SERANGOON JUNIOR COLLEGE

2014 JC2 PRELIMINARY EXAMINATIONS

MATHEMATICS

Higher 1 **8864**

Wednesday **20 August 2014**

Additional materials: Writing paper

List of Formulae (MF15)

**TIME : ** 3 hours

READ THESE INSTRUCTIONS FIRST

Write your name and class on the cover page and on all the work you hand in.
Write in dark or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphic calculator.
Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

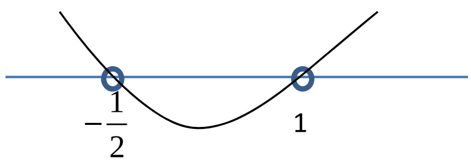
The number of marks is given in brackets **[ ]** at the end of each question or part question.
At the end of the examination, fasten all your work securely together.

Total marks for this paper is 95 marks

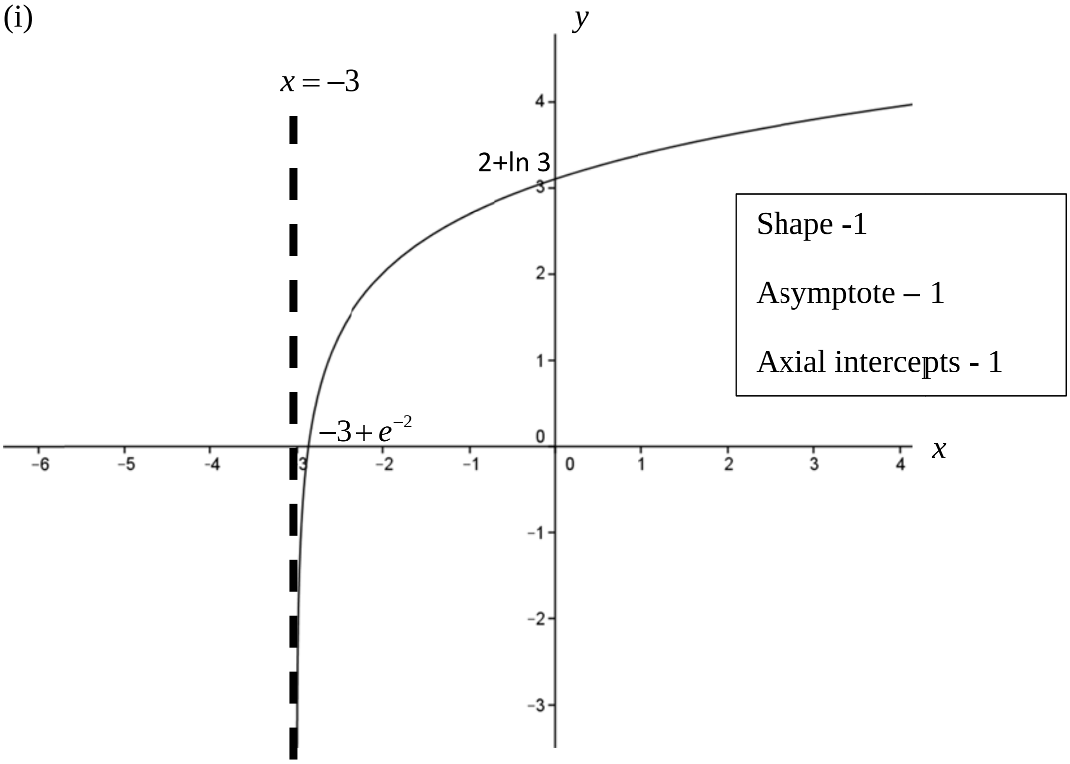
This question paper consists of 7 printed pages (inclusive of this page) and 1 blank page.

