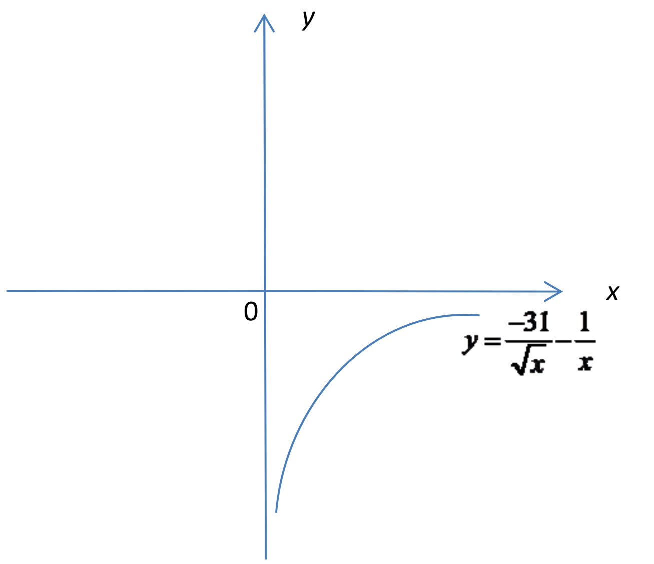
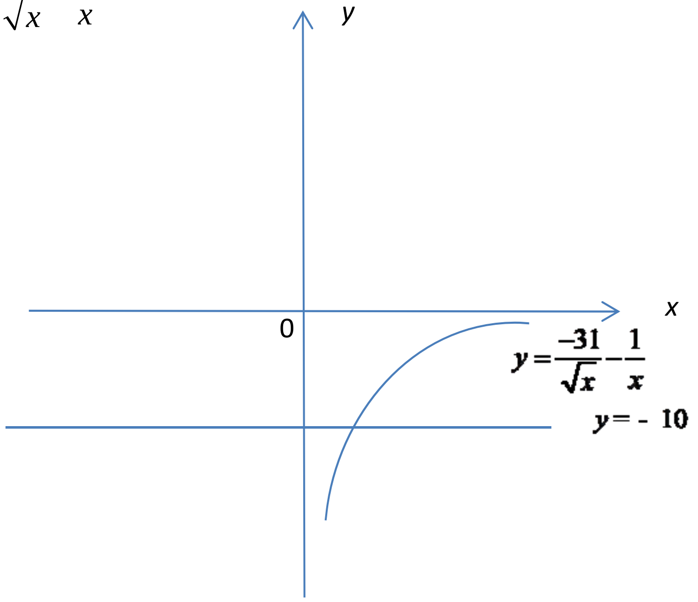
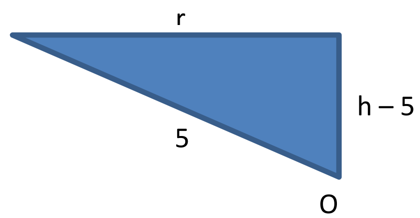
RVHS 2014 Year 6 Preliminary Examination: 8864 H1 Mathematics Solution

	Question 1 [3 marks]	
1	$e^{2x} - 3e^{x} + 12e^{-x} = 4$ Let $u = e^{x}$ $u^2 - 3u + \frac{12}{u} - 4 = 0$ $u^3 - 3u^2 - 4u + 12 = 0$ $u = 2, 3 \quad or \quad u = -2\ (NA)$ $e^{x} = 2$ or $e^{x} = 3$ $x = \ln 2$ or $x = \ln 3$	

