RAFFLES INSTITUTION
H1 Mathematics 8864
2014 Year 6 Preliminary Examination

Time Allowed: 3 hours **Total Marks: 95**

Section A: Pure Mathematics [35 marks]

1 Without using a calculator, solve the inequality

$$2x^2 - 3x - 2 > 0. \qquad [2]$$

Hence use a sketch of the graph of $x = \mathrm{e}^{\theta}$ to solve the inequality

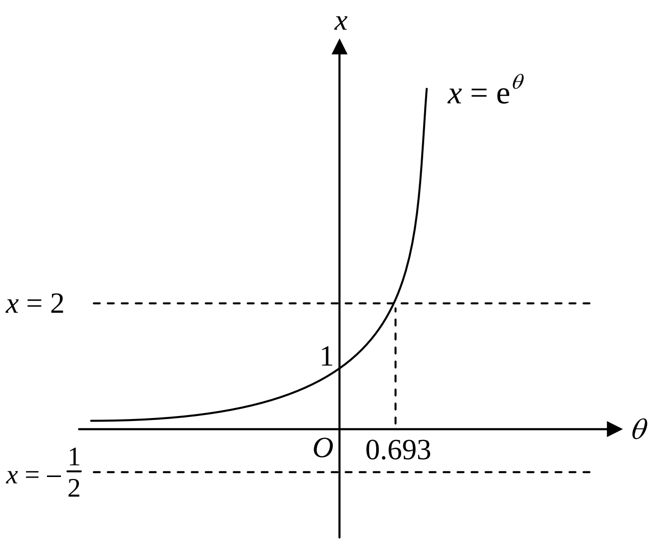
$$2\mathrm{e}^{2\theta} - 3\mathrm{e}^{\theta} - 2 > 0. \qquad [2]$$

Solution

$$2x^2 - 3x - 2 > 0 \;\Rightarrow\; (2x+1)(x-2) > 0$$

$$\Rightarrow\; x < -\frac{1}{2} \text{ or } x > 2$$

$$\therefore\; 2\mathrm{e}^{2\theta} - 3\mathrm{e}^{\theta} - 2 > 0 \;\Rightarrow\; \mathrm{e}^{\theta} < -\frac{1}{2} \text{ or } \mathrm{e}^{\theta} > 2$$



$$\Rightarrow\; \theta > 0.693 \text{ (3sf) } \text{ (or } \ln 2\text{)}$$

2 Given that both a and x are real numbers greater than 1, and

$$\log_a(4\log_a x) - \log_a(\log_x a) = \log_a 64,$$

find the numerical value(s) of $\log_a x$. [4]

Given further that $a = 3$, write down the exact value(s) of x. [1]

Solution

$\log_a(4\log_a x) - \log_a(\log_x a) = \log_a 64$

$\Rightarrow \log_a\left(\dfrac{4\log_a x}{\log_x a}\right) = \log_a 64$

$\Rightarrow \dfrac{4\log_a x}{\log_x a} = 64$

$\Rightarrow (\log_a x)^2 = 16$

$\Rightarrow \log_a x = 4$ or -4 (rejected, $\log_a x > 0$)

When $a = 3$, $x = 3^4 = 81$.

3 **(a)** Differentiate $\dfrac{(e^{1-3x})^2}{e^{x+1}}$. [2]

(b) Use a non-calculator method to find the exact value of $\displaystyle\int_{-5}^{1} \frac{8}{(7-4x)^2}\,dx$. [4]

Solution

(a) Let $y = \dfrac{(e^{1-3x})^2}{e^{x+1}} = \dfrac{e^{2-6x}}{e^{x+1}} = e^{1-7x}$

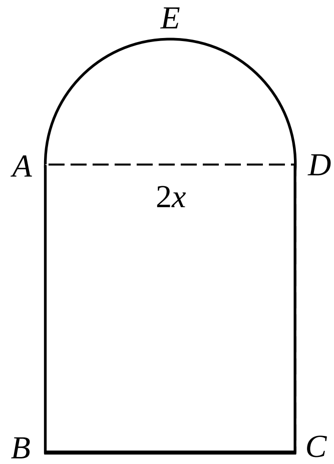
$\therefore \dfrac{dy}{dx} = -7e^{1-7x}$

(b) $\displaystyle\int_{-5}^{1} \frac{8}{(7-4x)^2}\,dx = 8\left[\frac{(7-4x)^{-1}}{(-1)(-4)}\right]_{-5}^{1}$

$= 2\left(\dfrac{1}{3} - \dfrac{1}{27}\right)$

$= \dfrac{16}{27}$

4



A window in a new building has the shape of a rectangle *ABCD* joined to a semicircle *ADE*, as shown in the diagram. It is given that $AD = 2x$ m and the total perimeter *ABCDEA* of the window is 7 m.

