

Bisection method to solve cubics

Year 11
Mathematical Methods

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Learning Objectives

By the end of the lesson, I hope that you understand and can apply the following to a range of questions from the Year 11 Mathematical Methods course. You need to understand:

- Know how to use the Bisection Method to solve cubics



RECAP

Just when you thought it was safe to get in the water, we come across another way to find the solutions for cubics. And, of course, this one seems a perfect vehicle to use in tests and exams.

This is an algorithm and leads into the idea of algorithmic thinking and pseudocode.

Note: There will be much more in Pseudocode later in the course.



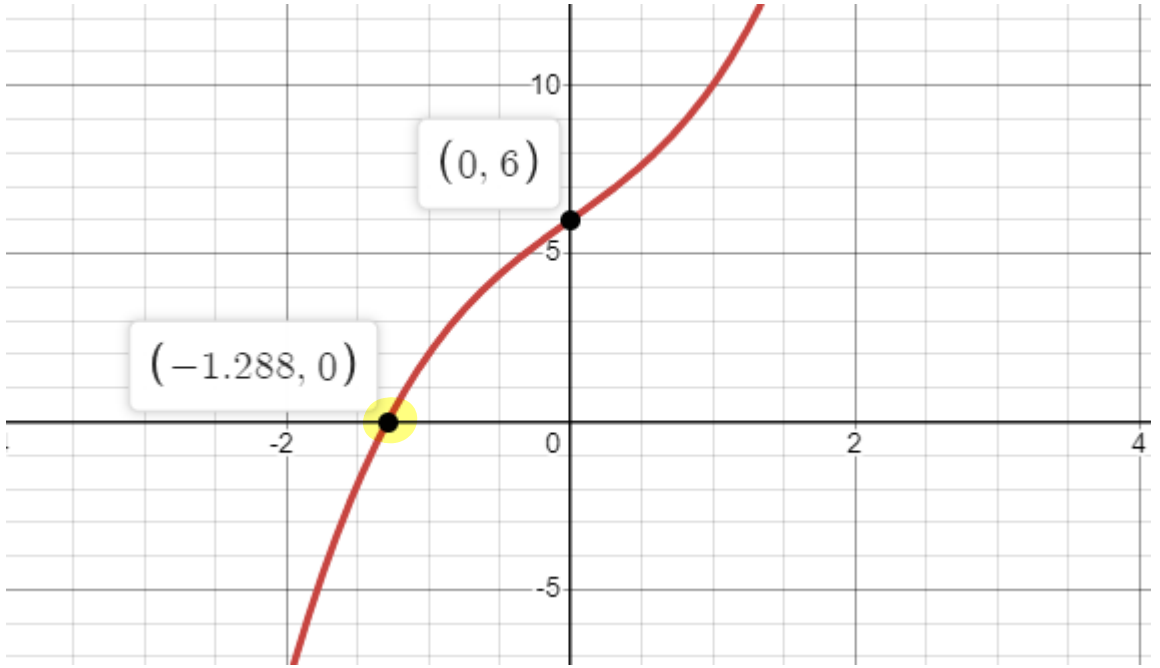
Finding the decimal solutions for a cubic

I think it's important to note that this method could be used for any polynomial!

Let's start with a simple cubic:

$$y = x^3 + 3x + 6$$

We can use the CAS or Desmos.com to graph it (as shown on the right).

