H1 Math (2014 JC2 Prelim Exam II) – Solution

Qn	Solution
1(a)	$\int\left(\frac{2}{x}-1\right)^2 \mathrm{d}x$ $=\int \frac{4}{x^2}-\frac{4}{x}+1\ \mathrm{d}x$ $=-\frac{4}{x}-4\ln(x)+x+C$
(b)	$\int_{\frac{2}{3}}^{a}\frac{1}{3x-1}\ \mathrm{d}x=\int_0^4\frac{1}{\sqrt{x}}\ \mathrm{d}x$ $\frac{1}{3}\left[\ln(3x-1)\right]_{\frac{2}{3}}^{a}=\left[\frac{x^{\frac{1}{2}}}{\left(\frac{1}{2}\right)}\right]_0^4$ $\frac{1}{3}\left[\ln(3a-1)-\ln 1\right]=\left[2\sqrt{x}\right]_0^4$ $\frac{1}{3}\ln(3a-1)=2(2)-2(0)$ $\ln(3a-1)=4\times 3$ $3a-1=\mathrm{e}^{12}$ $a=\frac{\mathrm{e}^{12}+1}{3}$

Qn	Solution
2 (i)	When $T = 18$, $18 = 24 - 29e^{-0.3t}$ $e^{-0.3t} = \frac{6}{29}$ $\ln e^{-0.3t} = \ln\frac{6}{29}$ $-0.3t = \ln\frac{6}{29}$ $\therefore t = 5.25$ (3 s.f.)
(ii)	$T = 24 - 29e^{-0.3t}$ $\frac{dT}{dt} = -29e^{-0.3t}(-0.3) = (8.7)e^{-0.3t}$ When $t = 10$, $\frac{dT}{dt} = (8.7)e^{-3} = 0.43315$ Required rate of change of T is 0.433 ^{o}C per hour.
(iii)	When t becomes very large, $e^{-0.3t}$ approaches to 0. $\therefore\ T \to 24 - 29(0)$ i.e., $T \to 24$ as $t \to \infty$. Therefore the temperature of the turkey can never reach 24 ^{o}C.

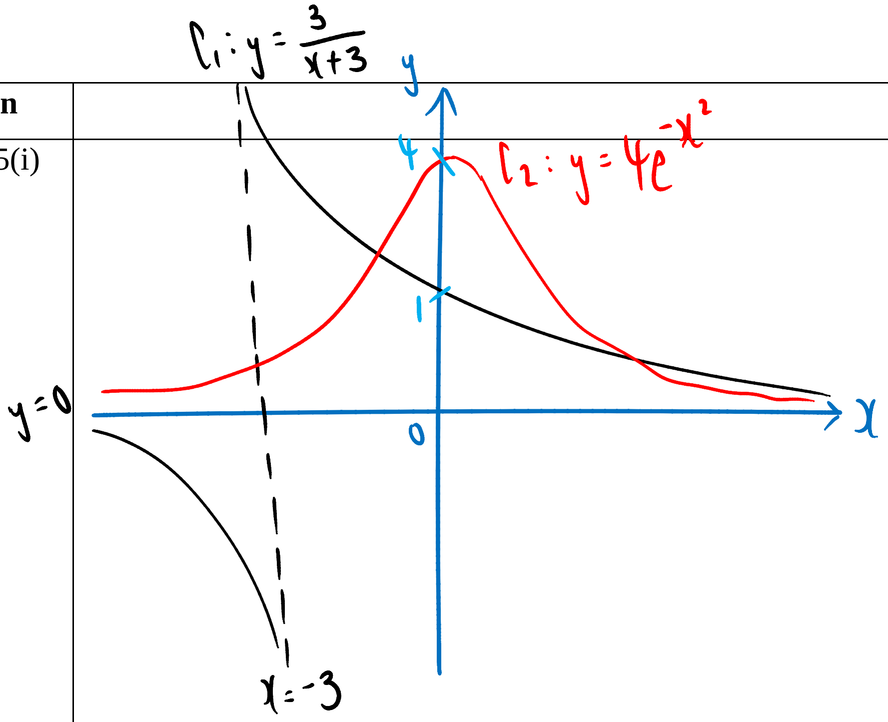
Qn	Solution
3(i)	$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x^2 - 3x + 2} \times (2x - 3)$ $= \frac{2x-3}{x^2 - 3x + 2}$ When $x = -1$, $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{5}{6}$ Thus, gradient of normal = $-1 \div \left(-\frac{5}{6}\right) = \frac{6}{5}$ When $x = -1,\ y = \ln(1+3+2) = \ln 6$ Equation of normal: $y - \ln 6 = \frac{6}{5}\left[x - (-1)\right]$ $y = \frac{6}{5}x + \frac{6}{5} + \ln 6,$ where $a = \frac{6}{5}, b = \frac{6}{5}, c = 6$
(ii)	At $A, y = 0$ $0 = \frac{6}{5}x + \frac{6}{5} + \ln 6$ $x = -1 - \frac{5}{6}\ln 6$ $\therefore A\left(-1 - \frac{5}{6}\ln 6, 0\right)$ At $B, x = 0$ $y = \frac{6}{5}(0) + \frac{6}{5} + \ln 6$ $= \frac{6}{5} + \ln 6$ $\therefore B\left(0, \frac{6}{5} + \ln 6\right)$