(i) Find the length of *AB* in terms of x. [2]

(ii) Show that the area of the window, S, is equal to $\left(7x - \dfrac{\pi + 4}{2}x^2\right)$ m^2. [3]

Use a non-calculator method to find the maximum value of S as x varies. Leave your answer in the form $\dfrac{P}{2\pi + Q}$, where P and Q are integers to be determined. [4]

Solution

(i) Let $AB = h$ m

$$\pi x + h + 2x + h = 7 \Rightarrow h = \frac{7 - (\pi + 2)x}{2}$$

(ii) Area, $S = \dfrac{1}{2}\pi x^2 + 2xh = \dfrac{1}{2}\pi x^2 + 7x - (\pi + 2)x^2$

$$= \left(7x - \frac{\pi + 4}{2}x^2\right) \text{m}^2 \quad \text{(shown)}$$

$$\frac{dS}{dx} = 7 - (\pi + 4)x$$

For stationary values, $\dfrac{dS}{dx} = 0 \Rightarrow x = \dfrac{7}{\pi + 4}$ m

$$\frac{d^2S}{dx^2} = -(\pi + 4) < 0 \text{ (maximum)}$$

Hence maximum area $= 7\left(\dfrac{7}{\pi + 4}\right) - \dfrac{\pi + 4}{2}\left(\dfrac{7}{\pi + 4}\right)^2$

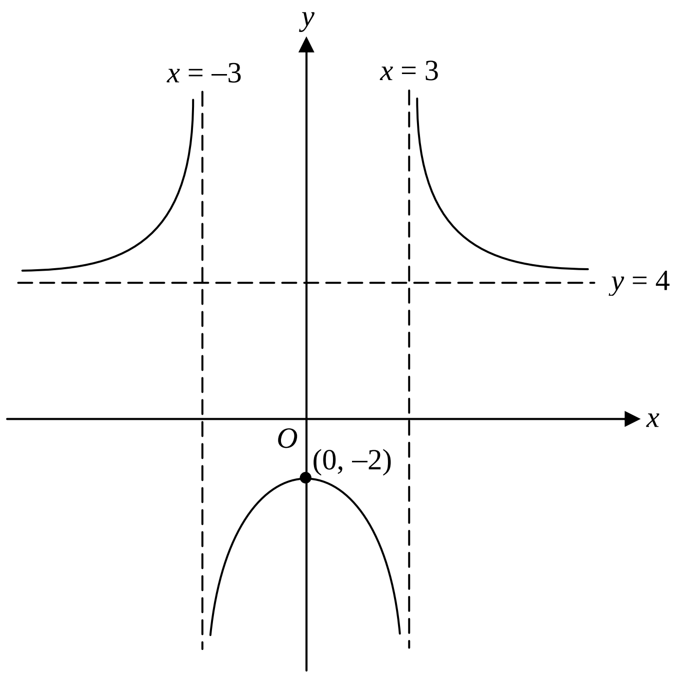
$$= \frac{49}{\pi + 4} - \frac{49}{2(\pi + 4)} = \frac{49}{2\pi + 8} \text{ m}^2$$

5 Let a, b and c be positive integers.

(a) Sketch the graph of $y = \dfrac{ax+b}{x-c}$, labelling clearly, in terms of a, b and c, the asymptotes and the intersections with the axes. [3]

Write down, in terms of a, b and c, an integral that gives the area of the region in the 3rd quadrant bounded by the curve $y = \dfrac{ax+b}{x-c}$ and the axes. [1]

(b) The graph of $y = \dfrac{ax^2+b}{x^2-c}$ is shown below. It passes through $(0, -2)$ and has asymptotes $x = -3$, $x = 3$ and $y = 4$.