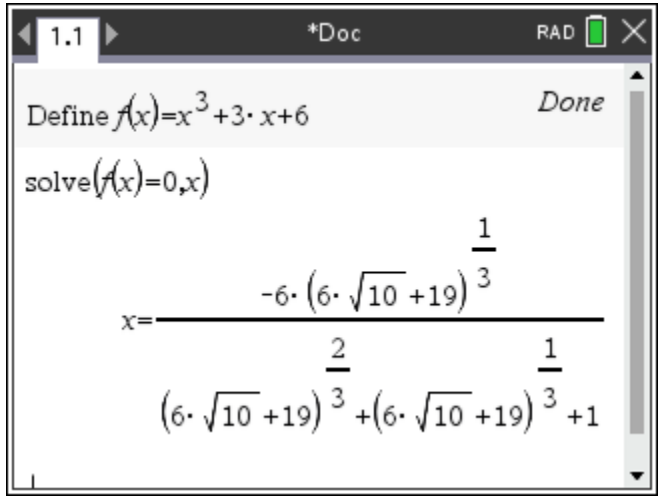


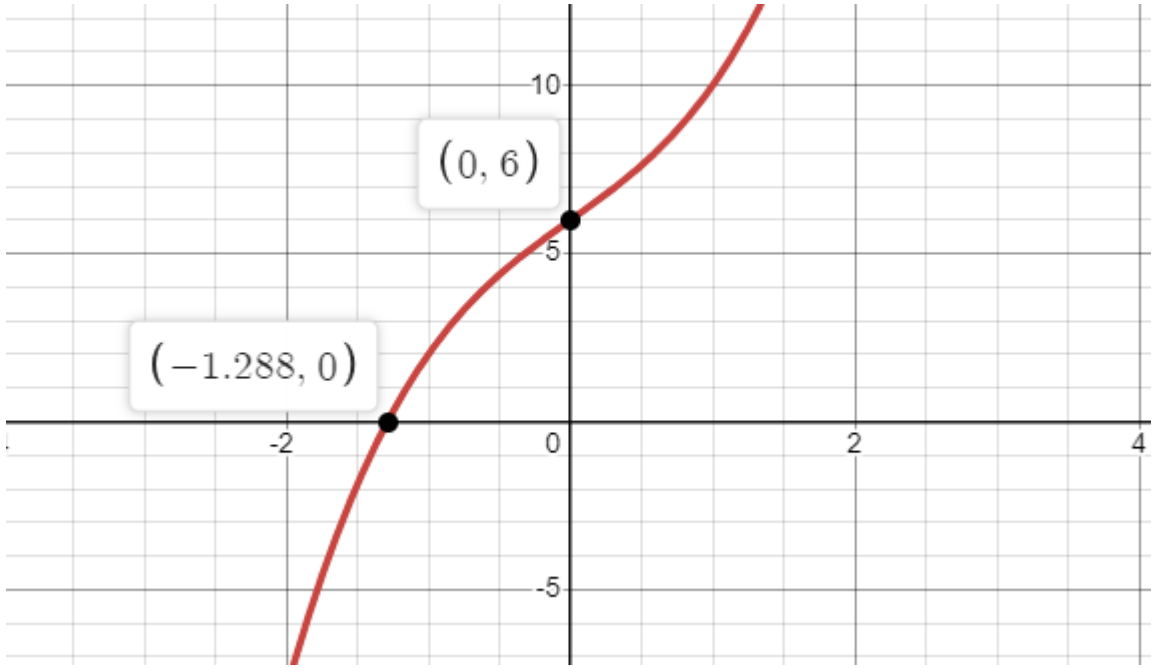
As expected, the CAS shows the x-axis intercept as a decimal value. But, that's OK, because we can use the CAS to find the exact value …



Can’t I just solve using the CAS?

Looking at the screen shot below, it doesn’t seem to be a very nice exact value now does it!





This does not seem very nice!



Can we guess and check in some way?

Let's remind ourselves of the equation:

