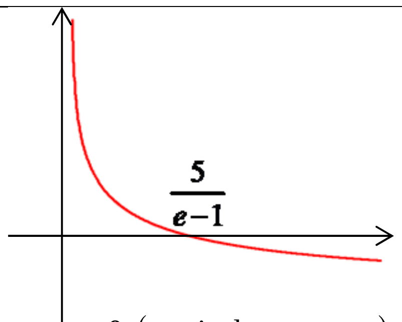
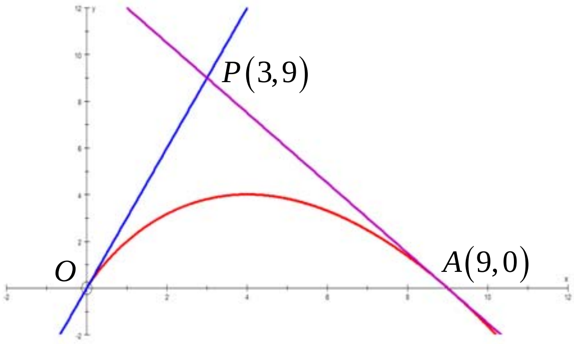
Solution to AJC 2014 JC2 H1 Math Prelim Exam

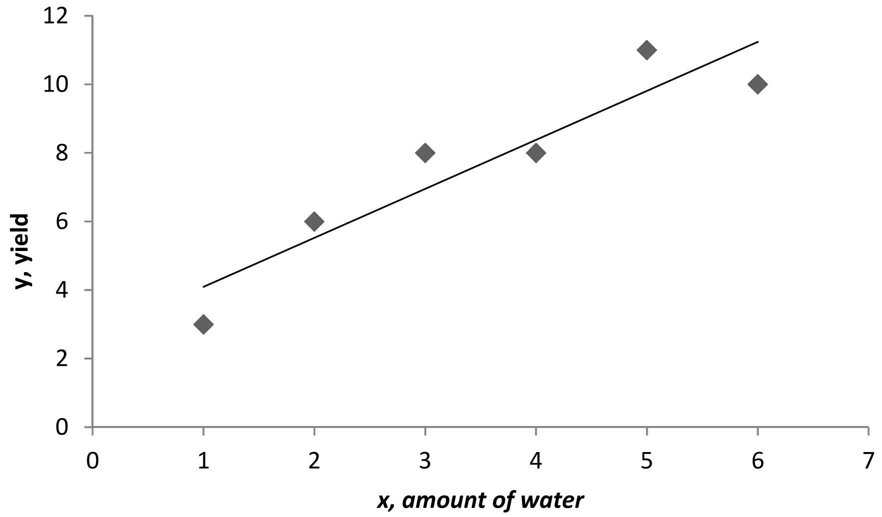
1	$p\left(x^2+2\right)<2x^2+6x+1 \Leftrightarrow \left(p-2\right)x^2-6x+2p-1<0$ $\left(-6\right)^2-4\left(p-2\right)\left(2p-1\right)<0$ and $p-2<0$ $-2p^2+5p+7<0$ and $p<2$ $p<-1$ or $p>3.5$ and $p<2$ Answer: $p<-1$	M1, M1 B1 A1
2(a)	$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\left(\ln\frac{x}{e^x}\right)^6\right\}=\frac{\mathrm{d}}{\mathrm{d}x}\left\{\left(\ln x-\ln e^x\right)^6\right\}$ $=\frac{\mathrm{d}}{\mathrm{d}x}\left\{\left(\ln x-x\right)^6\right\}$ $=6\left(\ln x-x\right)^5\left(\frac{1}{x}-1\right)$	M1 A1
(b)	$\int_{-2}^{0}\frac{1}{\sqrt{e^{3x-2}}}\,\mathrm{d}x=\int_{-2}^{0}e^{-\frac{1}{2}(3x-2)}\,\mathrm{d}x$ $=\int_{-2}^{0}e^{-\frac{3}{2}x+1}\,\mathrm{d}x$ $=\left[\frac{e^{-\frac{3}{2}x+1}}{-3/2}\right]_{-2}^{0}$ $=-\frac{2}{3}\left(e-e^4\right)=\frac{2}{3}\left(e^4-e\right)$	B1 M1 A1
3(a)	$y=\frac{1}{2-x}=\left(2-x\right)^{-1}$ $\frac{\mathrm{d}y}{\mathrm{d}x}=-\left(2-x\right)^{-2}\left(-1\right)=\frac{1}{\left(2-x\right)^2}$ $\frac{\mathrm{d}y}{\mathrm{d}t}=\frac{\mathrm{d}y}{\mathrm{d}x}\cdot\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{1}{\left(2-x\right)^2}\frac{\mathrm{d}x}{\mathrm{d}t}$ Given $\frac{\mathrm{d}y}{\mathrm{d}t}=0.5$, when $x=-1$, $0.5=\frac{1}{3^2}\frac{\mathrm{d}x}{\mathrm{d}t}\Rightarrow\frac{\mathrm{d}x}{\mathrm{d}t}=4.5$	A1 M1 A1
(b)(i)	Perimeter $=\pi r+2h+2r=7$ $h=\frac{7-\pi r-2r}{2}$ Area of shape, $A=\frac{1}{2}\pi r^2+2rh$ $=\frac{1}{2}\pi r^2+2r\left(\frac{7-\pi r-2r}{2}\right)=-\frac{1}{2}\pi r^2+7r-2r^2$	B1 B1
(ii)	When A is max., $\frac{\mathrm{d}A}{\mathrm{d}r}=0$ $-\pi r+7-4r=0$	M1

