

Limits and continuity

Year 11
Mathematical Methods

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Learning Objectives

By the end of the lesson I hope that you understand and can apply the following to a range of questions from the Unit 1 and 2 Mathematical Methods course.

- Know what it means to be a limit
- Know what it means to be continuous



Recap of past learning

We have been looking at the ideas of differentiation and how we can find the gradient of the tangent to a point of a function. We've done this with positive and negative numbers. We now take a bit of a break to look at limits again.

You have already met limits notation when we started differentiation. So it's not anything new. But it does have significant meaning!



Limits

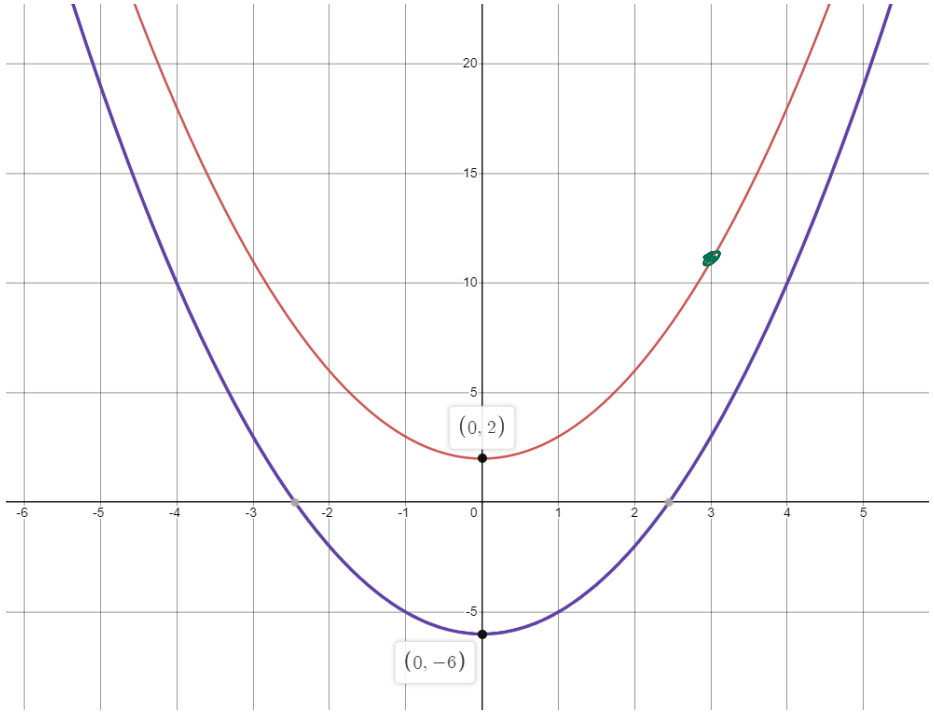
f(x)

This is a massive topic in its own right, but for the Methods course we simply need to understand what it means when we express something using limit notation.

When we look at the curves shown to the right, if we were to choose one point on either of the curves we find the value of f(x). We would hope that, if we were to take a point either side of the line and find their f(x), they would have a value either slightly larger or slightly smaller than the point we chose.

The further we move away from the point the greater or smaller the value that f(x) would be.

If we now **reverse** the process and take two points far away and get closer and closer to a chosen point, we would find that the value, either side, would get closer and closer to that of the chosen point. This is taking a limit of f(x).



$x \to 3$



Limits

So, when we have the following notation:

$$\lim_{x \to a} f(x) = k$$

$x \to a$

We are saying “as the value of x approaches a, f(x) approaches the value k”

We have used the idea of limits before when doing differentiation from first principles. We used:

$$\lim_{h \to 0} f'(x) = gradient$$

Making the value of h smaller and smaller hence taking the gradient of the secant and getting closer and closer to a given value of ‘x’.



Example: Limits

For $f(x) = \dfrac{2x^2 - 5x + 2}{x - 2}$, $x \neq 2$, find $\lim_{x \to 2} f(x)$.

