Rules for differentiation

Year 11
Mathematical Methods

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Learning Objectives

By the end of the lesson I hope that you understand and can apply the following to a range of questions from the Unit 1 and 2 Mathematical Methods course.

- Understand that, using the learning from the previous section, there are easier ways to find the equation of gradient of the tangent at a point.
- Know how to use the rules to find differentials of polynomials
- Know how to use the CAS to find differentials
- Know how to find the value of the gradient of a tangent to a point on a curve.
- Understand there are alternative notations which can be used (**thanks Barry!**)

Recap of past learning

In the previous lesson we looked at how we can find the gradient of a secant of a curve. As the horizontal difference between the two points became smaller, we noted that the secant got closer and closer point. This led us to the idea of **differentiation from first principles**.

$$f'(a) \approx \frac{f(a+h)-f(a)}{h}$$

We might also have noticed that there was a pattern between the differential and the original expression.

This lesson looks at formalising the pattern which will make life a LOT easier!

Looking for patterns

When we compare the original equations with the differential, we might notice that there is a bit of a pattern.

Original Equation: $y = x^2 - 2x$
Differential: $f'(x) = 2x - 2$

Original Equation: $y = x^3 + x$
Differential: $f'(x) = 3x^2 + 1$

The more of these we do, the more we can see that what appears to be happening is:

|| **Multiply each term by its power and then subtract one from the power.**

This turns out to be the "rule" for differentiation.

x^{1}

$1x^{\cancel{0}}$

$y = x^2 - 2x$

$f'(x) = 2x - 2$

$y = x^3 + x$

$f'(x) = 3x^2 + 1$

$x^{\boxed{2}}$

$\boxed{2x^1}$

$-\boxed{2}x^{\boxed{1}}$

$-2x^{\cancel{0}}$

The derivative of x^n

First off, let's make sure what we know this applies when n is **positive**.
We will look at negative numbers a little later!

For $f(x) = x^n$, $f'(x) = nx^{n-1}$, where $n = 1,2,3, \dots$

$$y = x^n$$

$$f'(x) = n.x^{n-1}$$

Derivative of polynomial functions

There are a number of important properties we need to know and use for differentiation and so let’s work through them now.

A constant function.

A constant will differentiate to zero.

$$y = 3x^0$$

$$f'(x) = \cancel{0 \cdot x^{-1}}$$

$$= 0$$

$$y = 10x^0$$

$$f'(x) = \cancel{0 \cdot x^{-1}} \; 0$$

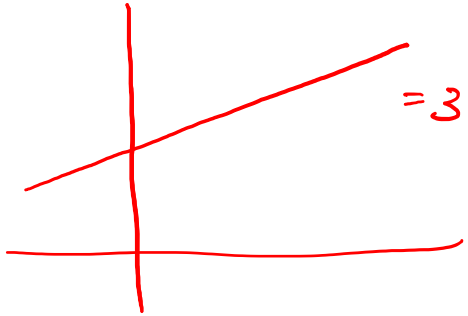
Derivative of polynomial functions

There are a number of important properties we need to know and use for differentiation and so let's work through them now.

A linear function.

When we have an equation of the form $y = mx + c$ *we note that the differential is m. Remember, when we are differentiating we are finding the gradient of the tangent to a point on the line or curve.*

Straight lines have the same gradient throughout.



$y = 3x + 2$

$y = 3x^{1} + \cancel{2x^{0}}$

$f'(x) = 3\cancel{x^{0}} + 0$

$= \underline{\underline{3}}$

Derivative of polynomial functions

There are a number of important properties we need to know and use for differentiation and so let's work through them now.

A multiple of function.

A multiple of a function has the same multiple of the derivative.

$$y = 3\boxed{x^2}$$

$$f'(x) = 3 \times 2x^1$$

$$= 3 \times (2x)$$

$$= \underline{\underline{6x}}$$

Derivative of polynomial functions

There are a number of important properties we need to know and use for differentiation and so let's work through them now.

A sum and difference of function.

The derivative of the sum of a function is the sum of the derivates (and the same is true of differences).

$$y = 2x^2 + x^1$$

$$f'(x) = 4x + 1$$

$$y = 3x^3 - 2x^2$$

$$f'(x) = 9x^2 - 4x$$

Example

Find the derivative of $x^5 - 2x^3 + 2$, i.e. differentiate $x^5 - 2x^3 + 2$ with respect to x.

$$y = x^5 - 2x^3 + 2$$

$$f'(x) = \underline{5x^4 - 6x^2}$$

Example

Find the derivative of $f(x) = 3x^3 - 6x^2 + 1$ and thus find $f'(1)$.

