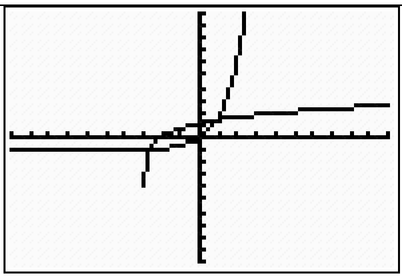
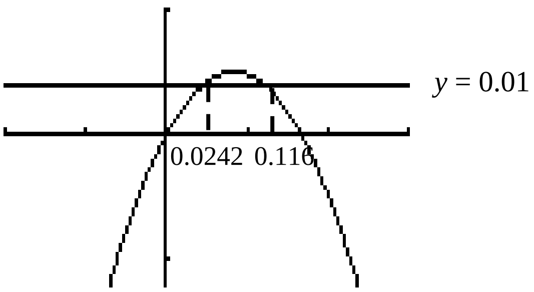
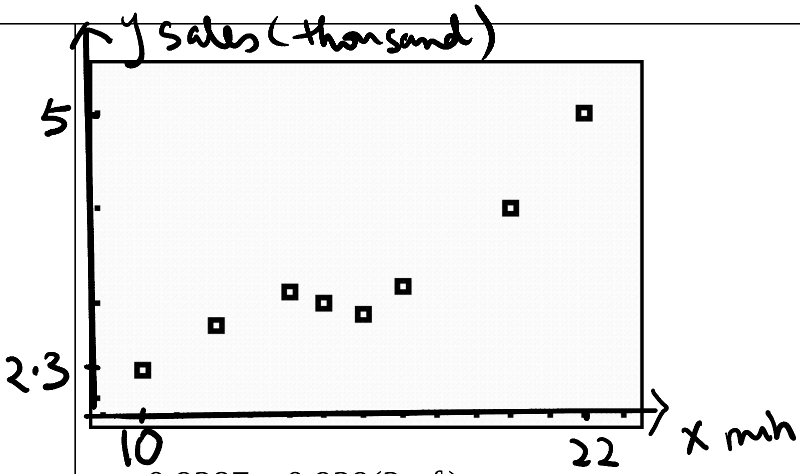
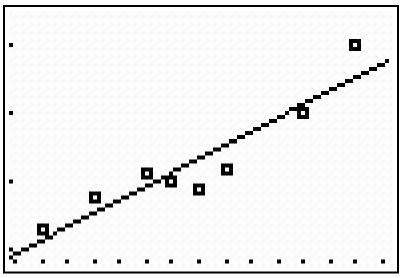
2014 C2 H1 Prelim Solutions

1	$\int_0^1 \sqrt{x}+\frac{1}{\sqrt{2-x}}\,\mathrm{d}x=\frac{x^{\frac{3}{2}}}{\frac{3}{2}}-\frac{(2-x)^{\frac{1}{2}}}{\frac{1}{2}}\Big\|_0^1$ $=\frac{2}{3}-2\left(1-\sqrt{2}\right)=2\sqrt{2}-\frac{4}{3}$
2	 Equations of asymptote: $x=-3,\quad y=-1$ x-coordinates of intersection $x=0.854,\ x=-2.60$ (ii) $\int_{-2.60387}^{0.85404}\{\ln(x+3)-(e^x-1)\}\,\mathrm{d}x=3.29(3\text{s.f})$
3(a)	$\sqrt{x}-\frac{3}{\sqrt{x}}=2$ $u-\frac{3}{u}=2$ $u^2-3-2u=0$ $u^2-2u-3=0$ $(u-3)(u+1)=0$ $u=3,\quad u=-1$ $\sqrt{x}=3,\quad \sqrt{x}=-1(\text{NA})$ $x=9$
3(b)	Coefficient of $x^2>0$ and $(-4+k)^2-4(k^2+4)<0$ $16-8k+k^2-4k^2-16<0$ $-3k^2-8k<0$ $3k^2+8k>0$ $k(3k+8)>0$ $k>0$ or $k<-\frac{8}{3}$ $\{k\in\mathbb{R}: k>0 \text{ or } k<-\frac{8}{3}\}$
4	$y=\frac{4}{x}-x \quad \therefore \frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{4}{x^2}-1$ When $x=1$, $\frac{\mathrm{d}y}{\mathrm{d}x}=-5$ $\therefore$ Gradient of normal at $x=1$ is $\frac{1}{5}$

	Substitute $x=1$ to (1) $\Rightarrow y=3$ $\therefore$ Equation of normal at $(1,3)$ is: $y-3=\frac{1}{5}(x-1)$ $5y-x=14$ $y=\frac{x+14}{5}$ - - - - (2) Substitute (2) into (1): $(x)(\frac{x+14}{5})+x^2=4 \Rightarrow x(x+14)+5x^2=20$ $6x^2+14x-20=0 \Rightarrow 3x^2+7x-10=0$ $\Rightarrow (3x+10)(x-1)=0$ $x=-\frac{10}{3}$ or $x=1$ $\{(1,3)$ is the given point$\}$ When $x=-\frac{10}{3}$, $y=\frac{32}{15}$ $\therefore$ The coordinates of the other point are $(-\frac{10}{3}, \frac{32}{15})$.
