The derivative

Year 11

Mathematical Methods

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Learning Objectives

By the end of the lesson I hope that you understand and can apply the following to a range of questions from the Unit 1 and 2 Mathematical Methods course.

- How to find the tangent to a curve at a point
- Understand what it means to differentiate using first principles
- Know how to use your CAS to differentiate
- Know how to approximate the value of the derivative

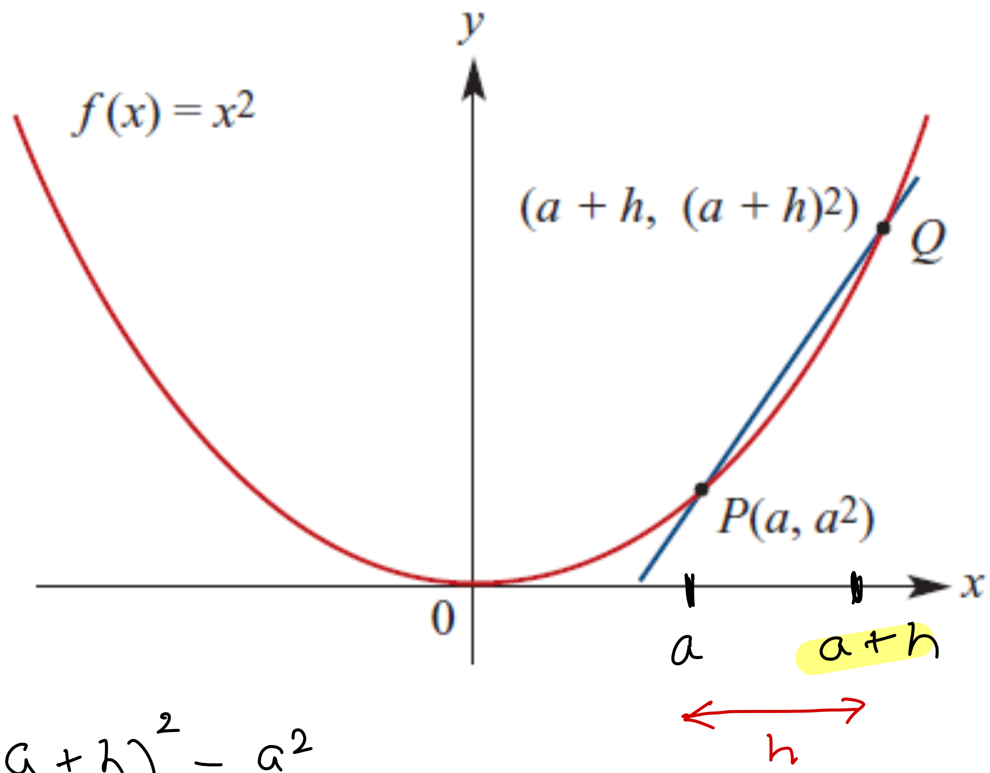
Recap of past learning

This is a new section of the Mathematical Methods Units 1 and 2 course but it builds heavily on the work covered in previous parts of the course.

It is incredibly important as it becomes the foundation of a significant part of the work your will do in Mathematical Methods Units 3 and 4. It also comprises the foundation for 4 sections of the Cambridge Textbook.

Finding the tangent to a curve at a point

Education in the UK is hard. Very hard. When I was at school we didn't have technology! We had to do everything by hand. My teacher loved to make things challenging and so, the work we are going to do today, we had to do by hand with a pencil and ruler.



$f(x) = x^2$

(a, a^2)

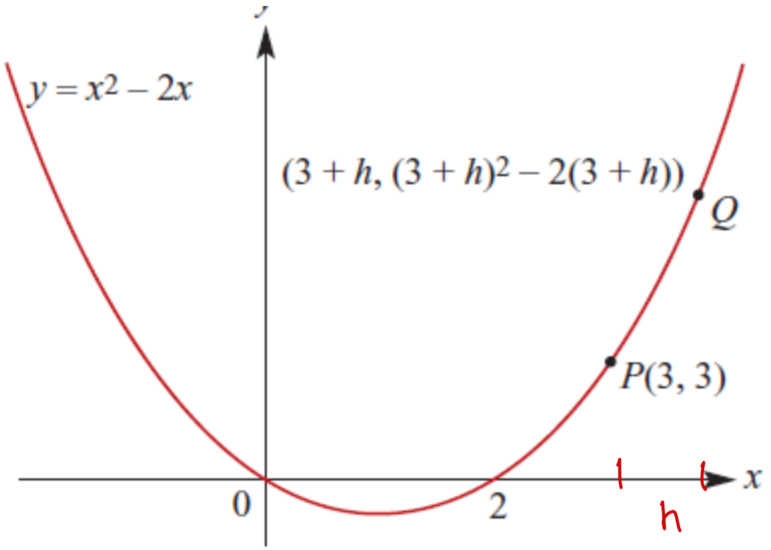
$(a+h, (a+h)^2)$

$$m_{PQ} = \frac{(a+h)^2 - a^2}{\cancel{a} + h - \cancel{a}} = \frac{\cancel{a^2} + 2ah + h^2 \cancel{- a^2}}{h}$$

