2014 PJC J2 H1 Maths Prelims Solutions

1 $$1+\log_3(x+2)=2\log_3(4-x)$$
$$\log_3 3+\log_3(x+2)=2\log_3(4-x)$$
$$\log_3 3(x+2)=\log_3(4-x)^2$$
$$3x+6=(4-x)^2$$
.
$$3x+6=16-8x+x^2$$
$$x^2-11x+10=0$$
$$(x-10)(x-1)=0$$
Hence, x = 10 (reject) or x =1

2 (a) $$\ln\sqrt{\frac{e^{2x}}{4x+1}}=\frac{1}{2}\left[\ln e^{2x}-\ln(4x+1)\right]$$

$$\frac{d}{dx}\ln\sqrt{\frac{e^{2x}}{4x+1}}=\frac{d}{dx}\frac{1}{2}\left[\ln e^{2x}-\ln(4x+1)\right]$$
$$=\frac{1}{2}\left(2-\frac{4}{4x+1}\right)$$
$$=1-\frac{2}{4x+1}$$

2 (b) $$\frac{(x+1)^2}{3\sqrt{x}}=\frac{x^2+2x+1}{3\sqrt{x}}=\frac{1}{3}\left(x^{\frac{3}{2}}+2x^{\frac{1}{2}}+x^{-\frac{1}{2}}\right)$$
$$\int_1^4 \frac{(x+1)^2}{3\sqrt{x}}\,dx=\int_1^4 \frac{1}{3}\left(x^{\frac{3}{2}}+2x^{\frac{1}{2}}+x^{-\frac{1}{2}}\right)dx$$
$$=\frac{1}{3}\left[\frac{2}{5}x^{\frac{5}{2}}+\frac{4}{3}x^{\frac{3}{2}}+2x^{\frac{1}{2}}\right]_1^4$$
$$=\frac{1}{3}\left(\frac{2}{5}(32)+\frac{4}{3}(8)+2(2)\right)-\frac{1}{3}\left(\frac{2}{5}(1)+\frac{4}{3}(1)+2(1)\right)$$
$$=\frac{1}{3}\left(\frac{356}{15}\right)$$
$$=\frac{356}{45}$$

3 Let y = Area of triangle OPQ.

y = Area of $OABC$ − Area of ΔOAP – Area of ΔBPQ – Area of Δ OCQ

$$y=(1)(1)-\frac{1}{2}(1)(x)-\frac{1}{2}(1-x)(kx)-\frac{1}{2}(1-kx)(1)$$

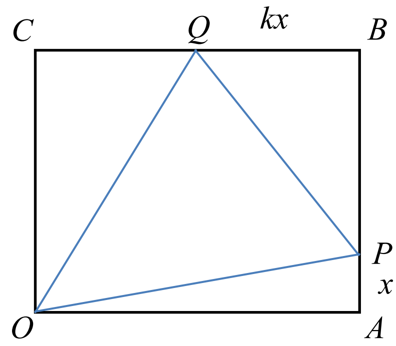
$$y=1-\frac{1}{2}x-\frac{k}{2}x+\frac{k}{2}x^2-\frac{1}{2}+\frac{k}{2}x$$

$$y=\frac{1}{2}-\frac{1}{2}x+\frac{k}{2}x^2$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{1}{2}+kx=0$$

$$x=\frac{1}{2k}$$

$$y=\frac{1}{2}-\frac{1}{2}\left(\frac{1}{2k}\right)+\frac{k}{2}\left(\frac{1}{2k}\right)^2=\frac{1}{2}-\frac{1}{4k}+\frac{1}{8k}=\frac{1}{2}-\frac{1}{8k}$$