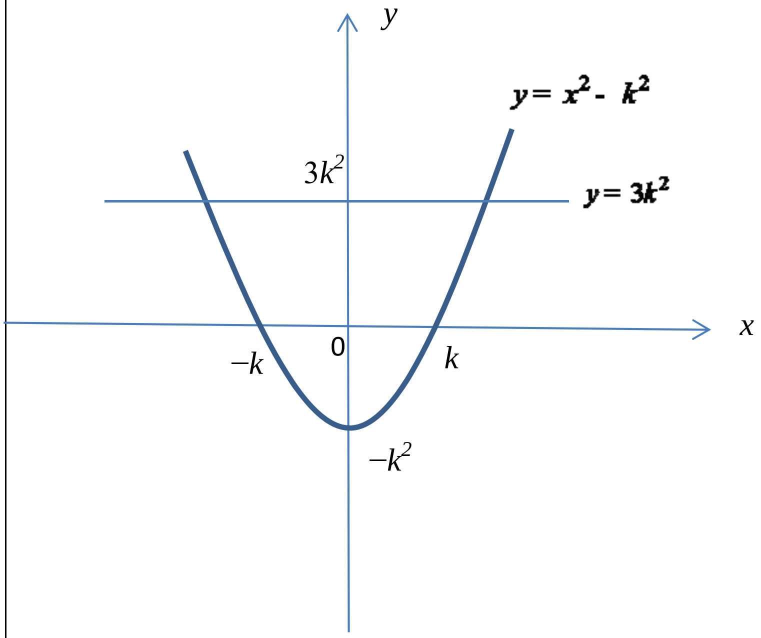
	Question 2 [6 marks]	
	$y = \ln\left(x\sqrt{x+1}\right)$ $= \ln x + \frac{1}{2}\ln\left(x+1\right)$ $\frac{dy}{dx} = \frac{1}{x} + \frac{1}{2}\left(\frac{1}{1+x}\right)$ $= \frac{2\left(1+x\right)+x}{2x\left(1+x\right)}$ $= \frac{3x+2}{2x\left(1+x\right)}$ $\int_3^8 \frac{3x+2}{x(x+1)}\ dx$ $= 2\int_3^8 \frac{3x+2}{2x(x+1)}\ dx$ $= \left[2\ln\left(x\sqrt{x+1}\right)\right]_3^8$ $= 2\ln\left(8\sqrt{9}\right) - 2\ln\left(3\sqrt{4}\right) = 4\ln 2$	

	Question 3 [8 marks]	
(i)	$y=\frac{a}{\sqrt{x}}-\frac{1}{x}$ $=ax^{-\frac{1}{2}}-x^{-1}$ $\frac{\mathrm{dy}}{\mathrm{dx}}=-\frac{1}{2}ax^{-\frac{3}{2}}+x^{-2}$ is the gradient of the tangent to the curve. $2y=10-x \Rightarrow y=5-\frac{x}{2}$. Gradient of normal is $-\frac{1}{2}$. Thus gradient of tangent is 2. At (4, *b*), $2=-\frac{1}{2}a(4)^{-\frac{3}{2}}+4^{-2}$ $2=-\frac{1}{16}a+\frac{1}{16}$ $32=-a+1$ $a=-31$ Substituting (4, *b*) into $y=\frac{-31}{\sqrt{x}}-\frac{1}{x}$ $b=\frac{-31}{2}-\frac{1}{4}=-\frac{63}{4}$	
(ii)		

(ii)	$10 = \dfrac{1 - a\sqrt{x}}{x}$ $10 = \dfrac{1}{x} - \dfrac{a}{\sqrt{x}}$ $\dfrac{a}{\sqrt{x}} - \dfrac{1}{x} = -10$  Number of real root = 1	

	Question 4 [8 marks]	
(i)	$5^2 = r^2 + (h-5)^2$ $25 = r^2 + h^2 - 10h + 25$ $r^2 = 10h - h^2$  $V = \dfrac{1}{3}\pi r^2 h$ $= \dfrac{1}{3}\pi(10h - h^2)h$ $= \dfrac{\pi h^2}{3}(10 - h)$	
(ii)	$V = \dfrac{\pi h^2}{3}(10 - h) = \dfrac{10}{3}\pi h^2 - \dfrac{1}{3}\pi h^3$ $\dfrac{\mathrm{d}V}{\mathrm{d}h} = \dfrac{20}{3}\pi h - \pi h^2$ For max volume, $\dfrac{\mathrm{d}V}{\mathrm{d}h} = 0$ $\dfrac{20}{3}\pi h - \pi h^2 = 0$	

