Estimating the area under a graph

Year 11

Mathematical Methods

www.maffsguru.com

Learning Objectives

By the end of the lesson, I hope that you understand and can apply the following to a range of questions from the Unit 1 and 2 Mathematical Methods course.

- Understand how to use the following methods to estimate the area under a curve
  - Left-endpoint method
  - Right-endpoint method
  - Trapezoidal estimate

Recap of past learning

This is the first lesson relating to the work we will cover on Integration.

We have, since Year 7, been asked to find the areas of example, shapes; squares, trapezium, circles and more. For example the area of a trapezium is given by:

$$area = \frac{1}{2} \times (a + b) \times h$$

We can combine these shapes to make more complex ones but still use the same building blocks.

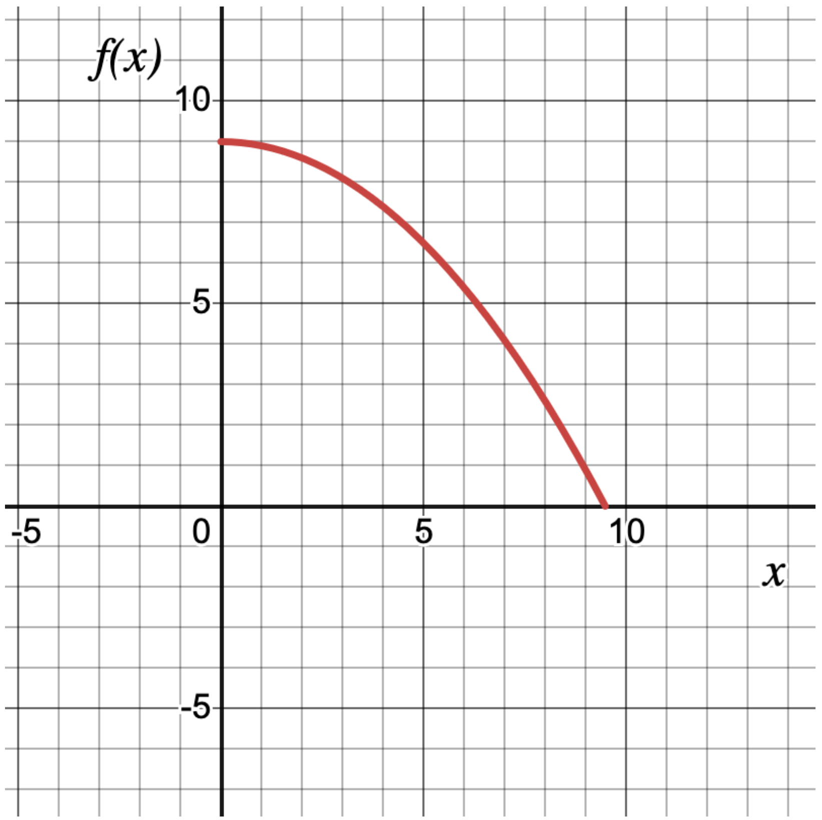
We haven't ever been asked to find the area under a curve. Until now.

A base functions to explore

For the duration of this video, we are going to look at the following function:

$$f(x) = 9 - 0.1x^2$$

As is normal with Methods, we might be best served looking at what the graph looks like

