2014 JC 2 H1 Prelim Suggested Solutions & Marking Scheme

Qn	Solution
1	**Techniques of Differentiation and Integration**
	$\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{7-2x^3} = \frac{\mathrm{d}}{\mathrm{d}x}\left(7-2x^3\right)^{\frac{1}{2}}$ $= \frac{1}{2}\left(7-2x^3\right)^{-\frac{1}{2}}\left(-6x^2\right)$ $= -3x^2\left(7-2x^3\right)^{-\frac{1}{2}}$ or $\frac{-3x^2}{\sqrt{7-2x^3}}$
	$\int_{-1}^{1}\frac{x^2}{\sqrt{7-2x^3}}\,\mathrm{d}x = -\frac{1}{3}\int_{-1}^{1}\frac{-3x^2}{\sqrt{7-2x^3}}\,\mathrm{d}x$ $= -\frac{1}{3}\left[\sqrt{7-2x^3}\right]_{-1}^{1}$ $= -\frac{1}{3}\left(\sqrt{7-2}-\sqrt{7+2}\right)$ $= -\frac{1}{3}\left(\sqrt{5}-3\right)$ $= \frac{1}{3}\left(3-\sqrt{5}\right)$

Qn	Solution
2	**Application of Differentiation (Max/Min Problem)**
(i)	area $A = xy$ $= x\left(6-2x^3\right)$ $= 6x-2x^4$
(ii)	$\frac{\mathrm{d}A}{\mathrm{d}x} = 6-8x^3$ For maximum area, $\frac{\mathrm{d}A}{\mathrm{d}x} = 6-8x^3 = 0$ $6 = 8x^3$ $x^3 = \frac{3}{4}$ $x = \sqrt[3]{\frac{3}{4}}$

x	$\left(\sqrt[3]{\frac{3}{4}}\right)^-$	$\sqrt[3]{\frac{3}{4}}$	$\left(\sqrt[3]{\frac{3}{4}}\right)^+$
$\frac{\mathrm{d}A}{\mathrm{d}x}$	+ve	0	-ve
Slope of Curve	/	—	\

Hence area A is a maximum.

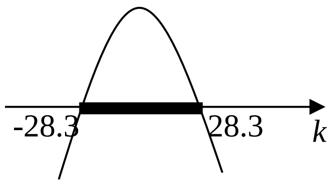
Alternative:

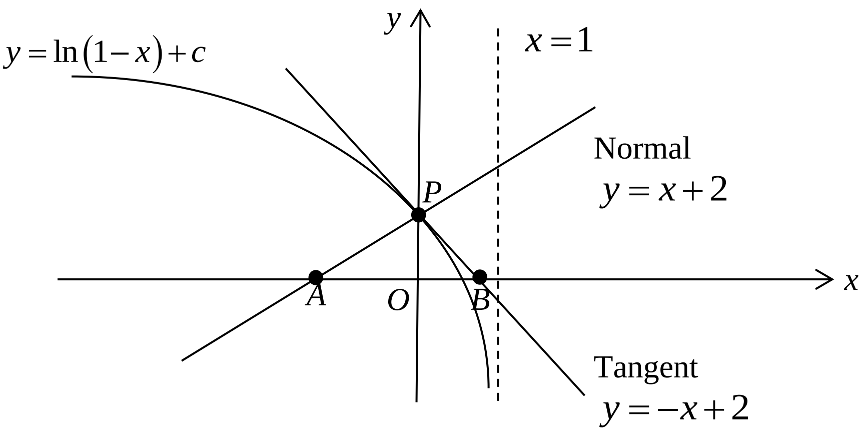
$\frac{\mathrm{d}^2A}{\mathrm{d}x^2} = -24x^2 < 0$

