Integration or Antidifferentiation

Year 12 Specialist Mathematics
Units 3 and 4

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Learning Objectives

By the end of the lesson I hope that you understand and can apply the following to a range of questions from the Unit 3 and 4 Specialist Mathematics course.

- Understand what is meant by a general antiderivative
- Understand and recall the basic antiderivatives
- Understand what the definite integral is and what it finds
- Know how to draw graphs of functions and their antiderivatives

Recap of past learning

Much of the work we are going to look at in this video has been covered before in Methods 1 and 2 or Methods 3 and 4 (dependent on where you are in your learning journey).

We know that antidifferentiation is the inverse to differentiation. We can find a function from its antiderivative using this process.

General antiderivative

When we differentiate a constant, we know it goes to zero.

This means that the following functions all have the same differential:

$$y = x^2 + 3$$
$$y = x^2 - 5$$
$$y = x^2 + 10$$

$y' = 2x \qquad y = x^2$

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This makes sense as they are the same function but translated by differing amounts vertically.

If, when we differentiate we **multiply by the power and then subtract one from the power** it makes sense that to reverse the process we **add one to the power and divide by the new power**.

If we do this with $y' = 2x$ we get back to $y = x^2$ but there seems to be no way to recover the constant.

In fact, we cannot recover it without more information and so we need to put a placeholder 'c'

$y' = 2x^1 \qquad y = \dfrac{\cancel{2}x^2}{\cancel{2}} = x^2$

General antiderivative

It’s important to note that when we have a function and we antidifferentiate it, we must place a ‘c’ at the end unless the question states it wants **an antiderivative** in which case we can have the ‘c’ values as zero.

So, $\int 2x\, dx = \{x^2 + c : c \in \mathbb{R}\}$

Whilst we don’t normally use the above set notation it’s important that you note that there is not one unique antiderivative for a given function.

$$\int 2x\, dx = x^2 + c$$

The above is called the general antiderivative of $2x$.

$y' = 2x$

$\int 2x \,.\, dx$

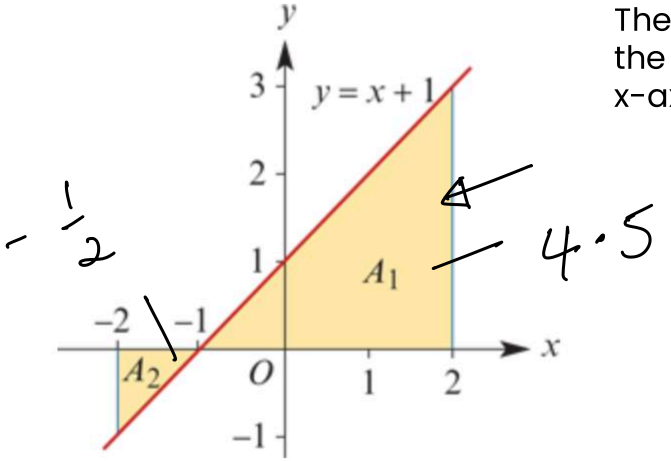
Basic antiderivatives

This table (from the Specialist Maths book) is pretty helpful and recaps the learning you should already have acquired.

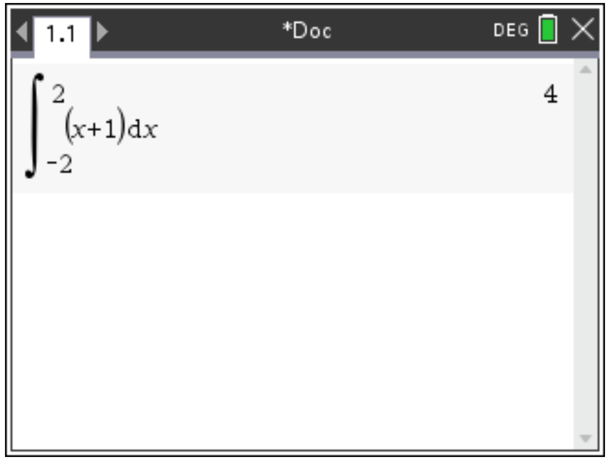
$f(x)$	$\int f(x)\,dx$	
x^n	$\dfrac{x^{n+1}}{n+1} + c$	where $n \neq -1$
$(ax+b)^n$	$\dfrac{1}{a(n+1)}(ax+b)^{n+1} + c$	where $n \neq -1$
x^{-1}	$\log_e x + c$	for $x > 0$
$\dfrac{1}{ax+b}$	$\dfrac{1}{a}\log_e(ax+b) + c$	for $ax + b > 0$
e^{ax+b}	$\dfrac{1}{a}e^{ax+b} + c$	
$\sin(ax+b)$	$-\dfrac{1}{a}\cos(ax+b) + c$	
$\cos(ax+b)$	$\dfrac{1}{a}\sin(ax+b) + c$	