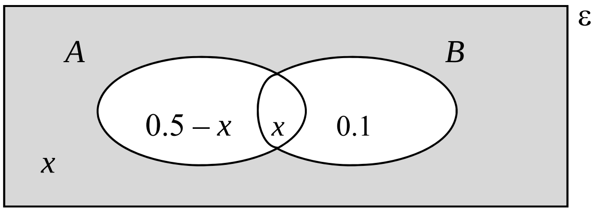
	$\pi h\left(\frac{20}{3}-h\right)=0$ $h\left(\frac{20}{3}-h\right)=0$ $h=0$ or $h=\frac{20}{3}$ Since $h>0$, $h=\frac{20}{3}$ $\frac{\mathrm{d}^2V}{\mathrm{d}h^2}=\frac{20}{3}\pi-2\pi h$ When $h=\frac{20}{3}$, $\frac{\mathrm{d}^2V}{\mathrm{d}h^2}<0$ <u>Alternatively</u> <table><tr><td>h</td><td>$\frac{20}{3}^{-}$</td><td>$\frac{20}{3}$</td><td>$\frac{20}{3}^{+}$</td></tr><tr><td>$\frac{\mathrm{d}V}{\mathrm{d}h}$</td><td>$>0$</td><td>$0$</td><td>$<0$</td></tr></table> Hence volume is maximum when $h=\frac{20}{3}$ Max V = $\frac{\pi}{3}\left(\frac{20}{3}\right)^2(10-\frac{20}{3})=\frac{4000}{81}\pi=49\frac{31}{81}\pi$	

<table>
<tr><td></td><td>Question 5 [10 marks]</td><td></td></tr>
<tr><td>(i)</td><td>
</td><td></td></tr>
<tr><td>(ii)</td><td>$x^2 - k^2 = 3k^2$
$x^2 = 4k^2$
$x = \pm 2k$</td><td></td></tr>
<tr><td>(iii)</td><td>When $k = 1$,
$\text{Area} = 2\int_0^2 3 - \left(x^2 - 1\right) \mathrm{d}x$
$= 2\int_0^2 4 - x^2 \mathrm{d}x$
$= 2\left[4x - \frac{x^3}{3}\right]_0^2$
$= 2\left[8 - \frac{8}{3}\right] = \frac{32}{3}$</td><td></td></tr>
<tr><td>(iv)</td><td>$\text{Area} = 2\int_0^{2k} 3k^2 - \left(x^2 - k^2\right) \mathrm{d}x > 200$
$\int_0^{2k} 4k^2 - x^2 \mathrm{d}x > 100$</td><td></td></tr>
</table>

	$\left[4k^2x - \dfrac{x^3}{3}\right]_0^{2k} > 100$ $8k^3 - \dfrac{8k^3}{3} > 100$ $\dfrac{16k^3}{3} > 100$ $k^3 > \dfrac{75}{4}$ $k > \sqrt[3]{\dfrac{75}{4}} \approx 2.66$ (3 sig fig)	

Question 6 [3 Marks]		
(i)	In order to use a stratified sampling, we need to know the composition of the shoppers according to the strata e.g gender, age-group which is difficult to collect.	
(ii)	Select <u>every 10th</u> (10% means every one out of 10 or $k = \dfrac{\text{population size}}{\text{sample size}} = \dfrac{N}{0.1N} = 10$) shopper as they leave the complex, starting from say the 5th shopper, where the number 5 is <u>randomly determined</u> from the first 10 shoppers who leave the complex. i.e choose the 5th, 15th, 25th,… until 10% of the shoppers are selected.	

<table>
<tr><td colspan="2">Question 7 [6 Marks]</td><td></td></tr>
<tr><td>(a)</td><td>Using Venn Diagram approach, let $\mathrm{P}(A \cap B) = x$:

