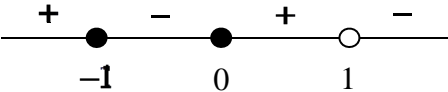
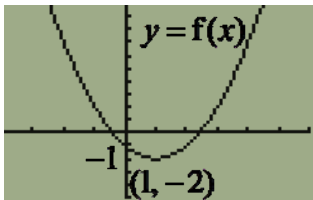


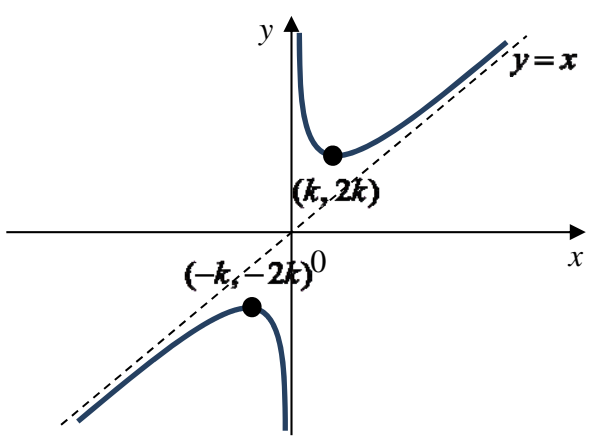
Paper 1 Solutions

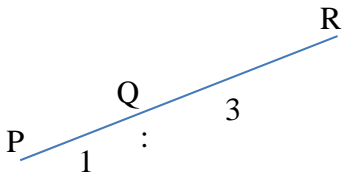
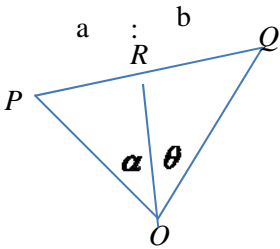
1		$f(x) = \frac{ax+b}{x+c}$ $f(0) = 4$ $\frac{b}{c} = 4$ $b - 4c = 0 \quad \dots(1)$ $f(2) = 2$ $\frac{2a+b}{2+c} = 2$ $2a + b - 2c = 4 \quad \dots(2)$ $f(-3) = -0.5$ $\frac{-3a+b}{-3+c} = -0.5$ $-3a + b + 0.5c = 1.5 \quad \dots(3)$ <p>From GC, $a = 1, b = 4, c = 1$</p> $f(x) = \frac{x+4}{x+1} = 1 + \frac{3}{x+1}$ <p><u>Step 1</u> Translation in the direction of the negative x-axis by 1 unit</p> <p><u>Step 2</u> Scale parallel to the y-axis by a factor of 3</p> <p><u>Step 3</u> Translation in the direction of the positive y-axis by 1 unit</p>
2	(i)	$\frac{x+x^2}{(1-x)^3} \geq 0$ $\frac{x(1+x)}{(1-x)^3} \geq 0$  <p>$\therefore x \leq -1$ or $0 \leq x < 1$</p>

	(ii)	$\frac{x+x^2}{(1-x)^3}$ $= (x+x^2)(1-x)^{-3}$ $= (x+x^2)\left(1+(-3)(-x)+\frac{(-3)(-4)}{2!}(-x)^2+\dots\right)$ $= (x+x^2)(1+3x+6x^2+\dots)$ $= x+4x^2+9x^3+\dots$
	(iii)	$\sum_{r=1}^{\infty} \frac{r^2}{2^r} = \frac{1^2}{2^1} + \frac{2^2}{2^2} + \frac{3^2}{2^3} + \dots$ $= \frac{1}{2} + 4\left(\frac{1}{2}\right)^2 + 9\left(\frac{1}{2}\right)^3 + \dots$ $= \frac{\frac{1}{2} + \left(\frac{1}{2}\right)^2}{\left(1 - \frac{1}{2}\right)^3}$ $= 6$
3	(i)	$y = \tan(e^x - 1)$ $\frac{dy}{dx} = \sec^2(e^x - 1) \times e^x$ $= (1 + \tan^2(e^x - 1))e^x$ $= (1 + y^2)e^x$
	(ii)	$\frac{d^2y}{dx^2} = (1 + y^2)e^x + 2y \frac{dy}{dx} e^x$ $= \frac{dy}{dx} + 2y \frac{dy}{dx} e^x$ $= \frac{dy}{dx} (1 + 2ye^x)$
		<p>When $x = 0$, $y = 0$</p> $\frac{dy}{dx} = 1$ $\frac{d^2y}{dx^2} = 1$ $\therefore \tan(e^x - 1) = 0 + (1)x + (1)\frac{x^2}{2!}$ $= x + \frac{1}{2}x^2 + \dots$
		Required equation of tangent is $y = x$

