



Use of trigonometric identities for integration

Year 12 Specialist Mathematics
Units 3 and 4

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Learning Objectives

By the end of the lesson I hope that you understand and can apply the following to a range of questions from the Unit 3 and 4 Specialist Mathematics course.

- Understand what it means by the products of sines and cosines



Recap of past learning

This section of the text book is pretty much teaching you all the new ways to integrate known functions. There are, of course, short cuts which can be used and this section seems to be about showing you one of those short cuts!

The overriding principle is to ensure that you remove any powers of sin or cos.

$$\int \cos^3 x \cdot dx$$

$$\int \cos^2 x \cdot dx$$



How do I know when to use substitution?

Evaluate the following:

$$\begin{aligned}\int_0^{\pi/2} \cos^3 x \, dx &= \int_0^{\pi/2} \cos^2 x \cdot \cos x \, dx \\ &= \int_0^{\pi/2} (1 - \sin^2 x) \cdot \cos x \, dx \\ &= \int (1 - u^2) \, du\end{aligned}$$

In this example, we can use the idea that sin differentiates to cos to help us. I know, for this example, I can use substitution.

$$\sin^2 x + \cos^2 x = 1$$

$$u = \sin x$$

$$\frac{du}{dx} = \cos x$$

$$dx = \frac{du}{\cos x}$$



What about this one?

Evaluate the following:

$$\int_0^{\frac{\pi}{2}} \cos^2 x \, dx$$

$$= \int_0^{\pi/2} (1 - \sin^2 x) \, dx$$

There is a problem! We have nothing to cancel out. So there must be a better (or different way of doing it).

$$u = \sin x$$

$$\frac{du}{dx} = \cos x$$



The formula sheet is there to help!

As we cannot do this by substitution, we can use a different identity to help up.

$$\int \cos^2 x \, dx$$

In this case, use the identity:

$$2 \cos^2 x = \cos(2x) + 1$$

$$\cos^2 x = \frac{1}{2} \cos 2x + \frac{1}{2}$$

Circular functions

$\cos^2(x) + \sin^2(x) = 1$	
$1 + \tan^2(x) = \sec^2(x)$	$\cot^2(x) + 1 = \operatorname{cosec}^2(x)$
$\sin(x + y) = \sin(x) \cos(y) + \cos(x) \sin(y)$	$\sin(x - y) = \sin(x) \cos(y) - \cos(x) \sin(y)$
$\cos(x + y) = \cos(x) \cos(y) - \sin(x) \sin(y)$	$\cos(x - y) = \cos(x) \cos(y) + \sin(x) \sin(y)$
$\tan(x + y) = \frac{\tan(x) + \tan(y)}{1 - \tan(x) \tan(y)}$	$\tan(x - y) = \frac{\tan(x) - \tan(y)}{1 + \tan(x) \tan(y)}$
$\sin(2x) = 2 \sin(x) \cos(x)$	
$\cos(2x) = \cos^2(x) - \sin^2(x) = 2 \cos^2(x) - 1 = 1 - 2 \sin^2(x)$	$\tan(2x) = \frac{2 \tan(x)}{1 - \tan^2(x)}$
$\sin^2(ax) = \frac{1}{2}(1 - \cos(2ax))$	$\cos^2(ax) = \frac{1}{2}(1 + \cos(2ax))$

Taken from the VCAA formula sheet 2025



Example 1

Evaluate the following:

$$\int \cos^2 x \, dx$$

$$= \frac{1}{2} \int (\cos 2x + 1) \cdot dx$$

$$= \frac{1}{2} \left[\frac{1}{2} \sin 2x + x \right] + C$$

$$= \frac{1}{4} \sin 2x + \frac{x}{2} + C$$

$$\frac{d}{dx}(\cos(ax)) = -a \sin(ax)$$

$$\int \cos(ax) \, dx = \frac{1}{a} \sin(ax) + c$$

$$2 \cos^2 x = \cos(2x) + 1$$

$$\cos^2 x = \frac{1}{2} \cos 2x + \frac{1}{2}$$



Products of sines and cosines

When we have products of sine and cosine, we can use the following to help us. This has been extracted from the Cambridge VCE Specialist Mathematics textbook.

Integrals of the form $\int \sin^m x \cos^n x \, dx$, where m and n are non-negative integers, can be considered in the following three cases.

Case A: the power of sine is odd

If m is odd, write $m = 2k + 1$. Then

$$\sin^{2k+1} x = (\sin^2 x)^k \sin x = (1 - \cos^2 x)^k \sin x$$

and the substitution $u = \cos x$ can now be made.

Case B: the power of cosine is odd

If m is even and n is odd, write $n = 2k + 1$. Then

$$\cos^{2k+1} x = (\cos^2 x)^k \cos x = (1 - \sin^2 x)^k \cos x$$

and the substitution $u = \sin x$ can now be made.

Case C: both powers are even

If both m and n are even, then the identity $\sin^2 x = \frac{1}{2}(1 - \cos(2x))$, $\cos^2 x = \frac{1}{2}(1 + \cos(2x))$ or $\sin(2x) = 2 \sin x \cos x$ can be used.