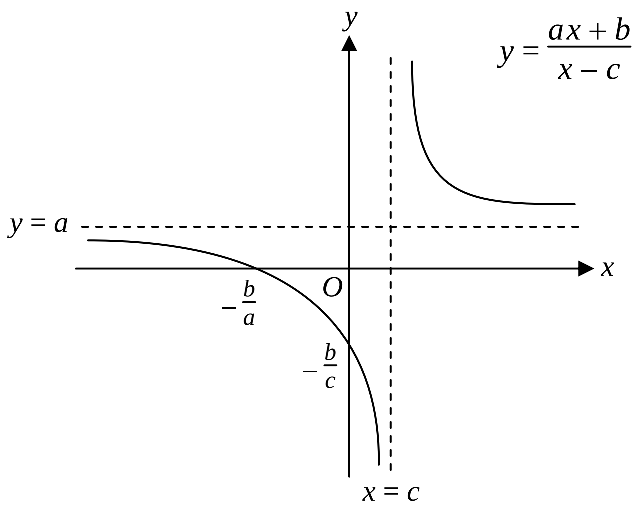


(i) Find the values of a, b and c. [3]

(ii) State the range of values of k for which the equation $\dfrac{ax^2+b}{x^2-c} = k$ has 2 distinct real roots. [2]

(iii) State the range of values of x for which $\left|\dfrac{ax^2+b}{x^2-c}\right| = \dfrac{ax^2+b}{c-x^2}$. [2]

Solution



Required area = $-\int_{\frac{-b}{a}}^{0} \dfrac{ax+b}{x-c}\,\mathrm{d}x$.

(i) Horizontal asymptote is $y = 4 \Rightarrow a = 4$

Vertical asymptotes are $x = -3$, $x = 3 \Rightarrow c = 9$

When $x = 0$, $y = -2 \Rightarrow -2 = \dfrac{b}{-9} \Rightarrow b = 18$

(ii) From the graph, $\dfrac{ax^2+b}{x^2-c} = k$ has 2 distinct real roots

$\Rightarrow k < -2$ or $k > 4$

(iii) Since $|\,\mathrm{f}(x)\,| = -\mathrm{f}(x)$ when $\mathrm{f}(x) \le 0$, $\left|\dfrac{ax^2+b}{x^2-c}\right| = \dfrac{ax^2+b}{c-x^2}$

$\Rightarrow -3 < x < 3$

Section B: Statistics [60 marks]

6 Consider events A and B, where $\mathrm{P}(A) = \frac{3}{8}$, $\mathrm{P}(B') = \frac{1}{4}$ and $\mathrm{P}(A \cup B) = \frac{7}{8}$.

(i) Write down $\mathrm{P}(B)$ and find $\mathrm{P}(A \cap B)$. [2]

(ii) Determine whether events A and B are independent. [1]

(iii) Find $\mathrm{P}(A' | B')$. [2]

Solution

$\mathrm{P}(B) = \frac{3}{4}$.

From $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$,

$\mathrm{P}(A \cap B) = \frac{3}{8} + \frac{3}{4} - \frac{7}{8} = \frac{1}{4}$.

Since $\mathrm{P}(A) \times \mathrm{P}(B) = \frac{3}{8} \times \frac{3}{4} \neq \frac{1}{4} = \mathrm{P}(A \cap B)$, events A and B are not independent.

$$\mathrm{P}(A' | B') = \frac{\mathrm{P}(A' \cap B')}{\mathrm{P}(B')} = \frac{1 - \frac{7}{8}}{\frac{1}{4}} = \frac{1}{2}.$$

7 In a particular country, public transport is organised through five transportation hubs. To regulate the transportation cost for commuters aged 21 to 60 years old, a study is to be conducted on the average amount of money spent per month on public transport. For the purpose of this study, 60 commuters from each of the 4 groups: Group I (aged 21 to 30 years old), Group II (aged 31 to 40 years old), Group III (aged 41 to 50 years old) and Group IV (aged 51 to 60 years old) were identified at a randomly chosen transportation hub, and were asked their expenditure per month on public transport.

(i) State the name given to this method of sampling. [1]

(ii) Explain, in the context of the question, an advantage and a disadvantage for this method of sampling. [2]

(iii) Suggest how you would improve the selection procedure in the context of this question. [1]

Based on the responses given by the commuters surveyed, the average expenditure for Group I and Group III are both \$100, and the average expenditure for Group II and Group IV are both \$50.