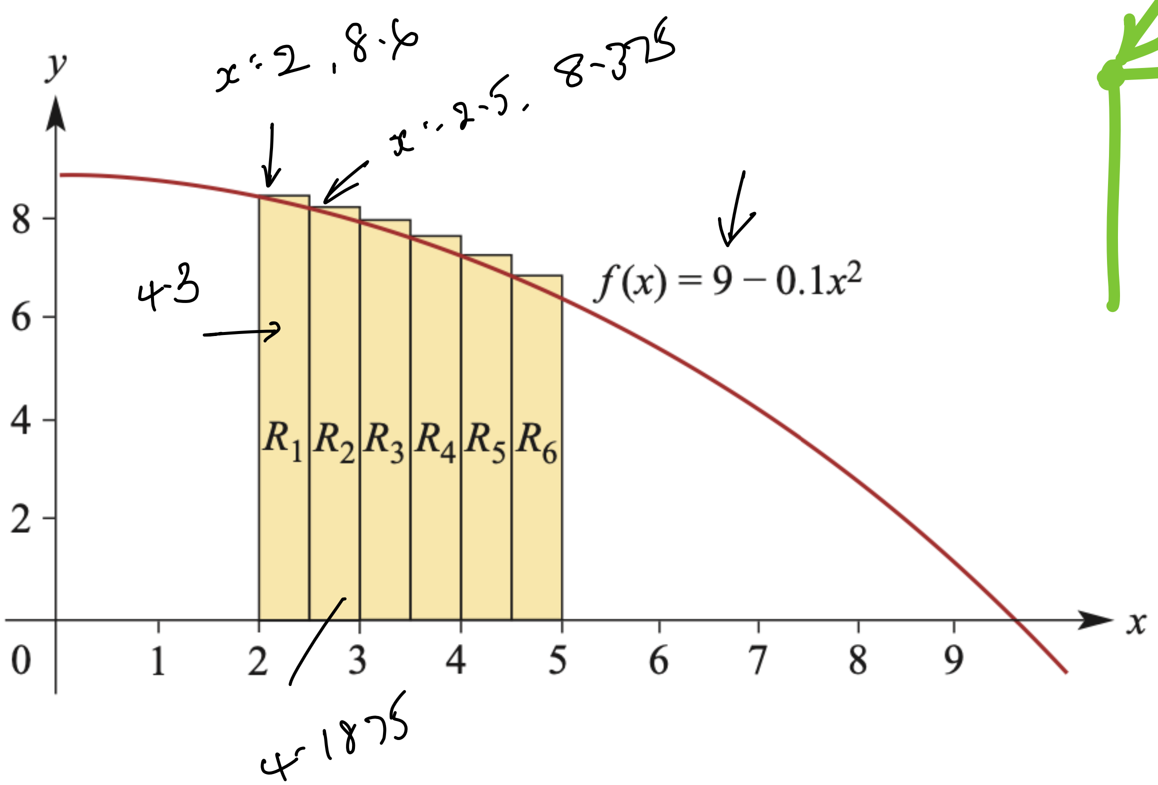


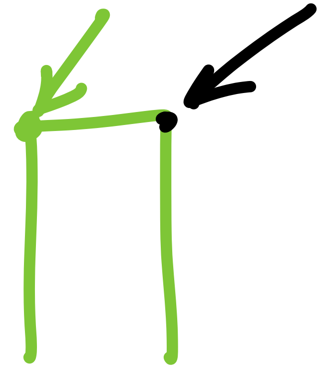
Using rectangles

The first shape we might consider using is the simple rectangle. We know how to find the area of a rectangle. All we need is the height and the width.

So, split the shape into equal width rectangles.

At the moment we will consider only the area between $x = 2$ and $x = 5$ with a bar width of 0.5.