4		$\frac{z^3 + 8}{8 - z^3} = i$ $z^3 + 8 = 8i - iz^3$ $z^3 = \frac{8(i-1)}{1+i}$ $z^3 = 8i$ $= 8e^{i\left(\frac{\pi}{2} + 2n\pi\right)}$ $z = 2e^{i\left(\frac{\pi}{6} + \frac{2n\pi}{3}\right)}, n = 0, \pm 1$ $= 2e^{\frac{\pi}{6}i}, 2e^{\frac{5\pi}{6}i}, 2e^{\frac{\pi}{2}i}$
		$\frac{1 + 8w^6}{8w^6 - 1} = i$ $\left(\frac{1}{w^6}\right) + 8$ $8 - \left(\frac{1}{w^6}\right) = i$ $\therefore \text{let } \frac{1}{w^2} = z$ $\text{Then } \frac{1}{w^2} = 2e^{\frac{\pi}{6}i}, 2e^{\frac{5\pi}{6}i}, 2e^{\frac{\pi}{2}i}$ $w^2 = \frac{1}{2}e^{-\frac{\pi}{6}i}, \frac{1}{2}e^{-\frac{5\pi}{6}i}, \frac{1}{2}e^{\frac{\pi}{2}i}$ $w = \pm \frac{1}{\sqrt{2}}e^{-\frac{\pi}{12}i}, \pm \frac{1}{\sqrt{2}}e^{-\frac{5\pi}{12}i}, \pm \frac{1}{\sqrt{2}}e^{\frac{\pi}{4}i}$
5	(i)	 <p>Since the line $y = 0$ cuts the curve $y = f(x)$ more than once, f is not one-to-one and thus f has no inverse.</p> <p>Least $k = 1$</p>
	(ii)	<p>Let $y = x^2 - 2x - 1$</p> $(x-1)^2 - 2 = y$ $x-1 = \sqrt{y+2} \quad (\because x \geq 1)$ $x = 1 + \sqrt{y+2}$ $\therefore f^{-1}(x) = 1 + \sqrt{x+2}, \quad x \geq -2$

	(iii)	$ff^{-1}(x) = f(f^{-1}(x))$ $= f(1 + \sqrt{x+2})$ $= (1 + \sqrt{x+2} - 1)^2 - 2$ $= x + 2 - 2$ $= x$
		$gf(x) = x^2$ $gf(f^{-1}(x)) = (f^{-1}(x))^2$ $g(x) = (1 + \sqrt{x+2})^2$ $= 1 + 2\sqrt{x+2} + x + 2$ $= 3 + 2\sqrt{x+2} + x$
6	(i)	$h(-x) = \frac{(-x)^2 + k^2}{-x}$ $= -\frac{x^2 + k^2}{x}$ $= -h(x)$ <p>Thus h is an odd function</p>
		$gh(-x) = g(h(-x))$ $= g(-h(x)) \quad (\because h \text{ is odd})$ $= -gh(x) \quad (\because g \text{ is odd})$ <p>Thus, gh is an odd function.</p>
	(ii)	$\frac{x^2 + k^2}{x} = y$ $x^2 + k^2 = xy$ $x^2 - yx + k^2 = 0$ $b^2 - 4ac \geq 0$ $(-y)^2 - 4(1)(k^2) \geq 0$ $y^2 \geq 4k^2$ $y \leq -2k \text{ or } y \geq 2k$ <p>Required set = $\{y \in \mathbb{R} : y \leq -2k \text{ or } y \geq 2k\}$</p>