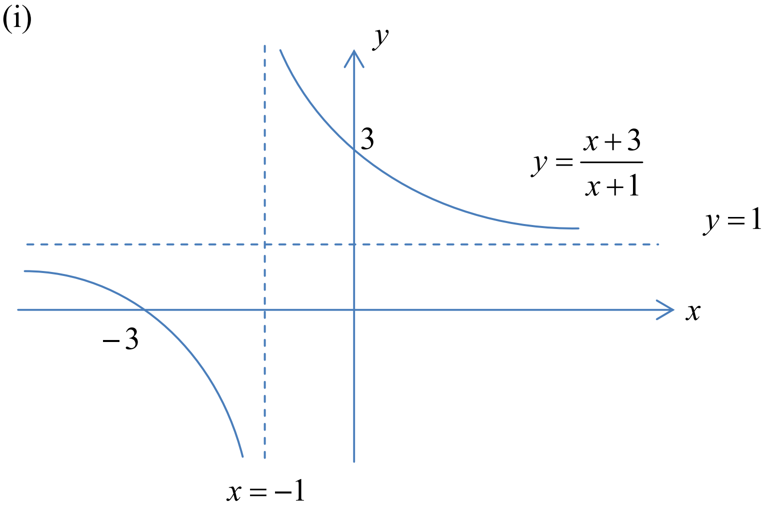


Using first derivative test,

x	$\frac{1}{2k}^-$	$\frac{1}{2k}$	$\frac{1}{2k}^+$
$\frac{\mathrm{d}y}{\mathrm{d}x}$	-	0	+
Outline	\	-	/

Conclusion: area is a minimum at $x=\frac{1}{2k}$

4 (i)

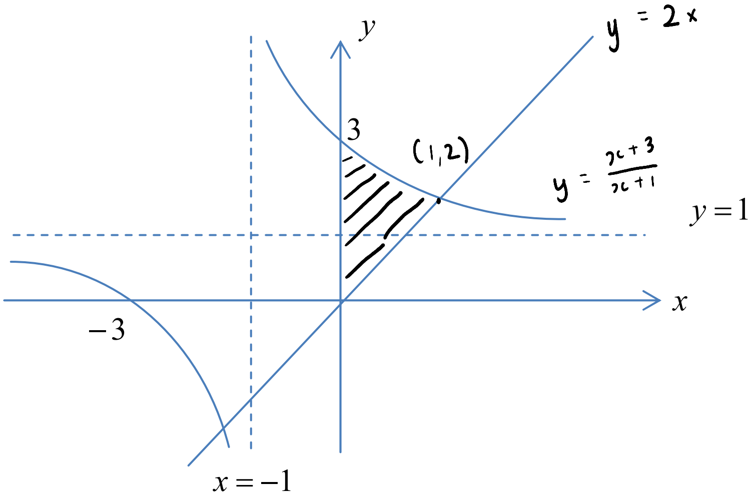


(ii) From graph, for $\frac{x+3}{x+1}\geq 0$, $x\leq -3$ or $x>-1$

For $\frac{3-x}{1-x}\geq 0$, replace x by $-x$: $-x\leq -3$ or $-x>-1$

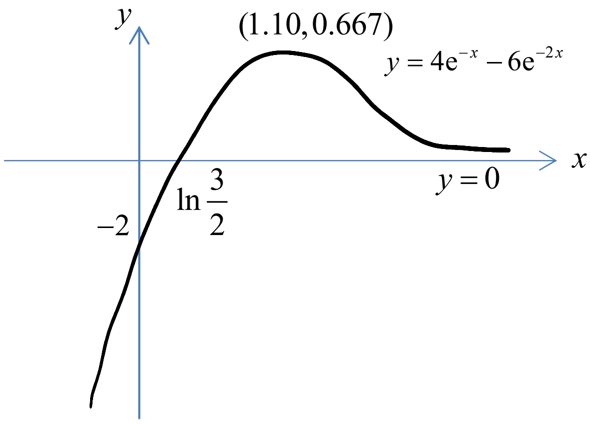
$x\geq 3$ or $x<1$

(iii)