	Area of triangle OAB $=\frac{1}{2}\times OA\times OB$ $=\frac{1}{2}\times\left(1+\frac{5}{6}\ln 6\right)\left(\frac{6}{5}+\ln 6\right)$ $=\frac{1}{2}\left(\frac{6+5\ln 6}{6}\right)\left(\frac{6+5\ln 6}{5}\right)$ $=\frac{1}{60}(6+5\ln 6)^2$, where $p=6,\ q=5$

Qn	Solution
4(i)	Area of fountain $=\frac{1}{2}(x)(x)\sin(60^\circ)$ (by sine rule) $=\frac{1}{2}x^2\left(\frac{\sqrt{3}}{2}\right)$ $=\frac{\sqrt{3}}{4}x^2$ [Shown]
(ii)	$2R=20+x \quad \Rightarrow \quad R=10+\frac{x}{2}$ $A=\frac{1}{2}\pi R^2-\frac{\sqrt{3}}{4}x^2$ $=\frac{\pi}{2}\left(10+\frac{x}{2}\right)^2-\frac{\sqrt{3}}{4}x^2$ $=\frac{\pi}{2}\left(100+10x+\frac{x^2}{4}\right)-\frac{\sqrt{3}}{4}x^2$ $=50\pi+5\pi x+\left(\frac{\pi}{8}-\frac{\sqrt{3}}{4}\right)x^2$ [Shown]
(iii)	$A=50\pi+5\pi x+\left(\frac{\pi}{8}-\frac{\sqrt{3}}{4}\right)x^2$ $\frac{\mathrm{d}A}{\mathrm{d}x}=5\pi+\left(\frac{\pi}{4}-\frac{\sqrt{3}}{2}\right)x$ When A is maximised, $\frac{\mathrm{d}A}{\mathrm{d}x}=0$. $5\pi+\left(\frac{\pi}{4}-\frac{\sqrt{3}}{2}\right)x=0$ $\left(\frac{\pi}{4}-\frac{\sqrt{3}}{2}\right)x=-5\pi$ $x=194.82$ $=195\ (3\text{ s.f.})$ By first derivative test,

x	195^-	195	195^+
$\frac{\mathrm{d}y}{\mathrm{d}x}$	+	0	-
slope	/	–	\

$\therefore$ When $x=195$, A is maximum. (*)

<table>
<tr><th>Qn</th><th></th></tr>
<tr><td>5(i)</td><td>
</td></tr>
<tr><td>(ii)</td><td>By GC,
$x = -0.99231 \approx -0.992$
or
$x = 1.3235 \approx 1.32$

$\frac{e^{x^2}}{x+3} > \frac{4}{3}$
$\frac{3}{x+3} > 4e^{-x^2}$
$\therefore -3 < x < -0.992 \text{ or } x > 1.32$</td></tr>
<tr><td>(iii)</td><td>Area
$= \int_{-0.99231}^{1.3235} 4e^{-x^2} - \frac{3}{x+3}\, dx$
$= 4.0024$
$\approx 4.00 \text{ units}^2$</td></tr>
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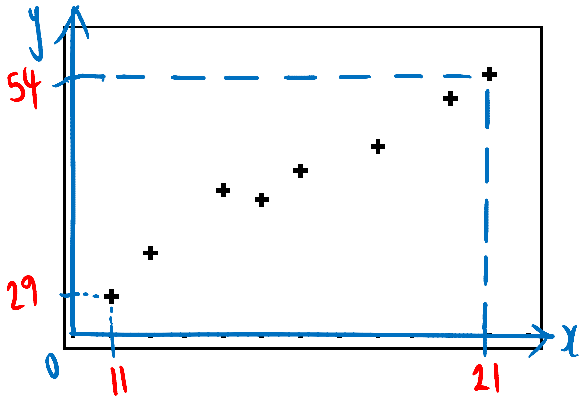
Qn	Solution
6(i)	(1) Number all the students in the college from 1 to 1000 according to the alphabetical order of their names in the college register. (2) Sampling interval $=\dfrac{1000}{100}=10$ (3) Select a student randomly from the first 10 students in the list. (4) Every 10^{th} student is selected from then on until the sample size of 100 is obtained.
(ii)	Quota sampling. Disadvantage: As the interviewer is free to choose who he/she would like to interview, the sample obtained **might be biased**.