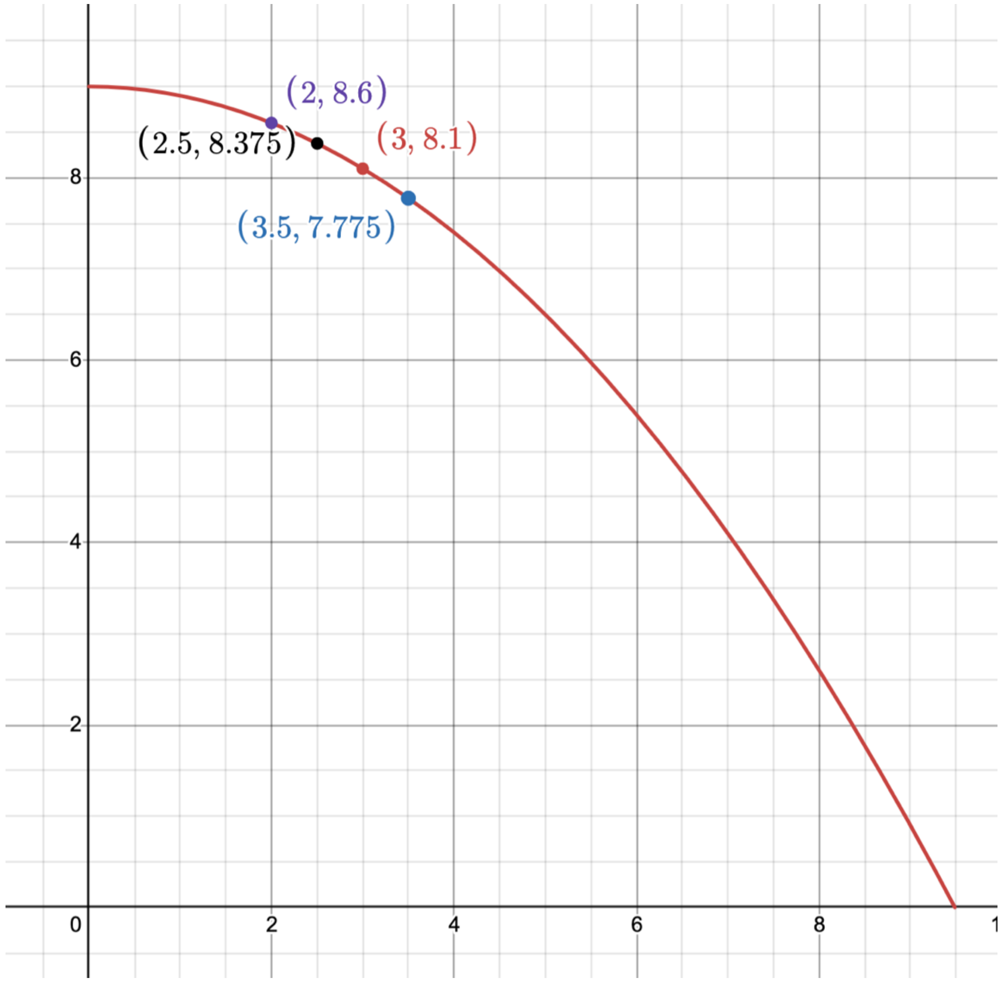
Here we can see that the left point of the bar intersects with the line. This method of finding the area if called the **left-endpoint estimate**.

Question: How do we find the height of each rectange?

The ‘y’ coordinate is the answer

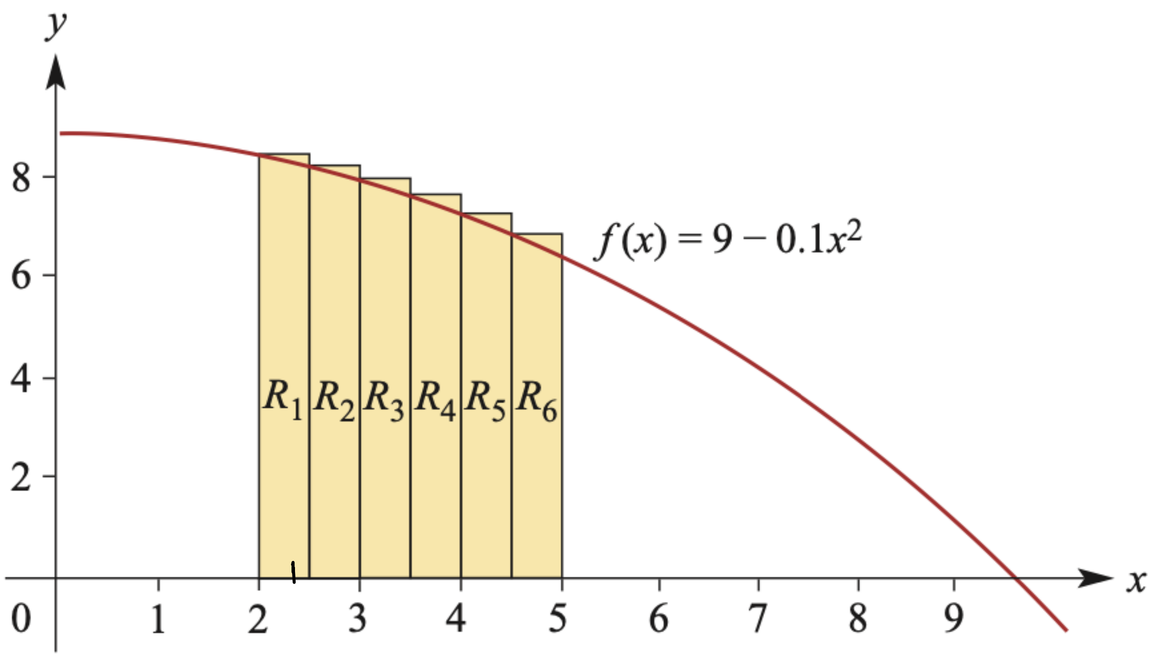
Knowing that the points on the line can be found using $f(x) = 9 - 0.1x^2$, for each x-value we can find a corresponding y-value.

