Matching and allocation problems

Year 12
Further Mathematics

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"This is fantastic! Especially in these current circumstances. You just saved me from making this exact video for my students."
Youtube comment
(from a current Teacher)

"Your british accent is so intriguing"
Youtube comment (Other YouTuber)

"Very helpful. I'm having trouble with mathematics during quarantine, I've found this channel off a tiktok one of your students has made and it's much easier now!"
Youtube comment
(from current Year 12 student)

"Mate. You're bloody awesome. To be a maths teacher and to make this stuff. You know... brilliant."
Youtube comment
(from current Year 12 student)

"Thank you so much for your videos! Especially now that we have had to move to remote learning they have been a life saver!!"
Youtube comment
(from current Year 12 student)

Learning Objectives

By the end of the lesson I hope that you understand and can apply the following to a range of questions from the Unit 3 and 4 Further Mathematics course.

- Understand what a bipartite graph is
- Understand what a directed bipartite graph is
- Understand what a complete weighted bipartite graph is
- Know how to apply the Hungarian algorithm

Recap of past learning

Just when you think you've learned all the language you need for this course, along comes lots more other words!

We've spent a lot of time looking at graphs and networks. We've found some practical applications of the theory.

Let's look at some more.

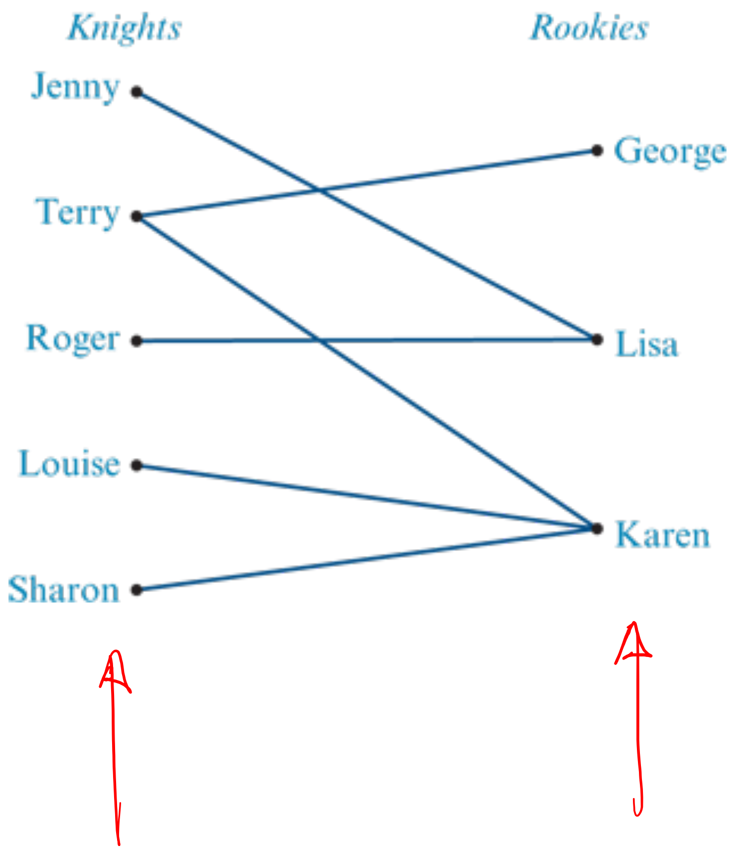
Bipartite graphs

I'm not a great lover of sports, but I do see the wonderful application of graphs and the theory to this area of life.

The diagram could represent different players on two chess teams. The lines which connect them show who needs to play whom.

This type of graph is called a **bipartite graph**.

Each edge in a bipartite graph joins one vertex from one group to a vertex in the other group.



