$$
$$= C\kappa^2 e^{\kappa x} + D(-\kappa)^2 e^{-\kappa x}$$
$$= \kappa^2\left(Ce^{\kappa x} + De^{-\kappa x}\right)$$ [1]
$$= \kappa^2\psi(x).$$

Since $\kappa = [2m(U_0 - E)]^{1/2}/\hbar$, this is equal to the right-hand side of the equation. The equation is satisfied and $\psi(x)$ is a solution. [1]

(ii) As U_0 approaches infinity, κ also approaches infinity. [1]
In the region $x < 0$, $\psi(x) = Ce^{\kappa x}$; as $\kappa \to \infty$, $\kappa x \to -\infty$ (since x is negative) and $e^{\kappa x} \to 0$, so the wave function approaches zero for all $x < 0$. Similarly, the wave function also approaches zero for all $x > L$. [1]
Hence, the wave function will be zero outside the well when $U_0 \to \infty$. (This is the particle in a box, for which the wave function must be zero outside the box.) [1]

(d) (i) It is mentioned in the question that, for a given level, the wavelength of the sinusoidal part of each wave function is longer than it would be with an infinite well.
According to the Broglie equation, $\lambda = h/p$, this means that, for a given level, **the momentum of a particle is smaller** than it would be with an infinite well. [1]
[1]
Consequently, the kinetic energy for a given level is **smaller** than it would be with an infinite well.

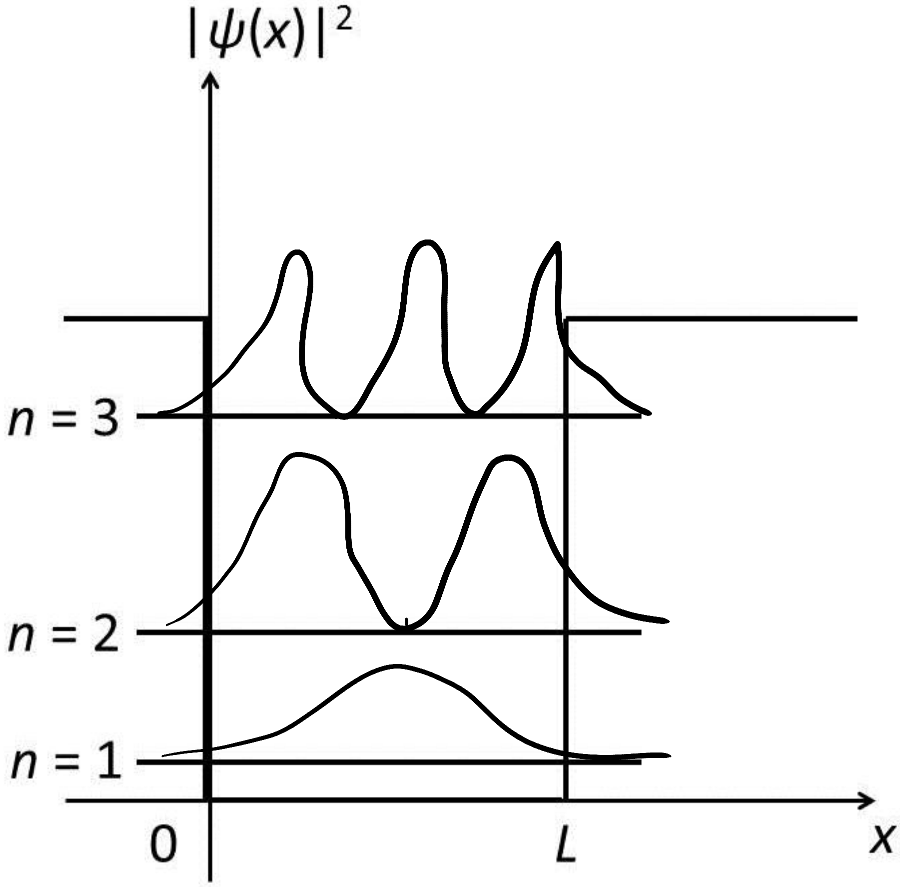
(ii)



Sinusoidal function, extending into the walls (and falling off exponentially in the walls). [1]

Half a wavelength for the n = 1 level, one wavelength for the n = 2 level, one and a half wavelengths for the n = 3 level. [1]

(iii)



$\sin^2$ function, i.e., non-negative everywhere, with $|\psi(x)|^2 = 0$ at $x = L/2$ for the $n = 2$ level and two zero points for the $n = 3$ level. [1]
One peak for the $n = 1$ level, two peaks for the $n = 2$ level, three peaks for the $n = 3$ level. [1]

9 **(a)** **(i)** $J = I / A$ [1]
$= V / (R A)$ [1]
$= V / (\rho L)$

(ii) By $\rho = \rho_0[1 + \alpha(T - T_0)]$,
$\rho = 5.920 \times 10^{-8} (1 + 3.7 \times 10^{-3} (55.0 - 20.0))$ [1]
$\rho = 6.69 \times 10^{-8}\ \Omega$ m [1]

(iii) $J = V / (\rho L)$ and $J = n e v$
$V / (\rho L) = n e v$ [1]
$0.032 / (6.69 \times 10^{-8} \times 0.100)$
$= (7.133 / 65.37)(2 \times 6.02 \times 10^{23} \times 10^{6}) (1.6 \times 10^{-19})\ v$ [1]
$v = 2.28 \times 10^{-4}$ m s^{-1} [1]

(b) **(i)** The assumptions for this model are:
- Motion of an electron after a collision is independent of its motion before the collision. The collision process is random. [1]
- Excess energy acquired by the electrons in the electric field is lost to the atoms of the conductor when the electrons and atoms collide. The given-up energy increases the vibrational energy of the atoms, resulting in an increase in the temperature of the conductor. [1]

(ii) $\rho = \dfrac{2m_e \langle v \rangle}{ne^2\lambda}$ and $J = V / (\rho L)$
$2 m \langle v \rangle / n e^2 \lambda = V / (J L)$ [1]
$2 m \langle v \rangle / n e^2 \lambda = E / (I / A)$
$2 m \langle v \rangle / n e^2 \lambda = E / (n e v)$ [1]
$2 m \langle v \rangle / e \lambda = E / (v)$ [1]

By $\frac{1}{2} m \langle v \rangle^2 = (3/2) k T$
$2 m [(3/m) kT]^{1/2} / e \lambda = E / (v)$ [1]
$[(12 m) kT]^{1/2} = E e \lambda / (v)$ [1]
$v = E e \lambda / [(12 m) kT]^{1/2}$

(iii) The resistivity of a metal in (a) has a linear relation (i.e. Y=MX+C) with its temperature. [1]
The resistivity of a metal in (b) is proportional to the square root of its temperature. [1]

The Drude model does not take into account that the metal lattice vibrates more vigorously at higher temperature and introduces greater hindrance to the flow of free electrons. Thus, the resistivity of a metal will increase at faster rate in (a) with the change in its temperature. [1]

(c) n = 3(2700/0.027)(6.02x10^{23}) = 1.80x10^{29} m^{-3} [1]

$$E_F = \frac{h^2}{8m}\left(\frac{3n}{\pi}\right)^{2/3}$$

= [(6.63 x 10^{-34})^2 / (8x9.11x10^{-31})] $(3n/\pi)^{2/3}$ [1]

= 11.7 eV [1]

End of Solutions