5	(i) By Pythagoras Theorem, $BC^2=h^2+5^2 \Rightarrow BC=\sqrt{h^2+5^2}=\sqrt{h^2+25}$ When the triangle is equilateral, $BC=AC=10$ $10=\sqrt{h^2+5^2}$ $100=h^2+25$ $h=\sqrt{100-25}=\sqrt{75}$ (shown) (ii) Let the length of BC be x cm. Given $\frac{\mathrm{d}h}{\mathrm{d}t}=2$ cm/s, find $\frac{\mathrm{d}x}{\mathrm{d}t}$ when $h=\sqrt{75}$ $x=\sqrt{h^2+25}$ $\frac{\mathrm{d}x}{\mathrm{d}h}=\frac{1}{2}\left(h^2+25\right)^{-\frac{1}{2}}(2h) \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}h}=h\left(h^2+25\right)^{-\frac{1}{2}}$ $\frac{\mathrm{d}x}{\mathrm{d}t}=\left(\frac{\mathrm{d}x}{\mathrm{d}h}\right)\left(\frac{\mathrm{d}h}{\mathrm{d}t}\right)$ $\Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t}=h\left(h^2+25\right)^{-\frac{1}{2}}(2)=\frac{2h}{\sqrt{h^2+25}}$ When $h=\sqrt{75}$, $\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{2\sqrt{75}}{\sqrt{75+25}}=\frac{\sqrt{75}}{5}=\frac{5\sqrt{3}}{5}$ $=\sqrt{3}$ or $1.73(3\text{s.f})$ cm/s
6(i)	Perimeter $=\pi r+2x+2r=0.5$ $2x=0.5-r(\pi+2) \Rightarrow x=\frac{0.5-r(\pi+2)}{2}=\frac{1-2r(\pi+2)}{4}$ $A=\frac{\pi r^2}{2}+(x)(2r)$

	$A=\frac{\pi r^2}{2}+\left[\frac{1-2r(\pi+2)}{4}\right](2r)$ $A=\frac{\pi r^2+r-2r^2\left(\pi+2\right)}{2}$ $A=\frac{r+\pi r^2-2\pi r^2-4r^2}{2}$ $A=\frac{r-\pi r^2-4r^2}{2}\Rightarrow A=\frac{r-r^2\left(\pi+4\right)}{2}$ (shown)
6(ii)	$A=\frac{r-r^2\left(\pi+4\right)}{2}$ $\frac{\mathrm{d}A}{\mathrm{d}r}=\frac{1}{2}\left(1-2r\left(\pi+4\right)\right)$ At the stationary point, $\frac{\mathrm{d}A}{\mathrm{d}r}=0 \quad \Rightarrow 1-2r(\pi+4)=0 \quad \Rightarrow r=\frac{1}{2\left(\pi+4\right)}$ Since $\frac{\mathrm{d}^2A}{\mathrm{d}r^2}=-(\pi+4)<0$ A is maximum when $r=\frac{1}{2\left(\pi+4\right)}$ Maximum value of A $A=\frac{r-r^2\left(\pi+4\right)}{2}$ $=\left(\frac{1}{2}\right)\left(\frac{1}{2\left(\pi+4\right)}-\frac{1}{4\left(\pi+4\right)^2}\left(\pi+4\right)\right)$ $=\frac{1}{8\left(\pi+4\right)}\,\mathrm{km}^2$
6(iii)	 $0.0242\le r\le 0.116$
7(i) 7(ii)	Simple random sample : Assign a number from 1 – 2000 to the 2000 students. Randomly generate 400 numbers and select the members who are assigned those numbers Systematic random sampling: First, obtain a list of all the students from 1 to 2000. Next, determine the sampling interval $\frac{2000}{400}=5$. Finally, use a graphing calculator to select a random number from 1 to 5(e.g.1) and subsequently select every 5^{th} student thereafter (i.e.6^{th}, 11^{th}, $16^{\text{th}}\ldots$) until 400 students are selected.