Required area = $\int_0^1 \frac{x+3}{x+1} - 2x \, dx = 1.3863 \approx 1.39$ units^2

5 (i)



(ii) $\frac{dy}{dx} = 12e^{-2x} - 4e^{-x}$

(iii) $\frac{dy}{dx} = 12e^{-2x} - 4e^{-x} = \frac{8}{3}$

$3e^{-2x} - e^{-x} = \frac{2}{3}$

$9e^{-2x} - 3e^{-x} - 2 = 0$

Let $u = e^{-x}$

$9u^2 - 3u - 2 = 0$

$(3u+1)(3u-2) = 0$

$u = -\frac{1}{3}$ (NA) or $u = \frac{2}{3}$

$e^{-x} = \frac{2}{3} \Rightarrow -x = \ln\frac{2}{3} \Rightarrow x = -\ln\frac{2}{3}$

$y = 4e^{-(-\ln\frac{2}{3})} - 6e^{-2(-\ln\frac{2}{3})} = 0$

Hence, P is $(-\ln\frac{2}{3}, 0)$

(iv) To find equation of tangent: $y = mx + c$

$$0 = \frac{8}{3}(-\ln\frac{2}{3}) + c$$

$$c = \frac{8}{3}(\ln\frac{2}{3})$$

Hence, equation of tangent is $y = \frac{8}{3}x + \frac{8}{3}(\ln\frac{2}{3})$

6 (i) $\mathrm{P}(A' \cap B') = \frac{2}{5} \Rightarrow \mathrm{P}(A \cup B) = \frac{3}{5}$

Since $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$,

$$\frac{3}{5} = \mathrm{P}(A) + \mathrm{P}(A) - \frac{1}{5}$$

$$\mathrm{P}(A) = \frac{2}{5}$$

(ii) $\mathrm{P}\ (A' \cap B) = \mathrm{P}(A \cup B) - \mathrm{P}(A) = \frac{3}{5} - \frac{2}{5} = \frac{1}{5}$

OR: $\mathrm{P}(A' \cap B) = \mathrm{P}(B) - \mathrm{P}(A \cap B) = \frac{2}{5} - \frac{1}{5} = \frac{1}{5}$

7 (i) Calculate the number of participants to survey from each strata as shown in the table below.

	14-15 years	16-17 years	18-19 years
Male	15	14	16
Females	4	6	5

Select the number of participants in each strata using simple random sampling.

(ii) It is difficult to obtain a full list of the spectators.
["difficult to divide the population between different strata" not accepted]

(iii) (1) Since we require 1% of the spectator, $k = 100$

(2) Choose a random spectator from the first 100 in the queue and choose every subsequent 100^{th} spectators until all of them are in the stadium.

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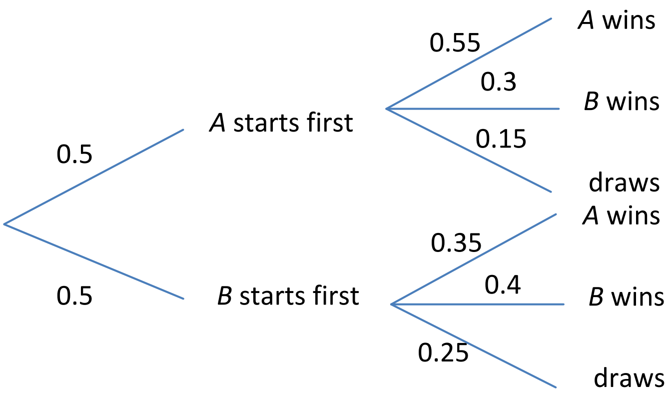

(i) P(A wins game) = (0.5)(0.55) + (0.5)(0.35) = 0.45

(ii) P(B wins game) = (0.5)(0.3) + (0.5)(0.4) = 0.35

Required probability = P(A wins) x P(B wins) x P (draw) x 6

= (0.45)(0.35)(0.2)(6)

