Differentiating x^n when n is a negative integer

Year 11
Mathematical Methods

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Learning Objectives

By the end of the lesson I hope that you understand and can apply the following to a range of questions from the Unit 1 and 2 Mathematical Methods course.

- Understand how to differentiate when the power is negative

Recap of past learning

In the previous lesson we looked at how we could differentiate when we had positive powers. The "short cut" basically told us to multiply by the power and subtract one from the power.

i.e. $f(x) = x^n, f'(x) = nx^{n-1}$ *where* $n = 1, 2, 3\ \ldots$

Does anything change when we have a negative power?

Let's take a look

$y = x^3$

$y' = \frac{dy}{dx} = f'(x) = \frac{d(x^3)}{dx} = \underline{\underline{3x^2}}$

Negative powers: What are they?

A negative power is an amazing thing.

When I was at school I was taught that it was a way of expressing that a term has moved from the denominator to the numerator (or vice versa).

So:

$$\frac{1}{x^2} = x^{-2}$$

x^{-2}

$\boxed{\frac{1}{x^2}}$

Likewise:

$$\frac{y^{-3}}{x^{-2}} = \frac{x^2}{y^3}$$

The derivative of x^{-n}

Let's look at using first principles to see if there is a more general pattern (or rule) we can use.

Let $f: \mathbb{R} \setminus \{0\} \to \mathbb{R}, \left\{f(x) = \frac{1}{x}\right\}$. Find $f'(x)$ by first principles.

$$f(x) = \frac{1}{x}$$

$$f(x+h) = \frac{1}{x+h}$$

$$f(x) = \frac{1}{x} = x^{-1}$$

$$f'(x) = \frac{f(x+h) - f(x)}{h}$$

$$= \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

$$= \frac{\frac{x - (x+h)}{x(x+h)}}{\frac{h}{1}}$$

$$f'(x) = \frac{\cancel{x} - \cancel{x} - h}{x(x+h)} \times \frac{1}{h}$$

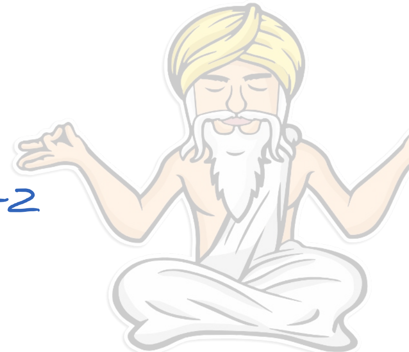
$$= \frac{-\cancel{h}}{x(x+h)} \times \frac{1}{\cancel{h}}$$

$$f'(x) = \frac{-1}{x(x+h)}$$

$$f'(x) = \frac{-1}{x^2}$$

$\lim h \to 0$

$$= -1.x^{-2}$$



The derivative of x^{-n}

Another example:

Let $f: \mathbb{R} \setminus \{0\} \to \mathbb{R}\ f(x) = x^{-3}$. Find $f'(x)$ by first principles.

$$f(x) = \frac{1}{x^3}$$

$$= x^{-3}$$

$$f'(x) = \frac{\frac{1}{(x+h)^3} - \frac{1}{x^3}}{h}$$

$$= \frac{\frac{x^3 - (x+h)^3}{(x+h)^3 \cdot x^3}}{\frac{h}{1}}$$

$$= \frac{x^3 - (x+h)^3}{(x+h)^3 \cdot x^3} \times \frac{1}{h}$$

$$= \frac{x^3 - (x^3 + 3x^2h + 3xh^2 + h^3)}{(x+h)^3\, x^3} \times \frac{1}{h}$$

$$= \frac{\cancel{x^3} - \cancel{x^3} - 3x^2\cancel{h} - 3xh^{\cancel{2}} - h^{\cancel{3}\,2}}{(x+h)^3 \cdot x^3} \cdot \frac{1}{\cancel{h}}$$

$$f'(x) = \frac{-3x^2 - \cancel{3xh} - \cancel{h^2}}{(x+h)^3\, x^3} \qquad -3x^{-4}$$

$$f'(x) = \frac{-3\cancel{x^2}}{x^{\cancel{6}\,4}}$$

$$= -3/x^4$$

Refining our previous rule for differentiation

We already had the following for positive values of n.

$$f(x) = x^n, f'(x) = nx^{n-1} \text{ where } n = 1, 2, 3 \ \ldots$$

It appears we now can refine this for positive and negative values for 'n' to be the following:

$$f(x) = x^n, f'(x) = nx^{n-1} \text{ where } n \text{ is a non} - \text{zero integer}$$

$$y = \frac{1}{x^2} = x^{-2}$$

$$y' = -2x^{-3} = \frac{-2}{x^3}$$

Example

Find the derivative f′ of $f: \mathbb{R} \setminus \{0\} \to \mathbb{R}, f(x) = 3x^2 - 6x^{-2} + 1$.

$$f(x) = 3x^2 - 6x^{-2} + 1$$

$$f'(x) = 6x + 12x^{-3}$$

$$= 6x + \frac{12}{x^3}$$

Example

Find the gradient of the tangent to the curve determined by the function $f: R \setminus \{0\} \to R, f(x) = x^2 + \frac{1}{x}$ at the point $(1,2)$.

$(1,2)$

x, y

$x = 1$

$$f(x) = x^2 + \frac{1}{x}$$

$$= x^2 + x^{-1}$$

$$f'(x) = 2x - x^{-2}$$

$$= \underline{\underline{2x - \frac{1}{x^2}}}$$

$$f'(1) = 2(1) - \frac{1}{(1)^2}$$

$$= 2 - 1$$

$$= \underline{\underline{1}}$$

Example

Show that the derivative of the function $f: \mathbb{R} \setminus \{0\} \to \mathbb{R}, f(x) = x^{-3}$ is always negative.

$$f(x) = x^{-3}$$

$$f'(x) = -3x^{-4}$$

$$= \underline{\underline{\frac{-3}{x^4}}}$$

x^4 is always positive

-3 constant

neg $\div$ pos $=$ neg