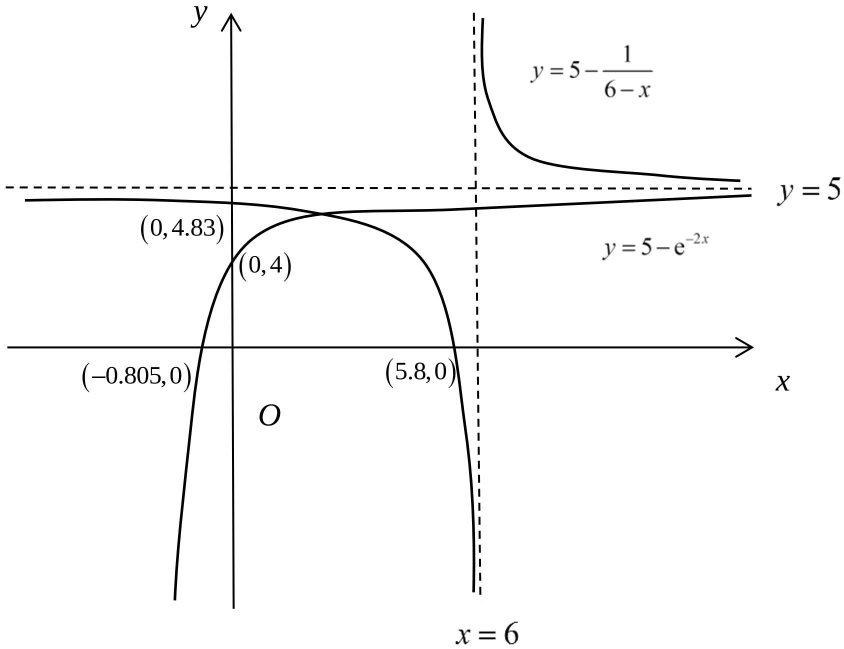
Hence area A is a maximum.

When $x = \sqrt[3]{\frac{3}{4}}$,

area A $= 6\left(\sqrt[3]{\frac{3}{4}}\right) - 2\left(\sqrt[3]{\frac{3}{4}}\right)^4 = \frac{9}{2}\left(\sqrt[3]{\frac{3}{4}}\right)$ or 4.09 units^2 (3s.f.)

Qn	Solution
3	**Equations and Inequalities**
(i)	Using Pythagoras' theorem, $x^2 + y^2 = (2\times5)^2$ $\Rightarrow x^2 + y^2 = 100$ (shown)
(ii)	$2x + 2y = k$ $\Rightarrow y = \frac{k-2x}{2}$ (sub into equation in (i)) $x^2 + \left(\frac{k-2x}{2}\right)^2 = 100$ $\Rightarrow 4x^2 + k^2 - 4kx + 4x^2 = 400$ $\Rightarrow 8x^2 - 4kx + k^2 - 400 = 0$ (shown)
	For the equation in **(ii)** to have real solutions, discriminant ≥ 0. $(-4k)^2 - 4(8)(k^2 - 400) \geq 0$ $16k^2 - 32k^2 + 12800 \geq 0$ $-16k^2 + 12800 \geq 0$ Using GC, $-28.3 \leq k \leq 28.3$ (3 s.f.) Since $k > 20$, $20 < k \leq 28.3$ (3 s.f.) 

Qn	Solution
4	**Applications of differentiation (Tangent/Normal)**
	Since gradient of normal at P is 1, gradient of tangent at point P is –1. $y=\ln(1-x)+c$ $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{-1}{1-x}$ Thus, $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{-1}{1-x}=-1$ $\Rightarrow x=0$ Since $x=0$, $y=0+2=2$. Coordinates of P are (0, 2). (shown) Equation of tangent to curve at P: $y-y_1=m(x-x_1)$ $y-2=-1(x-0)$ $\therefore y=-x+2$
	 Coordinates of A are (–2,0), Coordinates of B are (2,0), coordinates of P are (0,2) Area of triangle $ABP=\dfrac{1}{2}(\text{base})(\text{height})$ $=\dfrac{1}{2}(2-(-2))(2)$ $=4\text{units}^2$

Qn	Solution
5	**Application of Integration**
(i)	 Using GC, coordinates of point of intersection are $(0.82219, 4.80686) \approx (0.822, 4.81)$ (to 3 s.f.)
(ii)	Required area $= \int_{-0.80472}^{0.82219} \left(-e^{-2x}+5\right)\,dx + \int_{0.82219}^{5.8} \left(5-\frac{1}{6-x}\right)\,dx$ $= 27.366$ ≈ 27.4 (to 3 s.f.)