The definite integral and signed area

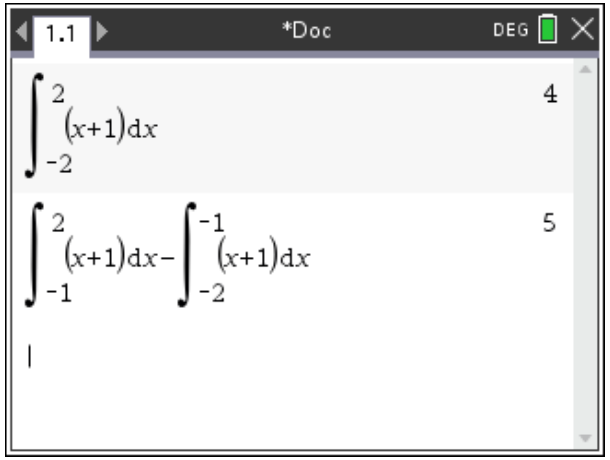
When we are given limits on an integral, we need to be careful as to whether we are being asked to find the **signed area** under the curve between the two limits given or the actual area. The limits are x values.



The signed area is the sum of the two areas. Areas below the x-axis are considered negative

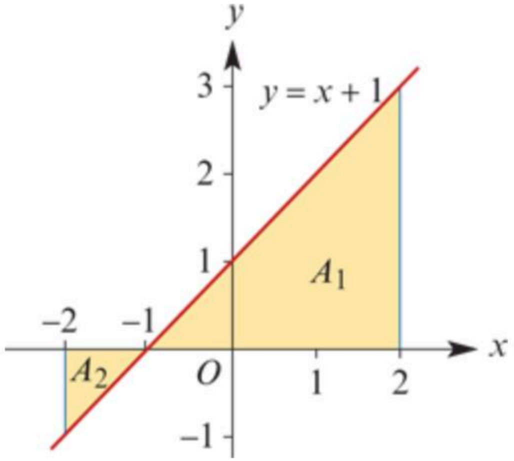


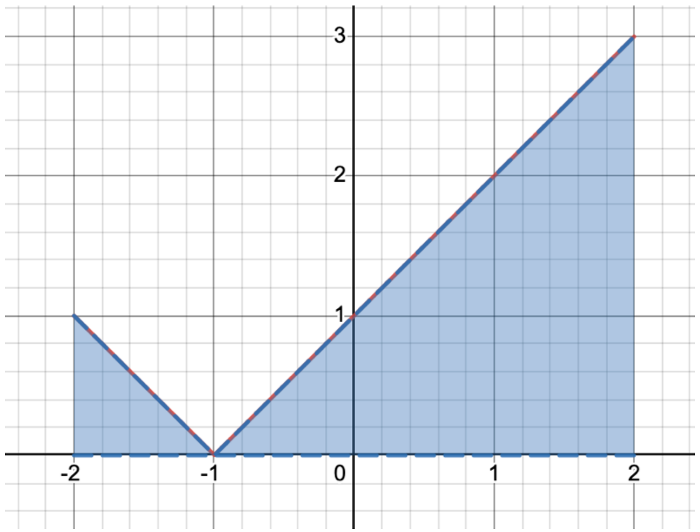
$$\int_{-1}^{2}(x+1).dx + \int_{-1}^{-2}(x+1).dx$$