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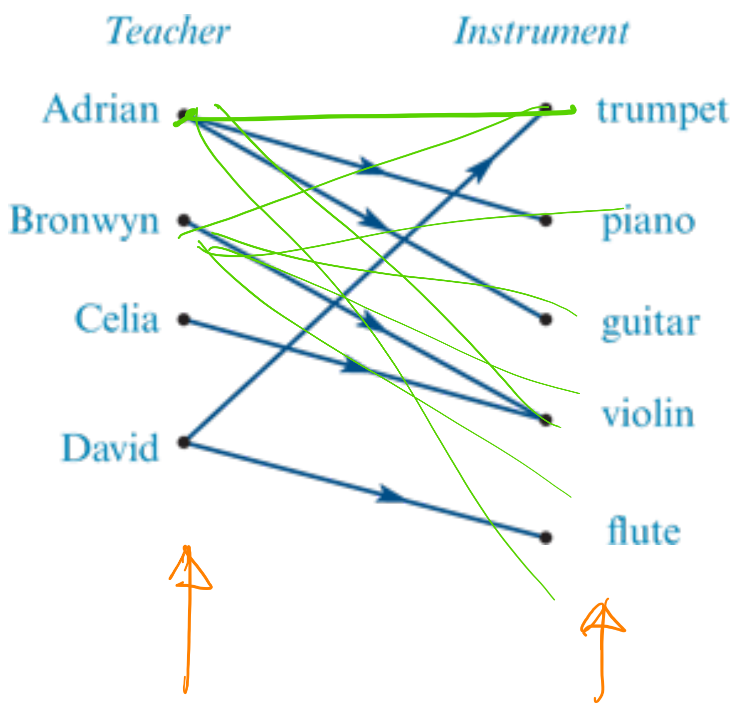
Directed bipartite graphs

I think it makes sense then that as soon as we add the word directed, there is going to be an arrow on the edge and this is going to stand for something important!

In matching problems, one vertex from one group is matched or allocated to one, or more, vertices of the second group and we use a directed bipartite graph to represent this matching.

This example looks at teachers and students.



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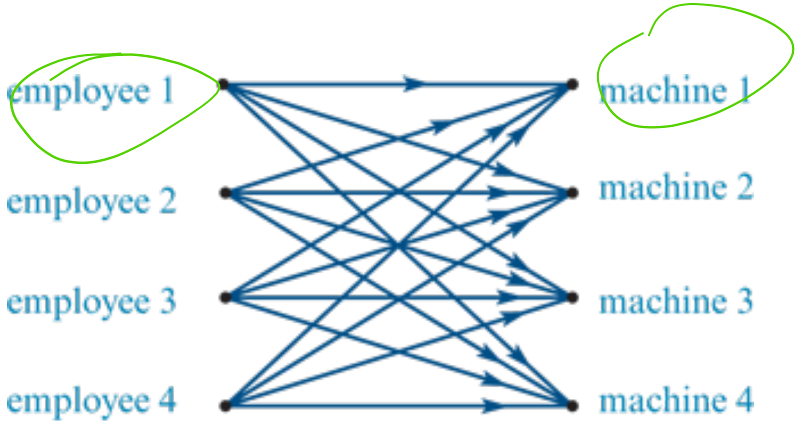
Complete weighted bipartite graphs

This is an example of a complete bipartite graph. It shows that each employee can work each machine in a factory. This would be good, but we know that people have strengths in different areas. It might well be that certain staff members are "quicker" at using one machine than others. To allocated the staff member to a machine they don't really know how to use would lead to productivity loses.

It would be much better if we can find a way to match the correct employee to the correct machine to give us the most productive outcome.



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The Hungarian Algorithm

Welcome to this awesome tool to allow us to solve complex matching problems.

The algorithm has a number of stages:

- Step 1: Subtract the lowest value in each row, from every value in each row
- Step 2: If the minimum number of lines required to cover all the zeros in the table is equal to the number of allocations to be made, jump to step 6. Otherwise, continue to step 3.
- Step 3: If a column does not contain a zero, subtract the lowest value in that column from every value in that column.
- Step 4: If the minimum number of lines required to cover all the zeros in the table is equal to the number of allocations to be made, jump to step 6. Otherwise, continue to step 5a.
- Step 5a: Add the smallest uncovered value to any value that is covered by two lines. Subtract the smallest uncovered value from all the uncovered values.
- Step 5b: Repeat from step 4.
- Step 6: Draw a directed bipartite graph with an edge for every zero value in the table.
- Step 7: Make the allocation and calculate minimum cost

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The table below shows the four employees: Wendy, Xenefon, Yolanda and Zelda. The machines in a factory are represented by the letters A, B, C and D.
The numbers in the table are the times, in minutes, it takes each employee to finish the task on each machine.
The table is called a cost matrix.