= 0.189

(iv)
$$\begin{aligned} \text{P} (A \text{ win}) &= \text{P} (B \text{ wins}) \\ p(0.55)+(1-p)(0.35) &= p(0.3)+(1-p)(0.4) \\ 0.55p+0.35-0.35p &= 0.3p+0.4-0.4p \\ 0.3p &= 0.05 \\ p &= \frac{1}{6} \end{aligned}$$

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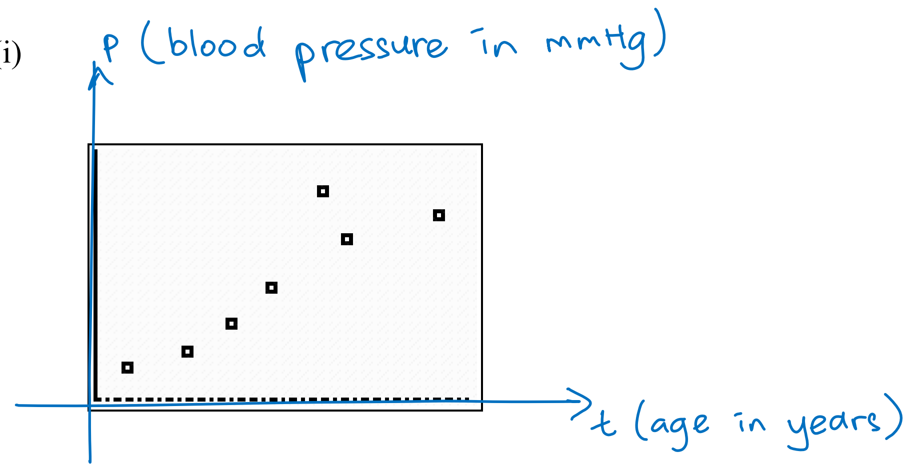
(i)



(ii) $r = 0.9001027 \approx 0.900$.

There is a strong positive linear correlation between the age of a person and his blood pressure.

(iii) Choice of line is p on t because age is the independent variable.

$p = 7.025032 + 2.386583t \approx 7.03 + 2.39t$

(iv)Sub in $t = 0$, to get p =7.03 (3sf)

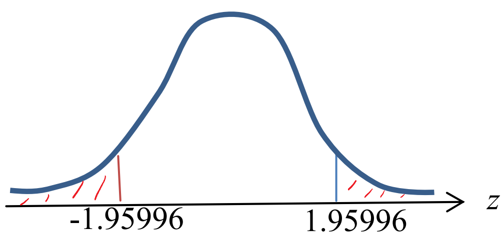
The model is not relevant in real life as 7.03 mmHg is way too low for blood pressure of a newborn.

10 Let M ~ mass of a piece of steak

M ~N(250, 10^2)

Test $H_0: \mu = 250$ vs $H_1: \mu \neq 250$

Under H_o, $\overline{M} \sim N\left(250, \frac{10^2}{n}\right)$ exactly



Critical region: $z \le -1.95996$ or $z \ge 1.95996$

Since owner's claim is accepted, we do not reject H_0.
$\Rightarrow$ Standardized test statistic lies outside critical region

$$-1.95996 \le \frac{247.5-250}{\sqrt{\frac{10^2}{n}}} \le 1.95996$$

$$-1.95996 \le -0.25\sqrt{n} \le 1.95996$$

$$7.8399 \ge \sqrt{n} \ge -7.8399$$

Since $\sqrt{n}$ is positive, $7.8399 \ge \sqrt{n} > 0$

$$61.464 \ge n > 0$$

Hence, largest value of n is 61.

Test $H_0 : \mu = 300$ vs $H_1 : \mu < 300$

Under H_o, $\overline{M} \sim N\left(300, \frac{10^2}{30}\right)$ exactly

Method1: From GC, p –value = 0.085452 (5 s.f.) > 0.05
Since the p –value is more than the level of significance, we do not reject H_o. There is insufficient evidence at 5% level of significance that the owner's claim is an overestimation.

