Division and Factorisation of Polynomials (4D)

Sunday, 25 February 2018 5:07 pm

Linear factors = 0 solve

( )( )( )

(CAS)

RECAP

Previously we have had examples of questions which have needed us to do polynomial division.
This has allowed us to express a complex fraction in a form which we can use to describe a series of transformations of commo n curves.
We recap and extend the work on polynomial division to show different ways of doing it.

Why do we need to divide polynomials? ???

We have already found there are a number of techniques which allow us to turn a **quadratic** into factorised form. This is helpful to us to be able to find key points on the graph e.g. turning points, intercepts etc.

We haven't got the same repertoire of techniques for cubics, quartics, quintics, etc.

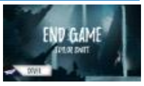
T method
Q F
CTS
X method

$(x-1)(x+2)(x-3)=0$

Except the use of polynomial division.

The end game

Whilst Mathematics has nothing much to do with Taylor Swift ... it's always good to know what the end game is.



If we know where we're going, we can use the correct tools.

When we have quadratics, we factorise them to help us find intercepts on the x - axis. This helps us sketch the graph.

The same is true for cubics, quartics and quintics etc.

We are looking to turn:

Poly

$$y = x^3 + x^2 - 14x - 24$$

$x-5$

$x=5$
$y=0$

$(-2)^3 + (-2)^2 - 14(-2) - 24$
$-8 + 4 + 28 - 24$
$= 0$

$\dfrac{x^3+x^2-14x-24}{x+2}$

$\boxed{r=0}$

$$y = (x+3)(x-4)(x+2) = 0$$

$\boxed{x=-3}$ $\boxed{x=4}$ or $x = \boxed{-2}$

There are three mains ways to do this:
- Polynomial Division
- Synthetic Division
- Using your CAS :)

$y = x^3 + x^2 - 14x - 24$ 1 Factor ...

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Polynomial Division

If we are not given one, we have to find a factor of $P(x) = x^3 + x^2 - 14x - 24$.
We need a factor to help us perform polynomial division.
If we're lucky we will be given a factor!

Otherwise we use a process of **trial and error**:

$1^3 + 1^2 - 14(1) - 24$
$2 - 14 - 24$

$(-1)^3 + (-1)^2 - 14(-1) - 24$

$(2)^3 + 2^2 - 14(2) - 24$

$(x-1)$
$(x+1)$

1
-1
2
-2
3
-3

$P(1) = -36$ zero
$P(-1) = -10$
$P(2) = -40$
$P(-2) = 0$

$x = -2$
$x + 2$

Having found a factor, we know that $x^3 + x^2 - 14x - 24$ can be divided by $x + 2$

Here is how it's done ...

$$\begin{array}{r}
x^2 - x - 12 \\
x+2 \,\overline{)\, x^3 + x^2 - 14x - 24} \\
-\,(x^3 + 2x^2) \qquad\qquad \\
\hline
-x^2 - 14x \qquad \\
-\,(-x^2 - 2x) \qquad \\
\hline
-12x - 24 \\
-\,(-12x - 24) \\
\hline
0
\end{array}$$

$\dfrac{x^3}{x} = x^2$

$\dfrac{-x^2}{x}$

$\dfrac{-12x}{x}$

Important Note:
A factor is something which, when divided into something else, leaves no remainder.
When something is not a factor it will divide by have a remainder.
This is the basis of remainder theorem (coming soon).

$x^3 + x^2 - 14x - 24 = (x+2)(x^2 - x - 12)$

$(x+2)(x+3)(x-4)$

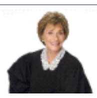
$x^2 - 1 \,\overline{)\, x^5 - 4x^4 + 3x^3 - 2x^2 + x + 4}$

$x^2 - x - 12$
$x^2 + 3x - 4x - 12$
$x(x+3) - 4(x+3)$
$(x+3)(x-4)$

Synthetic Division

I personally LOVE this, but have no idea WHY it works.

Again, you need to find a factor for this to work, but, once you have ... it's a breeze.



Don't judge me :)

$y = x^3 + x^2 - 14x - 24$ Poly

This time lets use the fact that $x = 4$ is a factor.

Step 1: Write the coefficients
Step 2: Write the divisor
Step 3: Copy down the first coefficient below the line.
Step 4: Multiply it by the divisor.
Step 5: Write the result in place.
Step 6: Add the two numbers
Step 7: Repeat until the end.
Step 8: If the last number is a zero, then it's a factor, if not it's a remainder.
Step 9: Look at the numbers of the left of the last one ... these are the coefficients of the quadratic.

	x^3	+	+	
	1	1	-14	-24
4		4	20	24
	1	5	6	0
	x^2	x	const	

< Factor

Remainder = 0 Factor!

$(x-4)(x^2+5x+6)$

$(x-4)(x+2)(x+3)$

$x^2 + 5x + 6$
$x^2 + 2x + 3x + 6$
$x(x+2) + 3(x+2)$
$(x+2)(x+3)$

The Remainder Theorem

I think textbooks always make this sound harder than it is!