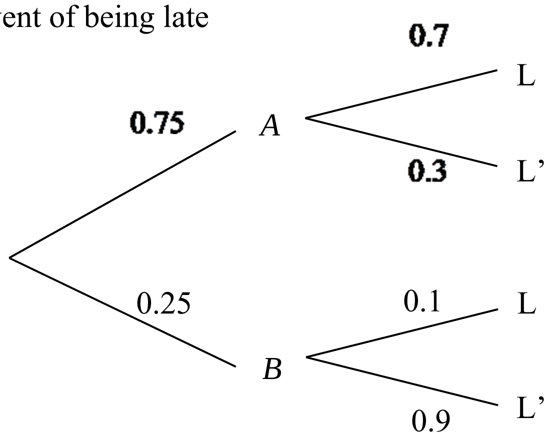


We then form the equation:
$x + 0.5 - x + x + 0.1 = 1$

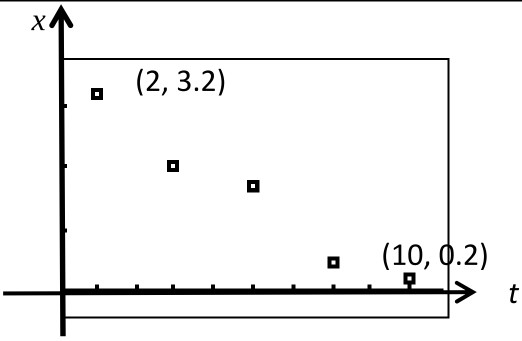
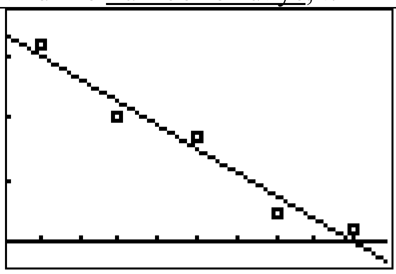
$\Rightarrow x = \mathrm{P}(A \cap B) = 0.4$

Alternatively,</td><td></td></tr>
</table>

	$P(B') = 0.4 + (0.5 - x)$ $0.5 = 0.4 + 0.5 - x$ $x = 0.4$ Alternatively, $P(A \cup B)' = x = 1 - P(A \cup B) = 0.4$	
(b)	(i) If X and Y are mutually exclusive, then X and Y are disjoint and $P(X \cup Y)$ =0.2 + 0.3 = 0.5. (ii) If X and Y are independent, then $P(X \cap Y) = P(X)P(Y) = 0.2 \times 0.3 = 0.06$. Hence, $P(X \cup Y) = P(X) + P(Y) - P(X \cap Y)$ $= 0.2 + 0.3 - 0.06$ $= 0.44$	

	Question 8 [7 marks]	
(i)	A: event of taking route A B: event of taking route B L: event of being late 	
(ii)	P(Alex late for school) = (0.75)(0.7) + (0.25)(0.1) = 0.55	
(iii)	P(Alex chooses route A | not late for school) $= \dfrac{\text{P(Alex chooses route } A \text{ and is not late for school)}}{\text{P(Alex is not late for school)}}$ $= \dfrac{0.75 \times 0.3}{1 - 0.55}$ = 0.5	
	Let Y be the random variable of the number of times he	

	is late out of 10 days, $Y \sim B(10, 0.55)$. $P(Y \le 2) = 0.0274$	

Question 9 [10 marks]		
(i)		
(ii)	$r = -0.97801 \approx -0.978$ Since the value of r is close to negative 1, there is a strong negative correlation between the amount of water retained, x and the number of days, t.	
(iii)	 $x = -0.377t + 3.77$	
(iv)	3.77 litres is the amount of water in the sponge at the beginning of the experiment. -0.377 litres is the amount of water decrease in the sponge for each day.	
(v)	$x = -0.377(5) + 3.77 \approx 1.89$ litres The estimate is reliable for the following reasons: 1. $t = 5$ is within data range for t 2. there is a strong linear correlation between x and t as $r = -0.978$ is very close to -1. 3. Since x is the independent variable, the line x on t is used.	