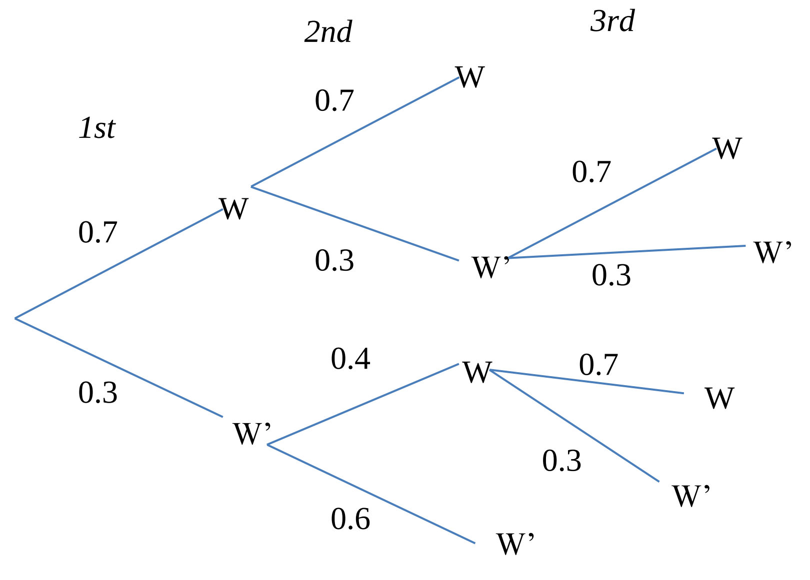
Qn	Solution
6	**Sampling Methods**
(a)	Random means that every member of the population has **an equal probability of being selected** to be in the sample.
(b)(i)	Method: Number all Singaporeans from 1 to N, where N is the population size. Then, 1000 numbers between 1 and $\underline{N}$ would need to be randomly generated by a computer or GC. The 1000 Singaporeans corresponding to the 1000 random numbers generated would make up the sample. Disadvantage: Any of the following or reasonable answer: 1. The sample chosen may not a good representation of the population as certain minority sub-groups (such as certain age groups, education level, race etc.) may be not included in the sample. 2. As the population is large, it is difficult and time consuming to identify every member of the population. 3. Sometimes, due to certain reasons, it is not possible to get access to some members who have been chosen for the sample.

(ii)	Stratified sampling using strata such as different age groups (alternatively ethnic groups, income groups, educational level and gender) may give a better representation of the Singapore population across the different age groups.

Qn	Solution
7	**Normal Distribution - Standardisation**
	$\mathrm{P}(X < -2a) = 0.2$ $\mathrm{P}\left(Z < \frac{-2a-\mu}{\sigma}\right) = 0.2$ $\frac{-2a-\mu}{\sigma} = -0.84162$ $-2a-\mu = -0.84162\sigma$ ----------(1) $\mathrm{P}(X > a) = 0.6$ $\mathrm{P}(X < a) = 0.4$ $\mathrm{P}\left(Z < \frac{a-\mu}{\sigma}\right) = 0.4$ $\frac{a-\mu}{\sigma} = -0.25335$ $a-\mu = -0.25335\sigma$ ----------(2) $(2)-(1): 3a = 0.58827\sigma$ $\sigma = 5.0997a$ $= 5.10a$ (3 s.f) Substitute $\sigma = 5.0997a$ into (2): $a-\mu = -0.25335(5.0997a)$ $\mu = 2.2929a$ $= 2.29a$ (3 s.f)

Qn	Solution
8	**Normal Distribution/Sampling**
	Let H and W be the mass of a randomly chosen honeydew melon and watermelon respectively, in kg. $H \sim \mathrm{N}(2.75, 0.45^2)$ $W \sim \mathrm{N}(6,1)$

