Graphs of the derivative function

Year 11
Mathematical Methods

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Learning Objectives

By the end of the lesson I hope that you understand and can apply the following to a range of questions from the Unit 1 and 2 Mathematical Methods course.

- Understand what is meant by the sign of the derivative
- Understand what is meant by strictly increasing and strictly decreasing in an interval.
- Understand what is meant by a derivative test
- Finding the angle associated with the gradient of a curve at a point

Recap of past learning

In the previous lesson we looked at the how to find the derivative of functions raised to a power (both positive and negative). We move ever onwards to build the knowledge of why we might find this stuff interesting.

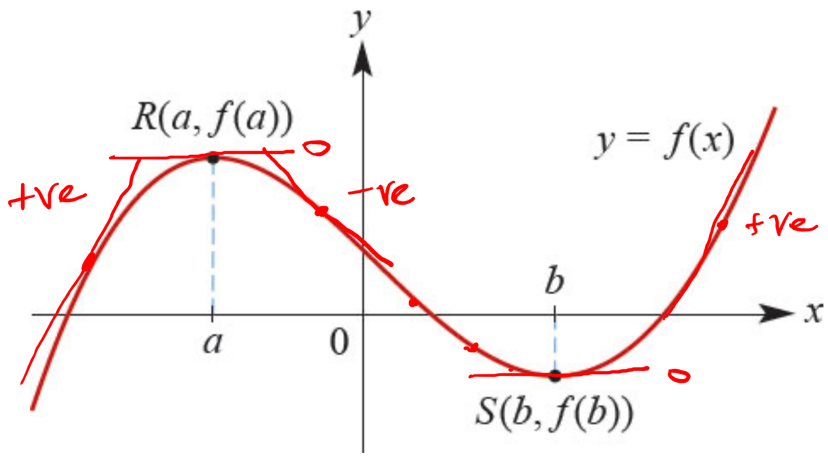
The sign of the derivative

When we look at a function we can see, from the shape, where the gradient might be positive or negative.

It is also going to be very important to know where there is a zero gradient.

There are many examples of exam questions where they use mathematical language to confuse students.

For example: $f'(x) > 0$



$f'(x) > \underbrace{0}_{\text{positive}}$

Example

As a point in fact, look at the following:

For the graph of $f\colon \mathbb{R} \to \mathbb{R}$, find:

$\{x\colon f'(x) > 0\}$

$\{x\colon f'(x) < 0\}$

$\{x\colon f'(x) = 0\}$

$\{x : f'(x) > 0\}$

positive

$x \in (-1, 5)$

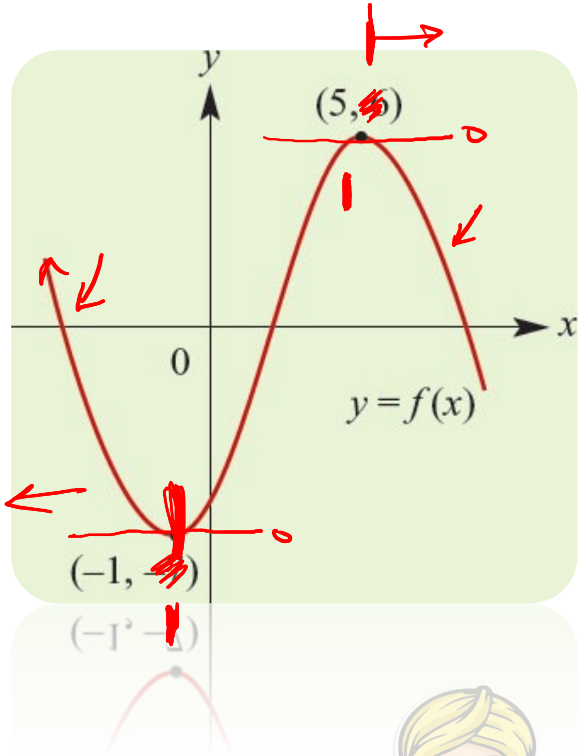
$f'(x) < 0$

neg

$x \in (-\infty, -1) \cup (5, \infty)$

$f'(x) = 0$

$x \in \{-1, 5\}$



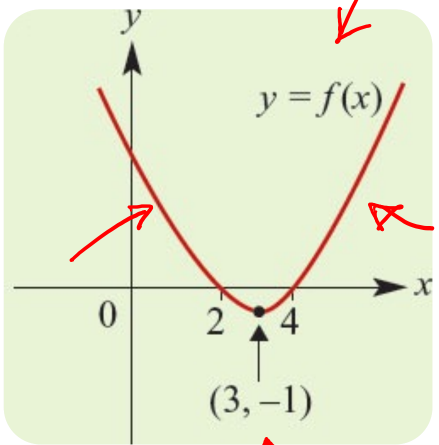
Example

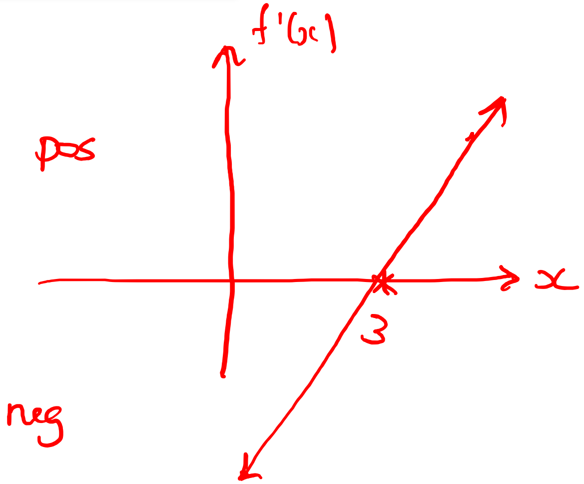
Sketch the graph of $y = f'(x)$ for each of the following.

Note: (It is impossible to determine all features.)

$y = x^2$

$y' = 2x$





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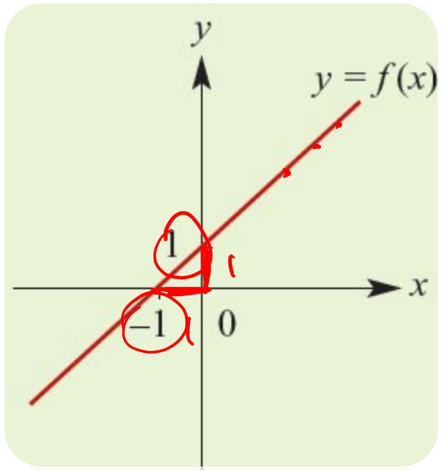
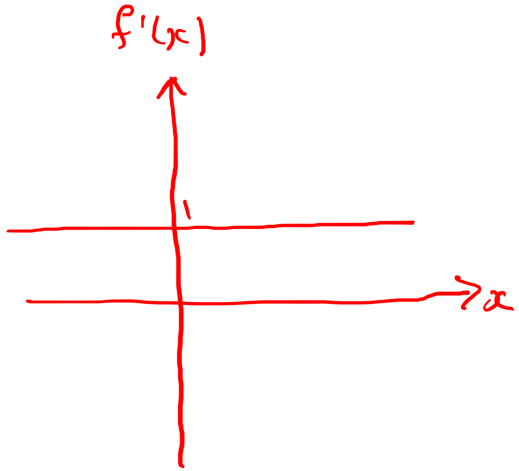


