TEMASEK JUNIOR COLLEGE, SINGAPORE
JC 2
Preliminary Examinations 2014
Higher 1

MATHEMATICS **8864/01**

Paper 1 **1 September 2014**

Additional Materials: Answer paper **3 hours**
List of Formulae (MF15)

READ THESE INSTRUCTIONS FIRST

Write your *Civics Group* and *Name* on all the work that you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

You are expected to use a graphic calculator.
Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.

Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.

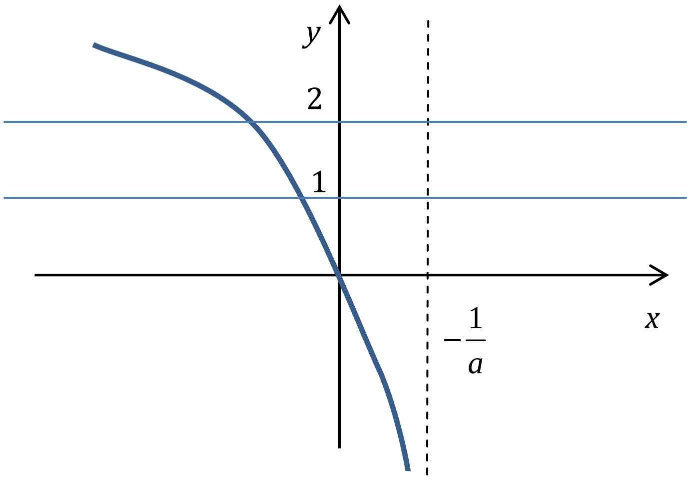
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

Pure Mathematics [35 marks]

1 Sketch the graph of $y = \ln(ax+1)$, for $a < 0$ and solve the inequalities $1 < \ln(ax+1) \leq 2$. Leave your answers in terms of a. [4]

[Solution]



When $y=1$, $\ln(ax+1)=1 \Rightarrow x = \frac{e-1}{a}$, when $y=2$, $\ln(ax+1)=2 \Rightarrow x = \frac{e^2-1}{a}$

Thus, $\frac{e^2-1}{a} < x \leq \frac{e-1}{a}$

2 Show that $\dfrac{d}{dx}\sqrt{3x^2-\ln x}=\dfrac{6x^2-1}{2x\sqrt{3x^2-\ln x}}$. [2]

Hence, find the exact value of $\displaystyle\int_1^e \frac{6x^2-1}{x\sqrt{3x^2-\ln x}}\,dx$. [3]

[Solution]

$$\frac{d}{dx}\sqrt{3x^2-\ln x}=\frac{1}{2}\left(3x^2-\ln x\right)^{-\frac{1}{2}}\left(6x-\frac{1}{x}\right)$$

$$=\frac{6x-\frac{1}{x}}{2\sqrt{3x^2-\ln x}}$$

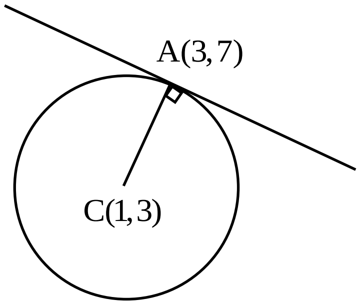
$$=\frac{6x^2-1}{2x\sqrt{3x^2-\ln x}}$$

$$\int_1^e \frac{6x^2-1}{x\sqrt{3x^2-\ln x}}\,dx=2\int_1^e \frac{6x^2-1}{2x\sqrt{3x^2-\ln x}}\,dx$$

$$=2\left[\sqrt{3x^2-\ln x}\right]_1^e$$

$$=\mathbf{2}\left[\sqrt{3e^2-1}-\sqrt{3}\right]$$

3



A curve has centre $\mathrm{C}(1,3)$ and passes through the point $\mathrm{A}(3,7)$ as shown in the figure above (not drawn to scale).