4

-1	-4
-2	-2
-4	-1

$2x^2 - 5x + 2$

$2x^2 - x - 4x + 2$

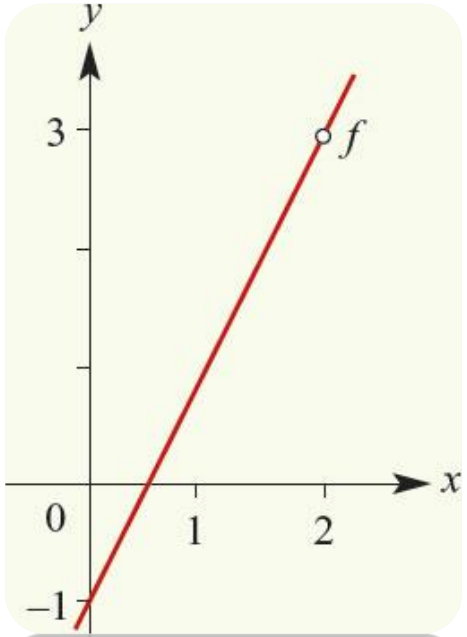
$x(2x - 1) - 2(2x - 1)$

$(2x - 1)(x - 2)$

$f(x) = \dfrac{(2x-1)(x-2)}{(x-2)}$

$f(x) = 2x - 1$

$\lim_{x \to 2} f(x) = 3$





Algebra of limits

There are a number of important results which relate to limits which might be nice in a summary book!

Sum:

$$\lim_{x\to a}\big(f(x)+g(x)\big)=\lim_{x\to a}f(x)+\lim_{x\to a}g(x)$$

Multiple:

$$\lim_{x\to a}kf(x)=k\lim_{x\to a}f(x)$$

Product:

$$\lim_{x\to a}\big(f(x)g(x)\big)=\lim_{x\to a}f(x)\lim_{x\to a}g(x)$$

Quotient:

$$\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right)=\frac{\lim_{x\to a}f(x)}{\lim_{x\to a}g(x)}$$



Examples

Find:

a $\lim_{h \to 0} (3h + 4)$ **b** $\lim_{x \to 2} 4x(x + 2)$ **c** $\lim_{x \to 3} \frac{5x + 2}{x - 2}$ **d** $\lim_{x \to 5} \frac{x^2 - 7x + 10}{x^2 - 25}$

a. $\lim_{h \to 0} (3h + 4)$

$= \lim_{h \to 0} 3h + \lim_{h \to 0} 4$

$= 0 + 4$

$= 4$

b. $\lim_{x \to 2} 4x(x + 2)$

$= \lim_{x \to 2} 4x \times \lim_{x \to 2} (x + 2)$

$= 8 \times 4$

$= 32$



Examples

Find:

a $\lim_{h \to 0} (3h + 4)$ **b** $\lim_{x \to 2} 4x(x + 2)$ **c** $\lim_{x \to 3} \frac{5x + 2}{x - 2}$ **d** $\lim_{x \to 5} \frac{x^2 - 7x + 10}{x^2 - 25}$

c. $\frac{\lim_{x \to 3} 5x + 2}{\lim_{x \to 3} x - 2}$

$= \frac{17}{1}$

$= 17$

d. $\lim_{x \to 5} \frac{(x - 2)(x - 5)}{(x + 5)(x - 5)}$

$= \frac{\lim_{x \to 5} x - 2}{\lim_{x \to 5} x + 5} = \frac{3}{10}$



Left and right limits

When we write the following notation we are implying that we will consider the limits **coming from both sides** of 'a'.

What if we only wanted to consider coming from the left or right sides only?

Well, there is a notation for that!