[Turn Over]

Section A: Pure Mathematics [35 marks]

1	Find the exact coordinates of the point(s) of intersection between the curve $y=\frac{x+2}{1-x}$ and the line $y=2x+3$.
	$\frac{x+2}{1-x}=2x+3$ $x+2=(1-x)(2x+3)$ $2x^2+2x-1=0$ $x=\frac{-2\pm\sqrt{4+8}}{4}$ $x=\frac{-1+\sqrt{3}}{2},\frac{-1-\sqrt{3}}{2}$ When $x=\frac{-1+\sqrt{3}}{2}$, $y=2+\sqrt{3}$ When $x=\frac{-1-\sqrt{3}}{2}$, $y=2-\sqrt{3}$ Therefore the coordinates are $\left(\frac{-1+\sqrt{3}}{2},\ 2+\sqrt{3}\right)$ and $\left(\frac{-1-\sqrt{3}}{2},\ 2-\sqrt{3}\right)$.
2	Find the range of values of x for which $2x^2-x-1<0$. Hence solve the inequality $2e^x-\frac{1}{e^x}<1$.
	$2x^2-x-1=(2x+1)(x-1)<0$ $-\frac{1}{2}<x<1$ $2e^x-\frac{1}{e^x}<1 \Rightarrow 2e^{2x}-e^x-1<0$ $-\frac{1}{2}<e^x<1 \Rightarrow 0<e^x<1$ ie $x<\ln 1=0$, ie $x<0$ 
3	**(i)** Sketch the curve whose equation is $y=2+\ln(x+3)$ indicating clearly the asympotes, if any, and the exact axial intercepts. **(ii)** Find the exact coordinates of the point P on the curve at which the gradient of the tangent is $\frac{1}{3}$. **(iii)** The tangent to the curve at the point P cuts the x-axis at A and the normal at the point P cuts the x-axis at B. Calculate exact area of the triangle ABP.

(i)



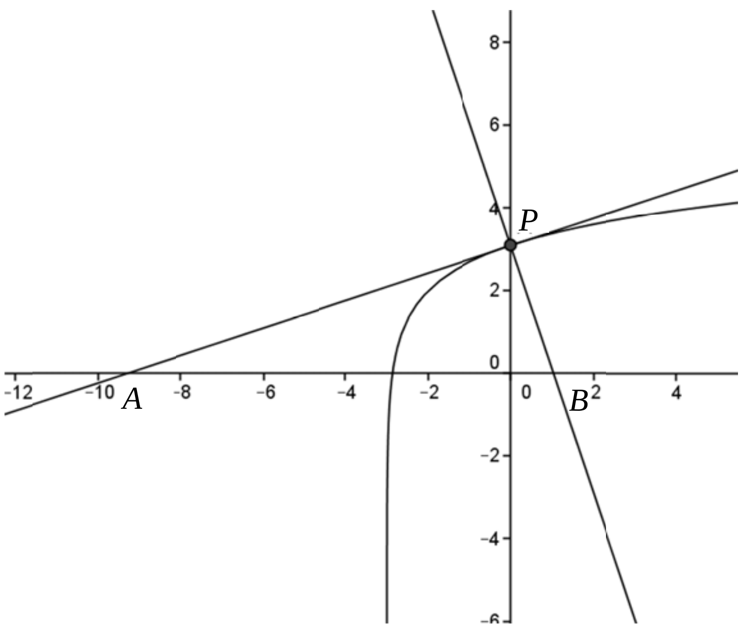
(ii) $\dfrac{dy}{dx}=\dfrac{1}{x+3}=\dfrac{1}{3}$

$x=0$

When $x=0$, $y=2+\ln 3$

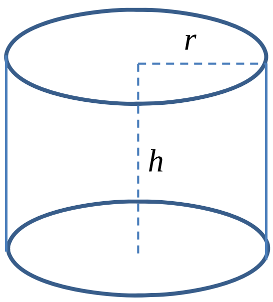
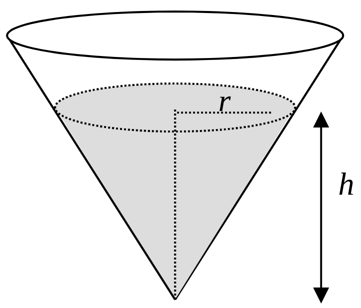
The exact coordinates of the point is (0, 2 + ln 3)

(iii)