8(i)	
8(ii)	$r = 0.9287 = 0.929(3\,\text{s.f})$ Since *r* is close to 1, we may say that the advertising time on television per week, *x* and the corresponding sales per week, *y* have a strong positive linear correlation.
8(iii)	The equation of the regression line : $y = 0.184 + 0.197x$ The interpretation of $m = 0.197$ is that for every minute increase of advertising time, number of sales increase by 197. $0.197 \times 1000 = 197\text{sales}$ 
8(iv)	When $x = 19 \Rightarrow y = 3.928 \approx 3.9$ Since $x = 19$ lies within the data range $10 \le x \le 22$ and $\lvert r \rvert$ is close to 1. Therefore this estimate is reliable
9(a) (i)	Let X be the number of cups without cracks. $X \sim \mathrm{B}(10, 0.9)$ $\mathrm{P}(X \ge 8) = 1 - \mathrm{P}(X < 8)$ $= 1 - \mathrm{P}(X \le 7)$ $= 0.929809$ ≈ 0.930
9(a)(ii)	Let Y be the number of cups with cracks. $Y \sim B(250, 0.1)$ $n = 250, \quad p = 0.1, \quad 1 - p = 0.9$ $np = 25 > 5, \quad n(1-p) = 225 > 5$ $Y \sim \mathrm{N}(25, 22.5)$ approximately $\mathrm{P}(5 \le Y \le 15) \xrightarrow{c.c.} \mathrm{P}(4.5 < Y < 15.5)$ $= 0.0226$

9(b)	Let U be the number of cracked cups in first sample. $U \sim \mathrm{B}(15, 0.1)$ Let W be the number of cracked cups in the second sample. $W \sim \mathrm{B}(10, 0.1)$ P(reject the batch) $= \mathrm{P}(U > 4) + \mathrm{P}(U = 4)\mathrm{P}(W \geq 1)$ $= 1 - \mathrm{P}(U \leq 4) + \mathrm{P}(U = 4)\left[1 - \mathrm{P}(W = 0)\right]$ $= 1 - 0.98728 + 0.042835(1 - 0.34868)$ $= 0.0406$
10(a) (i)	$n = 50$ is large, by central limit theorem $\overline{X} \sim N\left(11, \frac{3^2}{50}\right)$ *approx* $P\left(9.2 < \overline{X} < 12.2\right) = 0.998(3s.f)$
10(a) (ii)	$\mathrm{P}\left(\overline{X} < c\right) = 0.03 \Rightarrow c = 10.2(3\mathrm{s.f})$
10(b) (i)	Let S denote the volume of the small bottle of drink & L denote the volume of the larger bottle of drink $S \sim N\left(338, 3^2\right)$ & $L \sim N\left(1010, 12^2\right)$ $\mathrm{P}\left(L > S_1 + S_2 + S_3\right) = \mathrm{P}(\mathrm{L} - (S_1 + S_2 + S_3) > 0)$ $\mathrm{E}\left(L - (S_1 + S_2 + S_3)\right) = \mathrm{E}(L) - 3\mathrm{E}(S) = 1010 - 3 \times 338 = -4$ $\mathrm{Var}\left(L - (S_1 + S_2 + S_3)\right) = \mathrm{Var}(L) + 3\mathrm{Var}(S)$ $= 12^2 + 3 \times 3^2 = 171$ $L - S_1 + S_2 + S_3 \sim N(-4, \quad 171)$ $\mathrm{P}(\mathrm{L} - (S_1 + S_2 + S_3) > 0) = 0.3798 = 0.380(3\mathrm{s.f})$
10(b)(ii)	$\mathrm{P}(L > 3S) = \mathrm{P}(L - 3S > 0)$ $\mathrm{E}(L - 3S) = \mathrm{E}(L) - 3\mathrm{E}(S) = 1010 - 3 \times 338 = -4$ $\mathrm{Var}(L - 3S) = \mathrm{Var}(L) + 9\mathrm{Var}(S) = 12^2 + 9 \times 3^2 = 225$ $L - 3S \sim N(-4, \quad 225)$ $\mathrm{P}(L - 3S > 0) = 0.3949 = 0.395(3\mathrm{s.f})$