(i) Show that the equation of the tangent at A is $x+2y=17$. [3]

(ii) Find the range of k such that the tangent intersects the curve $y=kx^2$. [2]

[Solution]

(i)

Gradient of normal $= \dfrac{-1}{\frac{7-3}{3-1}} = -\dfrac{1}{2}$

Equation of tangent: $y-3=-\dfrac{1}{2}(x-3) \Rightarrow x+2y=17$

(ii) $2kx^2+x-17=0$

Since $1^2-4(2k)(-17)=1+136k\geq 0 \ \Rightarrow k\geq -\dfrac{1}{136}$

The equation $x+2kx^2-17=0$ has real roots for $\left\{k\in\mathbb{R} : k\geq -\dfrac{1}{136}\right\}$

4 The curve C has equation $y = \dfrac{2x}{x+3}$.

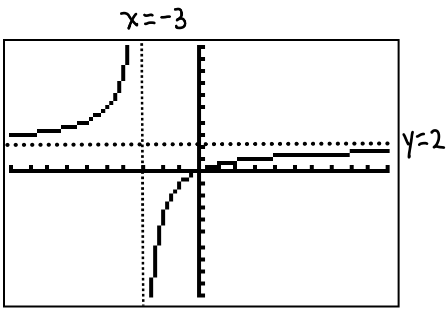
(i) Draw a sketch of C, showing clearly any axial intercepts, asymptotes and stationary points (if any). [3]

(ii) Verify that the line $y = m(x+3)+2$ passes through the point $(-3,\ 2)$.

Hence, find the value of m such that the equation $\dfrac{2x}{x+3} = m(x+3)+2$ has 2 real roots. [3]

[Solution]

(i)

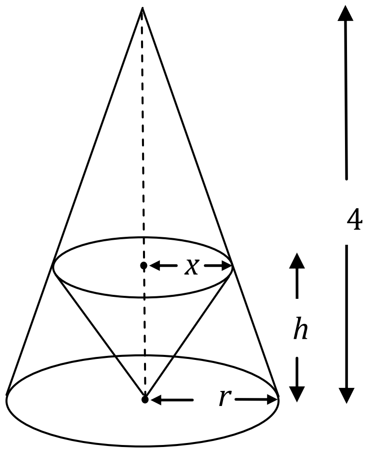


(ii) When $x = -3$, $y = m(0)+2 = 2$

Hence, the line $y = m(x+3)+2$ passes through the point $(-3,\ 2)$.

The range of values of m in which the curve will intersect the line at two points (and the equation has two real roots) is $m < 0$.

5 A paper weight is designed as shown below.



The paper weight consists of a right conical shell of height 4 metres and base radius r metres. Inside the cone is an inverted right solid cone of height h metres and base radius x metres (see above diagram) made of metal alloy. The circumference of the circular base of the inverted cone touches the conical shell. The vertex of the inverted cone is at the centre of the circular base of the conical shell.

Let V denote the volume of the inverted cone.

(i) Show that $V = \frac{1}{3}\pi r^2\left(h - \frac{h^2}{2} + \frac{h^3}{16}\right)$. [3]

(ii) Given that r has a fixed value and x varies, find the height h of the inverted cone that would maximise the amount of metal alloy used to create the paper weight, justifying your answer. [4]

[Solution]

(i) Using similar triangles, $\frac{x}{r} = \frac{4-h}{4}$

Volume of the inverted cone = $\frac{1}{3}\pi x^2 h$

$$= \frac{1}{3}\pi\left[r\left(1 - \frac{h}{4}\right)\right]^2 h$$

$$= \frac{1}{3}\pi r^2\left[1 - \frac{h}{2} + \frac{h^2}{16}\right]h$$

$$= \frac{1}{3}\pi r^2\left[h - \frac{h^2}{2} + \frac{h^3}{16}\right]$$

(ii)

$$\frac{dV}{dh}=\frac{1}{3}\pi r^2\left[1-h+\frac{3h^2}{16}\right]$$

$$\frac{dV}{dh}=0\Rightarrow\left[1-h+\frac{3h^2}{16}\right]=0$$

$$\Rightarrow 3h^2-16h+16=0$$

$$\Rightarrow h=4 \text{ or } h=\frac{4}{3}$$

$$\frac{d^2V}{dh^2}=\frac{1}{3}\pi r^2\left[-1+\frac{6h}{16}\right]$$

When $h=4,\frac{d^2V}{dh^2}=\frac{1}{3}\pi r^2\left[\frac{1}{2}\right]=\frac{1}{6}\pi r^2>0$; V is minimum