Tangent at P: $y-(2+\ln 3)=\dfrac{1}{3}x$

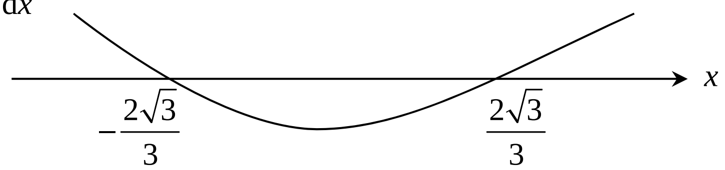
Normal at P: $y-(2+\ln 3)=-3x$

	A is $(-3(2+\ln 3), 0)$ and B is $\left(\frac{1}{3}(2+\ln 3), 0\right)$ $AB = \frac{20}{3} + \frac{10}{3}\ln 3 = \frac{10}{3}(2+\ln 3)$ Area of triangle ABP is $\frac{1}{2}\left(\frac{10}{3}(2+\ln 3)\right)(2+\ln 3)$ $= \frac{5}{3}(2+\ln 3)^2$ square units.
4 (a)	Find the set of values of x for which the curve $y = x^3 - 4x + 5$ is increasing, giving your answer in exact form.
(b)	A closed cylindrical jar has a fixed surface area of 100π cm^2. The height and the base radius of the jar are h cm and r cm respectively.  **(i)** Show that the volume, V cm^3, of the jar is $V = \pi r(50 - r^2)$. **(ii)** Find the value of r, for which the volume of the jar is maximum.
(c)	Water is poured into a small conical vessel at a rate of 10 cm^3/ sec. Given that the height of the water is twice as much as the radius of the surface area of the water at any instance, find the exact rate of increase in radius of the surface of the water when the height is 4 cm. [Volume of a cone with radius r and height h is given as $\frac{1}{3}\pi r^2 h$] 

(a)

$y = x^3 - 4x + 5$

$\frac{dy}{dx} = 3x^2 - 4 > 0$ for increasing function.



Solving $x \leq -\frac{2\sqrt{3}}{3}$ or $x \geq \frac{2\sqrt{3}}{3}$. Solution set = $\{x : x \in \mathbb{R}, x \leq -\frac{2\sqrt{3}}{3}$ or $x \geq \frac{2\sqrt{3}}{3}\}$

(b)

Total surface area is $2\pi r^2 + 2\pi rh = 100\pi$

$$r^2 + rh = 50$$

$$h = \frac{50 - r^2}{r}$$

Volume, $V = \pi r^2 h = \pi r^2 \left(\frac{50 - r^2}{r}\right)$

$$= \pi r\left(50 - r^2\right)$$

$\frac{dV}{dr} = \pi\left(50 - 3r^2\right) = 0$ for stationary points

$$r = \frac{5\sqrt{2}}{\sqrt{3}} = 4.08$$

Using sign test,

r	$\frac{5\sqrt{2}}{\sqrt{3}}^-$	$\frac{5\sqrt{2}}{\sqrt{3}}$	$\frac{5\sqrt{2}}{\sqrt{3}}^+$
$\frac{dV}{dr}$	$\left(50 - 3r^2\right) > 0$	0	$\left(50 - 3r^2\right) < 0$

Therefore V is maximum when $r = \frac{5\sqrt{6}}{3}$ or 4.08

(c)

Given $\frac{dV}{dt} = 10$

$V = \frac{1}{3}\pi r^2 h = \frac{2}{3}\pi r^3$. When $h = 4,\ r = 2$

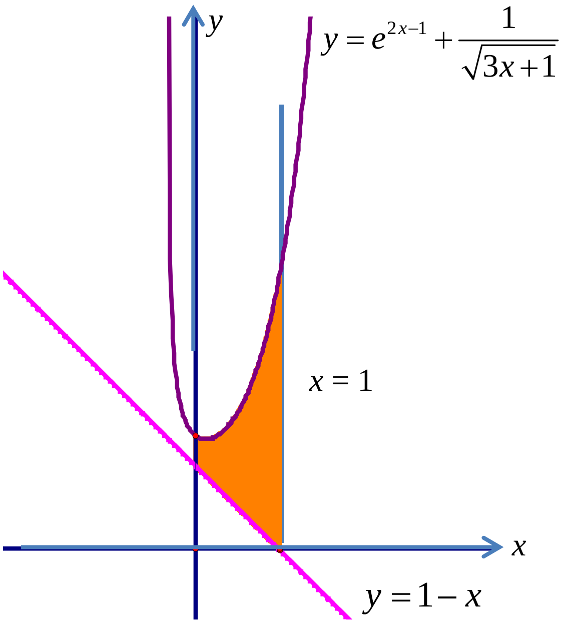
$\frac{dV}{dr} = 2\pi r^2 = 8\pi$, when $r = 2$

$\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt} = \frac{5}{4\pi}$ cm/sec

5(a)