	$(\pi+4)r = 7 \Rightarrow r = \dfrac{7}{\pi+4}$	B1
	when $r = \dfrac{7}{\pi+4}$, $h = \dfrac{7-\pi\left(\dfrac{7}{\pi+4}\right)-2\left(\dfrac{7}{\pi+4}\right)}{2}$	
	$= \tfrac{1}{2}\left[\dfrac{7(\pi+4)-7\pi-14}{\pi+4}\right] = \tfrac{1}{2}\left(\dfrac{14}{\pi+4}\right) = \dfrac{7}{\pi+4}$	B1

r	0.9	$\dfrac{7}{\pi+4} \approx 0.98$	1
$\dfrac{\mathrm{d}A}{\mathrm{d}r}$	0.57	0	- 0.14

	(or equivalent test)	
	therefore A is indeed maximum when $h = r = \dfrac{7}{\pi+4}$.	B1
4(i)	$\ln(x+5) = 1+\ln x \Leftrightarrow \ln(x+5)-\ln x = 1$	
	$\ln\left(\dfrac{x+5}{x}\right) = 1$	
	$\dfrac{x+5}{x} = e$	B1
	$x+5 = ex \Rightarrow (e-1)x = 5 \Rightarrow x = \dfrac{5}{e-1}$	A1
(ii)	 $x = 0$ (vertical asymptote)	curve A1 x-intercept A1 vertical asymptote A1
(iii)	$\ln(x+5) > 1+\ln x$	
	$\ln(x+5)-1-\ln x > 0$	
	From graph in (ii), $0 < x < \dfrac{5}{e-1}$.	A1
5(i)	$y = 3x - x^{\frac{3}{2}} \Rightarrow \dfrac{\mathrm{d}y}{\mathrm{d}x} = 3 - \tfrac{3}{2}x^{\frac{1}{2}}$	A1
	At origin, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 3$.	A1
	Equation of tangent at origin is $y = 3x$.	A1

(ii)	At $A(9,0)$, $\frac{\mathrm{d}y}{\mathrm{d}x}=3-\frac{3}{2}(9)^{\frac{1}{2}}=3-\frac{3}{2}(3)=-\frac{3}{2}$ Equation of tangent at A is $y=-\frac{3}{2}(x-9)=-\frac{3}{2}x+\frac{27}{2}$	A1 A1
(iii)	To find coordinates of P, solve $y=3x$ and $y=-\frac{3}{2}x+\frac{27}{2}$ $3x=-\frac{3}{2}x+\frac{27}{2}\Rightarrow\frac{9}{2}x=\frac{27}{2}\Rightarrow x=3$. Hence $y=9$. Hence $P\equiv(3,9)$.	A1
(iv)	$\int 3x-x^{\frac{3}{2}}\ \mathrm{d}x=\frac{3}{2}x^2-\frac{2}{5}x^{\frac{5}{2}}+c$	A1(for $\frac{3}{2}x^2-\frac{2}{5}x^{\frac{5}{2}}$), A1 (for c)
(v)	 Area of region bounded $=(\text{area of }\Delta\text{OAP})-(\text{area under curve from }x=0\text{ to }x=9)$ $=\frac{1}{2}(9)^2-\int_0^9 3x-x^{\frac{3}{2}}\ \mathrm{d}x$ $=\frac{81}{2}-\left[\frac{3}{2}x^2-\frac{2}{5}x^{\frac{5}{2}}\right]_0^9$ $=\frac{81}{2}-\left[\frac{243}{2}-\frac{486}{5}\right]=\frac{81}{2}-\frac{243}{10}=16\frac{1}{5}$	M1 M1 A1
6(i)	Reasons: Not all interviewed are residents and only those in the street are asked. (do not accept reasons like "this is not a random sample" or "it takes too long" for example)	B1, B1
(ii)	Simple random sampling (can be systematic) Obtain list of town population Number it sequentially Select using random numbers (If just mention "Take a random sample" – give B1)	} B1 B1
7(i)	$\text{P}(\text{household is a victim of both crimes})$ $=0.07\times0.12=0.0084$	A1
(ii)	$\text{P}(\text{household suffers only one of these misfortunes})$ $=0.07\times0.88+0.93\times0.12=0.1732$	M1, A1
(iii)	$\text{P}(\text{household suffers at least one of these misfortunes})$ $=0.1732+0.0084=0.1816$	B1