Employee	*A*	*B*	*C*	*D*
Wendy	30	40	50	60
Xenefon	70	30	40	70
Yolanda	60	50	60	30
Zelda	20	80	50	70

E	A	B	C	D
W	0	10	20	30
X	40	0	10	40
Y	30	20	30	0
Z	0	60	30	50

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Xenefon	40	0	10	40
Yolanda	30	20	30	0
Zelda	0	60	30	50

E	A	B	C	D
W	0	10	10	30
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Employee	*A*	*B*	*C*	*D*
Wendy	0	10	10	30
Xenefon	~~40~~ 50	0	0	40
Yolanda	~~30~~ 40	20	20	0
Zelda	0	60	20	50

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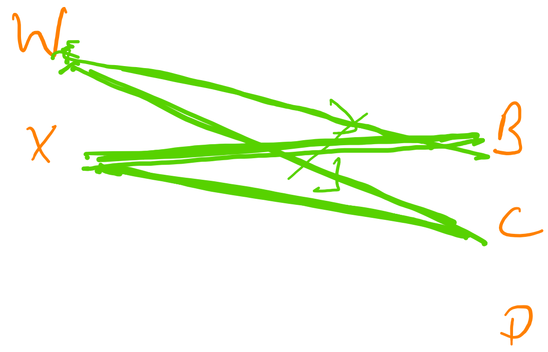
Employee	A	B	C	D
Wendy	0	0	0	30
Xenefon	50	0	0	50
Yolanda	30	10	10	0
Zelda	0	50	10	50

W → B

X → C

Y → D

Z → A



$W \xrightarrow{40} B$

$X \xrightarrow{40} C$ ‖ 80

$W \xrightarrow{50} C$

$X \xrightarrow{30} B$ ‖ 80

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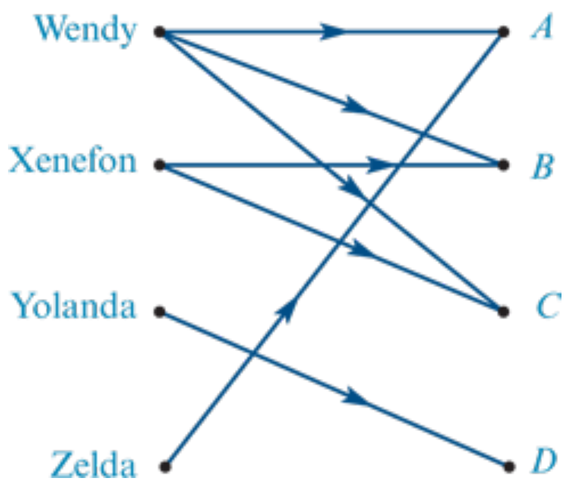

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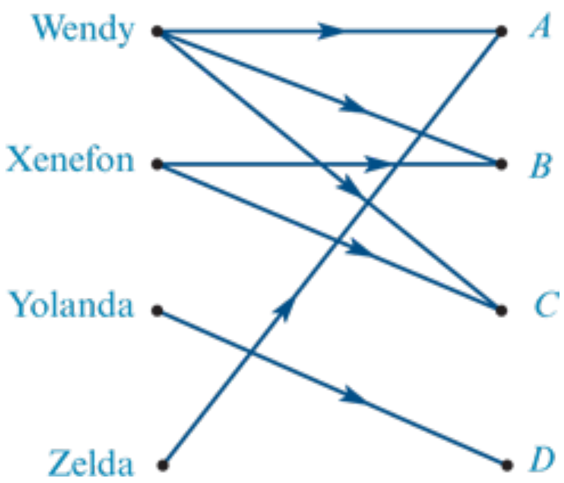

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VCAA Questions

Question 7

Four friends go to an ice-cream shop.

Akiro chooses chocolate and strawberry ice cream.

Doris chooses chocolate and vanilla ice cream.

Gohar chooses vanilla ice cream.

Imani chooses vanilla and lemon ice cream.

This information could be presented as a graph.

Consider the following four statements:

- The graph would be connected.
- The graph would be bipartite.
- The graph would be planar.
- The graph would be a tree.

How many of these four statements are true?

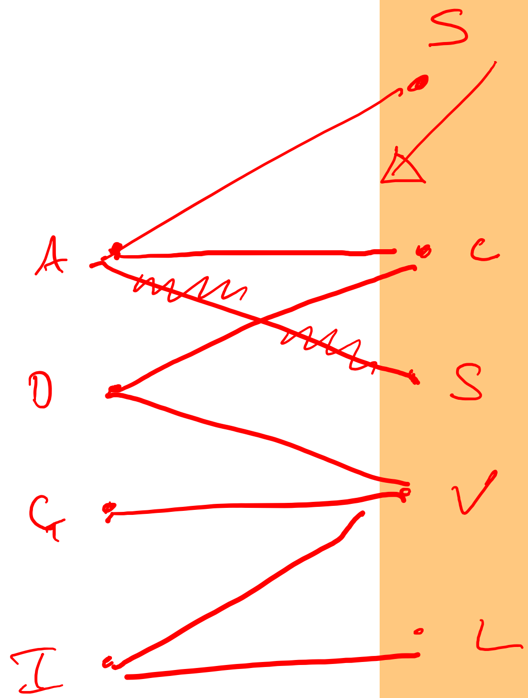
A. 0

B. 1

C. 2

D. 3

E. 4



VCAA Questions

Question 2 (1 mark)

A cricket team has 11 players who are each assigned to a batting position.