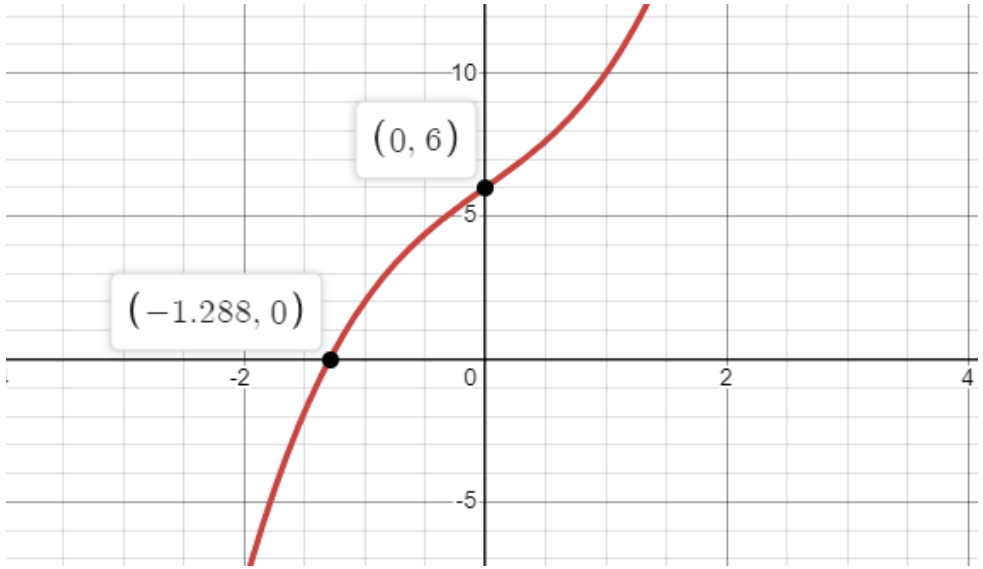
$$x^3 + 3x + 6 = 0$$

We can see, from the graph, that there is a solution somewhere between $x = -2$ and $x = -1$.

We know that we are looking for a solution where we put in an x-value and the y-value comes out as zero.

So, what we are going to do is guess and check.

$x = ?\ldots$, $y = 0$



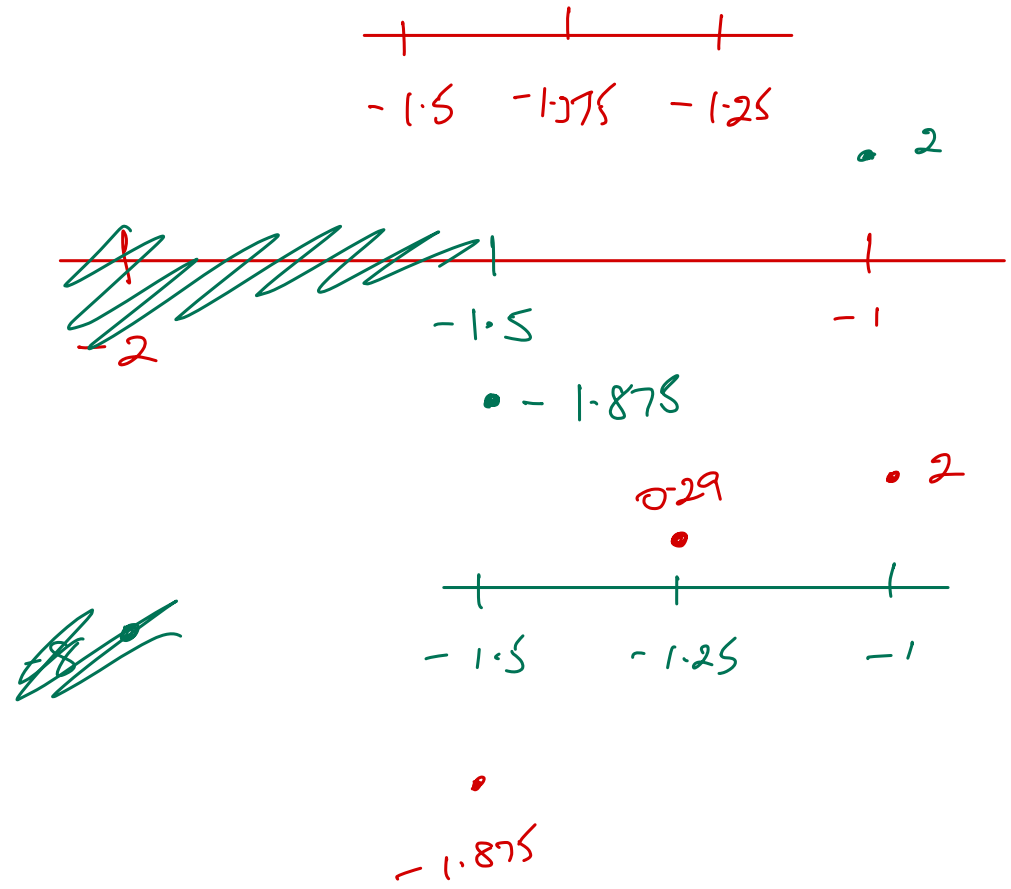


Visually what are we going to do?