	$\mathrm{P}\left(\text{household is a victim of both crimes} \mid \text{suffers at least one of the misfortunes}\right)$ $=\dfrac{0.0084}{0.1816}$ $=0.0463$ (3sf)	M1 A1
8(a)	$\mathrm{P}\left(A \mid B'\right)=\dfrac{\mathrm{P}\left(A\cap B'\right)}{\mathrm{P}\left(B'\right)}$ $\mathrm{P}\left(A\cap B\right)=\mathrm{P}\left(A\cup B\right)-\mathrm{P}\left(A\cap B'\right)-\mathrm{P}\left(A'\cap B\right)$ $=0.65-0.32-0.11=0.22$ $\mathrm{P}\left(B\right)=\mathrm{P}\left(A'\cap B\right)+\mathrm{P}\left(A\cap B\right)=0.11+0.22=0.33$ $\mathrm{P}\left(A \mid B'\right)=\dfrac{\mathrm{P}\left(A\cap B'\right)}{\mathrm{P}\left(B'\right)}=\dfrac{0.32}{1-0.33}=\dfrac{0.32}{0.67}=\dfrac{32}{67}$ or 0.478	M1 B1 B1 A1
(b)	$\mathrm{P}\left(A\right)=\mathrm{P}\left(A\cap B'\right)+\mathrm{P}\left(A\cap B\right)=0.32+0.22=0.54$ Since $\mathrm{P}\left(A \mid B'\right)\neq \mathrm{P}\left(A\right)$ (or equivalent like $\mathrm{P}\left(A\cap B\right)\neq \mathrm{P}\left(A\right)\cdot\mathrm{P}\left(B\right)$) therefore *A* and *B* are not independent.	A1 M1 A1
9(a)	(i) $r=-1$ (ii) $r=0$	B1, B1
(b)(i)		Must be *y* *vs x* B1 Correct number of data points B1
(ii)	Any of the following reasons is acceptable: - *x* is independent or controlled or changed - value of *y* was measured for each *x* - *x* is not dependent - water affects yield or yield is dependent on amount of water or yield does not control water supply (But not the following: - *y* is dependent; *x* goes up in equal intervals; *x* is not fixed)	B1
(iii)	The linear regression model is appropriate because the scatter diagram shows data	

	points are in a linear form (or close to a straight line) and that the correlation coefficient of 0.930 suggests a strong positive linear correlation between *x* and *y*.	B1, B1
(iv)	$y = 1.428571429x + 2.666666667$ $y = 1.4x + 2.7$ (correct to 1 d.p.) The gradient is 1.4 which can be interpreted that for every unit of water supplied, the yield increases by 1.4 units. The *y*-intercept is 2.7 which can be interpreted that without any water supplied, the yield will be 2.7 units	B1 B1 B1
(v)	Increase in yield $= 1.428571429 \times 5 = 7.1428... = 7.14$ (3sf)	A1
10 (i)	Let *X* denote the number of days cupcakes are sold out in a week. $X \sim B(7, 0.6)$ $P(X \geq 4) = 1 - P(X \leq 3)$ $= 0.710208$ $= 0.710(3sf)$	B1 A1
(ii)	Required probability = $2 \times 0.6^6 \times 0.4 + 0.6^7 = 0.0653184$ $= 0.0653(3sf)$	M1, M1
(iii)	$P(\text{at least 6 consecutive days} \mid X \geq 6)$ $= \dfrac{P(\text{at least 6 consecutive days})}{P(X \geq 6)}$ $= \dfrac{0.0653184}{1 - P(X \leq 5)}$ $= 0.41176$ $= 0.412(3sf)$	M1 A1
(iv)	L et *Y* denote the number of weeks, out of 52, which all his cupcakes will be sold out in at least 4 days in the week. $Y \sim B(52, 0.710208)$ Since *n=52* is large, $np = 36.9308 > 5$ and $nq = 15.069 > 5$ $\therefore Y \sim N(36.9308, 10.702)$ $P(25 < Y \leq 40) \xrightarrow{C.C.} P(25.5 < Y < 40.5)$ $= 0.862132$ $= 0.862(3sf)$	B1 M1 M1 A1
11 (i)	Let *X* denote the weight of a durian *X* $X \sim N(\mu, \sigma^2)$ $P(X < 0.7) = 0.1$ $P\left(Z < \dfrac{0.7 - \mu}{\sigma}\right) = 0.1$ From GC, $\dfrac{0.7 - \mu}{\sigma} = -1.281551567 ------------(1)$	M1