$$\lim_{x \to a^+} f(x) = k$$

This means that x is approaching the value of a from the **right hand side.**

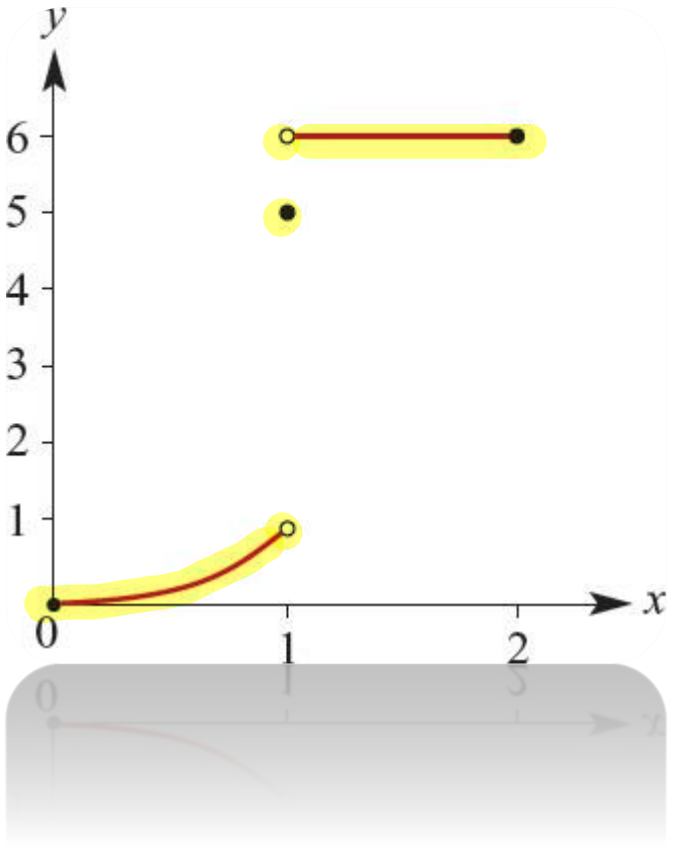
We can change the notation so that it comes from the left hand side in the following way:

$$\lim_{x \to a^-} f(x) = k$$



Example

Does the limit at $\lim_{x \to 1} f(x)$ exist?





Continuity at a point

A function with rule $f(x)$ is said to be continuous at x=a if the graph of $y = f(x)$ can be drawn through the point with coordinates $(a, f(a))$ without a break. Otherwise, there is said to be a discontinuity at $x = a$.

Writing this more formally:

A function f is continuous at the point $x = a$ if the following conditions are met:

- $f(x)$ is defined at $x = a$
- $\lim_{x \to a} f(x) = f(a)$

The function is **discontinuous** at a point if it is not continuous at that point.



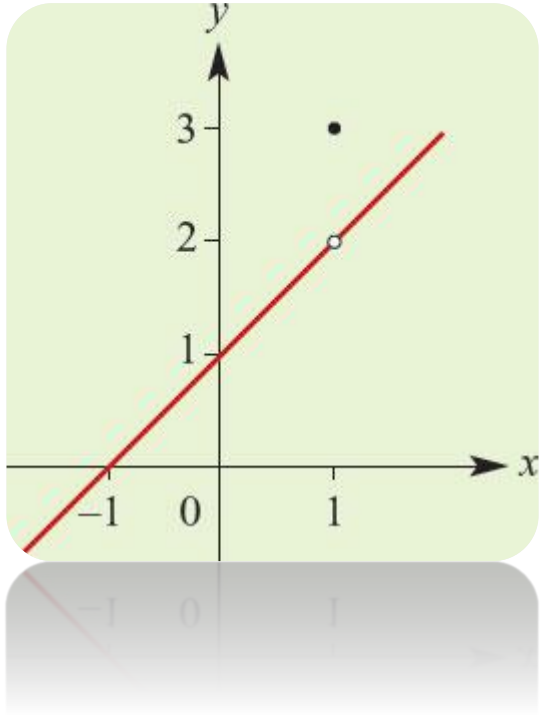
Example

State the values for x for which the function shown has a discontinuity.

Discontinuity at $x = 1$

since $f(1) = 3$

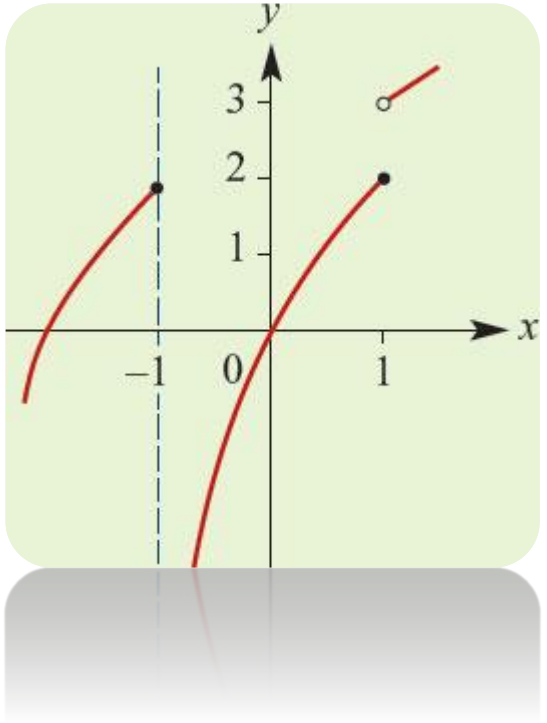
but $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x) = 2$





Example

State the values for x for which the function shown has a discontinuity.