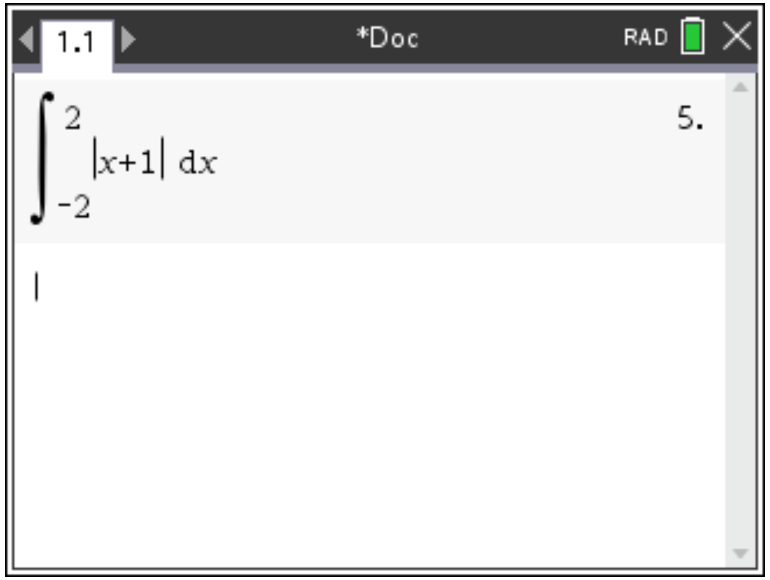
The definite integral and signed area

We might know that, when we take the modulus of a function, we are reflecting any part of the graph which falls below the x-axis to above the x-axis. This fact, can be used to find the total area under the graph in another way!





$|x+1|\{-2 \leq x \leq 2\}$



$|x+1|$

The definite integral

When we are given limits on an integral, we are finding the **signed area** under the curve between the two limits given. The limits are x values.

Hence,

$$\int_a^b f(x)dx = F(b) - F(a)$$

Where F is any antiderivative of f.

This is awesome! Think about the value of the constant

Examples

Find an antiderivative of each of the following:

- $\sin(3x - \frac{\pi}{4})$
- e^{3x+4}
- $6x^3 - \frac{2}{x^2}$

$$\therefore \int \sin\left(3x - \frac{\pi}{4}\right) . dx$$

$$= -\frac{1}{3} . \cos\left(3x - \frac{\pi}{4}\right)$$

$f(x)$	$\int f(x)\, dx$	
x^n	$\frac{x^{n+1}}{n+1} + c$	where $n \neq -1$
$(ax+b)^n$	$\frac{1}{a(n+1)}(ax+b)^{n+1} + c$	where $n \neq -1$
x^{-1}	$\log_e x + c$	for $x > 0$
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- $6x^3 - \frac{2}{x^2}$

$$\int e^{3x+4} \, . \, dx$$

$$= \frac{1}{3} . e^{3x+4}$$

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- e^{3x+4}
- $6x^3 - \frac{2}{x^2}$

$$\int (6x^3 - 2x^{-2})\,dx$$

$$= \frac{\overset{3}{\cancel{6}}x^4}{\underset{2}{\cancel{4}}} + \frac{2x^{-1}}{\cancel{4}\,1}$$

$$= \frac{3x^4}{2} + \frac{2}{x}$$

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Examples

Evaluate each of the following integrals:

$$\int_0^{\frac{\pi}{2}} \cos(3x)dx$$

$$\int_0^{1} (e^{2x} - e^{x})\, dx$$

$$\int_0^{\frac{\pi}{8}} \sec^2(2x)dx$$

$$\int_0^{1} \sqrt{2x+1}dx$$

$$\int_0^{\pi/2} \cos(3x).dx = \left[\frac{1}{3}\sin(3x)\right]_0^{\pi/2}$$

$$= \left(\frac{1}{3}\sin\overset{-1}{\cancel{\left(\frac{3\pi}{2}\right)}}\right) - \left(\frac{1}{3}.\sin(0)\right)$$

$$= -\frac{1}{3} - 0 = -\frac{1}{3}$$

$f(x)$	$\int f(x)\, dx$	
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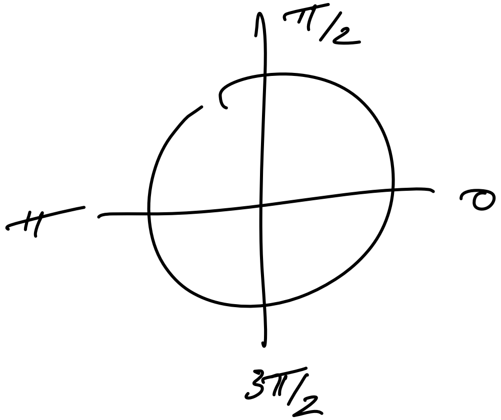