$m = \frac{1}{1} = 1$



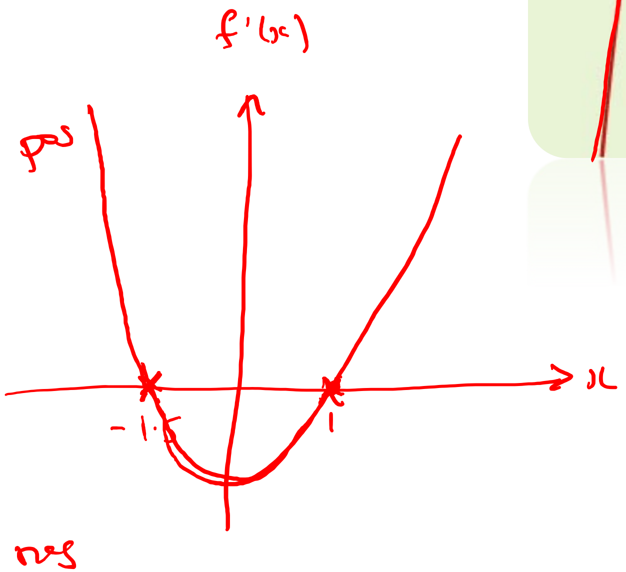
Example

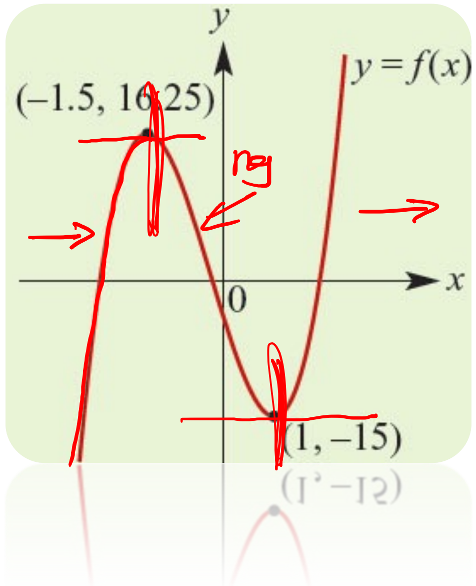
Sketch the graph of $y = f'(x)$ for each of the following.

Note: (It is impossible to determine all features.)

$y = x^3$

$y' = 3x^2$



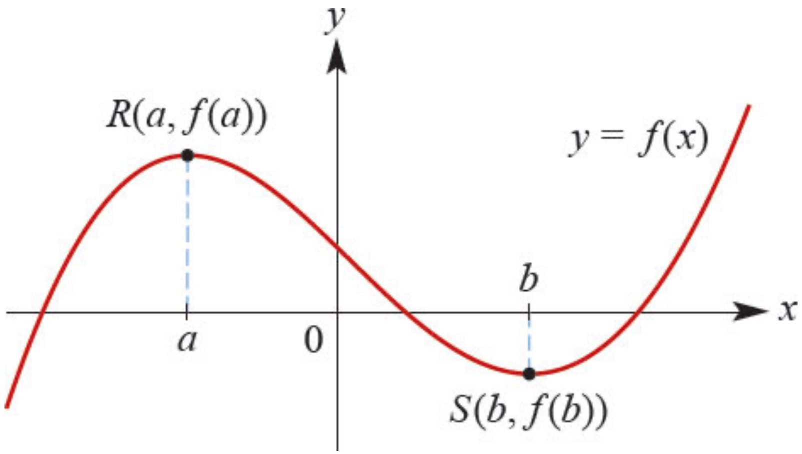


Strictly increasing and decreasing

We can describe sections of the graph as being strictly increasing or strictly decreasing. The way we decide is by using the following:

Strictly increasing: When $f(a) < f(b)$ over the interval [a, b]

Strictly decreasing when $f(a) > f(b)$ over the interval [a, b]



$f(a) < f(b)$

$x = a$ $\quad$ $x = b$

$f(a) = y_a$ $\quad$ $f(b) = y_b$

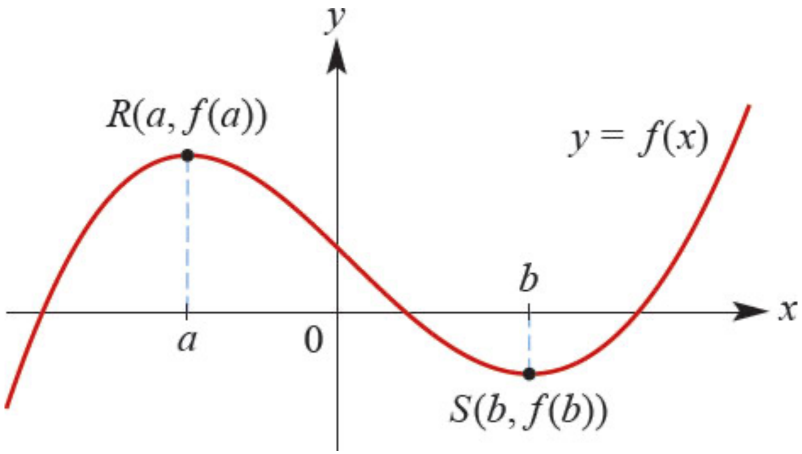
$f(a) > f(b)$

Strictly increasing and decreasing

We can sometimes just check the sign of the derivative as this may also tell us whether something is strictly increasing or strictly decreasing.