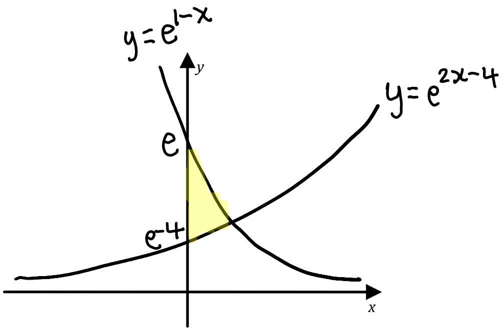
When $h=\frac{4}{3},\frac{d^2V}{dh^2}=\frac{1}{3}\pi r^2\left[-\frac{1}{2}\right]=-\frac{1}{6}\pi r^2<0$; V is maximum

Therefore $h=\frac{4}{3}$ maximises the amount of metal alloy used to create the paper weight.

6 Sketch the graphs of $y=e^{1-x}$ and $y=e^{2x-4}$ on the same diagram, indicating clearly their axes intercepts and their point of intersection. [3]

Shade the region bounded by the 2 curves and the y-axis and find the exact area of this region. [5]

[Solution]



Exact area

$$=\int_0^{\frac{5}{3}}(e^{1-x}-e^{2x-4})\ dx$$

$$=\left[-e^{1-x}-\frac{1}{2}e^{2x-4}\right]_0^{\frac{5}{3}}$$

$$=-e^{-\frac{2}{3}}-\frac{1}{2}e^{-\frac{2}{3}}+e+\frac{1}{2}e^{-4}$$

$$=-\frac{3}{2}e^{-\frac{2}{3}}+e+\frac{1}{2}e^{-4}$$

Statistics [60 marks]

7 A Country Club consisting of 10 000 members are asked to vote for or against building a new squash court. The Human Resource Manager is asked to conduct a poll prior to the vote.

(i) Describe how a sample of 100 members can be chosen using systematic sampling. [2]

(ii) Suggest another sampling method that will give the manager a better prediction of the result. Justify your answer. [1]

[Solution]

(i)

Number the members from 1 to 10, 000.

Randomly select a member **from the first** $\frac{10\ 000}{100} = 100$ **citizens.**

Select every 100th member thereafter.

(ii) <u>**Stratified sampling**</u> with appropriate <u>strata such as age group</u> is more appropriate because the sample is <u>**more representative of the population**</u>, as the choice to vote for or against building a new squash court may vary according to their age group.

8 A large sample of *n* observations was taken from a population of mean μ and variance 12. Find the least sample size required such that the probability of the sample mean lying between $\mu - 0.5$ and $\mu + 0.5$ is more than 0.75. State an assumption that you have to make in order to proceed with this calculation. [5]

[Solution]

Since n is large, by Central Limit Theorem, $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$ approximately.

$$\mathrm{P}(\mu - 0.5 < \bar{X} < \mu + 0.5) > 0.75$$

$$\mathrm{P}\left(\frac{-0.5}{\sqrt{12/n}} < \bar{\bar{X}} < \frac{0.5}{\sqrt{12/n}}\right) > 0.75$$

$$\frac{-0.5}{\sqrt{12/n}} > -1.150349379$$

$$n > 63.5$$

The least sample size is 64.

The observations are independent.

9 In a large batch of manufactured articles, the probability that an article is defective is 0.1, independent of all other articles. Samples, each of 10 articles, are taken at random from the batch.

(i) Find the probability that, of two samples,

(a) one contains 1 defective article and the other contains 2 defective articles, [2]

(b) there are less than 4 defective articles altogether. [2]

(ii) Find the least number of samples that must be taken such that the probability that at least one sample will contain at least 2 defective articles exceeds 0.99. [3]

[Solution]

(i) **(a)** Let X be the number of articles out of 10 which are defective.