This is then the height of the bar!



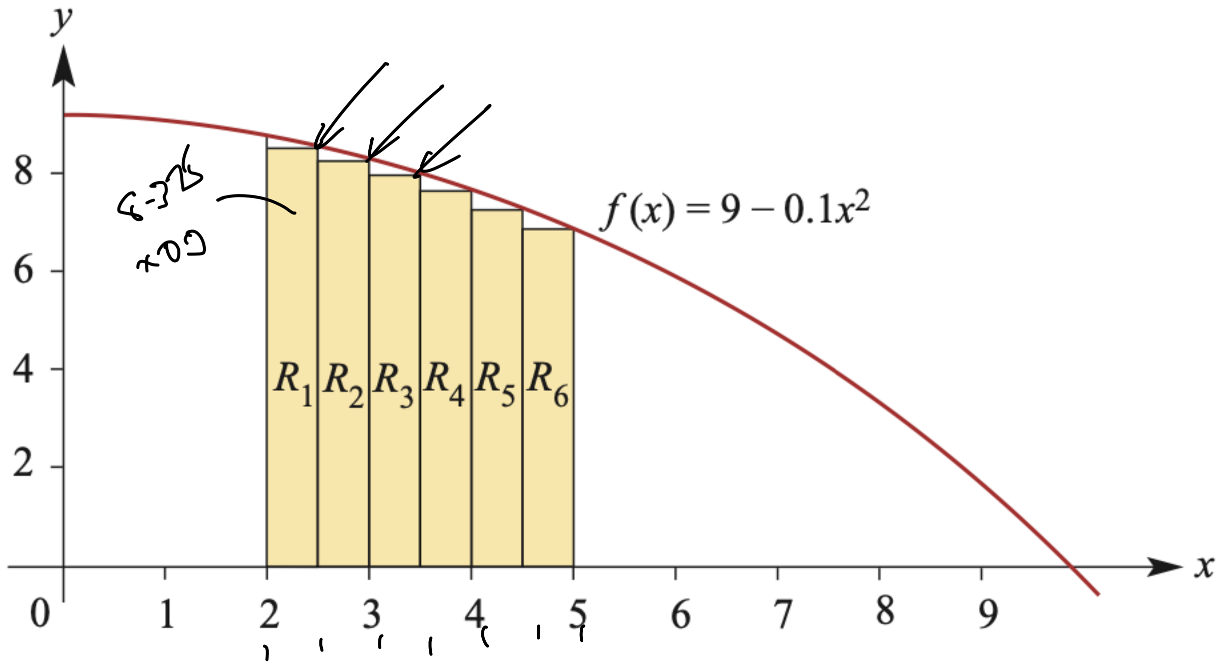
Not quite right

We can see from the diagram that any answer we get is going to be an overestimate



Right end-point estimate

Does it make a difference if we have the right point of the rectangle touching the curve?



