The Chain Rule: Differentiation

Year 11

Mathematical Methods

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Learning Objectives

By the end of the lesson, I hope that you understand and can apply the following to a range of questions from the Unit 1 and 2 Mathematical Methods course.

- Understand why we need to have something like the chain rule
- Understand what the chain rule is and when we need to use it
- Be able to apply the chain rule to a number of questions

Recap of past learning

We have, in previous lessons, looked at how we can differentiate expressions (or functions) to be able to give us a way to find the gradient of the tangent at a point. Most of the examples we used were (in hindsight) fairly trivial.

We know, to differentiate, you simply:

- Multiply the coefficient by the power
- Subtract one from the power

But what happens when we have something more complex to deal with? For example:

$$y = (x^3 + 1)^3$$

$y = \underline{(x^3+1)(x^3+1)}(x^3+1)$

$= \quad (x^3+1)$

$=$

We could expand …

Let’s make the expression a little simpler:

$$y = (x^3 + 1)^2$$

This would be pretty easy to expand. We could then differentiate each of the terms. Depending on the question, this might be enough. But what if we faced something like:

$$y = (x^3 + 1)^6$$

That seems a lot more complex!

Welcome to the chain rule

The chain rule is an amazing way to turn the complex into the trivial. It is best demonstrated using some examples. Then we can look at why.

Much of what I teach in Year 11 and 12 comes down to the same thing; **substitute out the complex thing and make the question easier.**

One thing we do need to know is …

$$\frac{dy}{dx} = \frac{dy}{\cancel{du}} \times \frac{\cancel{du}}{dx}$$

This seems somewhat intuitive and is something we have been using when we look at cross cancelling fractions.

Example 1

Find the derivative of $y = (3x + 4)^{20}$.

$y = u^{20}$ $\qquad$ $u = 3x + 4$

$\frac{dy}{du} = 20u^{19}$ $\qquad$ $\frac{du}{dx} = 3$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 20(3x+4)^{19} \cdot 3$$

$$= 60(3x+4)^{19}$$

✓

Example 2

Find the gradient of the curve with equation $y = \dfrac{16}{3x^2 + 1}$ at the point $(1, 4)$.

$$y = \frac{16}{u}$$

$$= 16u^{-1}$$

$$\frac{dy}{du} = -16u^{-2}$$

$$= \frac{-16}{u^2}$$

$$u = 3x^2 + 1$$

$$\frac{du}{dx} = 6x$$

$$\therefore \frac{dy}{dx} = \frac{-16}{(3x^2+1)^2} \times 6x$$

$$= \frac{-96x}{(3x^2+1)^2}$$

$$\therefore m = \frac{-96 \times 1}{(3 \cdot 1^2 + 1)^2}$$

$$= \frac{-96}{16}$$

$$= -6$$

Example 3

Differentiate $y = (4x^3 - 5x)^{-2}$.

$y = u^{-2}$ $\qquad$ $u = 4x^3 - 5x$

$\frac{dy}{du} = -2u^{-3}$ $\qquad$ $\frac{du}{dx} = 12x^2 - 5$

$\therefore \frac{dy}{dx} = -2(4x^3 - 5x)^{-3}(12x^2 - 5)$

Example 4

Given that $f(x) = (x^2 + 1)^3$, find $f'(x)$.

$y = u^3 \qquad u = x^2 + 1 \qquad (x^2+1)^3$

$\dfrac{dy}{du} = 3u^2 \qquad \dfrac{du}{dx} = 2x \qquad 3(x^2+1)^2 \cdot 2x$

$$f'(x) = \frac{dy}{dx} = 3(x^2+1)^2 \cdot 2x$$

$$= 6x(x^2+1)^2$$

A short cut if you will

Now! I hate short cuts, but it might have become somewhat obvious that the chain rule seems to work using the same rules as differentiation. We seem to multiply by the power and subtract one from the power. The only new thing is to then multiply by the differential of the bracket.

So … using the same example as before:

Find the derivative of $y = (3x + 4)^{20}$.

$$y = (3x + 4)^{20}$$

$$y' = 20(3x + 4)^{19} . 3$$

$$= \underline{\underline{60(3x + 4)^{19}}}$$

VCAA trying to trick us?

There is another notation I have seen time and again and it's important to note its existence.

We can think of the original question as being one of two functions:

$$f(x) = \underbrace{(x^2+1)}_{g(x)}{}^3$$

Can be thought of as:

$$f(x) = \left(\left(g(x)\right)^3, where\ g(x) = x^2 + 1\right.$$

$f(x) = g(x)^3$

$f'(x) = f'(g(x)) \cdot g'(x)$

$f(x)$

In which case, the differential would be: $f'(x) = f'\left(g(x)\right) \times g'(x)$

Note: I have seen a number of questions like this in VCAA exams

Question 16

Suppose that a differentiable function $f: R \to R$ and its derivative $f': R \to R$ satisfy $f(4) = 25$ and $f'(4) = 15$.

Determine the gradient of the tangent line to the graph of $y = \sqrt{f(x)}$ at $x = 4$.

A. $\sqrt{15}$

B. $\dfrac{1}{10}$

C. $\dfrac{15}{2}$

D. $\dfrac{3}{2}$

$(4, 25)$

$f(4) = 25$

$\to f'(4) = 15$

$\dfrac{1}{25^{1/2}} = \dfrac{1}{\sqrt{25}}$

$$y = \sqrt{f(x)}$$
$$= f(x)^{1/2}$$
$$m = y' = \frac{1}{2} \cdot f(x)^{-1/2} \cdot f'(x)$$
$$= \frac{1}{2} \cdot f(4)^{-1/2} \cdot f'(4)$$
$$= \frac{1}{2} \cdot 25^{-1/2} \cdot 15 = \frac{1}{2} \cdot \frac{1}{\cancel{5}} \cdot \cancel{15}^{\,3}$$
$$= 3/2$$

Example 5

Find the differential of the function shown below:

$$f(x) = \frac{1}{2x+3}$$

$$f(x) = (2x+3)^{-1}$$

$$f'(x) = -1(2x+3)^{-2} \cdot 2$$

$$= -2(2x+3)^{-2}$$

$$= \frac{-2}{(2x+3)^2}$$

Learning Objectives: Revisited

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