Three of the new players, Alex, Bo and Cameron, can bat in position 1, 2 or 3.

The table below shows the average scores, in runs, for each player for the batting positions 1, 2 and 3.

		Batting position		
		1	**2**	**3**
Player	**Alex**	22	24	24
	Bo	25	~~25~~	21
	Cameron	24	25	19

Each player will be assigned to one batting position.

To which position should each player be assigned to **maximise** the team's score? Write your answer in the table below.

minimise

Player	**Batting position**
Alex	3
Bo	1
Cameron	2

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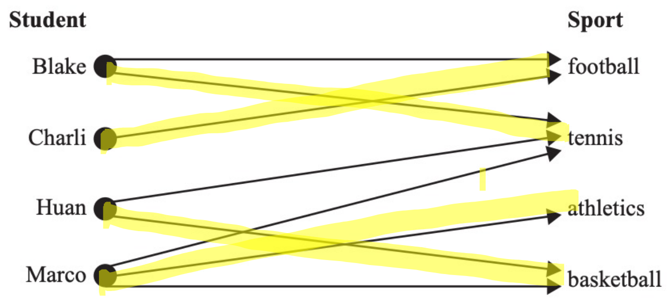
Question 2 (3 marks)

Fencedale High School offers students a choice of four sports, football, tennis, athletics and basketball.

The bipartite graph below illustrates the sports that each student can play.



Each student will be allocated to only one sport.

a. Complete the table below by allocating the appropriate sport to each student. 1 mark

Student	Sport
Blake	
Charli	
Huan	
Marco	

VCAA Questions

b. The school medley relay team consists of four students, Anita, Imani, Jordan and Lola.

The medley relay race is a combination of four different sprinting distances: 100 m, 200 m, 300 m and 400 m, run in that order.

The following table shows the best time, in seconds, for each student for each sprinting distance.

	Best time for each sprinting distance (seconds)			
Student	**100 m**	**200 m**	**300 m**	**400 m**
Anita	13.3	29.6	61.8	87.1
Imani	14.5	29.6	63.5	88.9
Jordan	13.3	29.3	63.6	89.1
Lola	15.2	29.2	61.6	87.9

The school will allocate each student to one sprinting distance in order to minimise the total time taken to complete the race.

To which distance should each student be allocated?

Write your answers in the table below. 2 marks

Student	**Spinting distance (m)**
Anita	
Imani	
Jordan	
Lola	

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VCAA Questions

Question 2 (2 marks)

Bai joins his friends Agatha, Colin and Diane when he arrives for the holiday in Seatown.
Each person will plan one tour that the group will take.
Table 1 shows the time, in minutes, it would take each person to plan each of the four tours.

Table 1

	Agatha	Bai	Colin	Diane
Tour 1	13	7	13	12
Tour 2	14	9	8	7
Tour 3	19	25	21	18
Tour 4	10	7	11	10

1 $\xrightarrow{7}$ B
2 - 8 C
3 - D 18
4 - A 10

The aim is to minimise the total time it takes to plan the four tours.
Agatha applies the Hungarian algorithm to Table 1 to produce Table 2.
Table 2 shows the final result of all her steps of the Hungarian algorithm.

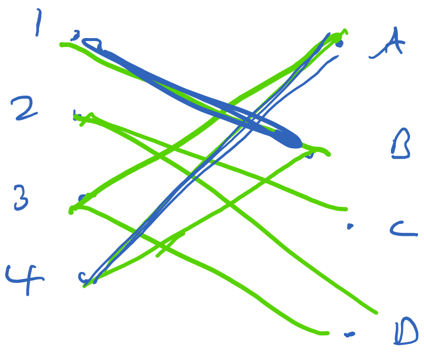

Table 2

	Agatha	Bai	Colin	Diane
Tour 1	3	0	3	3
Tour 2	6	4	0	0
Tour 3	0	9	2	0
Tour 4	0	0	1	1

a. In Table 2 there is a zero in the column for Colin.

When all values in the table are considered, what conclusion about minimum total planning time can be made from this zero? 1 mark

b. Determine the minimum total planning time, in minutes, for all four tours. 1 mark

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