The equation below has a solution between -2 and -1.

$$x^3 + 3x + 6 = 0$$

A solutions is where a particular x-value has a
y-value of 0. This is really important to remember!





Finding the decimal solutions for a cubic

What do we do then if this is in a technology free exam?

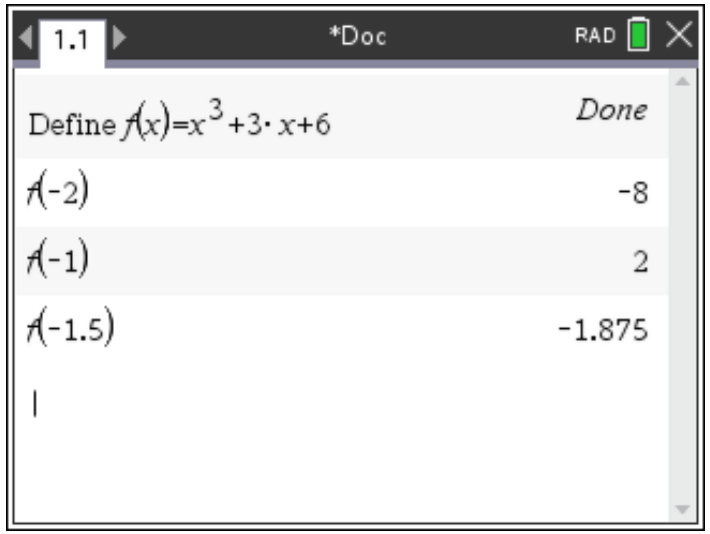
$$x^3 + 3x + 6 = 0$$

We can see, from the graph, that there is a solution somewhere between $x = -2$ and $x = -1$.

We know that we are looking for a solution where we put in an x-value and the y-value comes out as zero.

So, what we are going to do is guess and check.

We know $f(-2) = -8$ and $f(-1) = 2$.



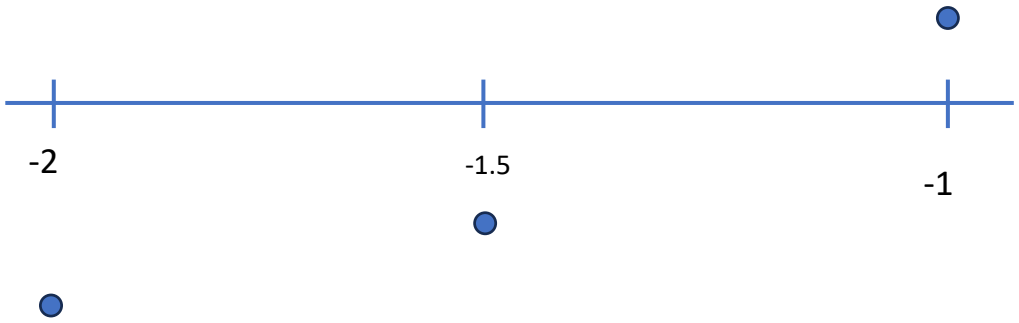
Left endpoint	Right endpoint	Midpoint, m	f(m)
-2 (-ve)	-1 (+ve)	-1.5	-1.875 (-ve)
-1.5 (-ve)	-1 (+ve)	-1.25	