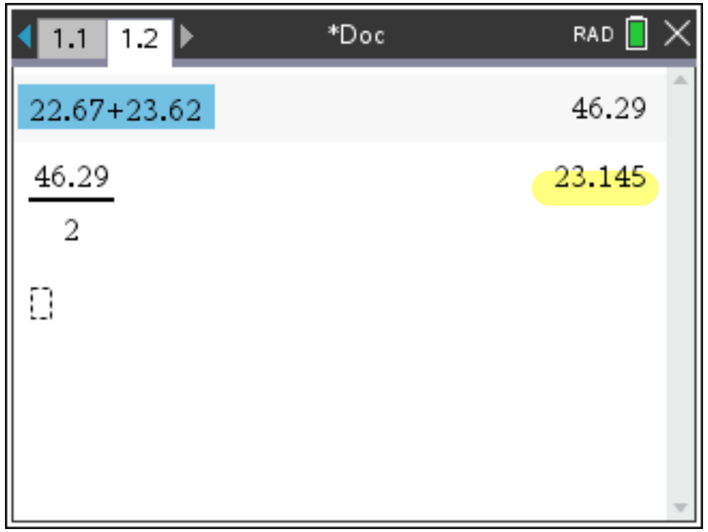
Sadly not! This time it is clear that we will have an underestimate.

Taking an average of the two areas?

There is an argument that we can take an average of the left-endpoint and right-endpoint estimates to find an even better estimate.

Let's see how close we get to the actual area of 23.1

Right-endpoint = 22.67
Left-endpoint = 23.62

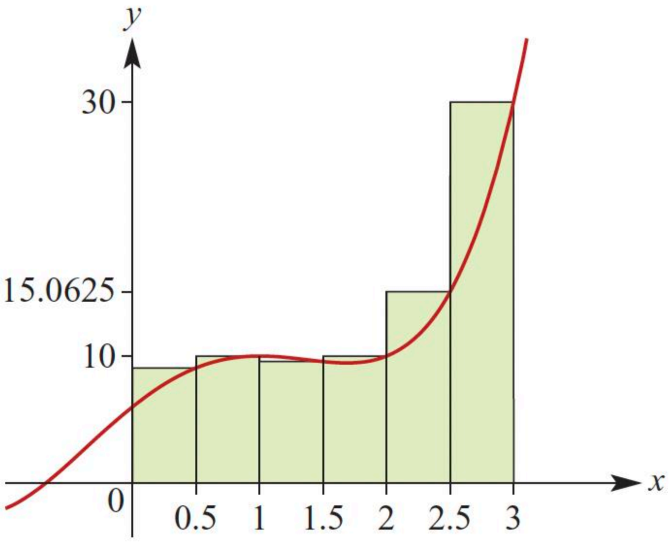


Example 1: Finding the area

Find the sum of the areas of the shaded rectangles to approximate the area under the graph of

$$f(x) = (x-2)(x+2)(x-1)^2 + 10$$

between $x = 0$ and $x = 3$.



Note: This seems to be suggesting the use of the right-endpoint estimate

$f(0.5) \times 0.5$
$+ f(1) \times 0.5$
$+ f(1.5) \times 0.5$
$+ f(2) \times 0.5$
$+ f(2.5) \times 0.5$
$+ f(3) \times 0.5$

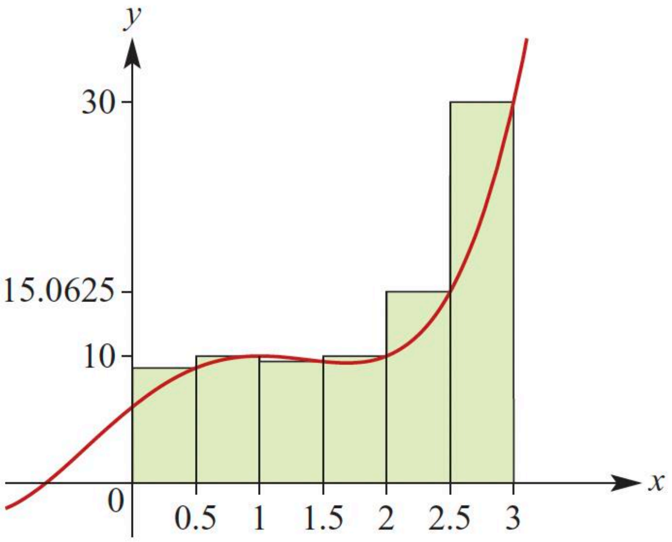
$\frac{1}{2}\left(f(0.5) + f(1) + f(1.5) \dots f(3)\right)$
$= 41.84$ units^2

Example 1: Finding the area

Find the sum of the areas of the shaded rectangles to approximate the area under the graph of

$$f(x) = (x-2)(x+2)(x-1)^2 + 10$$

between $x = 0$ and $x = 3$.



