Integration by substitution

Year 12 Specialist Mathematics
Units 3 and 4

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Learning Objectives

By the end of the lesson I hope that you understand and can apply the following to a range of questions from the Unit 3 and 4 Specialist Mathematics course.

- Understand how to integrate by substitution
    - Also called the change of variable rule
- Understand how to use linear substitutions
- Understand how to use the CAS

RECAP

This is the next video in the series relating to Techniques of Integration. These are effectively skills-based lessons showing lots of ways to integrate. The method you choose will depend on the question(s) given. These skills will be useful when we need to apply the learning to more “real world” problems.

I love this method!

I have never really understood why anyone would want to remember lots of different "short cuts" for integration when, to be honest, this rule seems to work for everything (except trigonometric or circular functions).

As it says on the tin, this is integration by taking out the most complex part of the question and turning it into something simpler. The best way to demonstrate this is to show you lots of examples.

Examples (simpler)

Find an antiderivative of each of the following:

$\sin x \cos^2 x$

$5x^2(x^3-1)^{\frac{1}{2}}$

$3xe^{x^2}$

$$\int \sin x \cdot \cos^2 x \cdot dx$$

$$u = \cos x$$

$$\frac{du}{dx} = -\sin x$$

$$dx = \frac{du}{-\sin x}$$

$$= \int \cancel{\sin x} \cdot u^2 \cdot \frac{du}{-\cancel{\sin x}}$$

$$= -\int u^2 \cdot du$$

$$= -\left[\frac{u^3}{3}\right] = -\frac{\cos^3 x}{3}$$

Examples (simpler)

Find an antiderivative of each of the following:

$\sin x \cos^2 x$

$\mathbf{5x^2(x^3-1)^{\frac{1}{2}}}$

$3xe^{x^2}$

$$\int 5x^2 \cdot (x^3 - 1)^{1/2} \cdot dx$$

$$u = x^3 - 1$$

$$\frac{du}{dx} = 3x^2$$

$$\therefore dx = \frac{du}{3x^2}$$

$$= \int 5 \cdot \cancel{x^2} \cdot u^{1/2} \cdot \frac{du}{3\cancel{x^2}}$$

$$= \frac{5}{3}\int u^{1/2} \cdot du$$

$$= \frac{5}{3}\left[\frac{u^{3/2}}{3/2}\right] = \frac{5}{3}\cdot\frac{2}{3}u^{3/2} = \frac{10}{9}\cdot(x^3-1)^{3/2}$$

Examples (simpler)

Find an antiderivative of each of the following:

$\sin x \cos^2 x$

$5x^2(x^3-1)^{\frac{1}{2}}$

$\boldsymbol{3xe^{x^2}}$

$$\int 3x \cdot e^{x^2} \cdot dx$$

$$= \int 3\cancel{x} \cdot e^{u} \cdot \frac{du}{2\cancel{x}}$$

$$= \frac{3}{2}\int e^{u} \cdot du$$

$$= \frac{3}{2} e^{u} = \frac{3}{2} e^{x^2}$$

$u = x^2$

$\frac{du}{dx} = 2x$

$dx = \frac{du}{2x}$

Examples (simpler)

Find an antiderivative of each of the following:

$\frac{2}{x^2+2x+6}$

$$= \int \frac{2}{x^2+2x+6}\,.dx$$

$$= 2\int \frac{1}{(x+1)^2+5}\,.dx$$

$$= 2\int \frac{1}{u^2+5}\,.du$$

$$= 2\int \frac{1}{5+u^2}\,.du$$

$$= \frac{2}{\sqrt{5}}\int \frac{\sqrt{5}}{5+u^2}\,.du$$

$$= \frac{2}{\sqrt{5}}\,.\tan^{-1}\left(\frac{u}{\sqrt{5}}\right)$$

$$= \frac{2}{\sqrt{5}}\,.\tan^{-1}\left(\frac{x+1}{\sqrt{5}}\right)$$

$$=$$

Note: Complete the square and then make a valid substitution. This will then have a circular function result

x^2+2x+6

$(x+1)^2+5$

$u = x+1$

$\frac{du}{dx} = 1$

$du = dx$

$a = \sqrt{5}$

Examples (simpler)

Find an antiderivative of each of the following:

$$\frac{3}{\sqrt{9-4x-x^2}} = 3\int \frac{1}{\sqrt{13-(x+2)^2}}\,dx$$

$$= 3\int \frac{1}{\sqrt{13-u^2}}\,du$$

$$= 3\sin^{-1}\left(\frac{u}{\sqrt{13}}\right)$$

$$= 3\sin^{-1}\left(\frac{x+2}{\sqrt{13}}\right)$$

$$-x^2-4x+9$$

$$-(x^2+4x-9)$$

$$-((x+2)^2-13)$$

$$13-(x+2)^2$$

$$u = x+2$$

$$\frac{du}{dx} = 1$$

$$du = dx$$

$$a = \sqrt{13}$$

Examples (simpler)

Find an antiderivative of each of the following:

