YISHUN JUNIOR COLLEGE
2014 JC2 PRELIMINARY EXAMINATION

MATHEMATICS **9740/01**

Higher 2

Paper 1 **19 AUGUST 2014**

TUESDAY 0800h – 1100h

Additional materials :
- Answer paper
- Graph paper
- List of Formulae (MF15)

TIME 3 hours

READ THESE INSTRUCTIONS FIRST

Write your name and CTG in the spaces provided on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphic calculator.
Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, write down the question number of the questions attempted, model of calculator used on the spaces provided on the cover page. Tie your cover page on top of the answer scripts before submission.
The number of marks is given in brackets [ ] at the end of each question or part question.

1 It is given that $\mathrm{f}(x)=\dfrac{ax+b}{x+c}$, where *a*, *b* and *c* are constants.

Given that the curve with the equation $y=\mathrm{f}(x)$ passes through the points with coordinates $(0,4), (2,2)$ and $(-3,-0.5)$, find the values of *a*, *b* and *c*. [3]

Hence describe fully a sequence of transformations which transform the curve $y=\dfrac{1}{x}$ onto the curve $y=\mathrm{f}(x)$. [3]

2 Let $\mathrm{f}(x)=\dfrac{x+x^2}{(1-x)^3}$.

(i) Without using a calculator, solve the inequality $\mathrm{f}(x)\geq 0$. [2]

(ii) Find the first three terms in the expansion of $\mathrm{f}(x)$ in ascending powers of *x*. [3]

(iii) By substituting a suitable value of *x* in the expansion in **(ii)**, find $\displaystyle\sum_{r=1}^{\infty}\frac{r^2}{2^r}$. [2]

3 Given that $y=\tan(e^x-1)$, show that

(i) $\dfrac{\mathrm{d}y}{\mathrm{d}x}=(1+y^2)e^x$, [2]

(ii) $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=\dfrac{\mathrm{d}y}{\mathrm{d}x}(1+2e^x y)$. [2]

Find the Maclaurin series for *y*, up to and including the term in x^2. [2]

Hence state the equation of tangent to the curve $y=\tan(e^x-1)$ at $x=0$. [1]

4 Find the roots of the equation $\dfrac{z^3+8}{8-z^3}=\mathrm{i}$, giving each root in the form $re^{\mathrm{i}\theta}$, where $r>0$ and $-\pi<\theta\leq\pi$. [4]

Hence solve $\dfrac{1+8w^6}{8w^6-1}=\mathrm{i}$. [3]

5 The function f is defined by

$$\mathrm{f}: x\mapsto x^2-2x-1,\ x\in\mathbf{R}.$$

(i) Give a reason why f does not have an inverse. The function f has an inverse if its domain is restricted to $x\geq k$. State the least value of *k*. [2]

(ii) Find $\mathrm{f}^{-1}(x)$ corresponding to the restricted domain in **(i)** and state the domain of f^{-1}. [3]

(iii) Verify that $\mathrm{ff}^{-1}(x)=x$. Hence, or otherwise, find $\mathrm{g}(x)$ such that $\mathrm{gf}(x)=x^2$. [3]

6 A function f is said to be odd if $\mathrm{f}(-x) = -\mathrm{f}(x)$ for all x in the domain of f.
The function h is defined by

$$\mathrm{h} : x \mapsto \frac{x^2 + k^2}{x}, \quad x \in \mathbb{R}, \ x \neq 0,$$

where k is a positive constant.

(i) Show that h is an odd function. If g is also an odd function, determine with a reason, whether the function gh is an odd function. [2]

(ii) Without using a calculator, find the set of values that $\mathrm{h}(x)$ can take. [3]

(iii) Sketch the curve $y = \mathrm{h}(x)$, stating the equations of any asymptotes and the coordinates of any stationary points. [3]

Hence find the range of values of m such that the equation $\dfrac{x^2 + k^2}{x} = mx + 2k$ has two distinct real roots. [1]

7 The rate of increase of a population of size P after t years is proportional to the population size and the rate of decrease is a constant m. Initially, the population size is P_0.

(i) By setting up and solving a differential equation, show that

$$P = \left(P_0 - \frac{m}{k}\right)\mathrm{e}^{kt} + \frac{m}{k}, \text{ where } k \text{ is a positive constant.}$$ [6]

(ii) State the condition for m, in terms of P_0 and k, that will lead to a decline in the population. [1]

(iii) In 1981, the population was 8 million and was increasing at a rate of 1.5% of the population. Due to a famine, the population was decreasing by 200 000 per year. Justify whether the population was expanding or declining in that year. [2]

8 Referred to the origin O, the position vectors of the points P, Q and R are $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ respectively.

(a) Given that the point R divides PQ externally in the ratio 4 : 3, show that the length of the projection of $\overrightarrow{OR}$ onto $\overrightarrow{OP}$ is given by $\left|\dfrac{4\mathbf{p}\cdot\mathbf{q}}{|\mathbf{p}|} - 3|\mathbf{p}|\right|$. [3]

(b) Given that the point R divides PQ internally in the ratio $a : b$, show that

$$\frac{\mathbf{r}\cdot\mathbf{p}}{\mathbf{r}\cdot\mathbf{q}} = \frac{b|\mathbf{p}|^2 + a(\mathbf{p}\cdot\mathbf{q})}{a|\mathbf{q}|^2 + b(\mathbf{p}\cdot\mathbf{q})}.$$ [2]

Deduce that, when the line OR bisects angle POQ, then $a : b = |\mathbf{p}| : |\mathbf{q}|$. [3]

9 **(i)** Use the method of mathematical induction to prove that $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$. [4]

(ii) Let $u_r = r^3$ and by considering $u_r - u_{r-1}$, show that $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$. [4]

(iii) Hence find $1^2 \times 2 + 2^2 \times 3 + \ldots + 50^2 \times 51$, showing your workings clearly. [3]

10 **(a)** **(i)** If *a*, *b* and *c* are three consecutive terms of an arithmetic progression, show that $a+b$, $a+c$ and $b+c$ form an arithmetic progression. [2]

(ii) If *a*, *b* and *c* are three consecutive terms of both an arithmetic and geometric progression, show that $a = b = c$. [3]

(b) A ball is dropped from a height of *a* metres above a horizontal surface and it rebounds to two-thirds of its previous height.

(i) Show that the total distance that the ball has travelled at the instant when it hits the surface for the *n*th time is $5a - 4a\left(\frac{2}{3}\right)^{n-1}$. [3]

(ii) If *L* is the total distance travelled by the ball until it stops bouncing, write down the value of *L* in terms of *a*. [1]

Hence find the least value of *n* if the total distance travelled exceeds 90% of *L*. [3]

11 **(i)** By differentiating both sides of the double-angle formula $\sin 2x = 2\sin x\cos x$, show that $\cos 2x = 1 - 2\sin^2 x$. [2]

(ii) A curve *C* has parametric equations

$$x = 2\cos\theta, \quad y = \sin\theta, \text{ where } 0 < \theta < \frac{\pi}{2}.$$

(a) Sketch *C*. [2]

(b) Without using a calculator, find the exact area of the region bounded by *C* and the *x*- and *y*- axes. [4]

(c) The tangent and the normal at the point $P(2\cos p, \sin p)$ meet the *y*-axis at *T* and *N* respectively. If *O* is the origin, prove that $OT \cdot ON$ is independent of *p*. [5]

(d) Find a cartesian equation of the locus of the point *Q* on *PN* produced such that *N* is the midpoint of *PQ*. [3]

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