Show that $\frac{d}{dx}\left(\sqrt{2x^2 - x^3}\right) = \frac{4 - 3x}{2\sqrt{2 - x}}$, given that $0 < x < 2$

Hence evaluate $\int \frac{4 - 3x}{\sqrt{2 - x}}\, dx$

	$\frac{d}{dx}\left(\sqrt{2x^2-x^3}\right) = \frac{d}{dx}\left(2x^2-x^3\right)^{\frac{1}{2}}$ $=\frac{1}{2}\left(2x^2-x^3\right)^{-\frac{1}{2}}\left(4x-3x^2\right)$ $=\frac{4x-3x^2}{2\sqrt{2x^2-x^3}}$ $=\frac{4-3x}{2\sqrt{2-x}}$ Therefore $\int \frac{4-3x}{2\sqrt{2-x}}\,dx = \sqrt{2x^2-x^3}+A$ Hence $\int \frac{4-3x}{\sqrt{2-x}}\,dx = 2\sqrt{2x^2-x^3}+2A$ $= 2\sqrt{2x^2-x^3}+C$
(b)	 Find the exact area of the shaded region, in the above figure, which is bounded by the curve $y=e^{2x-1}+\frac{1}{\sqrt{3x+1}}$, the y-axis, the line $y=1-x$ and the line $x=1$
	Area $=\int_0^1\left[e^{2x-1}+\frac{1}{\sqrt{3x+1}}-(1-x)\right]dx$ $=\left[\frac{e^{2x-1}}{2}+\frac{2}{3}\sqrt{3x+1}+\frac{x^2}{2}-x\right]_0^1$ $=\frac{e}{2}+\frac{4}{3}+\frac{1}{2}-1-\left(\frac{1}{2e}+\frac{2}{3}\right)$ $=\frac{e}{2}+\frac{1}{6}-\frac{1}{2e}$ sq units

Section B: Statistics [60 marks]

6	A machine grades mangoes according to their mass. Mangoes with a mass exceeding 385g are rejected as too large and mangoes with mass less than 325g are rejected as too small. A large batch of mangoes is graded and it is found that 13% are rejected as too large and 11% are rejected as too small. Assuming a normal distribution, find the mean mass and the variance of a randomly chosen mango from the batch.
	Let X denote the random variable representing the mass of a randomly chosen mango, in grams. $X \sim \mathrm{N}(\mu,\ \sigma^2)$, where μ is the mean mass and σ^2 is the variance of the distribution. P(X > 385) = 0.13 P($X \le$ 385) = 1 – 0.13 = 0.87 $\mathrm{P}\left(Z \le \frac{385-\mu}{\sigma}\right) = 0.87$ From GC, $\frac{385-\mu}{\sigma} = 1.12639$ $385 - \mu = 1.126391128\sigma$ $\mu + 1.126391128\sigma = 385$ ---------------------------------- (1) $\mathrm{P}(X < 325) = 0.11$ $\mathrm{P}\left(Z < \frac{325-\mu}{\sigma}\right) = 0.11$ From GC, $\frac{325-\mu}{\sigma} = -1.22652812$ $325 - \mu = -1.22652812\sigma$ $\mu - 1.22652812\sigma = 325$ ---------------------------------- (2) Using GC, μ = 356 and σ = 25.5002 Hence, the mean mass of a randomly chosen apple from the batch is 356 grams and the variance is 650.

7(a)	A professor is studying the sleep patterns of students at her university. There are 1800 male students and 1200 female students in that university. She decides to start by asking a random sample of 50 students how many hours of sleep they get on weekday nights. The professor assigns each student a unique number from 1 to 3000. She selects the sample by randomly choosing one of the first 60 numbers and every 60th number thereafter. **(i)** State one disadvantage of this sampling method in this case. **(ii)** Suggest a better sampling method and describe how the professor can choose a random sample of 50 students using this method.
	(i) The sample selected does not represent the proportion of male and female students. **(ii)** Stratified sampling. Male:Female = 3:2 Therefore choose 30 males and 20 females at random from 1800 males and 1200 females.
(b)	A sample of 100 college students was chosen and the students were given a test to measure their IQ. Following is the frequency table which consists of the students' IQ and the corresponding number of students.