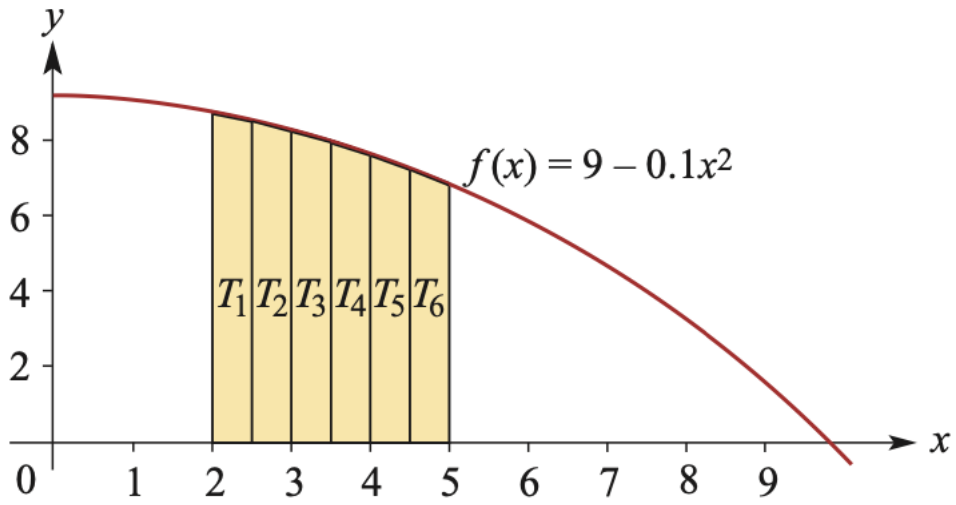
The left-endpoint estimate with the same intervals is 29.84375.
The actual area under the graph is 35.1.

Question: How do we find the actual area?

The trapezoidal estimate

What about it we use a better method?

Trapeziums would be a much better idea, wouldn't it? There is much less wasted space! It's not perfect, but it seems to be a lot better.



$$T_1 = \frac{1}{4} \times \left(f(2) + f(2.5)\right)$$

$$T_2 = \frac{1}{4} \times \left(f(2.5) + f(3)\right)$$

$$= \frac{1}{4}\left(f(2) + \underline{f(2.5) + f(2.5)} + f(3)\right)$$

$$= \frac{1}{4}\left(f(2) + \underline{2f}(2.5) + f(3)\right)$$

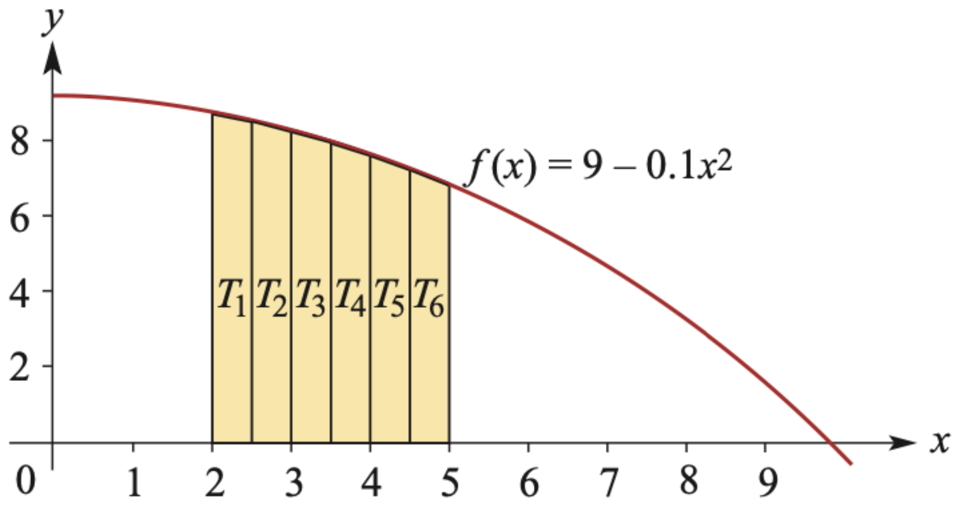
The trapezoidal estimate

What about it we use a better method?