10(b)(iii)	$\mathrm{P}\left(\frac{S_1 + S_2 + S_3 + \ldots\ldots + S_{12}}{12} < 340\right)$ $\quad \overline{S} \sim N\left(338, \frac{3^2}{12}\right)$ $P\left(\overline{S} < 340\right) = 0.990(3s.f)$
11(a)	$\mathrm{P}(\mathrm{A} \mid \mathrm{B}') = \frac{\mathrm{P}(\mathrm{A} \cap \mathrm{B}')}{\mathrm{P}(\mathrm{B}')}$ $\Rightarrow \mathrm{P}(\mathrm{A} \cap \mathrm{B}') = \mathrm{P}(\mathrm{A} \mid \mathrm{B}') \cdot \mathrm{P}(\mathrm{B}') = 0.2 \times 0.6 = 0.12$ $\mathrm{P}(\mathrm{A}) = \mathrm{P}(\mathrm{A} \cap \mathrm{B}) + \mathrm{P}(\mathrm{A} \cap \mathrm{B}')$ $= 0.3 + 0.12 = 0.42$
11(b) (i)	Required Probability $= 0.6 \times 0.9 \times 0.01 = 0.0054$
11(b) (ii)	Required Probability = P (men have continental car w/o turbo) + P (women have continental car w/o turbo) $= 0.6 \times 0.9 \times 0.99 + 0.4 \times 0.7 \times 0.99$ $= 0.8118$ or $0.812(3\mathrm{s.f})$
	P(non-continental car) $= 0.6 \times 0.1 + 0.4 \times 0.3 = 0.18$ P (woman owner | non-continental car)

	$=\dfrac{\text{P(woman owner } \cap \text{ non-continental car)}}{\text{P(non-continental car)}}=\dfrac{0.4\times 0.3}{0.18}$ $=\dfrac{2}{3}$ or 0.667
12(i)	Unbiased estimate of population mean $\bar{x}=\dfrac{635.78}{50}+2300=2312.7156=2310(3\text{sf})$ $s^2=\dfrac{1}{n-1}\left[\sum(x-2300)^2-\dfrac{\left(\sum(x-2300)\right)^2}{n}\right]$ $=\dfrac{1}{49}\left[130584.32-\dfrac{635.78^2}{50}\right]$ $=2499.999915$ or 2500 (3sf)
12(ii)	$\mathrm{H_0}: \mu=2300$ $\mathrm{H_1}: \mu\neq 2300$ at the 10% significance level For n = 50, $\bar{x}=2312.7156$, $s^2=2499.999915$ (min 4 s.f.to be used) Under $\mathrm{H_0}$, $\overline{X}\sim \mathrm{N}\left(2300,\dfrac{2499.999915}{50}\right)$ approximately by CLT, since n is large. Test Statistics, $Z=\dfrac{\overline{X}-2300}{\sqrt{\dfrac{2499.999915}{50}}}$ Since p-value = 0.0721 < 0.1, we reject $\mathrm{H_0}$ at 10% significance level and conclude that Company B's suspicion is supported.
12(a) 12(b)	$\mathrm{H_0}: \mu=2300$ $\mathrm{H_1}: \mu>2300$ at 5% level of significance. Under $\mathrm{H_0}$, $\overline{X}\sim \mathrm{N}\left(2300,\dfrac{2499.999915}{50}\right)$ Test Statistics, $Z=\dfrac{k-2300}{\sqrt{\dfrac{2499.999915}{50}}}$ For $\mathrm{H_0}$ to be rejected, $Z=\dfrac{k-2300}{\sqrt{\dfrac{2499.999915}{50}}}\geq \mathrm{InvNorm}(0.95)$ $k-2300\geq\sqrt{\dfrac{2499.999915}{50}}\times(1.64485)$ $k\geq 2311.6309$ Least value of k = \$2311.64