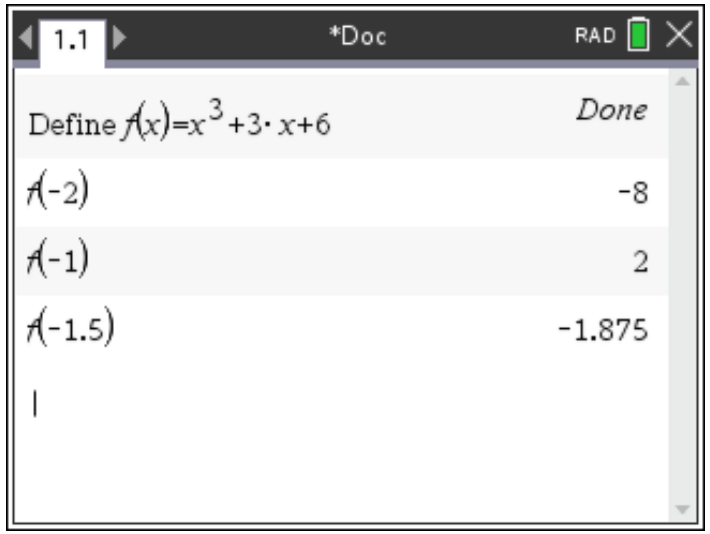
Note: We are looking for a negative and positive value



Showing it visually

The solution must now lie between -1.5 and -1 (as the line would have to cross there).





Left endpoint	Right endpoint	Midpoint, m	f(m)
-2 (-ve)	**-1 (+ve)**	-1.5	**-1.875 (-ve)**

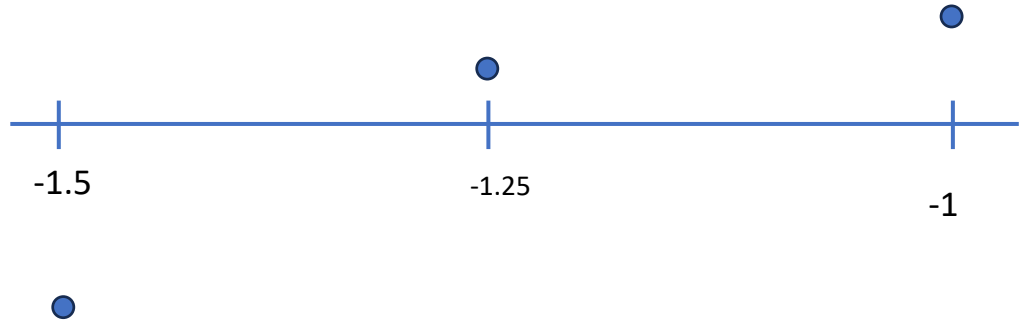
Note: We are looking for a negative and positive value



Finding the decimal solutions for a cubic

We now look at what this is telling us.
We can now see that the solution would lie between -1.5 and -1.25.
Find the midpoint between these two values and start again!

Note: The whole point of this process is to find a x-value correct to a given number of decimal places.



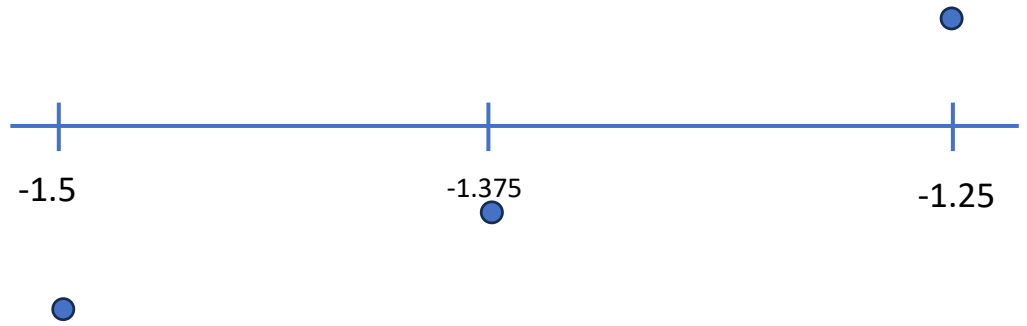
Left endpoint	Right endpoint	Midpoint, m	f(m)
-2 (-ve)	**-1 (+ve)**	-1.5	**-1.875 (-ve)**
-1.5 (-ve)	-1 (+ve)	-1.25	**0.296875 (+ve)**
-1.5 (-ve)	-1.25	-1.375	

Note: We are looking for a negative and positive value



Finding the decimal solutions for a cubic

We now look at what this is telling us.
We can now see that the solution would lie between -1.5 and -1.25.
So, test the solution at -1.375.