Examples

Evaluate each of the following integrals:

$$\int_0^{\frac{\pi}{2}} \cos(3x)dx$$

$$\int_0^1 (e^{2x} - e^x)\, dx$$

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$$\int_0^1 \left(e^{2x} - e^x\right) dx$$

$$= \left[\frac{1}{2}e^{2x} - e^x\right]_0^1$$

$$= \left(\frac{1}{2}e^2 - e^1\right) - \left(\frac{1}{2}e^0 - e^0\right)$$

$$= \frac{1}{2}e^2 - e - \frac{1}{2} + 1 = \frac{1}{2}e^2 - e + \frac{1}{2}$$

Examples

Evaluate each of the following integrals:

$$\int_0^{\frac{\pi}{2}} \cos(3x)dx$$

$$\int_0^{1} (e^{2x} - e^{x})\, dx$$

$$\int_0^{\frac{\pi}{8}} \sec^2(2x)dx$$

$$\int_0^{1} \sqrt{2x+1}dx$$

$$\int_0^{\pi/8} \sec^2(2x).dx$$

$$= \left[\frac{1}{2}\tan(2x)\right]_0^{\pi/8}$$

$$= \frac{1}{2}\tan\left(\frac{\pi}{4}\right) - \frac{1}{2}\tan 0$$

$$= \frac{1}{2} - 0$$

$$= \underline{\underline{\frac{1}{2}}}$$

$f(x)$	$\int f(x)\,dx$	
x^n	$\frac{x^{n+1}}{n+1} + c$	where $n \neq -1$
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$\cos(ax+b)$	$\frac{1}{a}\sin(ax+b) + c$	

$\sec^2(ax+b)$ $\quad \frac{1}{a}\tan(ax+b) + c$

Examples

Evaluate each of the following integrals:

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$$\int_0^{1} (e^{2x} - e^{x})\, dx$$

$$\int_0^{\frac{\pi}{8}} \sec^2(2x)dx$$

$$\int_0^{1} \sqrt{2x+1}dx$$

$$\int_0^1 (2x+1)^{1/2} \cdot dx$$

$$= \left[\frac{1}{\cancel{2} \cdot \frac{3}{\cancel{2}}} \cdot (2x+1)^{3/2} \right]_0^1$$

$$= \left(\frac{1}{3} \cdot 3^{3/2} \right) - \left(\frac{1}{3} \cdot 1^{3/2} \right)$$

$$= \frac{1}{3} \left(3\sqrt{3} - 1 \right)$$

$f(x)$	$\int f(x)\, dx$	
x^n	$\frac{x^{n+1}}{n+1} + c$	where $n \neq -1$
$(ax+b)^n$	$\frac{1}{a(n+1)}(ax+b)^{n+1} + c$	where $n \neq -1$
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$$3^{3/2} = \left(3^{1/2}\right)^3$$

$$= \left(3^3\right)^{1/2}$$

$$= 3 \cdot 3^{1/2}$$

Examples

Find an antiderivative of

$$\frac{1}{4x+2}$$

$$\int \frac{1}{4x+2} \cdot dx =$$

$$= \frac{1}{4} \cdot \log|4x+2|$$

$f(x)$	$\int f(x)\,dx$	
x^n	$\frac{x^{n+1}}{n+1} + c$	where $n \neq -1$
$(ax+b)^n$	$\frac{1}{a(n+1)}(ax+b)^{n+1} + c$	where $n \neq -1$
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Examples

Evaluate

$$\int_0^1 \frac{1}{4x+2}\,dx$$

$$= \left[\frac{1}{4}\log_e|4x+2|\right]_0^1$$

$$= \left[\frac{1}{4}\log_e(4x+2)\right]_0^1$$

$$= \frac{1}{4}\log_e 6 - \frac{1}{4}\cdot\log_e 2$$

$$= \frac{1}{4}\left(\log_e 6 - \log_e 2\right)$$

$$= \frac{1}{4}\log_e 3$$

$f(x)$	$\int f(x)\,dx$	
x^n	$\dfrac{x^{n+1}}{n+1} + c$	where $n \neq -1$
$(ax+b)^n$	$\dfrac{1}{a(n+1)}(ax+b)^{n+1} + c$	where $n \neq -1$
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Examples

