



Integration by parts

Year 12 Specialist Mathematics
Units 3 and 4

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Learning Objectives

By the end of the lesson I hope that you understand and can apply the following to a range of questions from the Unit 3 and 4 Specialist Mathematics course.

- Understand where the integration by parts formula comes from
- Know how to apply the formula to questions



Recap of past learning

When we differentiate, we know to use the correct tool for the question at hand.

When given a question like:

$$\frac{d(x \cdot e^x)}{dx}$$

We know that we can use product rule to find the differential.

Up until this year (when the syllabus changed) we were limited to using integration by inspection to solve the following types of questions:

$$\int x \cdot e^x dx$$

Now we are going to learn about a new tool; integration by parts.



Back to the product rule

We know that the product rule can be expressed in two form:

- The nice and easy to remember one,
- The not so nice and easy to remember one

If we look at the second case:

$$\frac{d}{dx}(uv) = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}$$

What happens if we integrate all of the above with respect to x?

$$\frac{d}{dx}(uv) = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}$$

$$y' = uv' + vu'$$



Integration by parts: The proof

We know that the product rule can be expressed in two form:

- The nice and easy to remember one,
- The not so nice and easy to remember one

If we look at the second case:

$$\frac{d}{dx}(uv) = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}$$

What happens if we integrate all of the above with respect to x ?

$$\frac{d}{dx}(uv) = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}$$

$$\int \frac{d}{dx}(uv) \cdot dx = \int u \cdot \frac{dv}{dx} dx + \int v \cdot \frac{du}{dx} dx$$

$$\int u \frac{dv}{dx} \cdot dx = uv - \int v \left(\frac{du}{dx} \right) dx$$

$u \quad v'$

$$\int x \cdot e^x \cdot dx$$



Example 1a

Find an antiderivative of each of the following:

a $x \cos x$

b xe^x

c $\arcsin x$

$$\begin{aligned}\int \overset{u}{x} \cdot \overset{v'}{\cos x} \cdot dx &= x \cdot \sin x - \int \sin x \cdot 1 \cdot dx \\ &= x \cdot \sin x + \cos x + c\end{aligned}$$

$$\int u \frac{dv}{dx} \cdot dx = uv - \int v \left(\frac{du}{dx} \right) dx$$

$$\begin{aligned}u &= x & v' &= \cos x \\ \frac{du}{dx} &= 1 & v &= \sin x\end{aligned}$$



Example 1b

Find an antiderivative of each of the following:

a $x \cos x$

b xe^x

c $\arcsin x$

$$\int u \frac{dv}{dx} \cdot dx = uv - \int v \left(\frac{du}{dx} \right) dx$$

$$\begin{aligned} & \overset{u}{x} \cdot \overset{v'}{e^x} \cdot dx \\ &= x \cdot e^x - \int e^x \cdot 1 \cdot dx \\ &= x e^x - e^x + c \\ &= \end{aligned}$$

$$\begin{aligned} u &= x & v' &= e^x \\ u' &= 1 & v &= e^x \end{aligned}$$

~~$$\begin{aligned} u &= e^x & v' &= x \\ u' &= e^x & v &= \frac{x^2}{2} \\ e^x \cdot \frac{x^2}{2} - \int \frac{x^2}{2} \cdot e^x \cdot dx \end{aligned}$$~~



Example 1c

Find an antiderivative of each of the following:

a $x \cos x$

b xe^x

c $\arcsin x$

$$\begin{aligned}\int \arcsin x \cdot dx \\&= \sin^{-1} x \cdot x - \int \frac{x}{\sqrt{1-x^2}} \cdot dx \\&= x \cdot \sin^{-1} x + \sqrt{1-x^2} + C\end{aligned}$$

$$\int u \frac{dv}{dx} \cdot dx = uv - \int v \left(\frac{du}{dx} \right) dx$$

$$u = \sin^{-1} x \quad v' = 1$$

$$u' = \frac{1}{\sqrt{1-x^2}} \quad v = x$$



Examples

Find $\int x^2 e^x dx$.

$$\begin{aligned} & \text{u. } v' \\ & \int x^2 \cdot e^x \cdot dx \\ &= x^2 \cdot e^x - 2 \int e^x \cdot x \cdot dx \\ &= x^2 \cdot e^x - 2(x e^x - e^x) + c \\ &= x^2 \cdot e^x - 2x e^x + 2e^x + c \\ &= \\ &= e^x (x^2 - 2x + 2) + c \\ &= \end{aligned}$$

$$\int u \frac{dv}{dx} \cdot dx = uv - \int v \left(\frac{du}{dx} \right) dx$$

$$u = x^2 \quad v' = e^x$$

$$\frac{du}{dx} = 2x \quad v = e^x$$



Examples

Find $\int e^x \cos x \, dx$.