IQ (x)	97	98	99	100	101	102	103	104	105	106	107	108
No. of Students (f)	5	7	12	13	15	11	10	8	6	7	3	3

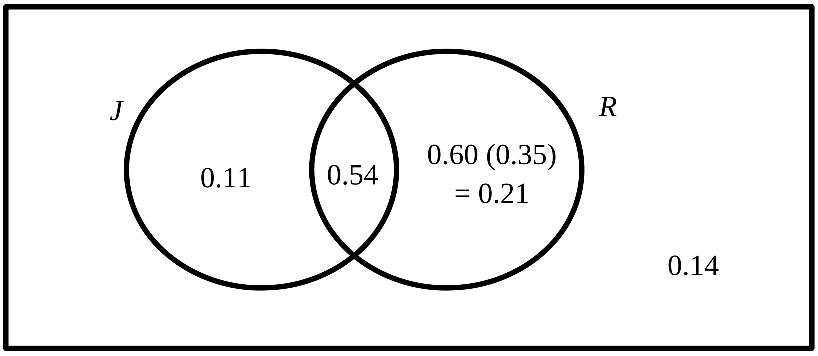
(i) Find the unbiased estimates of the population mean and population variance of distribution of IQ among college students.

(ii) Using the estimated mean and variance obtained in part **(i)**, find the probability that the mean IQ of 60 randomly chosen college students lies between 98 and 101.

	(i) From GC $\bar{x} = 101.75, \quad s = 2.836896$ $s^2 = 8.05$ (ii) Here n = 60. Since n is large, by Central Limit Theorem $\bar{X} \sim N\left(101.75, \frac{8.05}{60}\right)$ approximately $\mathrm{P}\left(98 < \bar{X} < 101\right) = 0.0203$ from GC.
8	On average 5% of computer chips manufactured in a firm are defective. A sample of twenty such computer chips is chosen randomly from the firm. **(i)** Evaluate the mean and variance of the number of defective computer chips. **(ii)** Find the probability that more than 2 computer chips are defective in this sample. **(iii)** Find the probability that at most 5 computer chips are defective, given that more than 2 computer chips are defective. **(iv)** One hundred such samples, each sample containing 20 computer chips, are taken from the firm. Using a suitable approximation, find the probability that there are at most 7 samples which contain more than 2 defective computer chips each.
	(i) Here n =20, p =0.05 Let X be the random variable , number of defective chips out of 20 chips. $X \sim \mathrm{B}(20, 0.05)$ Mean = np = 20(0.05) = 1 Variance = np(1-p) = 1(0.95) =0.95 **(ii)** $\mathrm{P}(X > 2) = 1 - \mathrm{P}(X \le 2) = 0.075484$ ≈ 0.0755 **(iii)** Let A be the event at most 5 chips are defective and B be the event more than 2 chips are defective. $P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(3 \le X \le 5)}{P(X > 2)}$ = 0.996

	(iv) Let Y be the random variable, number of samples which contains more than 2 defective chips out of 60 samples. Here n = 100 and p = 0.0755 Hence $Y \sim B(100, 0.0755)$ n = 100, large $np = 7.55 > 5,\ n(1-p) = 92.45 > 5$ and $np(1-p) = 6.98$ Therefore $Y \sim N(7.55,\ 6.98)$ approximately. $P(Y \le 7) \xrightarrow{cc} P(Y < 7.5)$ = 0.493
9 (a)	In a particular town, 65% of the people watch the show *Jumping Man* and 75% of the people watch the show *Running Woman*. 60% of those who do not watch the show *Jumping Man* watch the show *Running Woman*. Find the probability that a person chosen at random from the town **(i)** watches both shows, **(ii)** does not watch the show *Running Woman* given that the person watches the show *Jumping Man*.

<table>
<tr><td>(i)</td><td>Let J be the event that “a randomly chosen person watches Jumping Man” and R be the event that “a randomly chosen person watches Running Woman.”