Trapeziums would be a much better idea, wouldn't it? There is much less wasted space! It's not perfect, but it seems to be a lot better.

There is an easy way to do this … and a hard way!

Let's look at the hard way first!

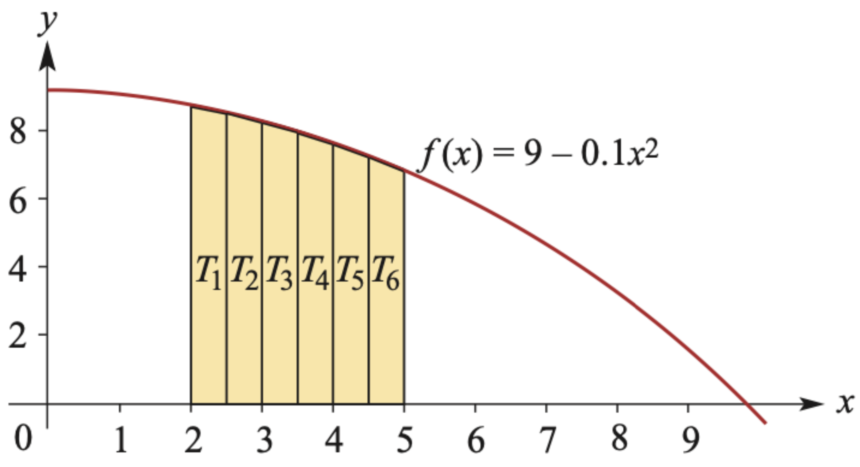


$$\therefore \frac{1}{2} \times \frac{1}{2}\Big(f(2) + 2f(2.5) + 2f(3) + 2f(3.5) + 2f(4) + 2f(4.5) + f(5)\Big)$$

$$= 23.0875$$

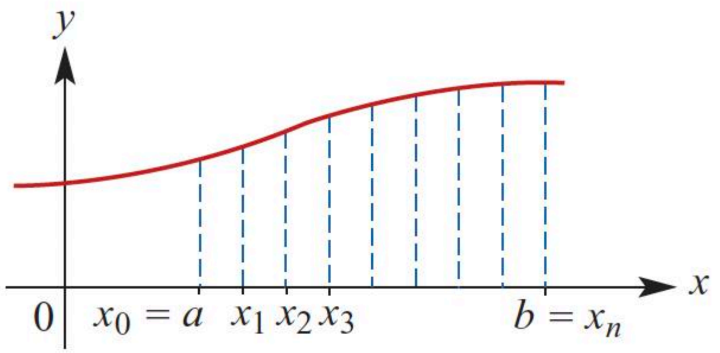
Example 2: Finding the area

Find the area of the shaded sections



A formula for those who like them!

Divide the interval $[a, b]$ on the x-axis into n equal subintervals $[x_0, x_1]$, $[x_1, x_2]$, $[x_2, x_3]$, $\ldots$, $[x_{n-1}, x_n]$ as illustrated.



Estimates for the area under the graph of $y = f(x)$ between $x = a$ and $x = b$:

- **Left-endpoint estimate**

$$L_n = \frac{b-a}{n}\left[f(x_0) + f(x_1) + \cdots + f(x_{n-1})\right]$$

- **Right-endpoint estimate**

$$R_n = \frac{b-a}{n}\left[f(x_1) + f(x_2) + \cdots + f(x_n)\right]$$

- **Trapezoidal estimate**

$$T_n = \frac{b-a}{2n}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right]$$

Learning Objectives

By the end of the lesson, I hope that you understand and can apply the following to a range of questions from the Unit 1 and 2 Mathematical Methods course.

- Understand how to use the following methods to estimate the area under a curve
  - Left-endpoint method
  - Right-endpoint method
  - Trapezoidal estimate

www.maffsguru.com