Strictly increasing for all x in an interval: $f'(x) > 0$

Strictly decreasing for all x in an interval: $f'(x) < 0$



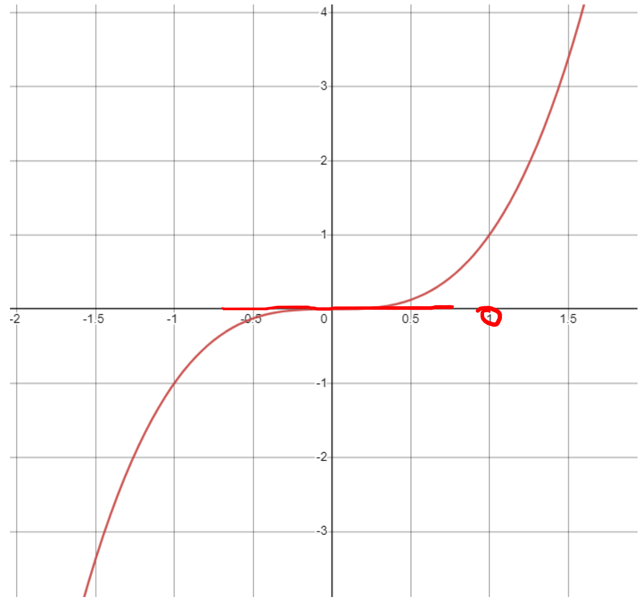
$f'(x) > 0$

Strictly increasing and decreasing: Warning

If we look at the graph of $y = x^3$ we can see that the below doesn't always work! And so, with Methods, we need to be careful that there are exceptions and to understand the work and not regurgitate.

Strictly increasing for all x in an interval: $f'(x) > 0$

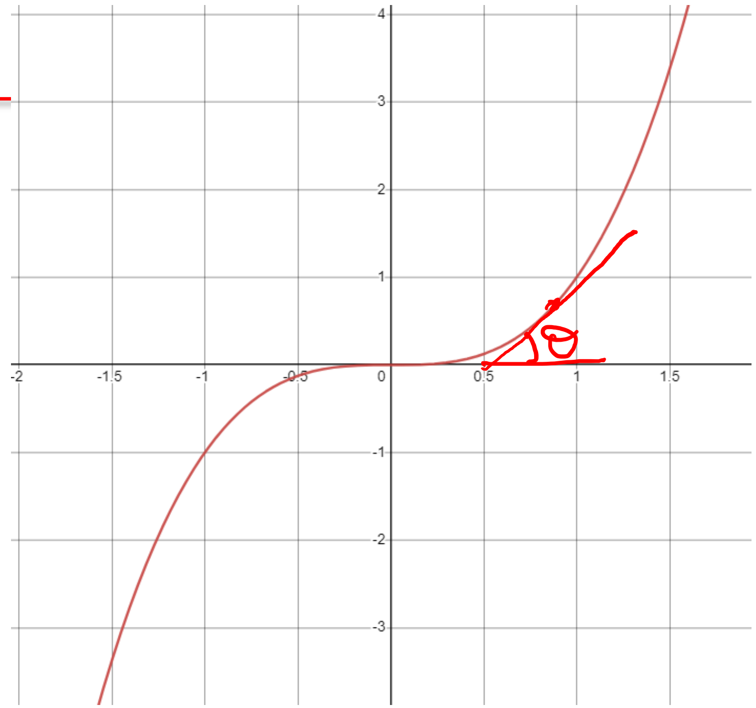
With this graph $f'(0) = 0$



$y = x^3$

$f'(x) > 0$

Angle of the gradient of a curve at a point



This is one of those things which takes 10 seconds to teach and 5 seconds to forget and, yet, seems to come up in the exam over and over again!

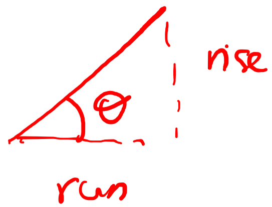
Be warned.

We know that we can find the value of the gradient at a point.

But we also know that gradient is rise over run.

So, rise over run must be equal to $\tan(\theta)$

Therefore $\boldsymbol{m = tan(\theta)}$ **… this can also be called the direction of motion.**



$$\boxed{\tan\theta = \frac{rise}{run}}$$

$$\boxed{m = \tan\theta}$$

$$m = \tan\theta$$

$$\theta = \tan^{-1}(m)$$

Example

Find the coordinates of the points on the curve with equation $y = x^2 - 7x + 8$ at which the tangent line:

- makes an angle of 45° with the positive direction of the x-axis
- is parallel to the line $y = -2x + 6$.