Qn	Solution
7(i)	Assume that the ability of any person in country *A* to roll his/her tongue is **independent** of another person's ability to do so.
(ii)	Let *R* be the number of people that can roll their tongues out of 16. $R \sim \mathrm{B}(16, 0.73)$ $\mathrm{P}(R = 12) = 0.222$ (3 s.f.)
(iii)	$\mathrm{P}(R \geq 10)$ $= 1 - \mathrm{P}(R \leq 9)$ $= 1 - 0.11248$ $= 0.888$ (3 s.f.)
	Let *Y* denote the number of people that can roll their tongues, out of 50. $Y \sim \mathrm{B}(50, 0.73)$ Since *n* is large, $np = 50 \times 0.73 = 36.5 > 5$ $nq = 50 \times (1 - 0.73) = 13.5 > 5$ $Y \sim N(36.5, 9.855)$ approximately $\mathrm{P}(Y > 38)$ $= \mathrm{P}(Y > 38.5)$ (with c.c.) $= 0.262$ (to 3 s.f.)

Qn	Solution
8(i)	Let X be the life time of a randomly chosen brand A cell phone. $X \sim N(\mu, \sigma^2)$ $\mathrm{P}(X < 6.8) = 0.369$ $\mathrm{P}\left(Z < \frac{6.8-\mu}{\sigma}\right) = 0.369$ $\frac{6.8-\mu}{\sigma} = -0.33450$ $-0.33450\sigma + \mu = 6.8$ ------(1) $\mathrm{P}(X > 7.8) = 0.522$ $\mathrm{P}(X < 7.8) = 0.478$ $\mathrm{P}\left(Z < \frac{7.8-\mu}{\sigma}\right) = 0.478$ $\frac{7.8-\mu}{\sigma} = -0.055174$ $-0.055174\sigma + \mu = 7.8$ -----------(2) Solving (1) and (2) using GC, $\mu = 8.00$ (to 3s.f.)
(ii)	Let $\overline{X}$ be the mean battery life time of the sample of 60 randomly chosen brand B cell phones. Since $n = 60$ is large, Central Limit Theorem can be applied. $\overline{X} \sim N\left(8.5, \frac{3.61^2}{60}\right)$ approximately $\mathrm{P}\left(7.8 < \overline{X} < 9.1\right) = 0.834$ (to 3s.f.)

Qn	Solution
9(i)	
(ii)	$r = 0.98326$ ≈ 0.983 Both the value of r and scatter diagram show a **strong positive linear correlation** between the number of thumb drives sold and the profits made from the sale of the thumb drives.
(iii)	From GC, $y = 2.2586x + 6.3948$ $y = 2.26x + 6.39$ m = 2.26, which means **for every 1 more thumb drive sold** in a week, the owner of the shop will make an **additional profit of $2.26**.
(iv)	Regression line of y on x: $y = 2.2586x + 6.3948$ When $y = 40$, $40 = 2.2586x + 6.3948$ $x = 14.879$ ≈ 14.9 Thus, 15 thumb drives need to be sold.
(v)	The estimation will be **not** be reliable because $x = 25$ is **out of the data range of x**.