$X \sim \mathrm{B}(10,\ 0.1)$

P (one contains 1 defective and the other contains 2 defectives)

$= \mathrm{P}(X=1)\times\mathrm{P}(X=2)\times 2$

$= 0.150$

(b) Let Y be the number of articles out of 20 which are defective.

$Y \sim \mathrm{B}(20,\ 0.1)$

$\mathrm{P}(Y<4)=\mathrm{P}(Y\le 3)=0.867$

(ii) Let W be the number of samples out of n that contains at least 2 defectives.

$W \sim \mathrm{B}(n,\ 0.263901070)$

$\mathrm{P}(W\ge 1)>0.99$

$\mathrm{P}(W=0)<0.01$

Using GC,

When n = 15, $\mathrm{P}(W=0)=0.01009>0.01$

When n = 16, $\mathrm{P}(W=0)=0.00743<0.01$

Least number of samples that must be taken is 16.

10 A manufacturer uses a machine to pack tea into packets of 50 g each. He decides to investigate if the mean mass of tea in a packet is more than 50 g. A sample of 50 packets is collected and the data of the 50 packets are shown below:

Mass(g)	49.8	49.9	50	50.1	50.2	50.3	50.4
Number with this mass	6	10	11	10	8	4	1

(i) Test, at the 5% significance level, whether the mean mass of tea in a packet is more than 50 g. State, giving a reason, whether you need to assume that the masses of packets of tea are normally distributed. [5]

(ii) Explain, in the context of the question, the meaning of "5% significance level". [1]

A random sample of another 50 packets is taken and the mean mass of the second sample is c g. A test at the 5% level of significance indicates that the mean mass differs from 50 g. Assuming that the population standard deviation is 1.5 g, find the range of values of c. [4]

[Solution]

(i)

Let μ be the mean mass of each packet of nuts.

$\mathrm{H_0}: \mu = 50$

$\mathrm{H_1}: \mu > 50$

$n = 50,\ \bar{x} = 50.4,\ s = 0.1564921593$

Level of significance: 5%

Since σ^2 is unknown and n $= 50$ **is** large

Test statistic : $\dfrac{\bar{X} - \mu}{s/\sqrt{n}} \sim \mathrm{N}(0,1)$ by Central Limit Theorem

If $\mathrm{H_0}$ is true, $\mu = 50$,

$z_{\text{cal}} = 1.80739$

p-value = 0.03535

Conclusion: Since p-value = 0.03535 < 0.05 (or z = 1.807 > 1.645), we reject $\mathrm{H_0}$ at 5 % level of significance and there is sufficient evidence to conclude that the mean mass is more than 50g.

It is not necessary to state, since n is large, by Central Limit Theorem, $\bar{X}$ is approximately normal.

(ii) In the context of this question, 'at 5% level of significance' means that there is 5% chance that we wrongly support the claim that the mean mass per packet is more than 50 g when it is not.

(iii)

$H_0: \mu = 50$

$H_1: \mu \neq 50$

$\mathbf{n} = 50, \bar{\mathbf{x}} = c, \sigma^2 = 1.5^2$

Level of significance: 5%

Since σ^2 is known and n = 50 **is** large

Test statistic : $\dfrac{\bar{X} - \mu}{\sigma / \sqrt{n}} \sim N(0,1)$ by Central Limit Theorem

Since the test at the 5% level of significance indicates that the mean mass differs from 50 g, i.e H_0 is rejected. Thus $z_{\text{cal}} > 1.95996$ or $z_{\text{cal}} < -1.95996$

$$\Rightarrow \quad \frac{\bar{x} - 50}{1.5 / \sqrt{50}} > 1.95996 \text{ or } \frac{\bar{x} - 50}{1.5 / \sqrt{50}} < -1.95996$$

$$\Rightarrow \quad \frac{c - 50}{1.5 / \sqrt{50}} > 1.95996 \text{ or } \frac{c - 50}{1.5 / \sqrt{50}} < -1.95996$$

$$\Rightarrow \quad c > 50.4 \text{ or } c < 49.6$$

11 **(a)** Peter and Kate each rolls a fair die once. The events A and B are defined as follows:
A: The score on Peter's die is a multiple of 3,
B: The sum of the two scores is 9.