$P(J' \cap R) = 0.6\ (0.35) = 0.21$

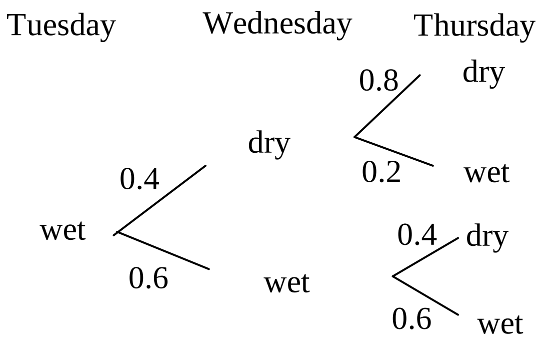
$P(J \cap R) = P(R) - P(J' \cap R)$
$= 0.75 - 0.21$
$= 0.54$</td></tr>
<tr><td>(ii)</td><td>$P(R' \mid J) = \dfrac{P(R' \cap J)}{P(J)}$

$= \dfrac{0.11}{0.65}$

$= \dfrac{11}{65}$ or 0.169 (to 3 sig fig)</td></tr>
<tr><td>9 (b)</td><td>The weather on any day is classified as either wet or dry. If it is wet on any particular day the probability that it will be wet the next day is 0.6. If it is dry on any particular day the probability that it will be dry the next day is 0.8. Tuesday was a wet day in a particular week.

(i) Draw a tree diagram to illustrate the possible outcomes from Tuesday to Thursday.

(ii) Find the probability that Wednesday is a wet day given that Thursday of that week is wet.</td></tr>
</table>

<table>
<tr><td>(i)</td><td>
1 mark per level</td></tr>
<tr><td>(ii)</td><td>P(Wednesday is wet | Thursday is a wet day)

$= \dfrac{\text{P(Wednesday and Thursday are wet days)}}{\text{P(Thursday is a wet day)}}$
$= \dfrac{0.6(0.6)}{0.4(0.2) + 0.6(0.6)}$
$= \dfrac{9}{11}$ or 0.818 (to 3 sig fig)</td></tr>
<tr><td>10</td><td>A company sells electric water heaters. A manager claims that their heaters will heat 50 litres of water from a temperature of $10^\circ C$ to a temperature of $35^\circ C$ in, on average, no longer than 12 minutes. In order to test this claim, 40 randomly chosen heaters are bought and the time (x minutes) taken by each heater to heat 50 litres of water from $10^\circ C$ to $35^\circ C$ is measured and summarised by
$\sum(x-15) = -106, \quad \sum(x-15)^2 = 318.$
(i) Find unbiased estimates of the population mean and variance.

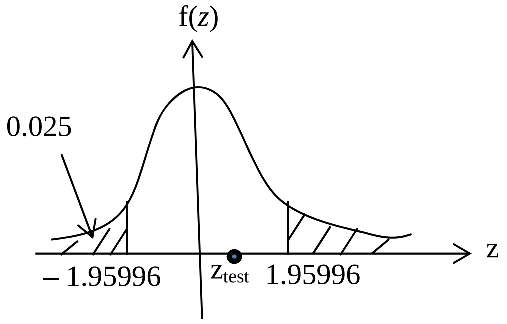
(ii) Test at the 1% significance level whether the manager’s claim is valid.

The company manufactures a new model of water heater and claims that the mean time of the new model of water heater to heat 50 litres of water from a temperature $10^\circ C$ to a temperature of $35^\circ C$ is 10 minutes. The time taken by the new model of water heater to heat 50 litres of water is known to have a normal distribution with standard deviation 0.98 minutes. A random sample of 20 water heaters is selected and found to have an average time of k minutes to heat 50 litres of water from $10^\circ C$ to $35^\circ C$. A test at the 5% significance level is carried out on this sample, and the company’s claim is accepted. Find the set of possible values of k.</td></tr>
<tr><td>(i)</td><td>Unbiased estimate of population mean
$= \dfrac{\sum(x-15)}{40} + 15$
$= \dfrac{-106}{40} + 15$</td></tr>
</table>

	= 12.35 (exact) Unbiased estimate of population variance $= \frac{1}{40-1}\left[\sum (x-15)^2 - \frac{\left(\sum (x-15)\right)^2}{40}\right]$ $= \frac{1}{39}\left[318 - \frac{(-106)^2}{40}\right]$ = 0.951282 = 0.951 (to 3 sig fig) or $\frac{371}{390}$ (exact)
	Let μ be the average time of water heaters to heat 50 litres of water from a temperature of $10^{\circ}C$ to a temperature of $35^{\circ}C$. To test Ho: $\mu = 12$ Against H_1: $\mu > 12$ using right tailed test at 1% level of significance. Under Ho, $\bar{X} \sim N(12, \frac{0.951282}{40})$ approximately by Central Limit Theorem since n = 40 is large Using GC, p – value = 0.0116 Since p – value > 0.01, we do not reject Ho and conclude that there is insufficient evidence to conclude that the average time to heat 50 litres of water from a temperature of $10^{\circ}C$ to a temperature of $35^{\circ}C$ is greater than 12 minutes at 1% level of significance. Therefore the manager’s claim is valid.
	Let μ be the average time of the new model of water heaters to heat 50 litres of water from a temperature of $10^{\circ}C$ to a temperature of $35^{\circ}C$. To test Ho: $\mu = 10$ Against H_1: $\mu \neq 10$ using 2 tailed test at 5% level of significance.