Method 2: Test statistic, $z = \frac{\bar{x}-\mu}{\sqrt{\frac{\sigma^2}{n}}} = -1.3693$ (5 s.f.)

-1.3693

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Critical region: $z \le -1.6449$
Since the standardized test statistic does not fall within the critical region, we do not reject H_o. There is insufficient evidence at 5% level of significance that the owner's claim is an overestimation.

From GC, p –value = 0.08545182.
To reject the owner's claim at α% level of significance,
0.08545182 (100) < α%
Hence, smallest level of significance is at 8.5452 % (4dp)

11 (i) $X \sim$ number of dark chocolates chosen, out of n. $X \sim B(n, 0.4)$
$P(X = 0) \ge 0.01$
Using GC,
$n = 8$, $P(X = 0) = 0.0168 \ge 0.01$
$n = 9$, $P(X = 0) = 0.01008 \ge 0.01$

$n = 10$, P ($X = 0$) = 0.00605
maximum number of chocolates = 9

(ii) $Y \sim$ number of dark chocolates, out of 10. $Y \sim B\ (10, 0.4)$
Required probability = P ($Y \le 4$) = 0.63310 ≈ 0.633

Alternative method:
$W \sim$ number of white chocolates, out of 10. $Y \sim B\ (10, 0.6)$
Required probability = P ($W \ge 6$) = 1 − P ($W \le 5$) = 0.63310 ≈ 0.633

(iii) Since half of the white chocolates contain almonds, 30% of chocolates are white chocolates with almonds
$A \sim$ number of white chocolates with almond, out of 10. $A \sim B\ (10, 0.3)$
Required probability = P ($W > 3$) = 1 − P ($W \le 3$) = 0.35039 ≈ 0.350

(iv) $V \sim$ number of boxes with more white than dark chocolates, out of 50.
$V \sim B\ (50, 0.63310)$
Since $n = 50$ is large, $np = 31.655 > 5$, $n(1-p) = 18.345 > 5$,
$V \sim N\ (31.655, 11.614)$
Required probability = P ($V < 30$) = P ($V < 29.5$) = 0.26358 ≈ 0.264

12 $X\sim$ mass of a randomly chosen cherry tomato $X\sim N(7, 0.81)$
$Y\sim$ mass of a randomly chosen lime $Y\sim N(8, 1.21)$

(i) $X_1 + X_2 + ... + X_{20} \sim N(140, 16.2)$
$P(X_1 + X_2 + ... + X_{20} \ge 145) = 0.10707 \approx 0.107$

(ii) $2Y - (X_1 + X_2) \sim N(2, 6.46)$
$P(2Y > X_1 + X_2) = P(2Y - (X_1 + X_2) > 0) = 0.784327 \approx 0.784$
OR: $X_1 + X_2 - 2Y \sim N(-2, 6.46)$
$P(2Y > X_1 + X_2) = P(X_1 + X_2 - 2Y < 0) = 0.784327 \approx 0.784$

(iii) $L\sim$ cost (in cents) of a randomly chosen packet of limes
$L\sim N\left(8\times10\times0.4\ ,\ \ 1.21\times10\times0.4^2\right) \sim N(32, 1.936)$
$P(L \le 30) = 0.0753024 \approx 0.0753$

(iv) $C\sim$ cost (in cents) of a randomly chosen packet cherry tomatoes
$C\sim N\left(7\times25\times0.6\ ,\ \ 0.81\times25\times0.6^2\right) \sim N(105, 7.29)$
$L_1 + L_2 + L_3 \sim N(32\times3\ ,\ \ 1.936\times3)$
Thus $L_1 + L_2 + L_3 - C \sim N(-9\ ,\ 13.098)$
$P(-10 \le L_1 + L_2 + L_3 - C \le 10) = 0.609$