Also note that $\int \sec^2(kx) \, dx = \frac{1}{k} \tan(kx) + c$. The identity $1 + \tan^2 x = \sec^2 x$ is used in the following example.

$$\sin^3 x = \sin^2 x \cdot \sin x$$

$$\cos^3 x = \cos^2 x \cdot \cos x$$

$$\sin 4x = 2 \sin 2x \cos 2x$$



Products of sines and cosines

$$\frac{d}{dx}(\tan(ax)) = a \sec^2(ax)$$

$$\int \sec^2(ax) dx = \frac{1}{a} \tan(ax) + c$$

Another example

$$\int \tan^2 x \, dx$$

This cannot easily be done using substitution.

In this case, use the identity: $1 + \tan^2 x = \sec^2 x$

$$= \int (\sec^2 x - 1) \cdot dx$$

$$= [\tan x - x] + c$$

$$= \underline{\tan x - x + c}$$



Products of sines and cosines

Another example

$$\int \sin(2x) \cos(2x) dx = \int \frac{1}{2} \sin 4x \cdot dx$$

In this case, use the identity ... $\sin(2x) = 2 \sin x \cos x$

This means that $\sin(4x) = 2 \sin(2x) \cos(2x)$

$$\begin{aligned} & \frac{1}{2} \int \sin 4x \cdot dx \\ &= \frac{1}{2} \left[-\frac{1}{4} \cos 4x \right] + C \\ &= -\frac{1}{8} \cos 4x + C \end{aligned}$$

Circular functions

$\cos^2(x) + \sin^2(x) = 1$	
$1 + \tan^2(x) = \sec^2(x)$	$\cot^2(x) + 1 = \operatorname{cosec}^2(x)$
$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$	$\sin(x-y) = \sin(x)\cos(y) - \cos(x)\sin(y)$
$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$	$\cos(x-y) = \cos(x)\cos(y) + \sin(x)\sin(y)$
$\tan(x+y) = \frac{\tan(x) + \tan(y)}{1 - \tan(x)\tan(y)}$	$\tan(x-y) = \frac{\tan(x) - \tan(y)}{1 + \tan(x)\tan(y)}$
$\sin(2x) = 2 \sin(x) \cos(x)$	
$\cos(2x) = \cos^2(x) - \sin^2(x) = 2 \cos^2(x) - 1 = 1 - 2 \sin^2(x)$	$\tan(2x) = \frac{2 \tan(x)}{1 - \tan^2(x)}$
$\sin^2(ax) = \frac{1}{2}(1 - \cos(2ax))$	$\cos^2(ax) = \frac{1}{2}(1 + \cos(2ax))$

$$\sin 2x = 2 \sin x \cos x$$

$$\sin 4x = 2 \sin 2x \cos 2x$$

$$\frac{1}{2} \sin 4x = \sin 2x \cos 2x$$



Products of sines and cosines

A different way of doing the example:

$$\int \sin(2x) \cos(2x) dx = \frac{1}{2} \int \sin 4x \cdot dx$$

Product-to-sum identities

$$2 \cos x \cos y = \cos(x - y) + \cos(x + y)$$

$$2 \sin x \sin y = \cos(x - y) - \cos(x + y)$$

$$2 \sin x \cos y = \sin(x + y) + \sin(x - y)$$

$$\sin x \cos y = \frac{1}{2} (\sin(x + y) + \sin(x - y))$$

$$\sin 2x \cos 2x$$

$$= \frac{1}{2} (\sin(-4x) + \sin(\cancel{0}))$$

$$= \frac{1}{2} \sin 4x$$



Products of sines and cosines

Another example

$$\int \cos^4 x \, dx$$

In this case we know that $(\cos^2 x)^2 = \cos^4 x$

We can use the same substitution as before
 $2 \cos^2 x = \cos(2x) + 1$ but note that you're
going to have to use it twice

$$\begin{aligned} &= \int \cos^2 x \cdot \cos^2 x \cdot dx \\ &= \int \frac{1}{2} (1 + \cos 2x) \frac{1}{2} (1 + \cos 2x) \cdot dx \\ &= \frac{1}{4} \int (1 + \cos 2x)^2 \cdot dx \\ &= \frac{1}{4} \int (1 + 2\cos 2x + \cos^2 2x) dx \\ &= \frac{1}{4} \int \left(1 + 2\cos 2x + \frac{1}{2} + \frac{1}{2} \cos 4x \right) \cdot dx \\ &= \frac{1}{4} \left[\frac{3x}{2} + \cancel{2 \cdot \frac{1}{2}} \cdot \sin 2x + \frac{1}{2} \cdot \frac{1}{4} \cdot \sin 4x \right] + c \\ &= \frac{3x}{8} + \frac{1}{4} \cdot \sin 2x + \frac{1}{32} \sin 4x + c \end{aligned}$$

Case C: both powers are even

If both m and n are even, then the identity $\sin^2 x = \frac{1}{2}(1 - \cos(2x))$, $\cos^2 x = \frac{1}{2}(1 + \cos(2x))$ or $\sin(2x) = 2 \sin x \cos x$ can be used.