Under Ho, $\bar{X} \sim N(10, \frac{0.98^2}{20})$

Since Ho is not rejected, z_{test} lies outside the critical region.



$-1.95996 < z_{test} < 1.95996$

$-1.95996 < \frac{k-10}{\frac{0.98}{\sqrt{20}}} < 1.95996$

$-0.429495 < k - 10 < 0.429495$

$9.5705 < k < 10.429495$

Hence, the set of values is $\{k \in \mathbb{R} : 9.57 < k < 10.4\}$.

11 A supermarket sells two types of curry powder. Type *A* and Type *B* curry powders are mixed from two ingredients *P* and *Q*. One packet of Type *A* curry powder contains 2 scoops of ingredient *P* and 3 scoops of ingredient *Q*. A packet of Type *B* curry powder contains 4 scoops of ingredient *P* and 1 scoop of ingredient *Q*. The masses, in grams, of ingredients *P* and *Q* in scoops have independent normal distributions with means and standard deviations as shown in the following table.

	Mean	Standard deviation
Ingredient *P*	55	3
Ingredient *Q*	25	2

(i) Find the probability that the total mass of two randomly chosen packets of Type *A* curry powder is within 100 grams of twice the mass of a packet of Type *B* curry powder.

(ii) Twenty packets of Type *A* curry powders are chosen at random. Find the probability that there are greater than 2 but not more than 10 packets of Type *A* curry powder each weighing more than 0.19 kg.

(iii) 10 packets of Type *A* curry powder are packed, in random, into ‘long’ bags. 5 packets of Type *B* curry powder are packed, in random, into ‘special’ bags. Elsa buys a ‘long’ bag at \$1.20 per 100 gram and Anna buys two ‘special’ bags at \$1.50 per 100 gram. Find the probability that Anna pays at least \$15 more than Elsa.

(i)	Let X be random variable "the mass of a scoop of ingredient P." Let Y be random variable "the mass of a scoop of ingredient Q." $X \sim N(55, 3^2)$ $Y \sim N(25, 2^2)$ Let W be random variable "the mass of a randomly chosen packet of Type A curry powder" Let V be random variable "the mass of a randomly chosen packet of Type B curry powder" $W = X_1 + X_2 + Y_1 + Y_2 + Y_3 \sim N(2(55) + 3(25), 2(3^2) + 3(2^2))$ $W \sim N(185, 30)$ $V = X_1 + X_2 + X_3 + X_4 + Y_3 \sim N(4(55) + 25, 4(3^2) + 2^2)$ $V \sim N(245, 40)$ $W_1 + W_2 - 2V \sim N(2(185) - 2(245), 2(30) + 2^2(40))$ $W_1 + W_2 - 2V \sim N(-120, 220)$ $P(-100 < W_1 + W_2 - 2V < 100)$ $= 0.088765 = 0.0888$ (to 3 sig fig)
(ii)	$P(W > 190) = 0.180655$ Let C be the random variable "the number of packets each weighing more than 0.19 kg out of 20 packets" $C \sim B(20, 0.180655)$ $P(2 < C \le 10)$ $= P(3 \le C \le 10)$ $= P(C \le 10) - P(C \le 2)$ $= 0.72746$ $= 0.727$ (to 3 sig fig)
(iii)	Let L be the random variable "the mass of a randomly chosen long bag" Let S be the random variable "the mass of a randomly chosen special bag" $L = W_1 + \dots + W_{10} \sim N(1850, 300)$ $S = V_1 + \dots + V_5 \sim N(1225, 200)$ $S_1 + S_2 \sim N(2450, 400)$ $\frac{1.5}{100}(S_1 + S_2) - \frac{1.2}{100}L \sim N(36.75 - 22.2, 0.09 + 0.0432)$