$$\int u \frac{dv}{dx} \cdot dx = uv - \int v \left(\frac{du}{dx} \right) dx$$

$$\int e^x \cdot \cos x = e^x \cdot \sin x - \int \sin x \cdot e^x \cdot dx$$

$$u = e^x \quad v' = \cos x$$

$$= e^x \cdot \sin x - \left[e^x \cdot (-\cos x) + \int \cos x \cdot e^x \cdot dx \right] \frac{du}{dx} = e^x \quad v = \sin x$$

$$= e^x \cdot \sin x + e^x \cos x - \int e^x \cdot \cos x \cdot dx$$

$$u = e^x \quad v' = \sin x$$

$$\frac{du}{dx} = e^x \quad v = -\cos x$$

$$2 \int e^x \cdot \cos x \cdot dx = e^x \sin x + e^x \cos x$$

$$\int e^x \cdot \cos x \cdot dx = \frac{1}{2} e^x \sin x + \frac{1}{2} e^x \cos x + c$$



Examples

a Find a recursive formula for $\int x^n e^x dx$, where $n \in \mathbb{N}$.

b Use your formula to find $\int x^3 e^x dx$.

$$\begin{aligned} I_n &= \int \overset{u}{x^n} \cdot \overset{v'}{e^x} dx = x^n \cdot e^x - \int e^x \cdot n \cdot x^{n-1} dx \\ &= x^n \cdot e^x - n \int e^x \cdot x^{n-1} dx \end{aligned}$$

$$I_n = \underline{\underline{x^n \cdot e^x - n \cdot I_{n-1}}}$$

$$I_3 = \int x^3 \cdot e^x \cdot dx$$

$$\int u \frac{dv}{dx} \cdot dx = uv - \int v \left(\frac{du}{dx} \right) dx$$

$$\begin{aligned} u &= x^n & v' &= e^x \\ \frac{du}{dx} &= nx^{n-1} & v &= e^x \end{aligned}$$

$$I_{n-1} = \int e^x \cdot x^{n-1} \cdot dx$$



Examples

a Find a recursive formula for $\int x^n e^x dx$, where $n \in \mathbb{N}$.

$$\int u \frac{dv}{dx} \cdot dx = uv - \int v \left(\frac{du}{dx} \right) dx$$

b Use your formula to find $\int x^3 e^x dx$.

$$I_3 = \int x^3 \cdot e^x \cdot dx \quad \therefore \int x^3 \cdot e^x \cdot dx = x^3 \cdot e^x - 3 \int x^2 \cdot e^x \cdot dx$$

$$\begin{aligned} &= x^3 \cdot e^x - 3 \left[x^2 \cdot e^x - 2 \int x \cdot e^x \cdot dx \right] \\ &= x^3 e^x - 3x^2 e^x + 6 \left[x \cdot e^x - \int x^0 \cdot e^x \cdot dx \right] \\ &= x^3 \cdot e^x - 3x^2 e^x + 6x e^x - 6e^x + C \\ &= \\ &= e^x (x^3 - 3x^2 + 6x - 6) + C \end{aligned}$$



Definite integrals

When dealing with definite integrals, the formula looks like the following:

$$\int_a^b u \left(\frac{dv}{dx} \right) dx = [uv]_a^b - \int_a^b v \left(\frac{du}{dx} \right) dx$$



Example

Evaluate $\int_1^2 \log_e x \, dx$.

$$\int_1^2 \overset{u}{\log_e x} \cdot dx$$

$$= \left[\log_e x \cdot x \right]_1^2 - \int_1^2 \cancel{x} \cdot \frac{1}{\cancel{x}} \cdot dx$$

$$= \left[x \cdot \log_e x \right]_1^2 - \int_1^2 1 \cdot dx$$

$$= 2 \log_e 2 - \cancel{\log_e 1} - \left[x \right]_1^2$$

$$= 2 \log_e 2 - [2 - 1] = 2 \log_e 2 - 1$$

$$\int_a^b u \left(\frac{dv}{dx} \right) dx = [uv]_a^b - \int_a^b v \left(\frac{du}{dx} \right) dx$$

$$u = \log_e x \quad v' = 1$$

$$\frac{du}{dx} = \frac{1}{x} \quad v = x$$



Learning Objectives: Reviewed

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Questions to complete

The following represents an indication of the minimum number of questions to complete for this exercise. If you choose to do more, then all good. Note that you should also aim to complete some questions from Chapter Reviews too.

Chapter 7H : Integration by parts

Questions: All questions



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