Differentiating Rational Powers

Year 11
Mathematical Methods

www.maffsguru.com

Learning Objectives

By the end of the lesson, I hope that you understand and can apply the following to a range of questions from the Unit 1 and 2 Mathematical Methods course.

- Understand how to different rational powers
- Apply the chain rule to various problems.

Examples have been extracted, with permission, from the Cambridge Mathematical Methods Units 1 and 2 Textbook

Recap of past learning

In the previous lesson we looked at how we can use the Chain Rule to greatly speed up the process of differentiating more complex functions. These are generally those which high powers.

It is clear, from years of teaching, that many students simply do not like working with fractions. Which means it becomes perfect for VCAA to use them in lots and lots of question.

We are now going to look at building on our differentiation knowledge to look at what to do when we have fractional powers.

$$f(x) = (x^2 + 4)^6$$

Recap: Differentiation by first principle

Just when you thought it was safe to go back into the water again … here comes first principles again!

$$f'(x) = \frac{f(x+h) - f(x)}{h}$$

Learn through example

Differentiate each of the following by first principles:

a $f(x) = x^{\frac{1}{2}}, \ x > 0$

b $g(x) = x^{\frac{1}{3}}, \ x \neq 0$

$$f'(x) = \frac{f(x+h) - f(x)}{h}$$

$$\text{a. } f'(x) = \frac{(x+h)^{1/2} - x^{1/2}}{h}$$

$$= \frac{\left(\sqrt{x+h} - \sqrt{x}\right)}{h} \times \frac{\left(\sqrt{x+h} + \sqrt{x}\right)}{\sqrt{x+h} + \sqrt{x}}$$

$$= \frac{\cancel{x} + \cancel{h}^{\,1} - \cancel{x}}{\cancel{h}_{1}\left(\sqrt{x+h} + \sqrt{x}\right)}$$

$$f'(x) = \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

$$f'(x) = \lim_{h \to 0} = \frac{1}{2\sqrt{x}}$$

$$f'(x) = \frac{1}{2} \cdot x^{-1/2}$$

Learn through example

Differentiate each of the following by first principles:

a $f(x) = x^{\frac{1}{2}},\ x > 0$ **b** $g(x) = x^{\frac{1}{3}},\ x \neq 0$

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

$$\left(a^{1/3}\right)^3 = a \qquad \left(b^{1/3}\right)^3 = b$$

$$\left(a^{1/3}\right)^3 - \left(b^{1/3}\right)^3 = \left(a^{1/3} - b^{1/3}\right)\left(a^{2/3} - a^{1/3}b^{1/3} + b^{2/3}\right)$$

$$a - b = \left(a^{1/3} - b^{1/3}\right)\left(a^{2/3} - a^{1/3}b^{1/3} + b^{2/3}\right)$$

$$a^{1/3} - b^{1/3} = \frac{a-b}{a^{2/3} - a^{1/3}b^{1/3} + b^{2/5}}$$

$$g'(x) = \frac{g(x+h) - g(x)}{h}$$

$$= \frac{(x+h)^{1/3} - x^{1/3}}{h}$$

$$= \frac{1}{\cancel{h}\cdot\left((x+h)^{2/3} + (x+h)^{1/3}\cdot x^{1/3} + x^{2/3}\right)}\cdot\left(\cancel{x} + \cancel{h} - \cancel{x}\right)$$

$$\lim_{h \to 0} g'(x) = \frac{1}{x^{2/3} + x^{2/3} + x^{2/3}} = \frac{1}{3x^{2/3}} = \frac{1}{3}\cdot x^{-2/3}$$

That seems like a LOT of work

Agreed!

Thankfully, the rules for rational powers are no different from those with integer powers.

Simply, multiply by the power, and subtract one from the power. Make sure you know your fractions though!

Example 1

Find the derivative of each of the following with respect to x:

a $4x^{\frac{2}{3}}$

b $x^{\frac{1}{5}} - 2x^{-3}$

a. $4x^{2/3} = 4 \cdot \frac{2}{3} \cdot x^{-1/3}$

$= \frac{8}{3} \cdot x^{-1/3}$

$= \frac{8}{3x^{1/3}}$

b. $\frac{1}{5} \cdot x^{-4/5} + 6x^{-4}$

A proof using the chain rule

$y = f(x)$ $\qquad$ $x = g(y)$

$y = f(g(y))$

$$1 = \frac{dy}{dy} = \frac{dy}{dx} \times \frac{dx}{dy}$$

$$1 = \frac{dy}{dx} \cdot \frac{dx}{dy}$$

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

This is quite interesting to watch and the technique is used is specialist mathematics.

A proof using the chain rule

This is quite interesting to watch and the technique is used is specialist mathematics.

$$y = x^{\frac{1}{n}}$$

$$y^n = \left(x^{1/n}\right)^n$$

$$y^n = x$$

$$x = y^n$$

$$\frac{dx}{dy} = n.y^{n-1}$$

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

$$= \frac{1}{n.y^{n-1}}$$

$$= \frac{1}{n.\left(x^{\frac{1}{n}}\right)^{n-1}}$$

$$= \frac{1}{n.x^{1-1/n}}$$

$$= \frac{1}{n}.x^{\frac{1}{n}-1} = \frac{1}{n.x}$$

$$\frac{1}{n}(n-1)$$

$$\frac{n-1}{n} = 1 - \frac{1}{n}$$

Learning Objectives

By the end of the lesson, I hope that you understand and can apply the following to a range of questions from the Unit 1 and 2 Mathematical Methods course.

- Understand how to different rational powers
- Apply the chain rule to various problems.

www.maffsguru.com