Also note that $\int \sec^2(kx) \, dx = \frac{1}{k} \tan(kx) + c$. The identity $1 + \tan^2 x = \sec^2 x$ is used in the following example.

$$\begin{aligned} &\cos^2 2x \\ &= \frac{1}{2} (1 + \cos 4x) \end{aligned}$$



Products of sines and cosines

Next example

$$\int \sin^3 x \cos^2 x \, dx = \int \sin^2 x \cdot \sin x \cdot \cos^2 x \cdot dx$$

This is going to be possible using substitution but we need to turn as many as we can into cos (or sin) leaving just one power of cos or sin to cancel.

$$= \int (1 - \cos^2 x) \cdot \sin x \cdot \cos^2 x \cdot dx$$

$$= - \int (1 - u^2) u^2 \cdot du$$

$$= - \int (u^2 - u^4) \cdot du$$

$$= - \left[\frac{u^3}{3} - \frac{u^5}{5} \right] + C$$

$$= \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C$$

Case B: the power of cosine is odd

If m is even and n is odd, write $n = 2k + 1$. Then

$$\cos^{2k+1} x = (\cos^2 x)^k \cos x = (1 - \sin^2 x)^k \cos x$$

and the substitution $u = \sin x$ can now be made.

$$u = \cos x$$

$$\frac{du}{dx} = -\sin x$$

$$dx = \frac{du}{-\sin x}$$



Products of sines and cosines

Product-to-sum identities

$$2 \cos x \cos y = \cos(x - y) + \cos(x + y)$$

$$2 \sin x \sin y = \cos(x - y) - \cos(x + y)$$

$$2 \sin x \cos y = \sin(x + y) + \sin(x - y)$$

Find:

a $\int \sin(3x) \sin(2x) dx$ **b** $\int \sin(5x) \cos(2x) dx$ **c** $\int_0^{\frac{\pi}{2}} \cos(3x) \cos(2x) dx$

$$\begin{aligned} \int \sin 3x \cdot \sin 2x \cdot dx &= \frac{1}{2} \int (\cos x - \cos 5x) dx \\ &= \frac{1}{2} \left[\sin x - \frac{1}{5} \sin 5x \right] + C \\ &= \frac{1}{2} \sin x - \frac{1}{10} \sin 5x + C \\ &= \end{aligned}$$



Products of sines and cosines

Product-to-sum identities

$$2 \cos x \cos y = \cos(x - y) + \cos(x + y)$$

$$2 \sin x \sin y = \cos(x - y) - \cos(x + y)$$

$$2 \sin x \cos y = \sin(x + y) + \sin(x - y)$$

Find:

a $\int \sin(3x) \sin(2x) dx$ **b** $\int \sin(5x) \cos(2x) dx$ **c** $\int_0^{\frac{\pi}{2}} \cos(3x) \cos(2x) dx$

$$\begin{aligned} \int \sin 5x \cdot \cos 2x \cdot dx &= \frac{1}{2} \int (\sin 7x + \sin 3x) dx \\ &= \frac{1}{2} \left[-\frac{1}{7} \cos 7x - \frac{1}{3} \cos 3x \right] + C \\ &= -\frac{1}{14} \cos 7x - \frac{1}{6} \cos 3x + C \end{aligned}$$



Products of sines and cosines

Find:

a $\int \sin(3x) \sin(2x) dx$

b $\int \sin(5x) \cos(2x) dx$

c $\int_0^{\frac{\pi}{2}} \cos(3x) \cos(2x) dx$

Product-to-sum identities

$$2 \cos x \cos y = \cos(x - y) + \cos(x + y)$$

$$2 \sin x \sin y = \cos(x - y) - \cos(x + y)$$

$$2 \sin x \cos y = \sin(x + y) + \sin(x - y)$$

$$\begin{aligned} &= \frac{1}{2} \int_0^{\pi/2} (\cos x + \cos 5x) dx \\ &= \frac{1}{2} \left[\sin x + \frac{1}{5} \sin 5x \right]_0^{\pi/2} \\ &= \frac{1}{2} \left[\left(\sin \frac{\pi}{2} + \frac{1}{5} \sin \frac{5\pi}{2} \right) - \left(\sin 0 + \frac{1}{5} \sin 0 \right) \right] \\ &= \frac{1}{2} \left(1 + \frac{1}{5} \right) = \frac{1}{2} \left(\frac{6}{5} \right) = \frac{6}{10} = \frac{3}{5} \end{aligned}$$



Learning Objectives: Revisited

By the end of the lesson I hope that you understand and can apply the following to a range of questions from the Unit 3 and 4 Specialist Mathematics course.

- Understand what it means by the products of sines and cosines



Questions to complete

The following represents an indication of the minimum number of questions to complete for this exercise. If you choose to do more, then all good. Note that you should also aim to complete some questions from Chapter Reviews too.

Chapter 9E : Use of trigonometric functions

Questions:

1adeghi, 2bdfg, 3acefh, 4acdfh



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