(i)	$H_1+H_2-W \sim \mathrm{N}\left(-0.5, 2\left(0.45^2\right)+1\right)$ $H_1+H_2-W \sim \mathrm{N}\left(-0.5, 1.405\right)$ Required probability = $\mathrm{P}\left(H_1+H_2-W>0\right)=0.337$ (to 3 s.f.)
(ii)	$\frac{H_1+H_2}{2}-H_3 \sim \mathrm{N}\left(0, \frac{0.45^2}{2}+0.45^2\right)$ $\frac{H_1+H_2}{2}-H_3 \sim \mathrm{N}\left(0, 0.30375\right)$ Required probability = $\mathrm{P}\left(\frac{H_1+H_2}{2}-H_3 \geq 0.1\right)=0.428$ (to 3 s.f.)
(iii)	Let S be the mass of a randomly chosen strawberry. Since n is large, by Central Limit Theorem, $X=S_1+S_2+\ldots+S_{50} \sim \mathrm{N}\left(18(50), 1(50)\right)$ approximately. $X \sim \mathrm{N}(900, 50)$ approximately. Required probability = $\mathrm{P}(X>880)=0.998$ (to 3 s.f.)

Qn	Solution
9	**Probability**
(i)	

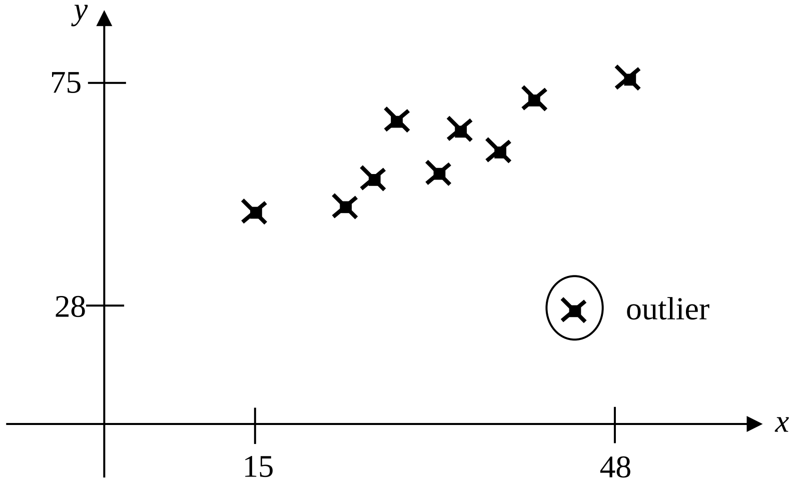
(ii)(a)	Required Probability $=(0.7)(0.3)+(0.3)(0.4)=0.33$ Alternative: Required Probability $=1-(0.7)(0.7)-(0.3)(0.6)=0.33$
(ii)(b)	Required Probability $=(0.7)(0.7)+(0.7)(0.3)(0.7)+(0.3)(0.4)(0.7)=0.721$
(ii)(c)	Required Probability $=(0.4)(0.7)$ $=0.28$ Alternative: Required Probability $=\dfrac{\mathrm{P}(\text{win the game and did not win first set})}{\mathrm{P}(\text{did not win first set})}$ $=\dfrac{(0.3)(0.4)(0.7)}{0.3}=0.28$

Qn	Solution
10	Binomial Distribution
(i)	Whether a calculator is defective is independent of whether another calculator is defective. The probability that a calculator is defective is constant for every calculator in the sample.
(ii)	Let X be the number of calculators, out of 20, that are defective. $X \sim \mathrm{B}(20, 0.05)$ $\mathrm{P}(X<3)=\mathrm{P}(X \le 2)$ $=0.92452$ Let Y be the number of batches, out of 80, which are acceptable. $Y \sim \mathrm{B}(80, 0.92452)$ Since n = 80 is large, np = 73.961 > 5 and nq = 6.0387 > 5, Y ~ N(73.961, 5.5829) approximately 90% of 80 batches is 72 batches. $\mathrm{P}(Y \ge 72)=\mathrm{P}(Y>71.5)$ after continuity correction $=0.851$ (3 sf)