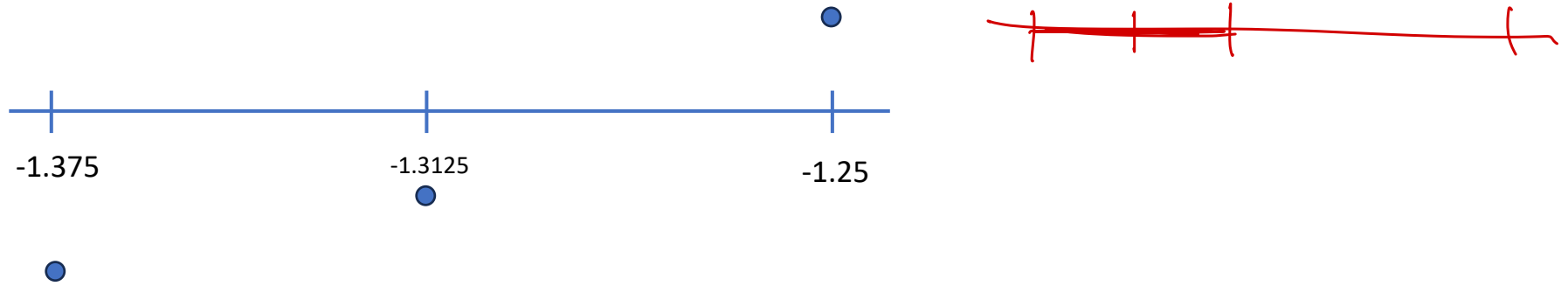
Left endpoint	Right endpoint	Midpoint, m	f(m)
-2 (-ve)	**-1 (+ve)**	-1.5	**-1.875 (-ve)**
-1.5 (-ve)	-1 (+ve)	-1.25	**0.296875 (+ve)**
-1.5 (-ve)	-1.25 (+ve)	-1.375	**-0.724609375 (-ve)**
-1.375 (-ve)	-1.25 (+ve)		

Note: We are looking for a negative and positive value



Finding the decimal solutions for a cubic

We now look at what this is telling us.
We can now see that the solution would lie between -1.5 and -1.25



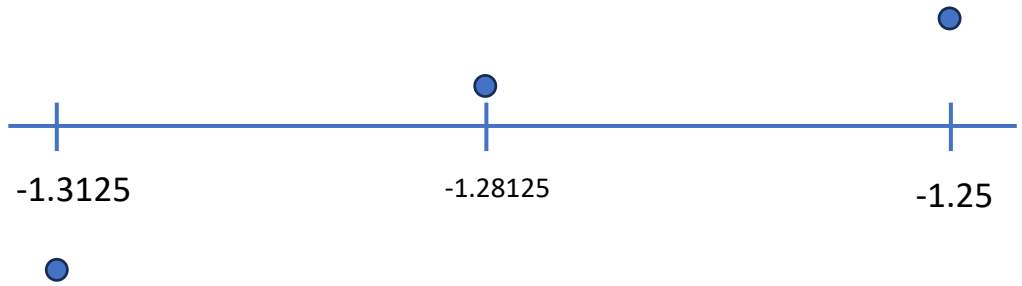
Left endpoint	Right endpoint	Midpoint, m	f(m)
-2 (-ve)	**-1 (+ve)**	-1.5	**-1.875 (-ve)**
-1.5 (-ve)	-1 (+ve)	-1.25	**0.296875 (+ve)**
-1.5 (-ve)	-1.25 (+ve)	-1.375	**-0.724609375 (-ve)**
-1.375 (-ve)	-1.25 (+ve)	-1.3125	**-0.198486328125 (-ve)**

Note: We are looking for a negative and positive value



Finding the decimal solutions for a cubic

We do this one more time … and then I'll show the whole table.
We can now see that the solution would lie between -1.3125 and -1.25