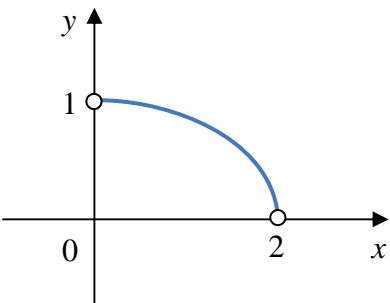
	(iii)	 <p style="text-align: center;">$m > 0, m \neq 1$</p>
7	(i)	$\frac{dP}{dt} = kP - m, (k, m > 0)$ $\int \frac{1}{kP - m} dP = \int 1 dt$ $\frac{1}{k} \ln kP - m = t + c$ $\ln kP - m = kt + kc$ $kP - m = \pm e^{kc} e^{kt}$ $kP - m = Ae^{kt} \quad (A = \pm e^{kc})$ <p>When $t = 0, P = P_0$</p> $A = kP_0 - m$ $kP - m = (kP_0 - m)e^{kt}$ $kP = (kP_0 - m)e^{kt} + m$ $P = \left(P_0 - \frac{m}{k} \right) e^{kt} + \frac{m}{k}$
	(ii)	<p>For decline in the population, $P_0 - \frac{m}{k} < 0$</p> <p>i.e. $m > kP_0$</p>
	(iii)	<p>$P_0 = 8000000, k = 0.015, m = 200000$</p> <p>$kP_0 = 0.015 \times 8000000 = 120000$</p> <p>Since $m > kP_0$, the population was declining in that year.</p>

8	(a)	<p>By Ratio Theorem,</p> $\frac{\mathbf{r} + 3\mathbf{p}}{4} = \mathbf{q}$ $\mathbf{r} = 4\mathbf{q} - 3\mathbf{p}$  <p>Required length of projection</p> $= \left \overrightarrow{OR} \cdot \frac{\overrightarrow{OP}}{ \overrightarrow{OP} } \right $ $= \left (4\mathbf{q} - 3\mathbf{p}) \cdot \frac{\mathbf{p}}{ \mathbf{p} } \right $ $= \left 4\mathbf{q} \cdot \frac{\mathbf{p}}{ \mathbf{p} } - 3\mathbf{p} \cdot \frac{\mathbf{p}}{ \mathbf{p} } \right $ $= \left 4\mathbf{q} \cdot \frac{\mathbf{p}}{ \mathbf{p} } - \frac{3 \mathbf{p} ^2}{ \mathbf{p} } \right $ $= \left \frac{4\mathbf{p} \cdot \mathbf{q}}{ \mathbf{p} } - 3 \mathbf{p} \right $
	(b)	 <p>By ratio theorem, $\mathbf{r} = \frac{a\mathbf{q} + b\mathbf{p}}{a + b}$</p> $\frac{\mathbf{r} \cdot \mathbf{p}}{\mathbf{r} \cdot \mathbf{q}} = \frac{(a\mathbf{q} + b\mathbf{p}) \cdot \mathbf{p}}{(a\mathbf{q} + b\mathbf{p}) \cdot \mathbf{q}}$ $= \frac{a\mathbf{p} \cdot \mathbf{q} + b\mathbf{p} \cdot \mathbf{p}}{a\mathbf{q} \cdot \mathbf{q} + b\mathbf{p} \cdot \mathbf{q}} = \frac{a\mathbf{p} \cdot \mathbf{q} + b \mathbf{p} ^2}{a \mathbf{q} ^2 + b\mathbf{p} \cdot \mathbf{q}}$ <p>When $\theta = \alpha$,</p> $\frac{ \mathbf{r} \mathbf{p} \cos \theta}{ \mathbf{r} \mathbf{q} \cos \theta} = \frac{a \mathbf{p} \mathbf{q} \cos 2\theta + b \mathbf{p} ^2}{a \mathbf{q} ^2 + b \mathbf{p} \mathbf{q} \cos 2\theta}$ $\frac{a \mathbf{q} \cos 2\theta + b \mathbf{p} }{a \mathbf{q} + b \mathbf{p} \cos 2\theta} = 1$ $a \mathbf{q} \cos 2\theta + b \mathbf{p} = a \mathbf{q} + b \mathbf{p} \cos 2\theta$ $a \mathbf{q} (\cos 2\theta - 1) = b \mathbf{p} (\cos 2\theta - 1)$ $\frac{a}{b} = \frac{ \mathbf{p} }{ \mathbf{q} }$