10(i)	Required probability $= \frac{5}{20} \cdot \frac{15}{19}$ $= \frac{15}{76}$
(ii)	Required probability $= \frac{5}{20} + \frac{5}{20} - \frac{5}{20} \cdot \frac{4}{19}$ or $\frac{5}{20} \times \frac{15}{19} \times 2 + \frac{5}{20} \times \frac{4}{19}$ $= \frac{17}{38}$
(iii)	Required probability $= \frac{\text{P(Bernie gets a camera and Charles gets a camera or watch)}}{\text{P(Charles gets a camera or watch)}}$ $= \frac{\frac{7}{20} \cdot \frac{14}{19}}{\frac{15}{20}}$ $= \frac{98}{285}$

Qn	Solution
11(i)	$\bar{x}=\dfrac{\sum(x-30)}{n}+30$ $=\dfrac{-15}{60}+30$ $=29.75$ $s^2=\dfrac{1}{n-1}\left[\sum(x-30)^2-\dfrac{\left[\sum(x-30)\right]^2}{n}\right]$ $=\dfrac{1}{59}\left[160-\dfrac{(-15)^2}{60}\right]$ $=2.6483$ $=2.65$
(ii)	Let X be the mass of a fruity bar, and μ be the population mean mass of a fruity bar. $\mathrm{H}_0: \mu=30$ $\mathrm{H}_1: \mu<30$ Level of significance: 0.05 Since n is large, by Central Limit Theorem, $\bar{X}\sim \mathrm{N}\left(30,\dfrac{2.6483}{60}\right)$ approximately. Test statistics: $Z=\dfrac{\bar{X}-30}{\sqrt{\dfrac{2.6483}{60}}}\sim \mathrm{N}(0,1)$ From GC, p-value = 0.11703 ≈ 0.117 Since p-value > 0.05, we do not reject H_0 and conclude that there is insufficient evidence, at 5% level of significance, that the manufacturer is overstating the average mass of a fruity bar.
(iii)	It is NOT necessary to assume a normal distribution. Since $n=60$ is large, by Central Limit Theorem, the distribution of sample mean, $\bar{X}$, can be approximated to a normal distribution.
(iv)	$\bar{X}\sim \mathrm{N}\left(35,\dfrac{1.6^2}{60}\right)$ For H_0 to be rejected,

	$z < -1.9600 \quad \text{or} \quad z > 1.9600$ $\dfrac{\overline{x}-35}{\sqrt{\dfrac{1.6^2}{60}}} < -1.9600 \quad \text{or} \quad \dfrac{\overline{x}-35}{\sqrt{\dfrac{1.6^2}{60}}} > 1.9600$ $\overline{x} < 34.595 \quad \text{or} \quad \overline{x} > 35.405$ $\therefore \overline{x} < 34.6 \quad \text{or} \quad \overline{x} > 35.4$

12(i)	Let B be the time taken by a randomly chosen boy to run 100 m. $B \sim N(19.6, 2.3^2)$ $P(15 < B < 18)$ $= 0.22057$ ≈ 0.221
(ii)	Let G be the time taken by a randomly chosen girl to run 100 m. $G \sim N(23.1, 3.2^2)$ $\mathrm{P}(G_1 + G_2 > 2B)$ $= \mathrm{P}(G_1 + G_2 - 2B > 0)$ $E(G_1 + G_2 - 2B) = 2E(G) - 2E(B)$ $= 2 \times 23.1 - 2 \times 19.6$ $= 7$ $Var(G_1 + G_2 - 2B) = 2Var(G) + 4Var(B)$ $= 2 \times 3.2^2 + 4 \times 2.3^2$ $= 41.64$ $G_1 + G_2 - 2B \sim N(7, 41.64)$ $\mathrm{P}(G_1 + G_2 > 2B)$ $= \mathrm{P}(G_1 + G_2 - 2B > 0)$ $= 0.861$ (to 3s.f.)
(iii)	Let T be the total time taken for the team to complete 400m. $E(T) = E(B_1 + B_2 + G_1 + G_2)$ $= 2 \times 19.6 + 2 \times 23.1$ $= 85.4$ $Var(T) = Var(B_1 + B_2 + G_1 + G_2)$ $= 2Var(B) + 2Var(G)$ $= 31.06$ $T \sim N(85.4, 31.06)$ $\mathrm{P}(55 < T < 80) = 0.16629$ $= 0.166$ (to 3s f)

(iv)	$\mathrm{P}(T<75 \mid 55<T<80)$ $=\dfrac{\mathrm{P}(T<75 \cap 55<T<80)}{\mathrm{P}(55<T<80)}$ $=\dfrac{\mathrm{P}(55<T<75)}{\mathrm{P}(55<T<80)}$ $=\dfrac{0.031014}{0.16629}$ $=0.187$ (to 3s.f.)