Left endpoint	Right endpoint	Midpoint, m	f(m)
-2 (-ve)	**-1 (+ve)**	-1.5	**-1.875 (-ve)**
-1.5 (-ve)	-1 (+ve)	-1.25	**0.296875 (+ve)**
-1.5 (-ve)	-1.25 (+ve)	-1.375	**-0.724609375 (-ve)**
-1.375 (-ve)	-1.25 (+ve)	-1.3125	**-0.198486328125 (-ve)**
-1.3125 (-ve)	-1.25 (+ve)	-1.28125	0.05294800 (+ve)

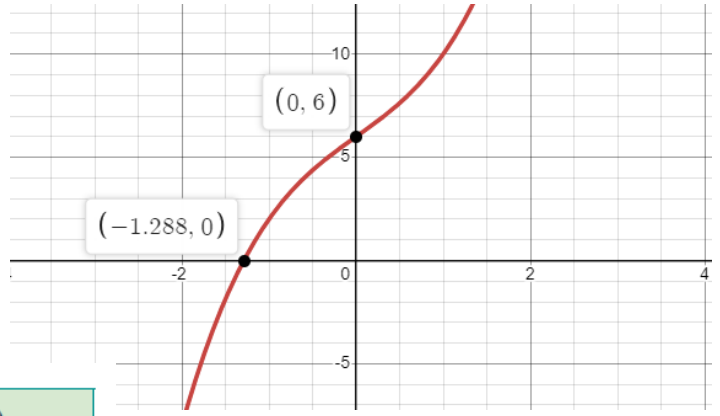
Note: We are looking for a negative and positive value



How often do you keep doing this?

The question generally tells you the number of decimal places it wants the solution to be correct to. This is the **midpoint** you are looking for it to be correct to.

If you notice, from the table below, we can see that from step 11 that the solution would be -1.288 correct to three decimal places.



	Left endpoint a_i	Right endpoint b_i	Midpoint m_i	$f(m_i)$
Step 1	−2	−1	−1.5	−1.87500000
Step 2	−1.5	−1	−1.25	0.29687500
Step 3	−1.5	−1.25	−1.375	−0.72460938
Step 4	−1.375	−1.25	−1.3125	−0.19848633
Step 5	−1.3125	−1.25	−1.28125	0.05294800
Step 6	−1.3125	−1.28125	−1.296875	−0.07181931
Step 7	−1.296875	−1.28125	−1.2890625	−0.00919962
Step 8	−1.2890625	−1.28125	−1.28515625	0.02193302
Step 9	−1.2890625	−1.28515625	−1.28710938	0.00638143
Step 10	−1.2890625	−1.28710938	−1.28808594	−0.00140541
Step 11	−1.28808594	−1.28710938	−1.28759766	0.00248893

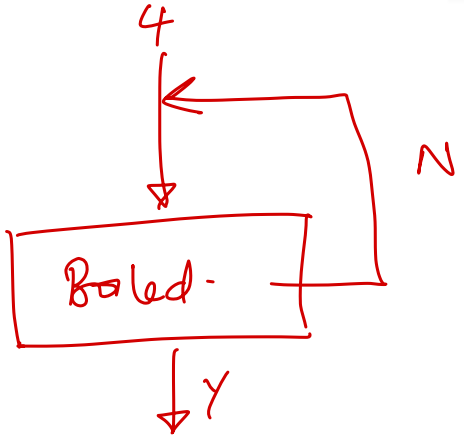


Pseudocode and Making a Cup of Tea!

If you were to make a flow chart of how to make a cup of tea … what would you include?



1. Put water in kettle and switch on
2. Get cup out of cupboard
3. Put sugar in cup
4. Put tea bag in cup
5. Pour hot water in cup
6. Add milk
7. Drink tea





Using Pseudocode

Pseudocode is a massive topic area. It is a way of writing code but making it language agnostic. We should be able to read the code using nothing more than an understanding of the English language (and a couple of control structures).

```
define f(x):
    return x^3 + 3x + 6

a ← −2
b ← −1
m ← −1.5
while b − a > 2 × 0.01
    if f(a) × f(m) < 0 then
        b ← m
    else
        a ← m
    end if
    m ← (a + b)/2
    print a, m, b, f(a), f(m), f(b)
end while
print m
```

This is the pseudocode which will perform the same as we have been doing with the table.