$(2x+1)\sqrt{x+4}$

$$= \int (2x+1)\sqrt{x+4}\,.\,dx$$

$$= \int (2u-7)u^{1/2}\,.\,du$$

$$= \int \left(2u^{3/2} - 7u^{1/2}\right) du$$

$$= \frac{2u^{5/2}}{5/2} - \frac{7u^{3/2}}{3/2}$$

$$= \frac{4u^{5/2}}{5} - \frac{14u^{3/2}}{3}$$

$$\therefore \frac{4(x+4)^{5/2}}{5} - \frac{14(x+4)^{3/2}}{3}$$

$u = x+4$

$\frac{du}{dx} = 1$

$du = dx$

$x = u-4$

$2x = 2u-8$

$2x+1 = 2u-7$

Examples (simpler)

Find an antiderivative of each of the following:

$$\frac{2x+1}{(1-2x)^2}$$

$$\int \frac{2x+1}{(1-2x)^2} \cdot dx$$

$$= -\frac{1}{2}\int \frac{2-u}{u^2} \cdot du$$

$$= -\frac{1}{2}\int (2-u) \cdot u^{-2} \cdot du$$

$$= -\frac{1}{2}\int \left(2u^{-2} - u^{-1}\right) \cdot du$$

$$\Rightarrow -\frac{1}{2}\left[\frac{2u^{-1}}{-1} - \ln|u|\right]$$

$$\Rightarrow \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{u} + \frac{1}{2}\ln|u|$$

$$= \frac{1}{u} + \frac{1}{2}\ln|u|$$

$$\Rightarrow \frac{1}{1-2x} + \frac{1}{2}\ln|1-2x|$$

$u = 1-2x$

$$\frac{du}{dx} = -2$$

$$dx = \frac{du}{-2}$$

$2x = 1-u$

$2x+1 = 2-u$

Examples (simpler)

Find an antiderivative of each of the following:

$x^2\sqrt{3x-1}$

$$= \int x^2\sqrt{3x-1}\,.\,dx$$

$$= \frac{1}{3}\int \frac{(u+1)^2}{9}\,u^{1/2}\,.\,du$$

$$= \frac{1}{3}\int (u^2+2u+1)\,u^{1/2}\,.\,du$$

$$= \frac{1}{27}\int \left(u^{5/2}+2u^{3/2}+u^{1/2}\right)du$$

$$= \frac{1}{27}\left[\frac{2u^{7/2}}{7}+\frac{4u^{5/2}}{5}+\frac{2u^{3/2}}{3}\right]$$

$$= \frac{2}{27}\left[\frac{(3x-1)^{7/2}}{7}+\frac{2(3x-1)^{5/2}}{5}+\frac{(3x-1)^{3/2}}{5}\right]$$

$u = 3x-1$

$\frac{du}{dx} = 3$

$dx = \frac{du}{3}$

$u = 3x-1$

$u+1 = 3x$

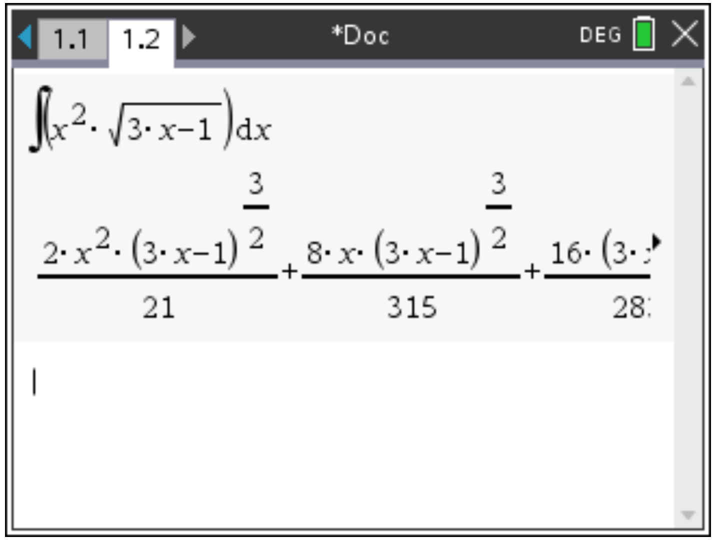
$x = \frac{u+1}{3}$

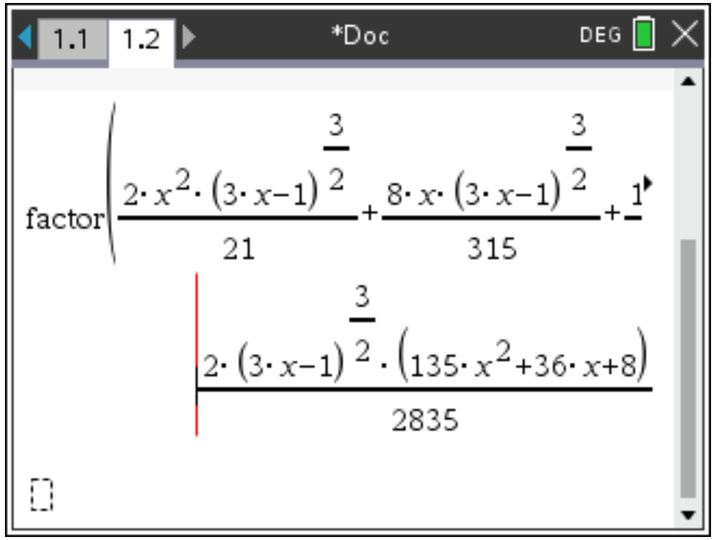
$x^2 = \frac{(u+1)^2}{9}$

Examples (simpler): Using the CAS

Find an antiderivative of each of the following:

$x^2\sqrt{3x-1}$





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