$$= \frac{2a\cancel{h} + h^{\cancel{2}}}{\cancel{h}} = 2a + h$$

Example 1

By first considering the gradient of the secant PQ, find the gradient of the tangent line to $y = x^2 - 2x$ at the point P with coordinates $(3, 3)$.



$$\text{grad}_{PQ} = \frac{(3+h)^2 - 2(3+h) - 3}{\cancel{3} + h - \cancel{3}}$$

$$= \frac{\cancel{9} + 6h + h^2 \cancel{-6} - 2h \cancel{-3}}{h}$$

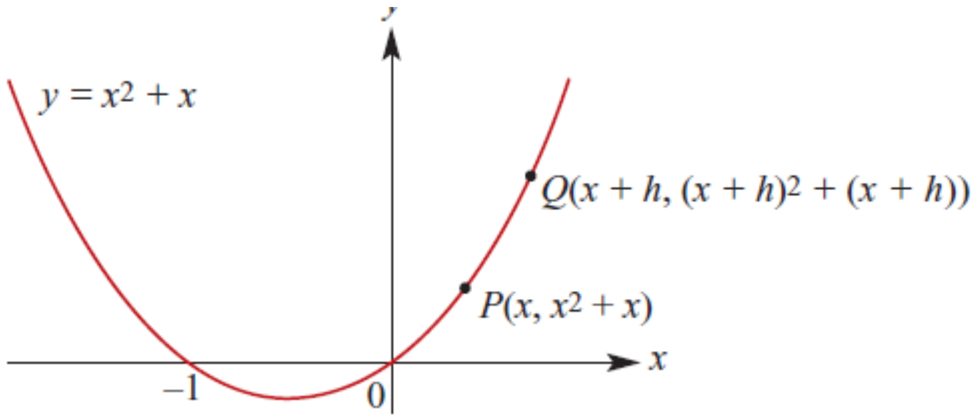
$$\text{grad}_{PQ} = \frac{4\cancel{h} + h^{\cancel{2}}}{\cancel{h}}$$

$$\text{grad}_{PQ} = 4 + h$$

$$\text{grad tangent } P = 4$$

Example 2

Find the gradient of the secant PQ and hence find the derivative of $x^2 + x$.



$$\text{grad } PQ = \frac{(x+h)^2 + (x+h) - (x^2 + x)}{x + h - x}$$

$$= \frac{\cancel{x^2} + 2xh + h^2 \cancel{+x} + h \cancel{-x^2} \cancel{-x}}{h}$$

$$\text{grad } PQ = \frac{2x\cancel{h} + h^{\cancel{2}} + \cancel{h}}{\cancel{h}}$$

$$= 2x + h + 1$$

$h \to 0$

tangent at P $= 2x + 1$

Side note: Expansion of brackets

It's useful to know how to expand brackets to various powers. We find that there is an "easy" to remember way to be able to expand brackets using Pascal's Triangle

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 1 & & 1 & & \\
 & 1 & & 2 & & 1 & \\
1 & & 3 & & 3 & & 1
\end{array}
$$

$$(x+h)^3 = 1x^3h^0 + 3x^2h^1 + 3x^1h^2 + 1x^0h^3$$

$$= 1x^3 + 3x^2h + 3xh^2 + h^3$$

Limit notation

It’s really helpful to have a mathematical way to explaining what happens when the value of ‘h’ gets smaller and smaller.

This effectively means that the points on the curve are getting closer and closer to each other (as ‘h’ gets smaller). At some point ‘h’ will be so small that the points will effectively be the same!

$$\lim_{h \to 0}$$

Example 3

Consider the function $f(x) = x^3$. By first finding the gradient of the secant through $P(2,8)$ and $Q(2+h, (2+h)^3)$, find the gradient of the tangent to the curve at the point $(2,8)$.