(iv) Suggest a value for the product moment correlation coefficient between the average amount of money spent and the age of the commuter. Give a reason for the choice of your value. [2]

Solution

(i) Quota sampling

(ii) Advantage: Easy to conduct

Disadvantage: Likely to be biased (because commuters were chosen only from one transportation hub).

(iii) Select commuters from all five transportation hubs.

This is because commuters from certain transportation hubs might have a lower (or higher) monthly average due to travel distance to and from town.

$r = -0.5$ (or any $-0.7 < r < 0$)

The data shows a negative trend, and the correlation is not likely to be strong.

8 Two fair dice are thrown. Write down the probability that the total score shown is more than 10. [1]

Box A contains 2 green balls and 6 blue balls. Box B contains 3 green balls and 1 blue ball. Two fair dice are thrown, and Box A is selected if the total score obtained is more than 10 or less than 4. Otherwise, Box B is selected. One ball is then chosen from the selected box and its colour noted.

Draw a tree diagram to represent this situation. [3]

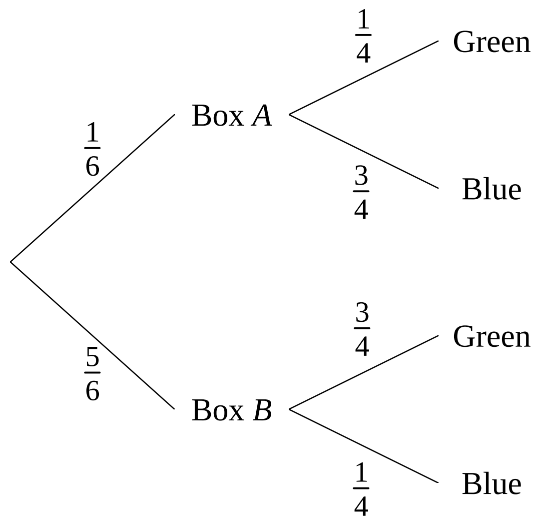
Find the probability that

(i) the ball chosen is green, [2]

(ii) the ball chosen is from Box A, given that its colour is blue. [2]

Solution

Required probability $= \dfrac{3}{36} = \dfrac{1}{12}$.



(i) $\text{P(Green)} = \dfrac{1}{6}\times\dfrac{1}{4}+\dfrac{5}{6}\times\dfrac{3}{4}=\dfrac{2}{3}$

(ii) $\text{P}(\text{Box } A \mid \text{Blue}) = \dfrac{\frac{1}{6}\times\frac{3}{4}}{\frac{1}{6}\times\frac{3}{4}+\frac{5}{6}\times\frac{1}{4}} = \dfrac{3}{8}$.

9 The test scores for Mathematics Paper 1 (x) and Paper 2 (y) for a certain class of students are given in the table below:

x	22	23	24	25	25	26	28	30	31	32	35	36	38	40	42	43
y	28	24	26	21	28	28	20	25	28	30	37	38	40	50	43	37

The full mark for each paper is 50.

(i) Draw a scatter diagram for these values. [2]

(ii) Calculate the product moment correlation coefficient between x and y. [1]

The regression line of y on x is given by $y = -0.120 + 1.010x$ (corrected to 3 decimal places) and the regression line of x on y is given by $x = a + by$.

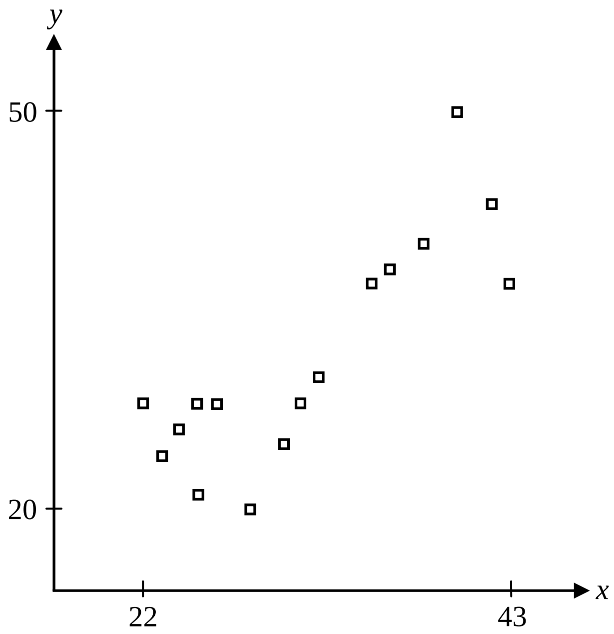
(iii) Find the values of a and b, giving your answer to 3 decimal places. [2]

