Antiderivatives involving inverse circular functions

Year 12 Specialist Mathematics Units 3 and 4

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Learning Objectives

By the end of the lesson I hope that you understand and can apply the following to a range of questions from the Unit 3 and 4 Specialist Mathematics course.

- Understand how to integrate the inverse circular functions

Recap of past learning

In a previous lesson we derived and used the following to allow us to differentiate inverse circular functions:

$$
\begin{array}{lll}
f\colon (-a, a) \to \mathbb{R}, & f(x) = \sin^{-1}\left(\dfrac{x}{a}\right), & f'(x) = \dfrac{1}{\sqrt{a^2 - x^2}} \\
f\colon (-a, a) \to \mathbb{R}, & f(x) = \cos^{-1}\left(\dfrac{x}{a}\right), & f'(x) = \dfrac{-1}{\sqrt{a^2 - x^2}} \\
f\colon \mathbb{R} \to \mathbb{R}, & f(x) = \tan^{-1}\left(\dfrac{x}{a}\right), & f'(x) = \dfrac{a}{a^2 + x^2}
\end{array}
$$

Hence, we can use them in reverse to integrate functions of the form given above.

Antiderivatives involving inverse circular functions

Stick these into your summary book and check if they are on the formula sheet!

$$\int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \sin^{-1}\left(\frac{x}{a}\right) + c \qquad \text{for } x \in (-a, a)$$

$$\int \frac{-1}{\sqrt{a^2 - x^2}}\, dx = \cos^{-1}\left(\frac{x}{a}\right) + c \qquad \text{for } x \in (-a, a)$$

$$\int \frac{a}{a^2 + x^2}\, dx = \tan^{-1}\left(\frac{x}{a}\right) + c \qquad \text{for } x \in \mathbb{R}$$

Examples

Find an antiderivative for each of the following:

$$\frac{1}{\sqrt{9-x^2}}$$

$$\frac{1}{\sqrt{9-4x^2}}$$

$$\frac{1}{9+4x^2}$$

$$\int \frac{1}{\sqrt{9-x^2}}\,dx = \sin^{-1}\left(\frac{x}{3}\right)$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + c \quad \text{for } x \in (-a, a)$$

$$\int \frac{-1}{\sqrt{a^2-x^2}}\,dx = \cos^{-1}\left(\frac{x}{a}\right) + c \quad \text{for } x \in (-a, a)$$

$$\int \frac{a}{a^2+x^2}\,dx = \tan^{-1}\left(\frac{x}{a}\right) + c \quad \text{for } x \in \mathbb{R}$$

$a = 3$

Examples

Find an antiderivative for each of the following:

$\dfrac{1}{\sqrt{9-x^2}}$

$\dfrac{1}{\sqrt{9-4x^2}}$

$\dfrac{1}{9+4x^2}$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right)+c \quad \text{for } x \in (-a,a)$$

$$\int \frac{-1}{\sqrt{a^2-x^2}}\,dx = \cos^{-1}\left(\frac{x}{a}\right)+c \quad \text{for } x \in (-a,a)$$

$$\int \frac{a}{a^2+x^2}\,dx = \tan^{-1}\left(\frac{x}{a}\right)+c \quad \text{for } x \in \mathbb{R}$$

$$\int \frac{1}{\sqrt{9-4x^2}}$$

$$= \int \frac{1}{\sqrt{\left(\frac{9}{4} - \frac{\cancel{4}x^2}{\cancel{4}}\right)4}}\,.dx$$

$$= \int \frac{1}{2\sqrt{\frac{9}{4}-x^2}}\,.dx$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{\frac{9}{4}-x^2}}\,.dx$$

$$= \frac{1}{2}.\sin^{-1}\left(\frac{x}{3/2}\right)$$

$$= \frac{1}{2}.\sin^{-1}\left(\frac{2x}{3}\right)$$

$\sqrt{ab} = \sqrt{a}.\sqrt{b}$

$a^2 = \frac{9}{4}$ $\quad a = \frac{3}{2}$

Examples

Find an antiderivative for each of the following:

$\dfrac{1}{\sqrt{9-x^2}}$

$\dfrac{1}{\sqrt{9-4x^2}}$

$\dfrac{1}{9+4x^2}$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right)+c \qquad \text{for } x \in (-a, a)$$

$$\int \frac{-1}{\sqrt{a^2-x^2}}\,dx = \cos^{-1}\left(\frac{x}{a}\right)+c \qquad \text{for } x \in (-a, a)$$

$$\int \frac{a}{a^2+x^2}\,dx = \tan^{-1}\left(\frac{x}{a}\right)+c \qquad \text{for } x \in \mathbb{R}$$

$$\int \frac{1}{9+4x^2}\cdot dx$$

$$= \int \frac{1}{\left(\frac{9}{4}+\frac{4x^2}{4}\right)4}\,dx$$

$$a = \frac{3}{2}$$

$$= \frac{1}{4}\int \frac{1}{\frac{9}{4}+x^2}\cdot dx = \frac{1}{\cancel{4}\,2}\cdot\frac{\cancel{2}}{3}\int \frac{\frac{3}{2}}{\frac{9}{4}+x^2}\cdot dx$$

$$= \frac{1}{6}\tan^{-1}\left(\frac{x}{3/2}\right) = \frac{1}{6}\tan^{-1}\left(\frac{2x}{3}\right)$$