$1 \; 2^3 h^0$
$3 \; 2^2 h^1$
$3 \; 2^1 h^2$
$1 \; 2^0 h^3$

$y = x^3$

$P\ (2, 8)$

$Q\ (2+h, (2+h)^3)$

$$\text{grad } PQ = \frac{(2+h)^3 - 8}{\cancel{2} + h - \cancel{2}}$$

$$= \frac{\cancel{8} + 12\cancel{h} + 6h^{\cancel{2}} + h^{\cancel{3}\,2} \cancel{- 8}}{\cancel{h}}$$

$$= 12 + 6h + h^2$$

$h \to 0$?

grad at P = 12

Example 4

Find:

$$\lim_{h \to 0}(22x^2 + 20xh) \quad = 22x^2$$

Example 4

Find:

$$\lim_{h\to 0}\left(\frac{3x^2\cancel{h}+2h^{\cancel{2}}}{\cancel{h}}\right) = \lim_{h\to 0}\left(3x^2 \cancel{+2h}\right) = 3x^2$$

Example 4

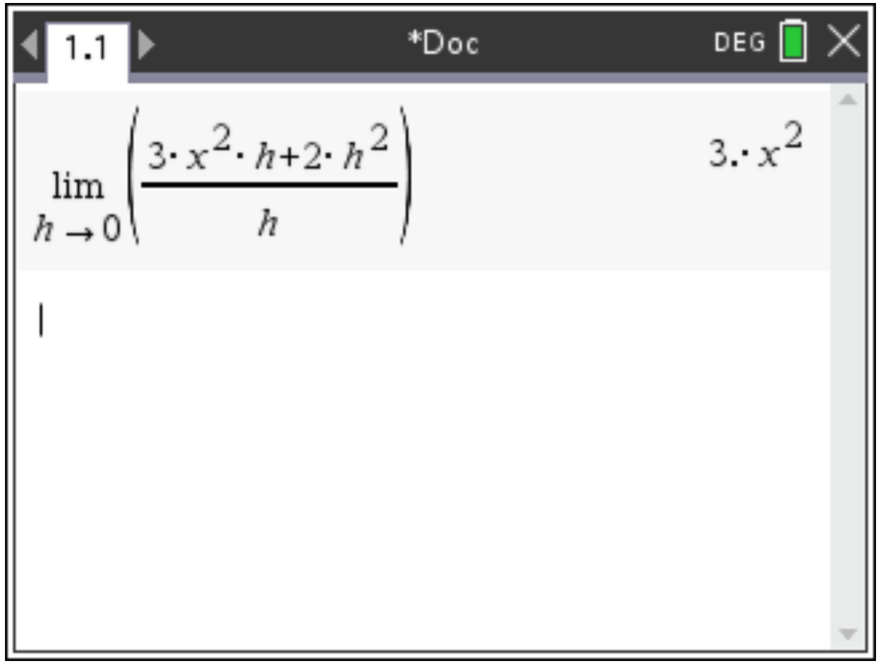
Find:

$$\lim_{h \to 0}(4) = \underline{4}$$

Using the CAS

Methods is a CAS enabled course and, as such, you are expected to know how to use the CAS effectively! Both CAS have the ability to find the limit of functions for you!

Where possible I'll show the CAS instructions for both calculators (unless they are similar).

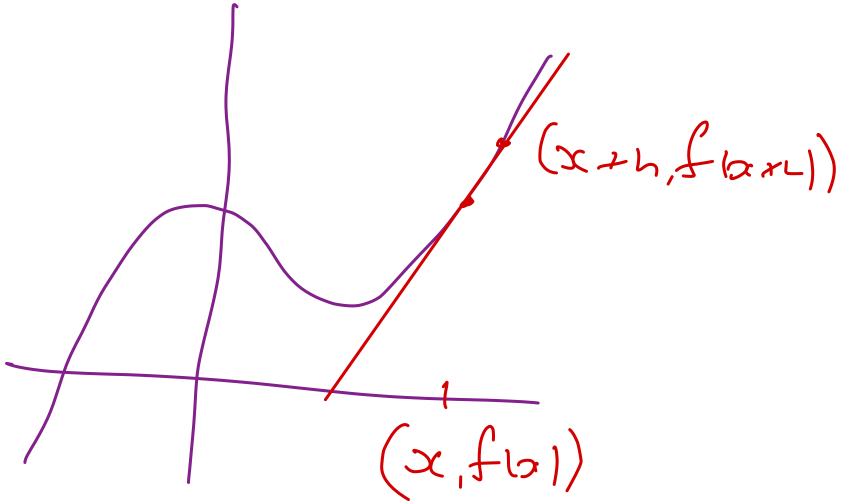


Menu -> Calculus -> Limit

Definition of the derivative

We can use the work we have been doing already to come up with a definition for the gradient of the tangent to a point on a curve.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$



Differentiation by first principles

When we use the limit we are, effectively, using a process called differentiation by first principles. Whilst it is pretty annoying, we do need to understand how it works.