$y = x^2 - 7x + 8$

$\boxed{\frac{dy}{dx} = 2x - 7}$

$\theta = 45^\circ$

$m = \tan\theta$

$m = \tan 45$

$m = 1$

$1 = 2x - 7$

$2x = 8$

$x = 4$

$y = 16 - 28 + 8$

$= -4$

$(4, -4)$

$y = -2x + 6$

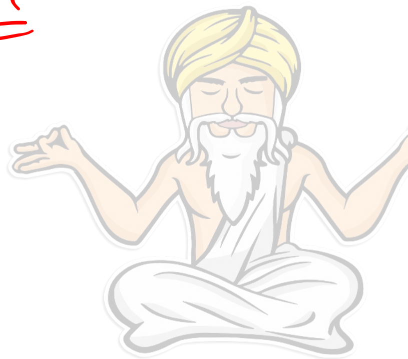
$m = -2$

$2x - 7 = -2$

$2x = 5$

$x = 5/2$

$\left(\frac{5}{2}, \frac{-13}{4}\right)$



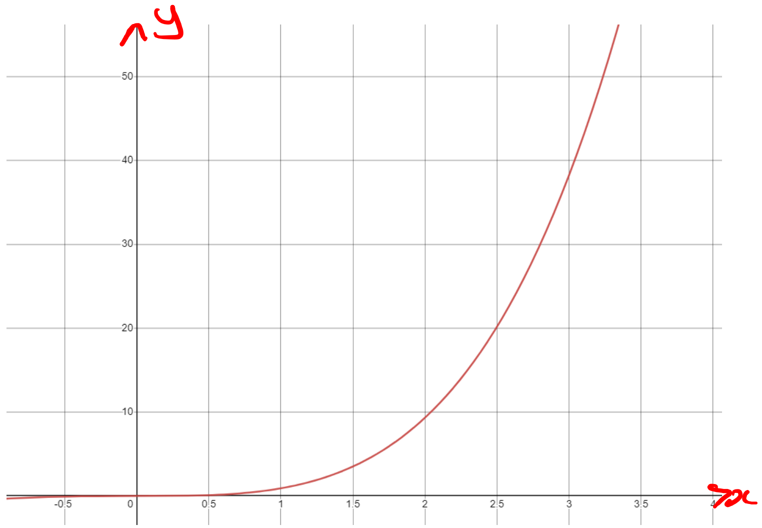
Example

The planned path for a flying saucer leaving a planet is defined by the equation

$$y = \frac{1}{4}x^4 + \frac{2}{3}x^3 for\ x > 0$$

The units are kilometres. (The x-axis is horizontal and the y-axis vertical.)

a. Find the direction of motion when the x-value is:
   i. 2
   ii. 3
b. Find a point on the flying saucer's path where the path is inclined at 45∘ to the positive x-axis (i.e. where the gradient of the path is 1).
c. Are there any other points on the path which satisfy the situation described in part b?



$y = \frac{1}{4}x^4 + \frac{2}{3}x^3$

$y' = x^3 + 2x^2$

$x = 2$ $\quad y' = m = 8 + 2(2)^2 = \underline{\underline{16}}$

$x = 3$ $\quad y' = m = 27 + 2(3)^2 = 45$

$\tan\theta = m$

$\theta = \tan^{-1}(16)$

$\theta = \underline{\underline{86.4^\circ}}$

$\theta = \tan^{-1}(45)$

$= 88.7^\circ$

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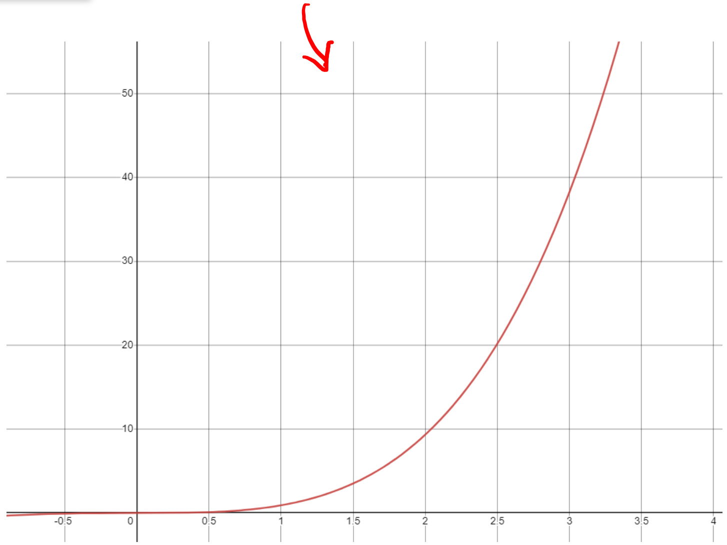

$m = \tan\theta$

$m = \tan 45$

$= 1$

$y' = x^3 + 2x^2$

$1 = x^3 + 2x^2$

$x = \underline{\underline{0.62}}$