	Question 10 [11 marks]	
(i)	$\bar{x} = 300 + \left(-\frac{60}{100}\right) = 299.4$ $s^2 = \frac{1}{99}\left(1240 - \frac{60^2}{100}\right) = \frac{1204}{99}$	
(ii)	Let μ be the mean length of the monkey tails $H_0 : \mu = 300$ $H_1 : \mu \neq 300$ Under the null hypothesis, $Z = \frac{\bar{X} - 300}{\frac{s}{\sqrt{100}}} \sim N(0,1)$ p-value = 0.0853 > 0.05 and do not reject H_0. Therefore there is insufficient evidence at 5% that the monkeys on the island have mean tail length different as the species known to her. Thus the monkeys on the island have tails of the same mean length as the species known to her. 5% significance level is the probability of 0.05 of rejecting that the mean tail length is 300 mm when in fact it is 300 mm.	
(iii)	The conclusion will not be the same as part(ii) because the new p-value is half of (ii). Thus the new p-value < 0.05 and reject the null hypothesis $H_0 : \mu = 300$.	
(iv)	Let μ' be the mean length of the monkeys from another island. $H_0 : \mu' = 300$ $H_1 : \mu' > 300$ Under the null hypothesis, $Z = \frac{\bar{X} - 300}{\frac{3.48}{\sqrt{100}}} \sim N(0,1)$ Since the null hypothesis is rejected at 10%	

	significance level, $\dfrac{k-300}{\dfrac{3.48}{\sqrt{100}}} > 1.28155$ $k > 300.44598$ The least integer value of k is 301.	

	Question 11 [11 marks]	
(i)	The probability of the number of people using InstaFace remains constant at p.	
(ii)	$\mathrm{P}(4 \le F < 8) = \mathrm{P}(F \le 7) - \mathrm{P}(F \le 3) = 0.39993 \approx 0.400$	
(iii)	n = 200 is large np = 60>5 nq=140>5 F can be approximated by normal. $F \sim N(60, 42)$ approximately. $\mathrm{P}(F > 60) = \mathrm{P}(F \ge 61) \overset{cc}{=} \mathrm{P}(F \ge 60.5) \approx 0.469$	
(iv)	$\mathrm{P}(F \le 1) = 0.15$ $\binom{20}{0} p^0 (1-p)^{20} + \binom{20}{1} p^1 (1-p)^{19} = 0.15$ $p = 0.158918 \approx 0.159$	
(v)	$F \sim B(40, 0.6)$ $E(F) = 24$ $Var(F) = 9.6$ Since the number of cities = 50 is large, by CLT, $\overline{F} \sim N(24, \frac{9.6}{50})$ $\mathrm{P}(\overline{F} > 25) = 0.0112$	

Question 12 [12 Marks]		
(i)	Let X g be the mass of a random chosen apple of variety A Then $X \sim \mathrm{N}(120, 25^2)$ $P(X > 150) = 0.115$ (3 sf) (from GC)	
(ii)	Let Y g be the mass of a random chosen apple of variety B Then let $Y \sim N(\mu, \sigma^2)$. Given $\mathrm{P}(Y < 100) = 0.1$ and $\mathrm{P}(Y \leq 150) = 0.9 \Rightarrow \mathrm{P}(Y > 150) = 0.1$, Applying symmetrical property of normal distribution, We have $\mu = \dfrac{100+150}{2} = 125\,\mathrm{g}$ Applying normal standardization, we have $P(Y < 100) = 0.1$ $\Rightarrow P\left(Z < \dfrac{100-\mu}{\sigma}\right) = 0.1$ $\Rightarrow \dfrac{100-\mu}{\sigma} = -1.281552$ Thus, $\sigma = 19.507597 = 19.5$ g (3 sf)	
(iii)	Let X_1, X_2, Y_1, Y_2, Y_3 be the 4 chosen apples of variety A and 5 chosen apples of variety B respectively. Let $U = X_1 + X_2$ and $V = Y_1 + Y_2 + Y_3$ Then $U \sim N(2\times120, 2\times25^2) = N(240, 1250)$ and $V \sim N(3\times125, 3\times19.507597^2) = N(375, 1141.64)$ We will then have $U - V \sim N(-135, 2391.64)$ So, $P(U > V) = P(U - V > 0) = 0.00289$ (3 sf) (from GC)	
(iv)	With $X \sim \mathrm{N}(120, 25^2)$ $P(X > 80) = 0.945201$ (from GC) Let W be the random variable of the number out of 10 random chosen apple of variety A that have mass at least 80 g each. Then $W \sim B(10, 0.945201)$ Thus, $\mathrm{P}(W \geq 8) = 1 - \mathrm{P}(W \leq 7)$ $= 1 - 0.0147587$ $= 0.985$ (3 sf) (from GC)	