	$P(X>2)=0.02$ $P(X<2)=0.98$ $P\left(Z<\frac{2-\mu}{\sigma}\right)=0.98$ From GC, $\frac{2-\mu}{\sigma}=2.053748911$ – – – – – – – – – – – – –(2) Solving, $\sigma=0.38976\approx 0.4(1dp)$, $\mu=1.1995\approx 1.2(1dp)$ (shown)	M1 M1
(ii)	Let X and Y denote the weight of a durian X and Y respectively. $X\sim N(1.2,0.4^2)$, $Y\sim N(2,0.3^2)$ $X_1+X_2+X_3-2Y\sim N(-0.4,0.84)$ $P(X_1+X_2+X_3>2Y)$ $=P(X_1+X_2+X_3-2Y>0)$ $=0.33126...=0.331(3sf)$	M1, M1 A1
(iii)	$X-Y\sim N(-0.8,0.25)$ $\mathrm{P}(-0.5\le X-Y\le 0.5)=0.26959...=0.270(3sf)$	M1,A1
	$5X+8Y\sim N(22,9.76)$ $P(5X+8Y\le 20)=0.26102=0.261(3sf)$	M1 M1,A1
12 (i)	unbiased estimates of the population mean $=\frac{\sum(x-55)}{n}+55=\frac{-38}{100}+55=54.62$ unbiased estimates of the population variance $=\frac{1}{n-1}\left[\sum(x-55)^2-\frac{\left(\sum(x-55)\right)^2}{n}\right]$ $=\frac{1}{99}\left[451-\frac{(-38)^2}{100}\right]$ $=4.40969=4.41(3sf)$	A1 A1
(ii)	Let X be the lifetime, in hours, of the AJBatteries, and μ be the mean time. $H_o:\mu=55$ (company's claim) $H_1:\mu\neq 55$ (retailer's suspicion) Under H_o, Since sample size is large, by Central Limit Theorem, $\overline{X}\sim N\left(55,\frac{s^2}{60}\right)$ Use a two-tailed test at 5%, and reject H_o if $p<0.05$. Using GC, with $\overline{x}=54.62$, $s=\sqrt{4.4096}$, n =100 $p-value=2P(\overline{X}<54.62)=0.070359$ not less than 0.04	B1 B1 A1 A1

	Do not reject H_o. There is insufficient evidence at 5% significance level to say that the company's claim is invalid.	
(iii)	$H_o: \mu = 55$ (company's claim) $H_1: \mu < 55$ (retailer's suspicion) $p-value = P(\overline{X} < 54.62) = \dfrac{0.070359}{2} = 0.035175 < 0.05$ The conclusion would not remain the same. We now reject H_0.	M1
(iv)	$H_o: \mu = \mu_0$ (manager's claim) $H_1: \mu < \mu_0$ (whether he has overstated the average time) Under H_o, $\overline{X} \sim N\left(\mu_0, \dfrac{3.9^2}{8}\right)$ Use a one-tailed test at 10%, and reject H_o if $p-value < 0.1$. Observed sample mean $\bar{x} = \dfrac{56.3+54.8+54.5+54.4+53.9+55.5+54.6+54.9}{8}$ $= \dfrac{438.9}{8}$ $= 54.8625$ If company has overstated the average time, then H_0 is rejected $p-\text{value} = \text{P}\left(\overline{X} < 54.8625\right) < 0.1$ Standardising $\overline{X}$, $\text{P}\left(Z < \dfrac{54.8625-\mu_0}{3.9/\sqrt{8}}\right) < 0.1$ $\dfrac{54.8625-\mu_0}{3.9/\sqrt{8}} < -1.281551567$ $54.8625-\mu_0 < -1.281551567\left(\dfrac{3.9}{\sqrt{8}}\right)$ $\mu_0 > 56.629$ Least possible value of μ_0 is 57 hours.	B1 M1 M1 A1