Evaluate

$$\int_{-2}^{-1} \frac{1}{4x+2}\,dx$$

$$= \left[\frac{1}{4}\log_e|4x+2|\right]_{-2}^{-1}$$

$$= \left(\frac{1}{4}\log_e 2\right) - \left(\frac{1}{4}\log_e 6\right)$$

$$= \frac{1}{4}\log_e \frac{1}{3}$$

$$= -\frac{1}{4}\log_e 3$$

$f(x)$	$\int f(x)\,dx$	
x^n	$\frac{x^{n+1}}{n+1} + c$	where $n \neq -1$
$(ax+b)^n$	$\frac{1}{a(n+1)}(ax+b)^{n+1} + c$	where $n \neq -1$
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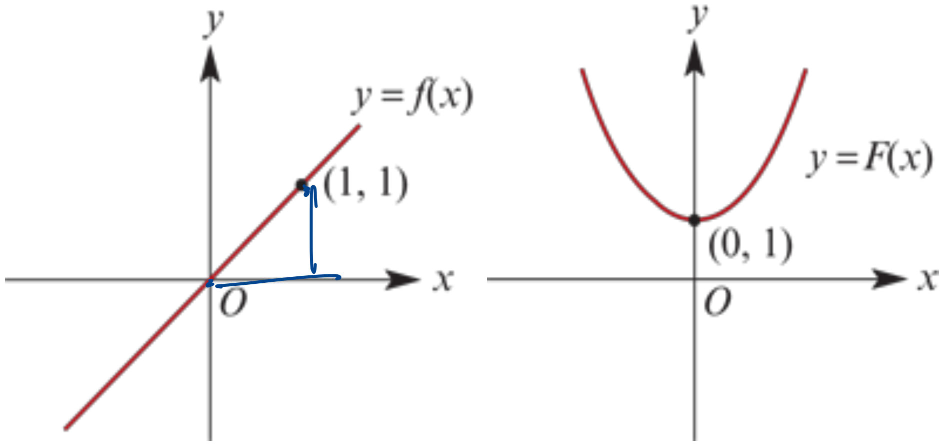
$$\frac{1}{3} = 3^{-1}$$

Graphs of functions and their antiderivatives

Using information from graphs we can find equations of the original function and the antiderivative. For example, given the two graphs we can find $f(x)$ and $F(x)$ assuming that F is an antiderivative of f