(iv) One student from the class is absent for Mathematics Paper 2. Her Paper 1 test score is 50. Use an appropriate equation to estimate her Paper 2 test score. Comment on your result. [2]

(v) Another student from the class is absent for Mathematics Paper 1. His Paper 2 test score is 26. Use an appropriate equation to estimate his Paper 1 test score. Comment on your result. [2]

Solution

(i)



(ii) From GC, $r = 0.837$ (3sf)

(iii) From GC, $x = a + by$
where $a = 9.423$, $b = 0.694$ (3dp)

(iv) Using regression line of y on x, $x = 50$ gives $y = 50.4$ (3sf).

The estimate is not reliable as $x = 50$ is beyond the range of the observed data. Furthermore, the estimate of 50.4 obtained does not make sense as it is more than the full mark.

(v) Using regression line of x on y, $y = 26$ gives $x = 27.5$ (3sf).

The estimate is reliable as $y = 26$ lies in the range of the observed data. Furthermore, $r = 0.837$ indicates a strong positive linear correlation between the 2 test scores.

10 The masses, in g, of small and large packets of paperclips are modelled as having independent normal distributions with means and standard deviations as shown in the table.

	Mean mass	Standard deviation
Small packets	50	2
Large packets	100	$\sqrt{10}$

One small packet and one large packet of paperclips are chosen at random. Find the probability that

(i) the total mass of the two packets is at most 154 g, [2]

(ii) the mass of the small packet is at most 52 g and the mass of the large packet is at most 102 g, [2]

(iii) Explain briefly why the answer to **(ii)** is smaller than the answer to **(i)**. [1]

Let p be the probability that the mass of the large packet exceeds twice the mass of the small packet by at least 5 g.

(iv) Find p, correct to 4 decimal places. [3]

(v) Write down, in terms of p, the probability that the mass of the large packet and twice the mass of the small packet differ by less than 5 g. [2]

Solution

Let X and Y be the masses of each small packet and each large packet of paperclips respectively.

$$X \sim \mathrm{N}(50,\ 4),\ Y \sim \mathrm{N}(100,\ 10)$$

(i) $X + Y \sim \mathrm{N}(150,\ 14)$

Required probability = $\mathrm{P}(X + Y \le 154) = 0.857$ (3sf)

(ii) Required probability = $\mathrm{P}(X \le 52) \times \mathrm{P}(Y \le 102) = 0.620$ (3sf)

(iii) The event described in **(ii)** is a subset of the event described in **(i)**.

(iv) $2X \sim \mathrm{N}(100,\ 16)$, $Y - 2X \sim \mathrm{N}(0,\ 26)$

$p = \mathrm{P}(Y - 2X \ge 5) = 0.1634$ (4 dp)

(v) Required probability = $\mathrm{P}\left(|Y - 2X| \le 5\right)$

$= \mathrm{P}(-5 \le Y - 2X \le 5) = 1 - 2p$

11 The traffic police department from a small town is proposing to pass a new law on speeding. To justify the passing of the law, they claim that the number of speeding offences has increased significantly over the last few years.

The number of speeding offences reported per day, x, for the period June - August 2014, is as follows:

x	≤ 2	3	4	5	6	7	8	9	10
Frequency, f	0	1	6	24	38	13	6	2	2

(i) Show that the mean for this data is 6 and calculate its variance, giving your answer as a fraction in its lowest term. [2]

Long-term records showed that the number of offences reported per day has mean 5.7 and variance 1.5.