Examples

Evaluate each of the following definite integrals:

$$\int_0^1 \frac{1}{\sqrt{4-x^2}}\,dx$$

$$= \left[\sin^{-1}\left(\frac{x}{2}\right)\right]_0^1$$

$$= \sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}(0)$$

$$= \frac{\pi}{6} - 0$$

$$= \underline{\underline{\frac{\pi}{6}}}$$

$$\int_0^2 \frac{1}{4+x^2}\,dx$$

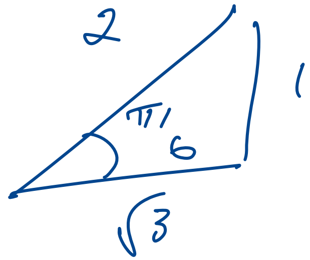
$$\int_0^1 \frac{3}{\sqrt{9-4x^2}}\,dx$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right)+c \quad \text{for } x \in (-a, a)$$

$$\int \frac{-1}{\sqrt{a^2-x^2}}\,dx = \cos^{-1}\left(\frac{x}{a}\right)+c \quad \text{for } x \in (-a, a)$$

$$\int \frac{a}{a^2+x^2}\,dx = \tan^{-1}\left(\frac{x}{a}\right)+c \quad \text{for } x \in \mathbb{R}$$

$a = 2$



Examples

Evaluate each of the following definite integrals:

$$\int_0^1 \frac{1}{\sqrt{4-x^2}}\,dx$$

$$\int_0^2 \frac{1}{4+x^2}\,dx$$

$$\int_0^1 \frac{3}{\sqrt{9-4x^2}}\,dx$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right)+c \quad \text{for } x \in (-a,a)$$

$$\int \frac{-1}{\sqrt{a^2-x^2}}\,dx = \cos^{-1}\left(\frac{x}{a}\right)+c \quad \text{for } x \in (-a,a)$$

$$\int \frac{a}{a^2+x^2}\,dx = \tan^{-1}\left(\frac{x}{a}\right)+c \quad \text{for } x \in \mathbb{R}$$

$$= \frac{1}{2}\int_0^2 \frac{\cancel{1}\,2}{4+x^2}\,dx$$

$$= \frac{1}{2}\left[\tan^{-1}\left(\frac{x}{2}\right)\right]_0^2$$

$$= \frac{1}{2}\left[\left(\tan^{-1}(1) - \tan^{-1}(0)\right)\right]$$

$$= \frac{1}{2}\left(\frac{\pi}{4} - 0\right) = \frac{\pi}{8}$$

Examples

Evaluate each of the following definite integrals:

$$\int_0^1 \frac{1}{\sqrt{4-x^2}}\,dx$$

$$\int_0^2 \frac{1}{4+x^2}\,dx$$

$$\int_0^1 \frac{3}{\sqrt{9-4x^2}}\,dx$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right)+c \quad \text{for } x \in (-a,a)$$

$$\int \frac{-1}{\sqrt{a^2-x^2}}\,dx = \cos^{-1}\left(\frac{x}{a}\right)+c \quad \text{for } x \in (-a,a)$$

$$\int \frac{a}{a^2+x^2}\,dx = \tan^{-1}\left(\frac{x}{a}\right)+c \quad \text{for } x \in \mathbb{R}$$

$$3\int_0^1 \frac{1}{\sqrt{9-4x^2}}\,dx$$

$$= 3\int_0^1 \frac{1}{\sqrt{\left(\frac{9}{4}-x^2\right)4}}$$

$$= \frac{3}{2}\int_0^1 \frac{1}{\sqrt{\frac{9}{4}-x^2}}\,dx$$

$$= \frac{3}{2}\left[\sin^{-1}\left(\frac{2x}{3}\right)\right]_0^1$$

$$= \frac{3}{2}\left(\sin^{-1}\left(\frac{2}{3}\right)-\sin^{-1}(0)\right)$$

$$= \underline{\underline{1.095}}$$

Learning Objectives: Revisited

By the end of the lesson I hope that you understand and can apply the following to a range of questions from the Unit 3 and 4 Specialist Mathematics course.

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