Determine whether A and B are independent, justifying your conclusion. [3]

(b) A school bus passes by traffic lights A, B and C on its journey to school. It stops at these traffic lights independently with probabilities 0.2, 0.7 and 0.4 respectively.

(i) Find the probability that it stops at exactly two traffic light. [2]
(ii) Find the probability that it stops at traffic light B given that it stops at exactly two traffic lights. [3]
(iii) Suppose it was known that the bus stops at traffic lights B and C, find the probability that it stops at exactly two traffic lights. [3]

[Solution]

(a) $\mathrm{P}(A \cap B) = \dfrac{2}{36} = \dfrac{1}{18}$

$\mathrm{P}(A) \times \mathrm{P}(B) = \dfrac{1}{3} \times \dfrac{4}{36} = \dfrac{1}{27}$

Since $\mathrm{P}(A \cap B) \neq \mathrm{P}(A) \times \mathrm{P}(B)$, A and B are not independent.

(b)(i) P(Bus stops at exactly two traffic lights)
= P(Bus stops at A & B but not C) + P(Bus stops at A & C but not B)
+ P(Bus stops at B & C but not A)
= (0.2)(0.7)(0.6) + (0.2)(0.3)(0.4) + (0.8)(0.7)(0.4)
= 0.332

(ii) Let events X: Bus stopped at traffic light B
Y: Bus stopped at exactly two traffic lights

$$\mathrm{P}(X \mid Y) = \frac{\mathrm{P}(X \cap Y)}{\mathrm{P}(Y)}$$

$$= \frac{\mathrm{P}(\text{Bus Stopped at } A \text{ \& } B \text{ but not } C) + \mathrm{P}(\text{Bus Stopped at } B \text{ \& } C \text{ but not } A)}{\mathrm{P}(Y)}$$

$$= \frac{(0.2)(0.7)(0.6) + (0.8)(0.7)(0.4)}{0.332}$$

$= 0.928$

(iii) Let events Y: Bus stopped at exactly two traffic lights
Z: Bus stopped at traffic lights B and C

$$\mathrm{P}(Y \mid Z) = \frac{\mathrm{P}(Y \cap Z)}{\mathrm{P}(Z)}$$

$$= \frac{\mathrm{P}(\text{Bus stopped at } B \text{ \& } C \text{ but not } A)}{\mathrm{P}(\text{Bus stopped at } B,\ C \text{ \& } A) + \mathrm{P}(\text{Bus stopped at } B \text{ \& } C \text{ but not } A)}$$

$$= \frac{(0.8)(0.7)(0.4)}{(0.2)(0.7)(0.4) + (0.8)(0.7)(0.4)} = 0.8$$

12 A researcher is interested to find out if there is a linear correlation between the score on a computer game (x) and the score on a mathematics test (y). The following table gives the data from 9 students.

x	40	56	66	78	68	52	96	110	84
y	38	51	48	58	55	50	77	80	69

(i) Draw a scatter diagram for the data. [2]

(ii) Find the product moment correlation coefficient and comment on its value in the context of the data. [2]

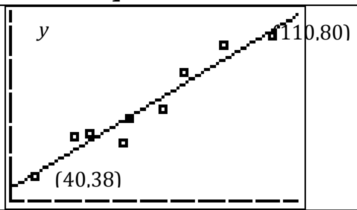
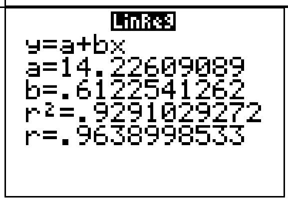
(iii) Find the least squares regression line of y on x in the form $y = mx + c$, giving the values of m and c correct to 2 decimal places. Sketch this line on your scatter diagram. [2]

(iv) Calculate an estimate of the mathematics test score of a student with a score of 89 on the computer game and comment on the reliability of your estimate. [2]