Discontinuity at $x = 1$ as $f(1) = 2$

but $\lim_{x \to 1^-} f(x) = 2$ and

$\lim_{x \to 1^+} f(x) = 3$

Discontinuity at $x = -1$

as $f(-1) = 2$

$\lim_{x \to -1^-} f(x) = 2$ and $\lim_{x \to -1^+} f(x) = -\infty$



Example

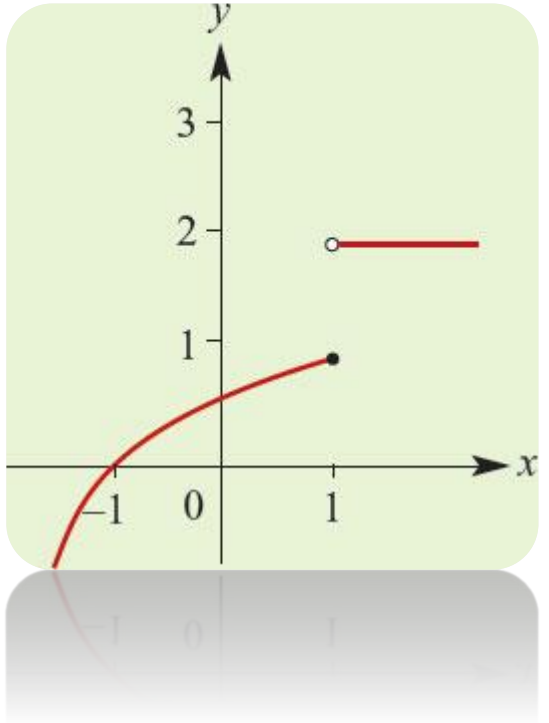
State the values for x for which the function shown has a discontinuity.

Discontinuity at $x = 1$

as $f(1) = 1$ and

$\lim_{x \to 1^-} f(x) = 1$

but $\lim_{x \to 1^+} f(x) = 2$





Example

For each function, state the values of x for which there is a discontinuity, and use the definition of continuity in terms of f(a), $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$ to explain why.

$$f(x) = \begin{cases} 2x, & x \geq 0 \\ -2x + 1, & x < 0 \end{cases}$$

$f(0) = 0$

$\lim_{x \to 0^+} f(x) = 0$

$\lim_{x \to 0^-} f(x) = -1$

$\therefore$ disc at $x = 0$



Example

For each function, state the values of x for which there is a discontinuity, and use the definition of continuity in terms of f(a), $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$ to explain why.

$$f(x) = \begin{cases} x^2, & x \geq 0 \\ -2x + 1, & x < 0 \end{cases}$$

$f(0) = 0$

$\lim_{x \to 0^+} f(x) = 0$

$\lim_{x \to 0^-} f(x) = 1$

$\therefore$ disc @ $x = 0$



Example

For each function, state the values of x for which there is a discontinuity, and use the definition of continuity in terms of f(a), $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$ to explain why.

$$f(x) = \begin{cases} x, & x \leq -1 \\ x^2, & -1 < x < 0 \\ -2x + 1, & x \geq 0 \end{cases}$$

$f(-1) = -1$

$\lim_{x \to -1^-} f(x) = -1$

$\lim_{x \to -1^+} f(x) = 1$

$\therefore$ disc $x = -1$

$f(0) = 1$

$\lim_{x \to 0^+} f(x) = 1$

$\lim_{x \to 0^-} f(x) = 0$

$\therefore$ disc $x = 0$



Example

For each function, state the values of x for which there is a discontinuity, and use the definition of continuity in terms of f(a), $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$ to explain why.

$$f(x) = \begin{cases} x^2 + 1, & x \geq 0 \\ -2x + 1, & x < 0 \end{cases}$$

$\therefore$ no discontinuity



Example

For each function, state the values of x for which there is a discontinuity, and use the definition of continuity in terms of f(a), $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$ to explain why.

$$f(x) = \begin{cases} x, & x \geq 0 \\ -2x, & x < 0 \end{cases}$$

$\therefore$ no discontinuity



Questions to complete

The following are the minimum number of questions you are expected to answer. There is nothing wrong with answering more!

Ex 17F
Questions: TBA





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