9	(i)	<p>Let P_n be the statement $\sum_{r=1}^n r^3 = \frac{1}{4}n^2(n+1)^2$, $n \in \mathbf{Z}^+$</p> <p>When $n = 1$, LHS = $1^3 = 1$</p> <p>RHS = $\frac{1}{4}(1)^2(1+1)^2 = 1$</p> <p>Since LHS = RHS, P_1 is true</p> <p>Assume that P_k is true for some $k \in \mathbf{Z}^+$</p> <p>i.e. $\sum_{r=1}^k r^3 = \frac{1}{4}k^2(k+1)^2$</p> <p>To prove P_{k+1} is true</p> <p>i.e. $\sum_{r=1}^{k+1} r^3 = \frac{1}{4}(k+1)^2(k+2)^2$</p> <p>LHS</p> $ \begin{aligned} &= \sum_{r=1}^{k+1} r^3 \\ &= \sum_{r=1}^k r^3 + (k+1)^3 \\ &= \frac{1}{4}k^2(k+1)^2 + (k+1)^3 \\ &= \frac{1}{4}(k+1)^2(k^2 + 4k + 4) \\ &= \frac{1}{4}(k+1)^2(k+2)^2 \\ &= \text{RHS} \end{aligned} $ <p>Since P_1 is true and P_k is true $\Rightarrow P_{k+1}$ is true, P_n is true for all $n \in \mathbf{Z}^+$ by Principle of Mathematical Induction.</p>
	(ii)	$ \begin{aligned} u_r - u_{r-1} &= r^3 - (r-1)^3 \\ &= r^3 - (r^3 - 3r^2 + 3r - 1) \\ &= 3r^2 - 3r + 1 \end{aligned} $ $ \sum_{r=1}^n (u_r - u_{r-1}) = \sum_{r=1}^n (3r^2 - 3r + 1) $ $ \begin{aligned} &\cancel{u_1} - u_0 \\ &+ \cancel{u_2} - \cancel{u_1} \\ &\quad \vdots \\ &+ \cancel{u_{n-1}} - u_{n-2} \\ &+ u_n - \cancel{u_{n-1}} \end{aligned} = 3 \sum_{r=1}^n r^2 - 3 \sum_{r=1}^n r + n $ $ n^3 = 3 \sum_{r=1}^n r^2 - 3 \times \frac{1}{2}n(n+1) + n $

		$3 \sum_{r=1}^n r^2 = n^3 + \frac{3}{2}n(n+1) - n$ $= \frac{1}{2}n(2n^2 + 3n + 3 - 2)$ $= \frac{1}{2}n(n+1)(2n+1)$ $\sum_{r=1}^n r^2 = \frac{1}{6}n(n+1)(2n+1)$
	(iii)	$1^2 \times 2 + 2^2 \times 3 + \dots + 50^2 \times 51$ $= \sum_{r=1}^{50} r^2(r+1)$ $= \sum_{r=1}^{50} (r^3 + r^2)$ $= \sum_{r=1}^{50} r^3 + \sum_{r=1}^{50} r^2$ $= \frac{1}{4}(50^2)(51^2) + \frac{1}{6}(50)(51)(101)$ $= 1668550$
10	(a) (i)	$(a+c) - (a+b) = c - b$ $(b+c) - (a+c) = b - a$ <p>Since a, b and c are consecutive terms of AP,</p> $c - b = b - a$ <p>Hence $a+b, a+c$ and $b+c$ are in AP</p>
	(a) (ii)	<p>Since a, b and c are in AP</p> $b - a = c - b$ $2b = a + c \quad \dots(1)$ <p>Since a, b and c are in GP</p> $\frac{b}{a} = \frac{c}{b}$ $b^2 = ac \quad \dots(2)$ <p>(1) gives $b = \frac{a+c}{2}$</p> <p>Subs. $b = \frac{a+c}{2}$ into (2):</p> $\left(\frac{a+c}{2}\right)^2 = ac$