SPOILER: We will find a much quicker way to short cut the process in later videos

Here is an example:

For $f(x) = x^2 + 2x$, find $f'(x)$ by first principles.

$$f(x) = x^2 + 2x$$

$$f(x+h) = (x+h)^2 + 2(x+h)$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 + 2(x+h) - (x^2 + 2x)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 \cancel{+ 2x} + 2h \cancel{- x^2} \cancel{- 2x}}{h}$$

$$= \lim_{h \to 0} \frac{2x\cancel{h} + h^{\cancel{2}} + 2\cancel{h}}{\cancel{h}}$$

$$= \lim_{h \to 0} 2x + h + 2$$

$$\therefore f'(x) = 2x + 2$$

Example

For $f(x) = 2 - x^3$, find $f'(x)$ by first principles.

$1 \quad x^3 \quad \cancel{h^0}$

$3 \quad x^2 \quad h$

$3 \quad x \quad h^2$

$1 \quad \cancel{x^0} \quad h^3$

$$f'(x) = \lim_{h \to 0} \frac{2 - (x+h)^3 - (2 - x^3)}{h}$$

$$= \lim_{h \to 0} \frac{2 - (x^3 + 3x^2h + 3xh^2 + h^3) - 2 + x^3}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{2} \cancel{- x^3} - 3x^2h - 3xh^2 - h^3 \cancel{- 2} \cancel{+ x^3}}{h}$$

$$= \lim_{h \to 0} \frac{-3x^2\cancel{h} - 3xh^{\cancel{2}} - h^{\cancel{3}\,2}}{\cancel{h}}$$

$$= \lim_{h \to 0} -3x^2 \cancel{-3xh} \cancel{-h^2}$$

$$f'(x) = -3x^2$$

$$f(x) = 2 - x^3$$

$$f(x+h) = 2 - (x+h)^3$$

Examples have been extracted, with permission, from the Cambridge Mathematical Methods Units 1 and 2 Textbook

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Using the CAS to differentiation from first principles

Let's use the CAS for the example we just did:

For $f(x) = 2 - x^3$, find $f'(x)$ by first principles.

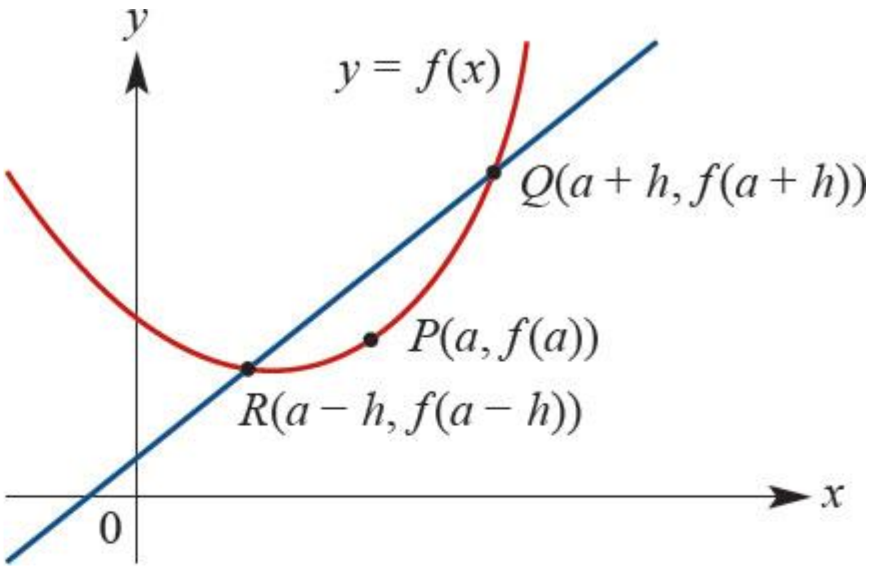
Approximating the value of the derivative

We have just decided that we can use:

$$f'(a) \approx \frac{f(a+h) - f(a)}{h}$$

It makes sense to think what we might gain a better approximation using the following:

$$f'(a) \approx \frac{f(a+h) - f(a-h)}{2h}$$



Learning Objectives: Revisited

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- Understand what it means to differentiate using first principles
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Questions to complete

The following represent the questions you need to complete at an absolute minimum. The extension work is highly recommended for those who wish to score highly in the end of semester examinations (and tests).

Sheet MM143
Ex 7.3 Q 1, 2, 5, Q4 a,c,e, 8a,b,g 9, 10

Exercise 17A
Q 9,11,12

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