	$\frac{1.5}{100}(S_1+S_2)-\frac{1.2}{100}L \sim \mathrm{N}(14.55,\ 0.1332)$ $\mathrm{P}(\frac{1.5}{100}(S_1+S_2)-\frac{1.2}{100}L \geq 15)$ = 0.1087893634 = 0.109 (to 3 sig fig)
12	A certain solution discolours when exposed to air. To protect the solution against discolouration, it is treated with a chemical. In an experiment, different quantities, *m* ml, of the chemical were applied to standard samples of the solution, and the times, *t* minutes, for the solution to discolour were measured. The results are given in the table.

m	1.3	2.1	2.8	3.5	4.7	5.6	6.5
t	3	13	17	20	22	28	33

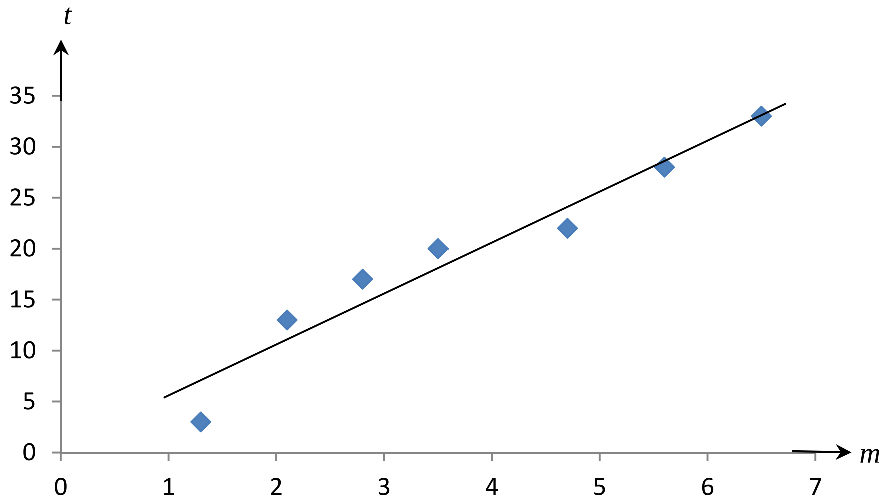
(i) Draw a sketch of the scatter diagram for the data, as shown on your calculator.

(ii) Find the product moment correlation coefficient and comment on its value in the context of the data.

(iii) Using a suitable regression line, estimate the quantities of the chemical applied when the time taken for the solution to discolour is 40 minutes. Comment on the reliability of the estimate. Sketch the regression line you have chosen on the scatter diagram.

(iv) A different type of chemical has been added to delay the process of discolouration of the solution. It is found that the time taken for the solution to discolour has been lengthened by 2 minutes for all the seven samples. Without any further calculations, state any change you would expect in the values of gradient and *y* – intercept of the regression line in (**iii**).

(v) A new pair of values where $m = 5$ and $t = a$ is added to form a new regression line t on m. Find the value of a if the new equation of the regression line t on m is $t = 0.635 + 4.817\ m$. Correct your answer to 1 decimal place.

(i)	
(ii)	$r = 0.970$. Since $r = 0.970$ is close to 1, there is a strong positive linear correlation between the amount of chemical applied and the time taken for the solution to discolour. As the amount of chemical applied to the solution increases, the time to discolour the solution increases.
(iii)	$t = 0.359584 + 5.03709m$ When $t = 40$ minutes, $40 = 0.359584 + 5.03709m$ $m = 7.8697 = 7.87$ (to 3 sig fig) Although $r = 0.970$ indicates there is a strong linear correlation between the amount applied and the time to discolour the solution, $t = 40$ is outside the data range [3 – 33], the linear correlation may not hold, hence the estimate obtained by extrapolation is not reliable.
(iv)	The gradient remains unchanged but the y – intercept will change from 0.359584 to 2.359584.
(v)	$\overline{m} = \dfrac{1.3+2.1+2.8+3.5+4.7+5.6+6.5+5}{8} = 3.9375$ $\overline{t} = 0.635 + 4.817\overline{m}$ $\overline{t} = 0.635 + 4.817(3.9375) = 19.6019375$ $\dfrac{3+13+17+20+22+28+33+a}{8} = 19.6019375$ $a = 20.8155 = 20.8$ (to 1 decimal place)

End of paper