$(x-1)$ -36

If we remember back to our original function: $y = x^3 + x^2 - 14x - 24$

Remember, when trying to find, by trial and error, a factor, we tried the following numbers:

$P(1) = -36$
$P(-1) = -10$
$P(2) = -40$
$P(-2) = 0$

$(x+1)$

Only $P(-2)$ gave a zero result. Hence, it's a factor.
All other values were remainders.

Therefore, when we divide $x^3 + x^2 - 14x - 24$ by $x - 1$ we know the remainder is -36

Example: Find the remainder when $x^3 + x^2 - 14x - 24$ is divided by by $2x - 1$

$2x - 1 = 0$
$2x = 1$
$x = \frac{1}{2}$

$\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^2 - 14\left(\frac{1}{2}\right) - 24$

$\Rightarrow \frac{1}{8} + \frac{1}{4} - 7 - 24$

$\neq 0$

The Factor Theorem

I'm beginning to sound like a broken record!!!

To be a factor a divisor has to have a remainder of zero.
Same as the remainder theorem .... When you divide a polynomial by a divisor if the result is zero it's a factor. Otherwise it's a remainder :)

FORWARDS AND BACKWARDS

A great example for factor theorem is when they give you a polynomial with missing values and ask you to find them.

Work to be completed at the end of teaching:
Methods 3&4 Textbook
Exercise 4D
Questions 7, 9, 10, 12, 13, 15-24(s)

$x = 2$

$-3x - 6$
$-3x - 6$
0

$6x^3 + 5x^2 - 17x - 6 = (x+2)(6x^2 - 7x - 3)$

	6	5	-17	-6
-2		-12	14	6
	6	-7	-3	0

$(x+2)(6x^2 - 7x - 3)$

	6	5	-17	-6
-2		-12	14	6
	6	-7	-3	0

$(x+2)(6x^2 - 7x - 3)$

8) $A(x+3) + B(x+2) = 4x + 9$

$Ax + 3A + Bx + 2B = 4x + 9$

$Ax + Bx + 3A + 2B = 4x + 9$

$x(A+B) + 3A + 2B = 4x + 9$

$A + B = 4$
$3A + 2B = 9$

$2 + 2 = 0$
$x = -2$

$x - 2$ $\boxed{3x - 2}$

	6	5	-17	-6
-2		-12		
	6			

$6(8) + 5(\ldots$
$48 + 20\ldots$
$6x^2$
$6x^3 + 5x^2$
$-\ 6x^3 + 12x^2$
$-7x^2$
$-7x^2$

$) - 17(2) - 6$

$- 34 - 6 = \times$

$- 7x - 3$

$- 17x - 6$

$- 17x$

$- 14x$

Example:
The polynomial $x^4 - ax^3 - 7x^2 + bx + 24$ have two factors of $x + 1$ and $x - 3$. Find the values of a and b.

$P(-1) = 0$
$P(3) = 0$

This can be done on the CAS!!!

$(-1)^4 - a(-1)^3 - 7(-1)^2 + b(-1) + 24 = 0$
$(3)^4 - a(3)^3 - 7(3)^2 + 3b + 24 = 0$

Simplify → CAS → solve!

Sums and Differences of cubes

It's really important what you make a note of the following.
~~Learn them.~~
Memorise them.

Important:
$a^2 - b^2 = (a+b)(a-b)$

$$\boxed{x^3 - a^3 = (x - a)(x^2 + ax + a^2)}$$

$$\boxed{x^3 + a^3 = (x + a)(x^2 - ax + a^2)}$$

Solving Polynomial Equations

In exactly the same way as you do for solving quadratics, you solve a polynomial by putting it equal to zero.
This is, in effect, finding the x-axis intercepts.

Like the quadratic factorisation, the NULL FACTOR LAW, still works.

$(\quad x \quad x \quad) = 0$

The Rational Root Theorem

This has to be the most long winded, convoluted way of doing anything in Mathematics.
It is more a process than a good way of doing it!

I've not yet seen this on an exam, but that isn't to say that it won't happen.

Example:
Use the rational root theorem to help factorise $P(x) = 3x^3 + 8x^2 + 2x - 5$

Factors

Factors

upto 3 factors

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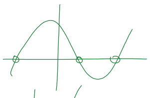
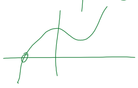
5	±1	±5
3	±1	±3



$\frac{\pm 1}{\pm 1}, \frac{\pm 1}{\pm 3}, \frac{\pm 5}{\pm 1}, \frac{\pm 5}{\pm 3}$



$\boxed{8}$ $\boxed{\pm 1}, \pm\frac{1}{3}, \pm 5, \pm\frac{5}{3}$

Factor
$+1 \rightarrow$
$-1 \rightarrow$ $\boxed{0}$
$+\frac{1}{3} \rightarrow$
$-\frac{1}{3} \rightarrow$

$+5 \rightarrow$
$-5 \rightarrow$
$+\frac{5}{3} \rightarrow$
$-\frac{5}{3} \rightarrow$

Syn division

18	±1	±2	±3	±6	±9	±18
6	±1	±2	±3	±6		

48 ouch!