(ii) Test, at the 1 % level of significance, whether the mean number of offences reported per day has increased. State appropriate hypotheses for the test, and define any symbols used. [4]

(iii) Explain, in the context of the question, what is meant by "1 % level of significance". [1]

Speeding above 40 km/h than the stipulated speed limit is considered a serious offence. To further support the passing of the law, the traffic police department also studied the number of serious offences, y, amongst those reported daily. The figure for the same period June - August 2014 is summarised as follow:

$$\sum (y - 3.2) = 27.6, \qquad \sum (y - 3.2)^2 = 123.28.$$

Long-term records showed that the number of serious offences reported per day follows a normal distribution with a mean of 3.2.

(iv) Find, to 1 decimal place, the minimum level of significance (in %) that would provide evidence for further support in the passing of the law. [3]

Solution

Mean = $\dfrac{\sum fx}{\sum f} = \dfrac{552}{92} = 6$ (shown), variance = $\dfrac{36}{23}$ (from GC).

Let μ denote the population mean of the offences reported.

Null hypothesis, $H_0 : \mu = 5.7$
Alternative hypothesis, $H_1 : \mu > 5.7$

Perform a one-tail test at the 1% level of significance.

Under H_0, $\overline{X} \sim N\left(\mu_0, \dfrac{\sigma^2}{n}\right)$ approx by CLT,

where $\mu_0 = 5.7$, $\sigma^2 = 1.5$, $n = 92$, $\overline{x} = 6$.

Using a z-test, p-value = 0.00940 (3sf).

Since p-value = 0.00940 < 0.01, we reject H_0 at the 1 % level of significance.

Hence there is sufficient evidence at the 1 % level of significance to conclude that the mean number offences reported per day has increased.

There is a probability of 0.01 that the test indicates that the mean number of speeding offences per day has increased when it is actually 5.7.

Let μ_y denote the population mean of the serious offences.

Null hypothesis, $H_0 : \mu_y = 3.2$
Alternative hypothesis, $H_1 : \mu_y > 3.2$

Under H_0, $\overline{Y} \sim N\left(\mu_y, \dfrac{s^2}{n}\right)$ approximately where $\mu_y = 3.2$, $n = 92$,

$\overline{x} = 3.5$ and $s^2 = \dfrac{115}{91}$ (or 1.2637 to 5sf).

Using a z-test, p-value = 0.0052383 (5sf)

Hence the minimum level of significance that would provide evidence for further support in the passing of the law is 0.6 %.

12 A factory produces 2000 writing pads daily. According to the records of the factory, the probability that a writing pad produced is substandard is 0.013.

(i) Find the expected number of substandard writing pads produced in a day. [1]

(ii) Use a suitable approximation to find the probability that there are not more than 31 substandard writing pads produced in a day. [4]

The writing pads are packed into boxes of 10 for retailing purpose.

(iii) Show that the probability that a randomly chosen box contains substandard writing pads is 0.1227, correct to 4 decimal places. [2]

On a particular day, the factory received an order of 52 boxes to be delivered to a local book store.

(iv) Find the probability that at most 3 of the boxes delivered contain substandard writing pads. [2]

(v) Find the probability that the mean number of substandard writing pads delivered per box is less than 0.1. [3]

Solution

(i) Expected number $= 2000 \times 0.013 = 26$

(ii) Let X denote the number of substandard writing pads produced in a day.

$$X \sim \mathrm{B}(2000, 0.013)$$

Since $n = 2000$ is large, $np = 26 > 5$, $n(1-p) = 1974 > 5$,

$$X \sim \mathrm{N}(26, 25.662) \text{ approx}$$

$\therefore$ $\mathrm{P}(X \le 31) = \mathrm{P}(X \le 31.5) = 0.861$ (3sf) (by continuity correction)

(iii) Let Y denote the number of substandard writing pads in a box.

$$Y \sim \mathrm{B}(10, 0.013)$$

$\mathrm{P}(Y \ge 1) = 1 - \mathrm{P}(Y = 0) = 1 - 0.8773 = 0.1227$ (4dp)

(iv) Let W denote the number of boxes that contain substandard writing pads.

$$W \sim \mathrm{B}(52, 0.1227)$$

$\mathrm{P}(W \le 3) = 0.105$ (3sf)

(v) $\overline{Y} = \frac{1}{52}(Y_1 + Y_2 + Y_3 + \cdots + Y_{52}) \sim \mathrm{N}(0.13, 0.0024675)$ approx by CLT

$\mathrm{P}(\overline{Y} < 0.1) = 0.273$ (3sf)