Question: What are you finding when you do $f'(1)$*?*

$$f(x) = 3x^3 - 6x^2 + 1$$

$$f'(x) = 9x^2 - 12x$$

$$\underline{f'(1)} = 9(1)^2 - 12(1)$$

$$= 9 - 12$$

$$= \underline{\underline{-3}}$$

grad $x = 1$ is $\underline{\underline{-3}}$

Examples have been extracted, with permission, from the Cambridge Mathematical Methods Units 1 and 2 Textbook **www.maffsguru.com**

Using the CAS

We can use the CAS to help us find the derivative of $f(x) = 3x^3 - 6x^2 + 1$ and thus find $f'(1)$.

Finding the gradient of the tangent line

It becomes more and more important to be able to find the gradient of the tangent line. This is effectively what you are doing when you sub in values to the differential.

Example: For the curve determined by the rule $f(x) = 3x^3 - 6x^2 + 1$, *find the gradient of the tangent line to the curve at the point* $(1, -2)$.

$f'(1)$

$(\boxed{1}, -2)$

$x \quad y$

$f'(x) = 9x^2 - 12x$

$f'(1) = 9(1)^2 - 12(1)$

$= -3$

Alternate Notations

Barry really is a pain in the …

It appears that we can use lots of ways to express the differential of an equation.

Firstly, Leibnitz notation:

$$\boxed{\frac{dy}{dx}}$$

This means a change in *y* with respect to a change in *x*.

$y = f(x)$

$y = 6x^2 + 4x$

$\frac{dy}{dx} = 12x + 4$

$y' = 12x + 4$

$f'(x) = 12x + 4$

Examples

If $y = t^2$, find dy/dt.

If $x = t^3 + t$, find dx/dt.

If $z = 13x^3 + x^2$, find dz/dx.

$y = t^2$

$\frac{dy}{dt} = 2t$

$x = t^3 + t$

$\frac{dx}{dt} = 3t^2 + 1$

$z = 13x^3 + x^2$

$\frac{dz}{dx} = 39x^2 + 2x$

Examples

For $y = (x+3)^2$, find $\frac{dy}{dx}$.

For $z = (2t-1)^2(t+2)$, find $\frac{dz}{dt}$.

For $y = \frac{x^2+3x}{x}$, find $\frac{dy}{dx}$.

Differentiate $y = 2x^3 - 1$ with respect to x..

$(x+3)(x+3)$

$y = (x+3)^2 = x^2 + 6x + 9$

$\frac{dy}{dx} = 2x + 6$

$z = (2t-1)^2(t+2)$

$= (4t^2 - 4t + 1)(t+2)$

$= 4t^3 + 8t^2 - 4t^2 - 8t + t + 2$

$= 4t^3 + 4t^2 - 7t + 2$

$\frac{dz}{dt} = 12t^2 + 8t - 7$

Examples

For $y = (x+3)^2$, find $\frac{dy}{dx}$.

For $z = (2t-1)^2(t+2)$, find $\frac{dz}{dt}$.

For $y = \frac{x^2+3x}{x}$, find $\frac{dy}{dx}$.

Differentiate $y = 2x^3 - 1$ with respect to x..

$$y = \frac{x^2+3x}{x}$$

$$= \frac{x^2}{x} + \frac{3x}{x}$$

$$y = x^1 + 3$$

$$\frac{dy}{dx} = 1$$

$$y = 2x^3 - 1$$

$$\frac{dy}{dx} = 6x^2$$

Operator Notation

Find:

$$\frac{d}{dx}(5x - 4x^3) = 5 - 12x^2$$

$$\frac{d}{dz}(5z^2 - 4z) = 10z - 4$$

$$\frac{d}{dz}(6z^3 - 4z^2) = 18z^2 - 8z$$

Finding the point given a gradient

For each of the following curves, find the coordinates of the points on the curve at which the gradient of the tangent line at that point has the given value:

$$y = x^3, \quad gradient \ = 8$$
$$y = x^2 - 4x + 2, \quad gradient \ = 0$$
$$y = 4 - x^3, \quad gradient \ = -6$$

$y = x^3$

$\frac{dy}{dx} = 3x^2$

$8 = 3x^2$

$x = -\frac{2\sqrt{6}}{3}$ $\quad x = \frac{2\sqrt{6}}{3}$

$y = x^2 - 4x + 2$

$\frac{dy}{dx} = 2x - 4$

$0 = 2x - 4$

$2x = 4$

$\underline{\underline{x = 2}}$

$y = 4 - x^3$

$\frac{dy}{dx} = -3x^2$

$-6 = -3x^2$

$x^2 = 2$

$\underline{\underline{x = \pm\sqrt{2}}}$

Using the CAS to find points with a given gradient

The CAS is there to make life easier for us! And it really does when finding points given gradients.

$$y = x^3, \quad gradient = 8$$

$$\boxed{\frac{dy}{dx} = 3x^2}$$

$$8 = 3x^2$$

$$3x^2 - 8 = 0$$

$$\frac{dy}{dx} - 8 = 0$$