(v) Explain why it might be unsuitable to use the equation in part (iii) to estimate a student's score on the computer game given that he scored 90 marks in the mathematics test. [1]

(vi) The mathematics test score is re-based to a maximum score of 50 marks instead of 100 marks as given in the data above. Without using your graphing calculator, state any change you would expect in the values of your constants m and c found in part (iii). [1]

[Solution]

(i)	
(ii)	 x $r = 0.964$ (3 s.f). There is a strong positive linear correlation between the score on the computer test and the score on the mathematics test.
(iii)	Least squares regression line is $y = 14.22 + 0.61x$ Correct sketch of the regression line.
(iv)	When $x = 89$, Mathematics score = 68.7 ≈ 69 Since the score of 89 is within the data range [40,110] used to calculate the regression line and $r = 0.964 \approx 1$, interpolation is used to estimate the mathematics score and hence the estimate is reliable.
(v)	Since both variables are subjected to experimental error and there is no clear indication that one variable if dependent on the other, thus to estimate the computer score given the mathematics score, we should use the line x on y.
(vi)	Given $y = mx + c$. Let $x = 2x' \Rightarrow y = m(2x') + c \Rightarrow \frac{y}{2} = mx' + \frac{c}{2}$ [Rewrite in terms of $Y = mX + c$] $Y = \frac{y}{2}, X = x'$ No change in m. c is divided by 2.

13 **(a)** The random variable X has the distribution $\mathrm{N}(90, \sigma^2)$.

Given that $\mathrm{P}(X < 85) = \frac{1}{5}\mathrm{P}(X < 95)$. Show that $\sigma = 5.17$. [2]

Y is another random variable such that $Y = \frac{1}{2}X + 2$.

Find E(Y) and Var(Y). [2]

(b) The lifetime of an ordinary light bulb follows a normal distribution with mean 600 hours and variance 225 hours^2. The lifetime of a "long-life" light bulb follows an independent normal distribution with mean 1850 hours and variance 169 hours^2.

(i) Find the probability that the lifetime of a randomly chosen long-life light bulb is more than three times the lifetime of a randomly chosen ordinary light bulb. [3]

(ii) Given that the probability that the total lifetime of two long-life light bulbs exceeding k hours is 0.138, find the value of k. [3]

100 ordinary light bulbs are packed in a box. Using a suitable approximation, find the probability that there are at least 14 but less than 29 ordinary light bulbs with lifetime of less than 590 hours in a box. [4]

[Solution]

(a) $\mathrm{P}(X < 85) = \frac{1}{6}$

$\mathrm{P}\left(Z < \frac{85-90}{\sigma}\right) = \frac{1}{6}$

Using GC, $-\frac{5}{\sigma} = -0.967421568$

$\sigma = 5.17$ (Shown)

E(Y) = $\frac{1}{2}\mathrm{E}(X) + 2 = 47$

Var(Y) = $\frac{1}{4}\mathrm{Var}(X) = 6.68$

(b) (i) Let X be the lifetime of an ordinary light bulb.

$X \sim \mathrm{N}(600, 225)$

Let Y be the lifetime of a long-life light bulb.

$Y \sim \mathrm{N}(1850, 169)$

$Y - 3X \sim \mathrm{N}(50,\ 2194)$

$\mathrm{P}(Y - 3X > 0) = 0.857$

(ii) $Y_1 + Y_2 \sim N(3700,\ 338)$

$\mathrm{P}(Y_1 + Y_2 > k) = 0.138$

$\mathrm{P}(Y_1 + Y_2 \le k) = 0.862$

Using GC, k = 3720

Let W be the number of ordinary lightbulbs out of 100 with lifetime of less than 590 hours.

$W \sim \mathrm{B}(100, 0.25249)$

Since n is large, $np = 25.249 \quad > 5$ and $nq = 74.751 \quad > 5$,

$\therefore W \sim \mathrm{N}(25.249, 18.87387999)$ approximately

$\mathrm{P}(14 \leq W < 29) \overset{c.c}{=} \mathrm{P}(13.5 < W < 28.5) = 0.769$ (3.s.f)