(iii)	Let W be the number of calculators, out of k, that are defective. $W \sim \mathrm{B}(k, 0.05)$ $\mathrm{P}(W \geq 3) > 0.8$ $1 - \mathrm{P}(W \leq 2) > 0.8$ When k = 84, $1 - \mathrm{P}(W \leq 2) = 0.79718 < 0.8$ When k = 85, $1 - \mathrm{P}(W \leq 2) = 0.80367 > 0.8$ $\therefore$ Least value of k = 85.

Qn	Solution
11	**Hypothesis Testing**
(i)	$n = 100$, Unbiased estimate of population mean is $\bar{x} = \frac{\sum x}{n} = \frac{29800}{100} = 298$ Unbiased estimate of population variance is $s^2 = \frac{1}{n-1}\left[\sum x^2 - \frac{\left(\sum x\right)^2}{n}\right] = \frac{1}{99}\left[8908058 - \frac{(29800)^2}{100}\right] = \frac{27658}{99}$
(ii)	$\mathrm{H_0}$: $\mu = 300$ $\mathrm{H_1}$: $\mu < 300$ Under $\mathrm{H_0}$, using GC, $p\text{-value} = 0.11573 = 0.116$ (3s.f.) p-value = 0.116 is probability of obtaining a sample mean volume less than or equal to the observed value of 298 ml, given that the population mean volume is 300ml. For consumers' complaint to be justified i.e. reject $\mathrm{H_0}$, $p\text{-value} = 0.11573 < \frac{\alpha}{100}$ Therefore, set of values of α for consumers' complaint be justified is $\{\alpha \in \mathbb{R} : 11.6 < \alpha \leq 100\}$
(iii)	Let X be the volume of apple juice in a randomly chosen packet (in ml). Let μ denote the population mean volume of apple juice in packets. $\mathrm{H_0} : \mu = \mu_0$ $\mathrm{H_1} : \mu \neq \mu_0$ Since n = 100 is large, by Central Limit Theorem, $\overline{X} \sim \mathrm{N}\left(\mu, \frac{\sigma^2}{n}\right)$ approximately. Test Statistic: $Z = \frac{\overline{X} - \mu}{\sigma / \sqrt{n}}$ Level of significance: 12% Reject $\mathrm{H_0}$ if z-value > 1.5548 or z-value < -1.55478

	For H_0 not to be rejected, $-1.5548 < \dfrac{298-\mu_0}{11/\sqrt{100}} < 1.5548$ $-1.71025 < 298-\mu_0 < 1.71025$ $296.29 < \mu_0 < 299.71$ $296 < \mu_0 < 300$ (3 sig fig)
(iv)	It is not necessary to assume that X follows a normal distribution in both parts. Since n is large, by Central Limit Theorem, $\overline{X}$ follows a normal distribution approximately.

Qn	Solution
12	**Correlation and Regression**
	(i)(ii) 
(iii)	$y = 36.055 + 0.82494x = 36.1 + 0.825x$ For every one more hour spent revising Mathematics, the score for the Mathematics paper is increased by 0.825.
(iv)	$\bar{x} = \dfrac{15+22+25+28+31+33+36+39+42+48}{10} = 31.9$ $\bar{y} = 36.2 + 0.817\bar{x}$ $\bar{y} = 36.2 + 0.817(31.9) = 62.2623$ $\bar{y} = \dfrac{49+50+55+70+57+65+60+72+k+75}{10} = 62.2623$ $k + 553 = 622.623$ $k = 69.623 = 70$ (to the nearest integer) The rectified score is 70.

(v)	$r = 0.857$
(vi)	$x = 60$, $y = 36.2 + 0.817(60) = 85.22 = 85$ (to nearest integer). Since $x = 60$ hours is outside the data range of x, the variables may not be linear correlated, thus the estimate obtained by extrapolation may not be valid.