Finding the exact area: The definite Integral

Year 11

Mathematical Methods

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Learning Objectives

By the end of the lesson I hope that you understand and can apply the following to a range of questions from the Unit 1 and 2 Mathematical Methods course.

- Understand what it means to be a definite integral
- Know how to use it to find the exact area under a curve and between two limits and the x-axis.
- Understand what it means by the fundamental theorem of calculus

Recap of past learning

In previous lessons we have looked at how to integrate/antidifferentiate polynomial functions. We can use this to find the position function of an object when given then velocity function. It's a really important skill to learn and to apply.

To integrate you simply:

1. Add one to the power
2. Divide by the new power

Integration has a far more greater application as it helps us find the areas under curves!

$y = 3x^2$

$y' = 6x^1$

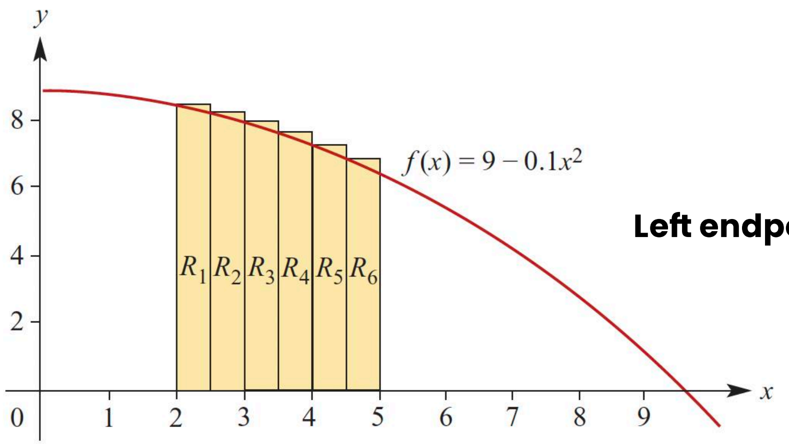
Left Endpoint estimate, Right endpoint estimate and trapezoidal estimate

In a previous lesson we have looked at how we can use the above three methods to find an **approximation** for the area under a curve between two limits and the x-axis.

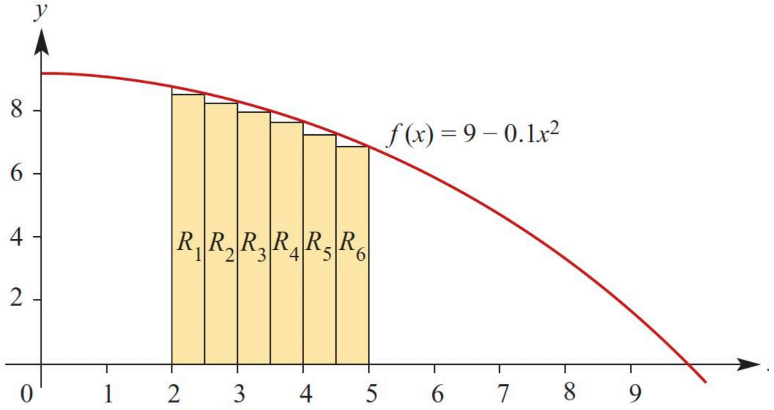
There has to be a better way right?

Surely, if we make the widths of the bars smaller and smaller we will get a much better approximation? If we make the bars infinitesimally small then we should be able to get the actual area.

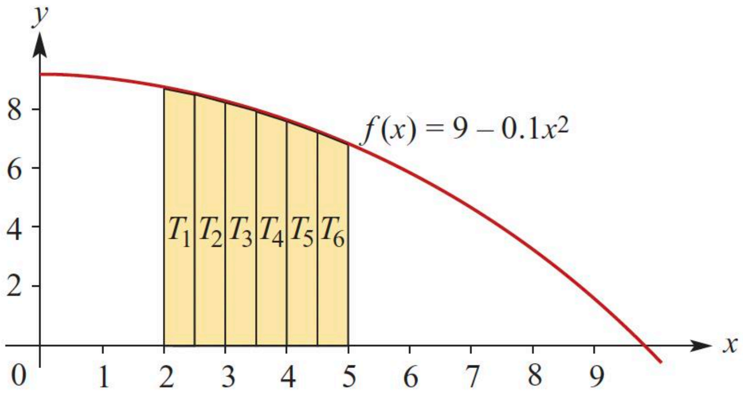
This is the theory we are looking at now!



Left endpoint estimate



Right endpoint estimate

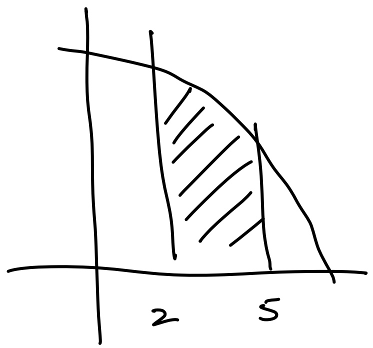


Trapezoidal estimate

Definition of the integral

We can use the following notation to find the area under a continuous function. It is called the **definite integral.**

$f(x) = 9 - 0.1x^2$



The symbol is called the "integral sign" and really means "sum"

5

$$\int_a^b f(x)\ dx$$

2

This is stating, "by considering really small widths"

This is the function we are trying to find the area under

Limits of the integral denote the start and end of the area we are trying to find

The Fundamental Theorem of Calculus

A general method to find the area under a graph $y = f(x)$ between $x = a$ and $x = b$ is given by the following important theorem.