Using Pseudocode

And this is the output which (might) have been produced had we programmed the CAS or spreadsheet to do it.

	a	m	b	$f(a)$	$f(m)$	$f(b)$
Initial	−2	−1.5	−1	−8	−1.875	2
Pass 1	−1.5	−1.25	−1	−1.875	0.296875	2
Pass 2	−1.5	−1.375	−1.25	−1.875	−0.724609	0.296875
Pass 3	−1.375	−1.3125	−1.25	−0.724609	−0.198486	0.296875
Pass 4	−1.3125	−1.28125	−1.25	−0.198486	0.052948	0.296875
Pass 5	−1.3125	−1.296875	−1.28125	−0.198486	−0.071819	0.052948
Pass 6	−1.296875	−1.289063	−1.28125	−0.071819	−0.009200	0.052948

```
define f(x):
  return x^3 + 3x + 6

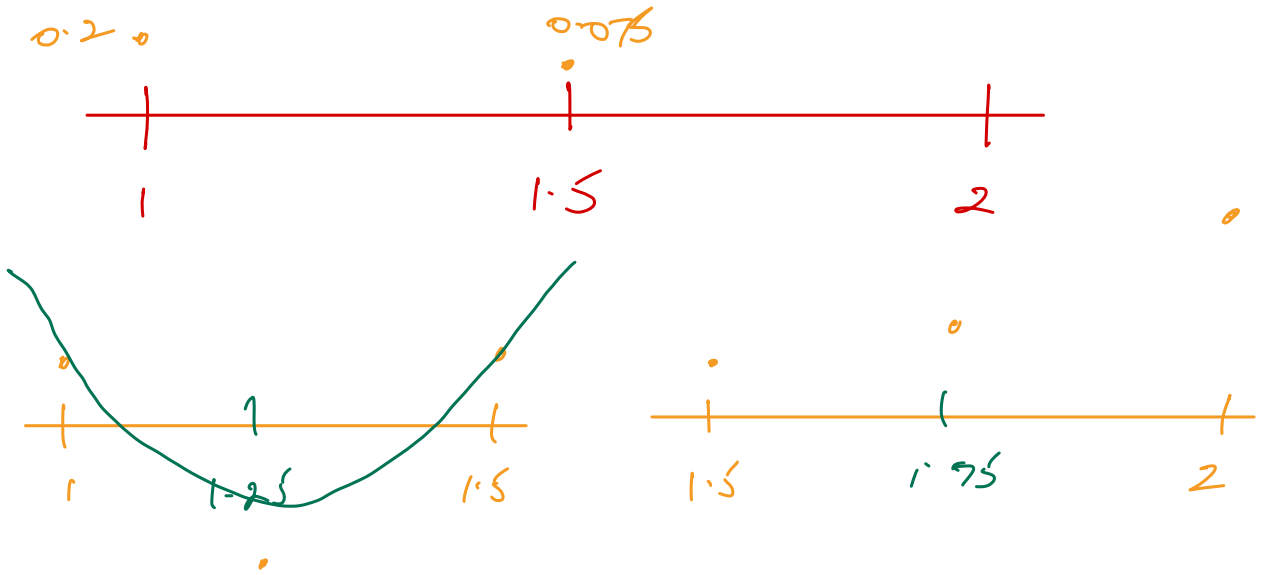
a ← −2
b ← −1
m ← −1.5
while b − a > 2 × 0.01
    if f(a) × f(m) < 0 then
        b ← m
    else
        a ← m
    end if
    m ← (a + b)/2
    print a, m, b, f(a), f(m), f(b)
end while
print m
```



An interesting example

Find the solution in the interval [1, 2] for the following equation:

$$f(x) = x^3 - 5x + 4.2$$





Learning Objectives: Revisited

By the end of the lesson, I hope that you understand and can apply the following to a range of questions from the Year 11 Mathematical Methods course. You need to understand:

- Know how to use the Bisection Method to solve cubics





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Questions to complete

These are the questions which you are asked to complete as a **minimum**. Answering more can never be a bad thing.

Questions:
Sheet of questions
6K: 1ac, 3