$y' = mx + c$

$= 1x + 0$

$y' = x$

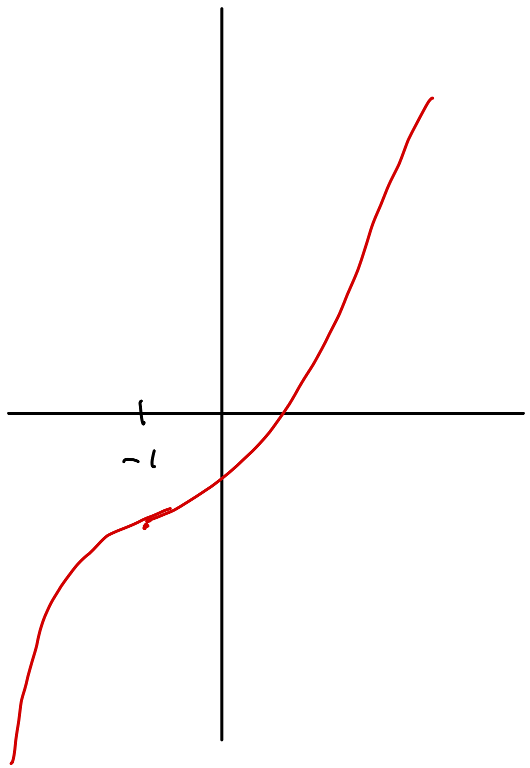
$y = \dfrac{x^2}{2} + c$

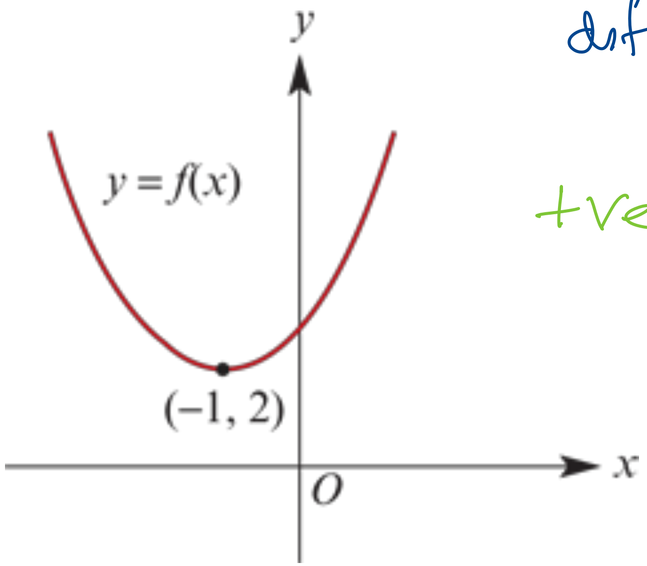
$\therefore y = \dfrac{x^2}{2} + 1$

Example

The graph of $y=f(x)$ is as shown.

Sketch the graph of $y=F(x)$, given that $F(0)=0$.





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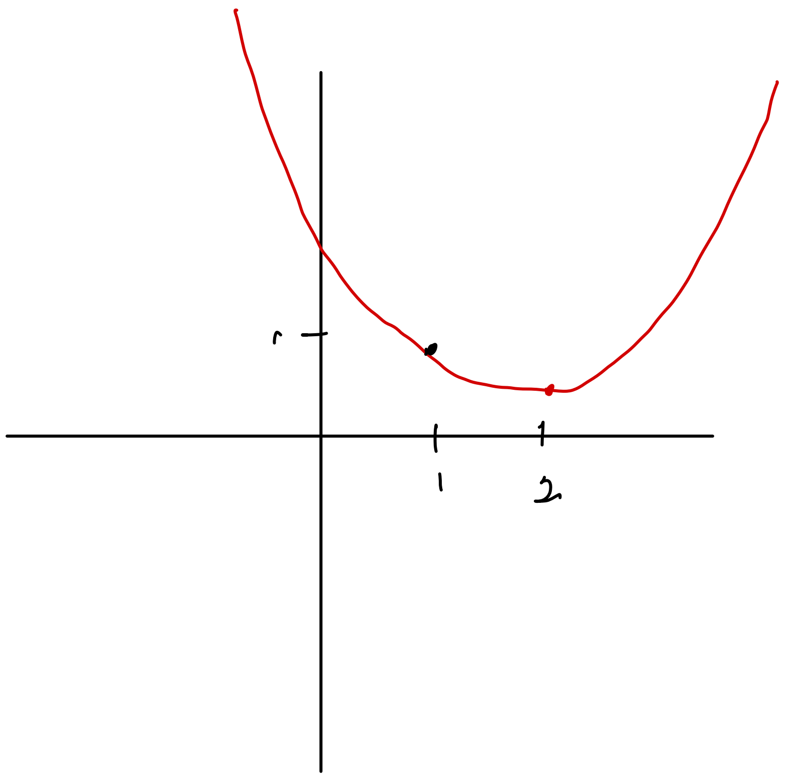
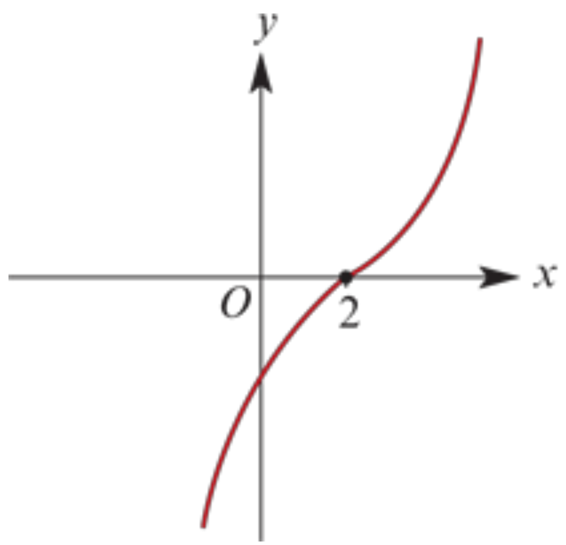
Example

The graph of $y=f(x)$ is as shown.

Sketch the graph of $y=F(x)$, given that $F(1)=1$.





Learning Objectives: Revisited

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