		$(a+c)^2 = 4ac$ $a^2 + 2ac + c^2 = 4ac$ $a^2 - 2ac + c^2 = 0$ $(a-c)^2 = 0$ $a-c = 0$ $a = c$ <p>Subs. $a = c$ into (1):</p> $2b = a + a$ $b = a$ <p>Hence $a = b = c$</p>
(b)	(i)	<p>Required distance</p> $= a + \frac{2}{3}a \times 2 + \left(\frac{2}{3}\right)^2 a \times 2 + \dots + \left(\frac{2}{3}\right)^{n-1} a \times 2$ $= a + \frac{\frac{4}{3}a \left(1 - \left(\frac{2}{3}\right)^{n-1}\right)}{1 - \frac{2}{3}}$ $= a + 4a \left(1 - \left(\frac{2}{3}\right)^{n-1}\right)$ $= 5a - 4a \left(\frac{2}{3}\right)^{n-1}$
(b)	(ii)	<p>Total distance travelled before the ball stops bouncing, $L = 5a$</p> $5a - 4a \left(\frac{2}{3}\right)^{n-1} > 0.9 \times 5a$ $4 \left(\frac{2}{3}\right)^{n-1} < 0.5$ $\left(\frac{2}{3}\right)^{n-1} < 0.125$ $n-1 > \frac{\ln 0.125}{\ln \frac{2}{3}}$ $n > 6.13$ <p>Hence the required least n is 7.</p>

11	(i)	$\sin 2x = 2 \sin x \cos x$ <p>Differentiate both sides with respect to x</p> $2 \cos 2x = 2[\sin x(-\sin x) + \cos x(\cos x)]$ $\cos 2x = \cos^2 x - \sin^2 x$ $\cos 2x = 1 - \sin^2 x - \sin^2 x$ $= 1 - 2 \sin^2 x$
	(ii) (a)	
	(ii) (b)	<p>Required area $= \int_0^2 y \, dx$</p> $= \int_{\frac{\pi}{2}}^0 \sin \theta \cdot 2(-\sin \theta) \, d\theta$ $= -2 \int_{\frac{\pi}{2}}^0 \sin^2 \theta \, d\theta$ $= 2 \int_0^{\frac{\pi}{2}} \frac{1 - \cos 2\theta}{2} \, d\theta$ $= \int_0^{\frac{\pi}{2}} (1 - \cos 2\theta) \, d\theta$ $= \left[\theta - \frac{1}{2} \sin 2\theta \right]_0^{\frac{\pi}{2}}$ $= \frac{\pi}{2}$
	(ii) (c)	$\frac{dx}{d\theta} = -2 \sin \theta, \frac{dy}{d\theta} = \cos \theta$ $\frac{dy}{dx} = \cos \theta \times \frac{1}{-2 \sin \theta} = -\frac{1}{2} \cot \theta$ <p>Eq. of tangent at P:</p> $y - \sin p = -\frac{1}{2} \cot p (x - 2 \cos p)$ $y = -\frac{1}{2} x \cot p + \frac{\cos^2 p}{\sin p} + \sin p$

		<p>Eq. of normal at P:</p> $y - \sin p = 2 \tan p (x - 2 \cos p)$ $y = 2x \tan p - 3 \sin p$ $OT \cdot ON$ $= \left \frac{\cos^2 p}{\sin p} + \sin p \right \cdot -3 \sin p $ $= -3(\cos^2 p + \sin^2 p) $ $= -3 $ $= 3 \quad (\text{independent of } p)$
	(ii) (d)	<p>Let Q be (x, y)</p> $\frac{x + 2 \cos p}{2} = 0 \qquad \frac{y + \sin p}{2} = -3 \sin p$ $x = -2 \cos p \qquad y = -7 \sin p$ $\cos p = -\frac{x}{2} \qquad \sin p = -\frac{y}{7}$ $\sin^2 p + \cos^2 p = 1$ $\frac{x^2}{4} + \frac{y^2}{49} = 1 \quad (\text{eq. of locus of } Q)$