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$$

Where F is any antiderivative of f

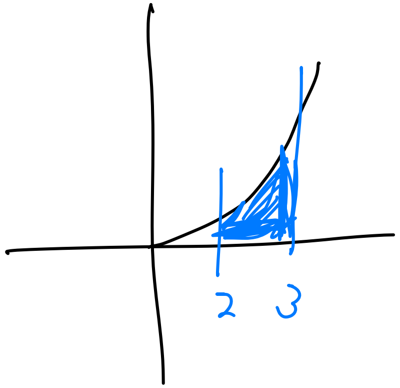
An example

a $\int_2^3 x^2\,dx$ **b** $\int_1^2 3x^3 + 2\,dx$ **c** $\int_0^1 x^{\frac{1}{2}} + x^{\frac{3}{2}}\,dx$

$$\int_2^3 x^2\,dx = \left[\frac{x^3}{3}\right]_2^3 = \left(\frac{3^3}{3}\right) - \left(\frac{2^3}{3}\right)$$

$$= \frac{27}{3} - \frac{8}{3}$$

$$= \frac{19}{3} \text{ units}^2$$



An example

a $\int_2^3 x^2\,dx$ **b** $\int_1^2 3x^3 + 2\,dx$ **c** $\int_0^1 x^{\frac{1}{2}} + x^{\frac{3}{2}}\,dx$

$2x^0$

$\frac{2x^1}{1}$

$$\int_1^2 (3x^3+2).dx = \left[\frac{3x^4}{4} + 2x\right]_1^2$$

$$= \left(\frac{3.2^4}{4} + 2.2\right) - \left(\frac{3.1^4}{4} + 2.1\right)$$

$$= \left(3.\frac{\overset{4}{\cancel{16}}}{\cancel{4}} + 4\right) - \left(\frac{3}{4} + \frac{\overset{8}{\cancel{2}}}{4}\right)$$

$$= \frac{\overset{64}{\cancel{16}}}{4} - \frac{11}{4} = \frac{53}{4}$$

An example

a $\int_2^3 x^2\,dx$ **b** $\int_1^2 3x^3 + 2\,dx$ **c** $\int_0^1 x^{\frac{1}{2}} + x^{\frac{3}{2}}\,dx$

$$\int_0^1 \left(x^{1/2} + x^{3/2}\right) dx = \left[\frac{x^{3/2}}{3/2} + \frac{x^{5/2}}{5/2}\right]_0^1$$

$$= \left[\frac{2x^{3/2}}{3} + \frac{2x^{5/2}}{5}\right]_0^1$$

$$= \left(\frac{2}{3} + \frac{2}{5}\right) = \frac{10}{15} + \frac{6}{15} = \frac{16}{15}$$

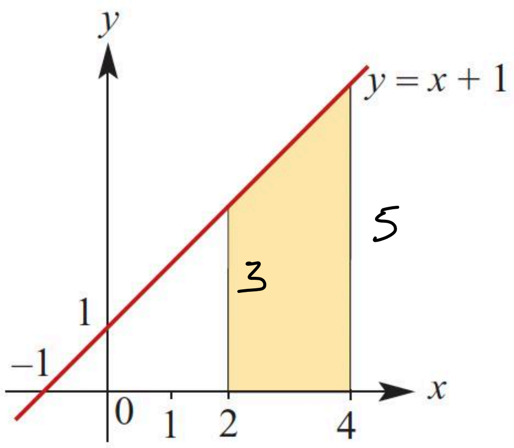
$$x^{3/2} \div \frac{3}{2}$$

$$x^{3/2} \times \frac{2}{3}$$

$$= \frac{2x^{3/2}}{3}$$

An example

Find the area of the shaded region.



$$\therefore \text{Area} = \frac{1}{2}(3+5)\cdot 2$$

$$= \underline{\underline{8}}$$

$$\int_2^4 (x+1)\,dx$$

$$= \left[\frac{x^2}{2} + x\right]_2^4$$

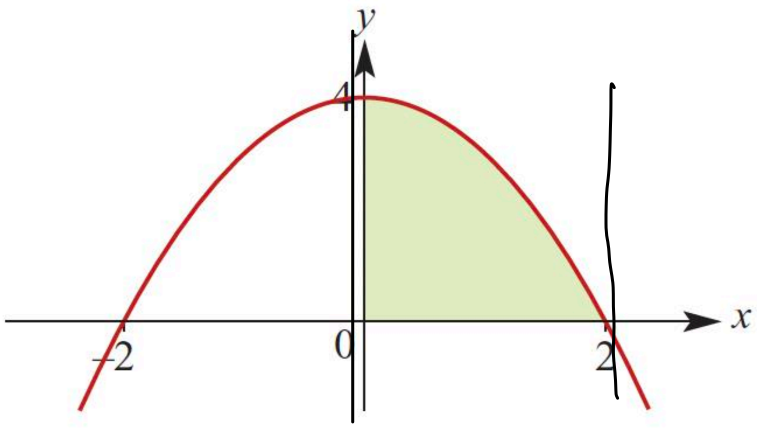
$$= \left(\frac{4^2}{2} + 4\right) - \left(\frac{2^2}{2} + 2\right)$$

$$= 12 - 4$$

$$= \underline{\underline{8}}$$

An example

Part of the graph of $y = 4 - x^2$ is shown on the right. Find the area of the shaded region.



$$\int_0^2 (4 - x^2)\,.\,dx = \left[4x - \frac{x^3}{3}\right]_0^2$$

$$= \left(4.2 - \frac{2^3}{3}\right)$$

$$= \frac{24\cancel{8}}{3} - \frac{8}{3} = \frac{16}{3}$$

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$$\int_2^3 x^2 \, dx$$

$$\int x^2 \, dx$$

$$= \frac{x^3}{3} + c$$

Questions to complete

The following are the minimum number of questions you are expected to answer. There is nothing wrong with answering more!

Ex 21B
Questions: 1acdf, 2, 3, 4, 5

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